Uniqueness of the $\mathbf{Diff^+(S^1)}$ Symmetry for Local Nets of von Neumann Algebras

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In this short report we briefly explain a recent uniqueness result for the diffeomorphism symmetry in chiral 2-dimensional conformal field theory and some of its consequences. We refer the reader to [4] for more details and for the proofs.

Our framework is the algebraic approach to Quantum Field Theory (see [12]) where operator algebraic methods are extensively used. In this approach the chiral component of a 2-dimensional conformal field theory is usually described by means of a (local) Möbius covariant net on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (the "compactified light ray") namely a map $\mathcal{A}$

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which assigns to every open nonempty nondense interval (arc) $I \subset S^1$ a von Neumann algebra $\mathcal{A}(I)$ acting on a fixed complex, separable Hilbert-space $\mathcal{H}_\mathcal{A}$ ("the vacuum Hilbert-space of the theory"), together with a given strongly continuous representation U of Mob $\simeq$ PSL$(2,\mathbb{R})$, the group of Möbius transformations[1] of the unit circle S^1 satisfying, for all I_1, I_2, I, contained in the family $\mathcal{I}$ of open nonempty nondense intervals of S^1 and every $\varphi \in$ Mob, the following properties:

(i) *Isotony.*
$$I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2), \tag{1}$$

(ii) *Locality.*
$$I_1 \cap I_2 = \emptyset \Rightarrow [\mathcal{A}(I_1), \mathcal{A}(I_2)] = 0, \tag{2}$$

(iii) *Covariance.*
$$U(\varphi)\mathcal{A}(I)U(\varphi)^{-1} = \mathcal{A}(\varphi(I)), \tag{3}$$

(iv) *Positivity of the energy.* The representation U is of positive energy type: the conformal Hamiltonian L_0, defined by $U(R_\alpha) = e^{i\alpha L_0}$ where $R_\alpha \in$ Mob is the anticlockwise rotation by degree α, is positive.

(v) *Existence and uniqueness of the vacuum.* There exists a unique (up to phase) unit vector $\Omega \in \mathcal{H}_\mathcal{A}$ called the "vacuum-vector of the theory" which is globally invariant under the action of U. (Equivalently: up to phase there exists a unique unit vector Ω that is of *zero-energy* for U; i.e. eigenvector of L_0 with eigenvalue zero.)

(vi) *Cyclicity of the vacuum.* Ω is cyclic for the algebra $\mathcal{A}(S^1) = \bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

The above properties are sufficient to approach many general features of chiral quantum field theories in a model independent way such as PCT symmetry, spin and statistic relations, superselection structure and its relations with subfactor theory, the existence of point-like localized fields (see e.g. [7, 8, 10, 19]) and other important issues like modular invariance and the characterization of rationality, have been successfully studied starting from the basic properties of Möbius covariant nets on S^1 (see e.g. [1, 15, 21]).

Diffeomorphsim symmetry (which corresponds to the Virasoro algebra symmetry at the Lie algebra level, see [9, 18, 22]) plays a fundamental role

[1] diffeomorphisms of S^1 of the form $z \mapsto \frac{w_1 z + w_2}{\overline{w}_2 z + \overline{w}_1}$ where $w_1, w_2 \in \mathbb{C}$, $|w_1|^2 - |w_2|^2 = 1$

in various approaches to 2-dimensional Conformal Field Theory (see e.g. [6]) and has been also quite early considered in the operator algebraic framework (see e.g. [2, 8, 18]). It seems however that its full power when combined with operator algebras methods has become apparent only recently (see e.g. [3, 5, 13, 14, 17, 20, 23]).

Let us now explain how diffeomorphism symmetry enters in the structure of local nets of von Neumann algebras on the circle. A Möbius covariant net $\mathcal{A}$ on S^1 is said to be diffeomorphism covariant if the corresponding representation U of Mob extends to a continuous projective unitary representation of the group $\mathrm{Diff}^+(S^1)$ of orientation preserving (smooth) diffeomorphisms of the circle such that for all $\gamma \in \mathrm{Diff}^+(S^1)$ and $I, J \in \mathcal{I}$ the following hold

$$\gamma|_I = \mathrm{id}_I \Rightarrow \mathrm{Ad(V(\gamma))}|_{\mathcal{A}(\mathrm{I})} = \mathrm{id}_{\mathcal{A}(\mathrm{I})} \tag{4}$$

$$\gamma(I) = J \Rightarrow V(\gamma)\mathcal{A}(I)V(\gamma)^* = \mathcal{A}(J). \tag{5}$$

The central result in [4] is about the uniqueness of the representation V once the net $\mathcal{A}$ and the corresponding representation U of Mob are fixed. Note the representation U is completely determined by the net $\mathcal{A}$ (i.e. the map $\mathcal{I} \ni I \to \mathcal{A}(I) \subset B(\mathcal{H}_{\mathcal{A}})$) and the vacuum vector Ω *via* the modular structure (in the sense of Tomita and Takesaki) of the local von Neumann algebras [8]. Recall that a Möbius covariant net on S^1 is said to be 4-regular if

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \vee \mathcal{A}(I_3) \vee \mathcal{A}(I_4) = B(\mathcal{H}_{\mathcal{A}}) \tag{6}$$

whenever the intervals $I_1, I_2, I_3, I_4 \in \mathcal{I}$ are obtained by removing four points from S^1 we have the following.

Theorem 1. *([4]) Let $\mathcal{A}$ be a 4-regular Möbius covariant net on S^1 and let U be the corresponding unitary representation of* Mob. *Then there exists at most one continuous projective unitary representation V of* $\mathrm{Diff}^+(S^1)$ *which extends U and makes $\mathcal{A}$ diffeomorphism covariant.*

In the talk given by one of the authors at this conference only a somewhat weaker theorem was announced. The first version of the result was obtained with the assumption of strong additivity. The latter property means that

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I) \tag{7}$$

whenever the intervals I_1, I_2 are obtained by removing one point from a given $I \in \mathcal{I}$ and it is stronger then 4-regularity. The relevance of the new result comes from the fact that, in contrast with strong additivity, which does not always hold[2], 4-regularity seems to be a general property of diffeomorphism covariant nets. Infact the only examples for a net with no 4-regularity, up to the knowledge of the authors, are the ones given in [11] and [16], and they are not diffeomorphism covariant.

The new stronger version stated above has been obtained thanks to a rather detailed inspection of the analytic properties of the positive energy representations of $\mathrm{Diff}^+(S^1)$ which shows that these representation can be extended to certain nonsmooth diffeomorphisms, see [4]. The reason for considering nonsmooth transformations is that they permit us to work with local transformations that are picewise Möbius. (A "true" nontrivial Möbius transformation is non-local, while a nontrivial local picewise Möbius transformation is not smooth.)

An important consequence of Theorem 1 is that the the central charge c of the theory is completely determined by the pair $(\mathcal{A}, U)$ in the case where $\mathcal{A}$ is 4-regular and hence it is invariant under isomorphisms of 4-regular Möbius covariant nets. Another rather direct consequence is the following.

Corollary 2. *([4]) Let W be an (unbroken) internal symmetry of a 4-regular diffeomorphism covariant net on S^1 $\mathcal{A}$, namely W is a unitary operator on $\mathcal{H}_{\mathcal{A}}$ leaving the vacuum vector Ω invariant and such that $W\mathcal{A}(I)W^* = \mathcal{A}(I)$ for all $I \in \mathcal{I}$. Then the unique representation V of $\mathrm{Diff}^+(S^1)$ making the net diffeomorphism covariant must commute with W.*

Finally as a further application of our uniqueness result we can give many new examples of Möbius covariant nets on S^1 which are not diffeomorphism covariant (see [11] and [16] for previously known examples). The idea is to consider the infinite tensor product of diffeomorphism covariant nets which is defined as a Möbius covariant net in the following way.

Let $\mathcal{A}_n$, $n = 1, 2, ...$ be a sequence of Möbius covariant nets on S^1 and let Ω_n, U_n, $n = 1, 2, ...$, be the corresponding sequence of vacuum vectors and representations of Mob respectively. The infinite tensor product net

$$\mathcal{A} \equiv \bigotimes_n \mathcal{A}_n \tag{8}$$

[2]the Virasoro net with central charge $c > 1$ is not strongly additive, see [2] for the proof

on the infinite tensor product Hilbert space

$$\mathcal{H}_{\mathcal{A}} := \bigotimes_{n}^{(\Omega_n)} \mathcal{H}_{\mathcal{A}_n} \tag{9}$$

is defined by

$$\mathcal{A}(I) := \bigotimes_{n} \mathcal{A}_n(I). \tag{10}$$

It is a Möbius covariant net on S^1 (the corresponding representation of Mob is obtained by tensoring the representations U_n) which is 4-regular whenever every net $\mathcal{A}_n$, $n = 1, 2...$ is 4-regular.

The result is the following.

Theorem 3. *([4]) Let $\mathcal{A}_n$, $n \in \mathbb{N}$ be a sequence of 4-regular diffeomorphism covariant nets on S^1. Then the infinite tensor product net $\otimes_n \mathcal{A}_n$ is not diffeomorphism covariant.*

With the help of Theorem 3 one can give many examples of Möbius covariant nets which are not diffeomorphism covariant but, differently fom those which where previously known, are strongly additive.

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