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On shear and torsion factors in the theory of linearly elastic rods

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Abstract Lower bounds for the factors entering the standard notions of shear and torsion stiffness for a linearly elastic rod are established in a new and simple way. The proofs are based on the following criterion to identify the stiffness parameters entering rod theory: the rod's stored-energy density per unit length expressed in terms of force and moment resultants should equal the stored-energy density per unit length expressed in terms of stress components of a Saint-Venant cylinder subject to either flexure or torsion, according to the case. It is shown that the shear factor is always greater than one, whatever the cross section, a fact that is customarily stated without proof in textbooks of structure mechanics; and that the torsion factor is also greater than one, except when the cross section is a circle or a circular annulus, a fact that is usually proved making use of Saint-Venant's solution in terms of displacement components.

Keywords Rod theory · shear stiffness · torsion stiffness · shear factor · torsion factor

1 Introduction

When a direct approach is chosen to expound the standard theory of linearly elastic rods with a straight axis $\mathcal{L}$, the *Principle of Virtual Working* is laid down:

$$\int_{\mathcal{L}} (T\gamma + N\varepsilon + M\psi + M_t\psi_t)dx_3 =: \mathcal{W}^i = \mathcal{W}^e := \int_{\mathcal{L}} (pv + qw + c\varphi + c_t\vartheta)dx_3. \quad (1)$$

The *internal working* $\mathcal{W}^i$ collects four pairs (T,γ), ..., (M_t,ψ_t) of kinematic and dynamic fields in duality, with T, N, M, and M_t the *shear force, normal force, bending*

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moment and *twisting moment* and with $\gamma, \varepsilon, \psi$ and ψ_t the *shear, extension, flexion* and *torsion measures*. The geometric compatibility conditions are:

$$\gamma = v' + \varphi, \quad \varepsilon = w', \quad \psi = \varphi' \ \text{ and } \ \psi_t = \vartheta',$$

where v and w are the *transverse* and *axial displacements*, φ and ϑ are the rotations about, respectively, the x_2-axis and the x_3-axis (a superscript prime denotes differentiation with respect to x_3; see Fig. 1); the *external working* $\mathcal{W}^e$ sets these kinematic
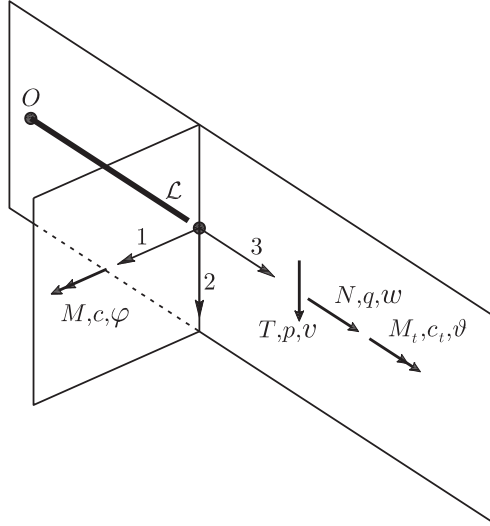

Fig. 1

variables in duality with the applied loads per unit length p, q, c and c_t.[1] The constitutive equations are:

$$T = \mathfrak{s}_s \gamma, \dots, M_t = \mathfrak{s}_t \psi_t,$$

all the *stiffness moduli* $\mathfrak{s}_s, \dots, \mathfrak{s}_t$ being assumed positive; the *stored-energy density* per unit length is:

$$\frac{1}{2}\frac{T^2}{\mathfrak{s}_s} + \dots + \frac{1}{2}\frac{M_t^2}{\mathfrak{s}_t}$$

(see, e.g., [1]).

To make use of such a theory — in fact, to make use of any structure theory formulated by way of a direct approach — the issue of *parameter identification* must be dealt with. In particular, one would like to gather from some suitable equilibrium theory for a rod-like three-dimensional body subject to flexure and torsion enough information to choose the 'right' shear and torsion stiffness parameters $\mathfrak{s}_s, \mathfrak{s}_t$, that is to say, to choose those parameters so as to guarantee that the qualitative and quantitative predictions of the ensuing rod theory allow for a quick, but technically sufficient, evaluation of the behavior of the corresponding three-dimensional body.

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[1] To keep things simple, in laying down (1) we have ignored the contributions to external and internal workings of concentrated loads, axial pin-junctions *et similia*.

Customarily, the three-dimensional theory one picks is the treatment of the cases of flexure and torsion of the *Saint–Venant Problem* in classic linearly isotropic elasticity. With this choice, one is led to set:

$$\mathfrak{s}_s = \frac{GA}{\chi_s}, \quad \mathfrak{s}_t = \frac{GJ_o}{\chi_t}, \tag{2}$$

where G is the shear modulus of the material Saint–Venant's prismatic cylinder is comprised of, A and J_o are the area and the polar moment of inertia of the cylinder's cross section, and the dimensionless coefficients χ_s and χ_t are the *shear* and *torsion factors*. Generally, these identifications are justified by inspection of the Saint–Venant solutions for flexure and torsion in terms of displacement components. Each of these solutions is constructed by means of Saint–Venant's *semi-inverse method*, starting from an educated guess about a representation for the displacement field depending on an as-short-as-possible list of parameter functions. Since the analytic forms of these functions depend solely on the shape of the cross section, the same is automatically true for the shear and torsion factors.

In textbooks of structural mechanics, the shear factor is interpreted as a measure of the non-uniformity in the distribution over the cross section of Saint–Venant's cylinder of the tangential stresses induced by the shear force; it is stated, but as a rule not proved, that

$$\chi_s > 1. \tag{3}$$

As to the torsion factor, it is generally stated, and proved, that

$$\chi_t \geq 1, \tag{4}$$

with equality holding if and only if the cross section is a circle or a circular annulus. While this facts suggests that $(\chi_t - 1)$ measures the deviation from central symmetry of the cross section, a precise interpretation is generally non attempted; at best, Saint-Venant's conjecture that a circular section minimizes the twist angle among all simply-connected sections with the same area is mentioned.

In this note we approach the issue of parameter identification as is done in [2]: we require that a linearly elastic rod stores the same energy per unit length as a Saint-Venant cylinder subject to end loads having the same force and moment resultant. The expression we use for the rod energy is in terms of either the shear resultant force or the twisting moment, that for Saint-Venant's cylinder is in terms of stress components. Both to prove (3) and to characterize the torsion optimality of center-symmetric cross sections, we make no use of Saint-Venant's solutions in terms of displacement; the only assumptions we borrow from Saint-Venant's problem are those restricting the admissible external forces and the class of stress fields.

2 The shear and torsion factors

Given a cartesian reference frame $\{O\,;x_1, x_2, x_3\}$ and the associated orthonormal basis $\{e_1, e_2, e_3\}$, we consider a Saint–Venant's cylinder whose axis is parallel to e_3 and whose cross section has its barycenter O on the x_3-axis. The vector $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2$ yields the position of a point of a typical cross section $\mathcal{A}$ with respect to O. On denoting by S_{ij} $(i, j = 1, 2, 3)$ the cartesian components of the stress tensor, with Saint–Venant

we restrict attention to cases when both external distance forces and external contact forces over the cylinder's mantle are null and, moreover,

$$S_{\alpha\beta} \equiv 0 \quad (\alpha, \beta = 1, 2). \tag{5}$$

Consequently, in order to satisfy the equilibrium equations, not only the fields $S_{3\alpha}$ $(\alpha = 1, 2)$ must be independent of x_3 but also

$$\boldsymbol{s} \cdot \boldsymbol{n} = S_{3\alpha} n_\alpha = 0\,, \quad \boldsymbol{x} \in \partial\mathcal{A}\,, \tag{6}$$

where $\boldsymbol{s} = S_{3\alpha}\boldsymbol{e}_\alpha$ is the cross-sectional traction, $\boldsymbol{n} = n_\alpha\boldsymbol{e}_\alpha$ is the normal to the mantle, and $\partial\mathcal{A}$ is the boundary curve of $\mathcal{A}$.

When expressed in terms of stress components, the stored-energy density for unit volume of an isotropic material is:

$$w(\boldsymbol{S}) = \frac{1}{4\mu}\left(|\boldsymbol{S}|^2 - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{S})^2\right), \tag{7}$$

where λ and $\mu \equiv G$ are the Lamé coefficients, $|\boldsymbol{S}| = (S_{ij}S_{ij})^{1/2}$, and $\operatorname{tr}\boldsymbol{S} = S_{11} + S_{22} + S_{33}$; under Saint-Venant's assumption (5), (7) becomes:

$$w_{SV} = \frac{1}{2G}(S_{31}^2 + S_{32}^2) + \frac{1}{2E}S_{33}^2, \quad E := \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},$$

where E is the Young modulus. Thus, in Saint-Venant's cylinder, the stored-energy density splits additively into two terms: the first one is non-null if and only if the applied loads induce cross-sectional stress, while the second one is non-null if and only if the applied loads induce axial stress. On the other hand, the stored-energy density per unit length of a linearly elastic rod is:

$$w_s = \frac{1}{2}\frac{T^2}{\mathfrak{s}_s}$$

when the rod is subjected to a shear force of magnitude T, and is:

$$w_t = \frac{1}{2}\frac{M_t^2}{\mathfrak{s}_t}$$

when the rod is subjected to a torsion moment of magnitude M_t.

Following [2], we identify the shear-stiffness parameter by imposing that

$$\frac{1}{2}\frac{T^2}{\mathfrak{s}_s} = \left(w_s = \int_\mathcal{A} w_{SV}\,\mathrm{d}A\right) = \frac{1}{2G}\int_\mathcal{A}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A\,,$$

with the integrand on the right side proportional to T^2; hence,

$$\boxed{\mathfrak{s}_s := G\frac{T^2}{\int_\mathcal{A}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A}}\,. \tag{8}$$

Next, we compare this relation with the first of (2): since the factor multiplying the shear modulus G has the dimension of an area, we find it natural to give $\mathfrak{s}_s$ the form (3) by defining

$$\chi_s := \frac{A\int_\mathcal{A}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A}{T^2}\,, \quad A = \int_\mathcal{A}\mathrm{d}A\,. \tag{9}$$

Quite similarly, we identify the torsion-stiffness parameter by imposing that

$$\frac{1}{2}\frac{M_t^2}{\mathfrak{s}_t} = \frac{1}{2G}\int_{\mathcal{A}}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A\,,$$

with the integrand on the right side proportional to M_t^2; hence,

$$\boxed{\mathfrak{s}_t := G\frac{M_t^2}{\int_{\mathcal{A}}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A}}\,. \tag{10}$$

This time, the factor multiplying G has the dimension of a moment of inertia; to recover the second of (2), we set:

$$\chi_t := \frac{J_o \int_{\mathcal{A}}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A}{M_t^2}\,, \quad J_o = \int_{\mathcal{A}}\|\boldsymbol{x}\|^2\,\mathrm{d}A\,. \tag{11}$$

3 Lower bounds

In this section we prove the lower bounds (3) and (4) for, respectively, χ_s and χ_t.

3.1 $\chi_s > 1$.

Consider the following chain of inequalities:

$$A\int_{\mathcal{A}}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A \geq A\int_{\mathcal{A}}S_{32}^2\,\mathrm{d}A \geq \left(\int_{\mathcal{A}}S_{32}\,\mathrm{d}A\right)^2. \tag{12}$$

Note that the first inequality in (12) holds true with the equality sign if and only if S_{31} is identically null on $\mathcal{A}$; and that the second, which is established by making use of Jensen's inequality for convex functions, reduces to an equality if and only if S_{32} is identically constant on $\mathcal{A}$.[2] Next, without any loss of generality, take the shear force parallel to $\boldsymbol{e}_2$, so that

$$T = \int_{\mathcal{A}}S_{32}\,\mathrm{d}A,$$

(12) yields:

$$A\int_{\mathcal{A}}(S_{31}^2 + S_{32}^2)\,\mathrm{d}A \geq T^2,$$

and (9) reduces to

$$\chi_s \geq 1,$$

with

$$\chi_s = 1 \quad \Leftrightarrow \quad S_{31} \equiv 0 \text{ and } S_{32} \equiv \text{const}\,.$$

This set of conditions on the stress field is incompatible with the boundary condition (6) no matter the shape of the cross section, because the stress field must be continuous up to the boundary of the cross section itself. We conclude that the strict inequality (3) must hold.

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[2] A use of this second inequality in similar circumstances is found on p. 475 of [3].

3.2 $\chi_t \geq 1$.

By definition,

$$M_t = \int_{\mathcal{A}} \left(x_1 S_{32} - x_2 S_{31} \right) \mathrm{d}A.$$

With an application of Fubini's theorem, the denominator in (11) can be written as

$$M_t^2 = \int_{\mathcal{A} \times \mathcal{A}} \left(x_1 S_{32}(\boldsymbol{x}) - x_2 S_{31}(\boldsymbol{x}) \right) \left(y_1 S_{32}(\boldsymbol{y}) - y_2 S_{31}(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A,$$

or rather, equivalently, given that

$$\int_{\mathcal{A} \times \mathcal{A}} x_1 y_2 \, S_{32}(\boldsymbol{x}) S_{31}(\boldsymbol{y}) \, \mathrm{d}A \, \mathrm{d}A = \int_{\mathcal{A} \times \mathcal{A}} y_1 x_2 \, S_{32}(\boldsymbol{y}) S_{31}(\boldsymbol{x}) \, \mathrm{d}A \, \mathrm{d}A \, ,$$

as

$$M_t^2 = \int_{\mathcal{A} \times \mathcal{A}} \left(x_1 y_1 \, S_{32}(\boldsymbol{x}) S_{32}(\boldsymbol{y}) + x_2 y_2 \, S_{31}(\boldsymbol{x}) S_{31}(\boldsymbol{y}) - 2 x_1 y_2 \, S_{32}(\boldsymbol{x}) S_{31}(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A \, . \tag{13}$$

An iterated use of the inequality:

$$\pm 2ab \leq a^2 + b^2 \tag{14}$$

yields:

$$\begin{cases} \displaystyle \int_{\mathcal{A} \times \mathcal{A}} x_1 y_1 \, S_{32}(\boldsymbol{x}) S_{32}(\boldsymbol{y}) \, \mathrm{d}A \, \mathrm{d}A \leq \frac{1}{2} \int_{\mathcal{A} \times \mathcal{A}} \left(x_1^2 S_{32}^2(\boldsymbol{y}) + y_1^2 S_{32}^2(\boldsymbol{x}) \right) \mathrm{d}A \, \mathrm{d}A = \\ \displaystyle \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_{\mathcal{A} \times \mathcal{A}} \left(x_1^2 S_{32}^2(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A \, , \\ \displaystyle \int_{\mathcal{A} \times \mathcal{A}} x_2 y_2 \, S_{31}(\boldsymbol{x}) S_{31}(\boldsymbol{y}) \, \mathrm{d}A \, \mathrm{d}A \leq \frac{1}{2} \int_{\mathcal{A} \times \mathcal{A}} \left(x_2^2 S_{31}^2(\boldsymbol{y}) + y_2^2 S_{31}^2(\boldsymbol{x}) \right) \mathrm{d}A \, \mathrm{d}A = \\ \displaystyle \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_{\mathcal{A} \times \mathcal{A}} \left(x_2^2 S_{31}^2(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A \, , \\ \displaystyle \int_{\mathcal{A} \times \mathcal{A}} -2 x_1 S_{32}(\boldsymbol{x}) \, y_2 S_{31}(\boldsymbol{y}) \, \mathrm{d}A \, \mathrm{d}A \leq \int_{\mathcal{A} \times \mathcal{A}} \left(x_1^2 S_{31}^2(\boldsymbol{y}) + y_2^2 S_{31}^2(\boldsymbol{x}) \right) \mathrm{d}A \, \mathrm{d}A = \\ \displaystyle \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \int_{\mathcal{A} \times \mathcal{A}} \left(x_1^2 S_{31}^2(\boldsymbol{y}) + x_2^2 S_{31}^2(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A \, . \end{cases}$$

With this and another use of Fubini's Theorem, it follows from (13) that

$$M_t^2 \leq \int_{\mathcal{A} \times \mathcal{A}} \left(x_1^2 S_{32}^2(\boldsymbol{y}) + x_2^2 S_{31}^2(\boldsymbol{y}) + x_1^2 S_{31}^2(\boldsymbol{y}) + x_2^2 S_{32}^2(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A =$$

$$= \int_{\mathcal{A} \times \mathcal{A}} \|\boldsymbol{x}\|^2 \left(S_{31}^2(\boldsymbol{y}) + S_{32}^2(\boldsymbol{y}) \right) \mathrm{d}A \, \mathrm{d}A = \left(\int_{\mathcal{A}} \|\boldsymbol{x}\|^2 \, \mathrm{d}A \right) \int_{\mathcal{A}} \left(S_{31}^2(\boldsymbol{x}) + S_{32}^2(\boldsymbol{x}) \right) \mathrm{d}A =$$

$$= J_0 \int_{\mathcal{A}} \left(S_{31}^2 + S_{32}^2 \right) \mathrm{d}A \, ,$$

which is tantamount to having from (11) that $\chi_t \geq 1$ whatever the shape of the cross section.[3]

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[3] A reviewer of our paper remarked that the inequality $M_t^2 \leq J_0 \int_{\mathcal{A}} (S_{31}^2 + S_{32}^2) \, \mathrm{d}A$ can be proved by a straightforward application of the Cauchy-Bunyakovsky-Schwarz inequality.

Now, given that equality holds in (14) if and only if $a = \mp b$, $\chi_t = 1$ if and only if all of the following conditions are identically satisfied in $\mathcal{A} \times \mathcal{A}$:

$$x_1 S_{32}(\boldsymbol{y}) = y_1 S_{32}(\boldsymbol{x}), \quad x_2 S_{31}(\boldsymbol{y}) = y_2 S_{31}(\boldsymbol{x}), \quad x_1 S_{31}(\boldsymbol{y}) = -y_2 S_{32}(\boldsymbol{x});$$

the first two are implied by the last, which can be written as:

$$\frac{S_{32}(\boldsymbol{x})}{x_1} = -\frac{S_{31}(\boldsymbol{y})}{y_2} = c,$$

or rather, equivalently,

$$\boldsymbol{s} = c\,(-x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2) = c\,\boldsymbol{e}_3 \times \boldsymbol{x}, \tag{15}$$

with c a constant. Hence, the cross-sectional traction $\boldsymbol{s}$ must be orthogonal to the position vector $\boldsymbol{x}$ all over $\mathcal{A}$ up to the boundary, where on the other hand it has to be orthogonal to the normal $\boldsymbol{n}$ to satisfy (6). Because of the assumed continuity of the stress field up to the boundary, $\boldsymbol{x}$ must then be parallel to $\boldsymbol{n}$ all over $\partial\mathcal{A}$, which is possible if and only if $\mathcal{A}$ is a circle or a circular annulus. Thus, as anticipated in the introduction, $(\chi_t - 1)$ quantifies the reduction in torsional stiffness due to the defect in polar symmetry of the cross section and the accompanying deviation from the form (15) of the cross-sectional traction.

4 Final remarks

Remark 1 One can define the *extension* and *bending stiffnesses* $\mathfrak{s}_e$ and $\mathfrak{s}_b$ and the relative factors χ_e and χ_b of a rod by posing, respectively,

$$\boxed{\frac{1}{2}\frac{N^2}{\mathfrak{s}_e} := \frac{1}{2E}\int_{\mathcal{A}} S_{33}^2\,\mathrm{d}A}, \quad \chi_e := \frac{A\int_{\mathcal{A}} S_{33}^2\,\mathrm{d}A}{N^2} \tag{16}$$

for

$$N = \int_{\mathcal{A}} S_{33}\,\mathrm{d}A,$$

and

$$\boxed{\frac{1}{2}\frac{M^2}{\mathfrak{s}_b} := \frac{1}{2E}\int_{\mathcal{A}} S_{33}^2\,\mathrm{d}A}, \quad \chi_b := \frac{J\int_{\mathcal{A}} S_{33}^2\,\mathrm{d}A}{M^2} \tag{17}$$

for

$$M = \int_{\mathcal{A}} x_2 S_{33}\,\mathrm{d}A, \quad J = \int_{\mathcal{A}} x_2^2\,\mathrm{d}A.$$

To prove that

$$\chi_e \geq 1 \quad \text{and} \quad \chi_b \geq 1 \tag{18}$$

is left as an exercise for the interested reader. We note that equality is realized in relations (18) whatever the shape of the cross section $\mathcal{A}$ if and only if, respectively, the field S_{33} is constant-valued or linear in x_2 over $\mathcal{A}$, just as it happens to be in Saint-Venant's cases of normal force and pure bending.

Remark 2 The recipe for parameter identification embodied in our definitions (8), (10), $(16)_1$, and $(17)_1$ for rod stiffnesses is applicable as such to any one-dimensional counterpart of a three-dimensional rod-like material body, no matter its axis were straight, its cross section constant or its mechanical response spatially uniform, as is the case for Saint-Venant's prismatic cylinder; while linearity in the elastic response is crucial, isotropy is not: all those definitions make sense also for *transverse isotropy* with respect to the axial direction.

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