Classification of Subsystems for the Haag-Kastler Nets Generated by $c = 1$ Chiral Current Algebras

Sebastiano Carpi
Dipartimento di Matematica,
Università di Roma "La Sapienza"
P.le A. Moro 2, 00185 Roma, Italy.
e-mail: carpi@mat.uniroma1.it

Abstract

Let $\mathcal{F}$ be the Haag-Kastler net generated by the $\hat{su}(2)$ chiral current algebra at level 1. We classify the SL$(2,\mathbb{R})$-covariant subsystems $\mathcal{B} \subset \mathcal{F}$ by showing that they are all fixed points nets $\mathcal{F}^H$ for some subgroup H of the gauge automorphisms group SO(3) of $\mathcal{F}$. Then using the fact that the net $\mathcal{F}_1$ generated by the $\hat{u}(1)$ chiral current can be regarded as a subsystem of $\mathcal{F}$ we classify the subsystems of $\mathcal{F}_1$. In this case there are two distinct proper subsystems: the one generated by the energy-momentum tensor and the gauge invariant subsystem ${\mathcal{F}_1}^{\mathbb{Z}_2}$.

[0]Original article published in *Letters in Mathematical Physics* **47** (1999), no. 4, 353-364.

1 Introduction

In the algebraic approach to quantum field theory [18] the natural objects under investigation are local nets $\mathcal{R}$ of von Neumann algebras (Haag-Kastler nets). These are inclusion preserving maps $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ from (suitably chosen) bounded sets of space-time to von Neumann algebras satisfying certain structural assumptions. The algebra $\mathcal{R}(\mathcal{O})$ associated to a region $\mathcal{O}$ can be interpreted either as the algebra of observables that can be measured within this region (in this case the net is assumed to satisfy locality), or as the algebras of local charged fields that create charged states from the vacuum vector.

In this framework it is natural to consider subsystems $\mathcal{B}$ of a given Haag-Kastler net $\mathcal{R}$ (cf. [1, 7, 31]). These are Haag-Kastler nets such that for every region $\mathcal{O}$, $\mathcal{B}(\mathcal{O})$ is a subalgebra (with the same identity) of $\mathcal{R}(\mathcal{O})$ subjected to some restriction such as Poincaré or conformal covariance (we simply write $\mathcal{B} \subset \mathcal{R}$). The study of a net $\mathcal{R}$ together with a subsystem $\mathcal{B}$ is a typical situation in algebraic quantum field theory [12, 13, 25].

The study of the subsystems of concrete models goes back to the work of Langerholc and Schroer in the middle sixties [23]. These authors considered the Haag-Kastler nets generated by the Wick polinomials in the massive scalar free field theory in four space-time dimensions and showed that there are only two possible results: the net generated by the basic field $\phi(x)$ or the one generated by $:\phi^2:(x)$. In a subsequent paper [24] the same authors proved that the net generated by $:\phi^2:(x)$ coincides with the net of fixed points under the gauge transformation $\phi(x) \to -\phi(x)$. As far as we know, the natural conjecture that this is the only proper (Poincaré-covariant) subsystem of this model has been confirmed only recently by Davidson [9].

In this letter we will be concerned with the classification of the $\mathrm{SL}(2,\mathbb{R})$-

covariant subsystems of the nets constructed from two models of chiral conformal field theories; the $\hat{u}(1)$ and the $\hat{su}(2)_1$ chiral current algebras [10]. In particular the $\hat{u}(1)$ model is the chiral version of the scalar free field but the results here obtained are not covered by those cited above which crucially depends on the existence of a mass gap. Actually, the situation is here drastically different since there are two distinct proper $\mathrm{SL}(2,\mathbb{R})$-covariant subsystems that, as we will prove, exhaust all the possibilities. Moreover, if one relax $\mathrm{SL}(2,\mathbb{R})$-covariance to Poincaré-covariance it is known that there are infinite many subsystems [17].

Other results in this direction have been given by Rehren in [26] who proved that for the net generated by the $\hat{su}(2)_1$ chiral current algebra the subsystem of the gauge invariant local operators coincide with the one generated by the energy-momentum tensor. Finally the present author in [6] proved that the net generated by the (chiral) energy-momentum tensor has no proper $\mathrm{SL}(2,\mathbb{R})$-covariant subsystems.

As in [5, 6] we use scaling limit techniques based on the results of Fredenhagen and Jörß [14] (see also [21]) This is reminiscent of the idea in [23]. Two other important ingredients are used: the result of Rehren cited above and the work of Izumi, Longo and Popa on Galois correspondence for compact groups of automorphisms of von Neumann factors [20].

We conclude this introduction with some comments to make contact with the conventional approach to chiral conformal field theory based on quantum fields. From this point of view our result is not surprising. In fact we find that the subsystems of the net corresponding to the $\hat{su}(2)_1$ chiral current algebra (the classification in the $\hat{u}(1)$ case follows from this result) are all obtained by taking the local algebras generated by local operators invariant under a closed subgroup H of the model's gauge symmetries group $\mathrm{SO}(3)$. Thus, these subsystems correspond to well known submodels such as the $\hat{u}(1)$ model and orbifold models (see eg. [10]). What is new here, to the best of our knowledge, is a rigorous proof that these are the only possible submodels in the algebraic approach to quantum field theory. Similar results have been recently obtained in the purely algebraic framework of vertex operator algebras in [11].

This letter is organized as follows. In section 2 we consider some basic properties of chiral nets. In section 3 we discuss our classification results for the $\hat{su}(2)_1$ model. The results for the $\hat{u}(1)$ model are discussed in section 4. Finally in section 5 we make some comments on possible generalizations to other models.

2 Chiral Nets on $\mathbb{R}$

Let $\mathcal{K}_0$ be the set of nonempty, open, bounded intervals of the real line $\mathbb{R}$.

A chiral net on $\mathbb{R}$ is a family $\mathcal{A} = \{\mathcal{A}(I) | I \in \mathcal{K}_0\}$ of von Neumann algebras, acting on a separable Hilbert space $\mathcal{H}$ with the following properties (see [14]):

(i) Isotony.

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \text{ for } I_1 \subset I_2, \quad I_1,\ I_2 \in \mathcal{K}_0. \tag{1}$$

(ii) Locality.

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \text{ for } I_1 \cap I_2 = \emptyset,\ I_1,\ I_2 \in \mathcal{K}_0. \tag{2}$$

(iii) Conformal covariance. There exists a strongly continuous unitary representation U of $\mathrm{SL}(2,\mathbb{R})$ in $\mathcal{H}$ such that

$$U(-1) = 1 \tag{3}$$

and

$$U(\alpha)\mathcal{A}(I)U(\alpha)^{-1} = \mathcal{A}(\alpha I) \text{ for } I,\ \alpha I \in \mathcal{K}_0, \tag{4}$$

with the usual action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}\cup\{\infty\}$.

(iv) Positivity of the energy. The conformal Hamiltonian L_0, which generates the restriction of U to the one-parameter group of rotations has non-negative spectrum.

(v) Existence of the vacuum. There is a unique (up to a phase) U-invariant unit vector $\Omega\in\mathcal{H}$ (the vacuum vector).

(vi) Cyclicity of the vacuum. $\mathcal{H}$ is the smallest closed subspace containing Ω which is invariant for U and $\mathcal{A}(I)$ for every $I\in\mathcal{K}_0$.

We will use the following notation: $T(a)$ for the group of translations; $D(\lambda)$ for the group of dilations.

A chiral net $\mathcal{A}$ can always be extended to the family $\mathcal{K}$ of open, nonempty, nondense, intervals of the circle $S^1 = \mathbb{R}\cup\{\infty\}$ [14]. Then the family $\{\mathcal{A}(I)|I\in\mathcal{K}\}$ satisfies the axioms of a conformal precosheaf (net) on S^1, see eg. [15, 16].

From the previous assumptions several properties of the net $\mathcal{A}$ can be proved (see [15, 16, 14] and references there). Among others, we mention:

(vii) Reeh-Schlieder property. For every $I\in\mathcal{K}$, Ω is cyclic and separating for $\mathcal{A}(I)$.

(viii) Bisognano-Wichmann property. If Δ_I is the modular operator associated to $\mathcal{A}(I)$ and Ω [29] then

$$\Delta^{it}_{(0,\infty)} = D(e^{-(\pi/2)t}). \tag{5}$$

(ix) Haag duality (on S^1). For every $I\in\mathcal{K}$

$$\mathcal{A}(I)' = \mathcal{A}(I^c), \tag{6}$$

where I^c is the interior of $S^1\backslash I$.

(x) Factoriality. The algebras $\mathcal{A}(I)$ are type III_1 factors.

A conformal subsystem of a chiral net $\mathcal{A}$ on $\mathbb{R}$ is a family $\mathcal{B} = \{\mathcal{B}(I) | \; I \in \mathcal{K}_0\}$ of nontrivial von Neumann algebras acting on $\mathcal{H}$ such that:

$$\mathcal{B}(I) \subset \mathcal{A}(I) \;\; \text{for } I \in \mathcal{K}_0; \tag{7}$$

$$U(\alpha)\mathcal{B}(I)U(\alpha)^{-1} = \mathcal{B}(\alpha I) \;\; \text{for } I, \; \alpha I \in \mathcal{K}_0; \tag{8}$$

$$\mathcal{B}(I) \subset \mathcal{B}(J) \;\; \text{for } I \subset J, \;\; I, \; J \in \mathcal{K}_0. \tag{9}$$

Given a conformal subsystem $\mathcal{B}$ we consider the smallest closed subspace $\mathcal{H}_{\mathcal{B}}$ of $\mathcal{H}$ which contains Ω and is invariant for U and all the algebras $\mathcal{B}(I)$, $I \in \mathcal{K}_0$. If we denote by $\pi_0(\mathcal{B})(I)$ the restriction of $\mathcal{B}(I)$ to $\mathcal{H}_{\mathcal{B}}$, then it is quite easy to verify that the family $\pi_0(\mathcal{B}) = \{\pi_0(\mathcal{B})(I) | I \in \mathcal{K}_0\}$ is a chiral net on $\mathbb{R}$ with respect to the restriction of U to $\mathcal{H}_{\mathcal{B}}$.

If $\mathcal{B}$ is a conformal subsystem of $\mathcal{A}$ we write $\mathcal{B} \subset \mathcal{A}$. Moreover if $\mathcal{B}_1$, $\mathcal{B}_2$ are conformal subsystems of $\mathcal{A}$ such that, for every $I \in \mathcal{K}_0$,

$$\mathcal{B}_1(I) \subset \mathcal{B}_2(I) \tag{10}$$

then, in a obvious way, $\mathcal{B}_1$ can be considered as a conformal subsystem of $\pi_0(\mathcal{B}_2)$. In this case when no confusion arises, we write $\mathcal{B}_1 \subset \mathcal{B}_2$.

Let $\mathcal{A}$ a chiral net on $\mathbb{R}$ and let V be a strongly continuous unitary representation of a compact group G (gauge group) leaving the vacuum invariant, and such that

$$V(g)\mathcal{A}(I)V(g)^{-1} = \mathcal{A}(I) \;\; \text{for } I \in \mathcal{K}_0, \; g \in G. \tag{11}$$

As usual we denote by $\mathcal{A}(I)^G$ the algebra of the G-invariant elements of $\mathcal{A}(I)$. Since V must commute with U (see [15]) it is quite easy to verify that the net $\mathcal{A}^G$ of fixed points, defined by

$$\mathcal{A}^G(I) = \mathcal{A}(I)^G \;\; \text{for } I \in \mathcal{K}_0, \tag{12}$$

is a conformal subsystem of $\mathcal{A}$ (the nontriviality follows from the fact that a compact group cannot act ergodically on a type III factor [19]). We conclude this section with the following proposition (cf. [2, 8] for related results).

Proposition 2.1. *Let $\mathcal{A}$ be chiral net on $\mathbb{R}$ and V a representation of a compact group as described above. Then for every $I \in \mathcal{K}$*

$$\mathcal{A}^G(I)' \cap \mathcal{A}(I) = \mathbb{C}1. \tag{13}$$

Proof. By conformal covariance we can restrict ourselves to the case $I = (0, \infty)$. Let

$$\mathcal{M} = \mathcal{A}(0,\infty),\ \mathcal{N} = \mathcal{A}^G(0,\infty),\ \mathcal{R} = \mathcal{A}^G(0,\infty)' \cap \mathcal{A}(0,\infty).$$

By the Bisognano-Wichmann property the modular group of $\mathcal{M}$ (with respect to Ω) is given by

$$\sigma^t_{\mathcal{M}} = \mathrm{Ad}D(e^{-(\pi/2)t}). \tag{14}$$

Since the only dilation invariant vectors are proportional to Ω (see eg. [16]) the Reeh-Schlieder property implies that $\sigma^t_{\mathcal{M}}$ acts ergodically on $\mathcal{M}$. Moreover, from the fact that U and V commute we find

$$D(e^{-(\pi/2)t})\mathcal{N}D(e^{-(\pi/2)t})^{-1} = \mathcal{N} \tag{15}$$

and thus

$$D(e^{-(\pi/2)t})\mathcal{N}'D(e^{-(\pi/2)t})^{-1} = \mathcal{N}' \tag{16}$$

so that $\sigma^t_{\mathcal{M}}(\mathcal{R}) = \mathcal{R}$. By a well known result of Takesaki [29] this implies that the modular group $\sigma^t_{\mathcal{R}}$ of $\mathcal{R}$, is the restriction to $\mathcal{R}$ of $\sigma^t_{\mathcal{M}}$. Therefore the $\sigma^t_{\mathcal{R}}$ acts ergodically on $\mathcal{R}$ which is then either a type III factor or equal to the multiples of the identity. On the other hand we have

$$\mathcal{R}^G \subset \mathcal{R} \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{N} = \mathbb{C}1. \tag{17}$$

This means that G acts ergodically on $\mathcal{R}$ so that $\mathcal{R}$ is not a type III factor [19]. *q.e.d.*

3 Classification for the $\hat{su}(2)_1$ Model

The model under investigation is generated by the local currents $j^a(x)$ ($a = 1,2,3$; $x \in \mathbb{R}$) of the $\hat{su}(2)$ chiral current algebra at level 1, defined by the local commutation relations

$$\left[j^a(x), j^b(y)\right] = i\sum_{c=1}^{3} \epsilon_{abc} j^c(x)\delta(x-y) + \frac{i}{4\pi}\delta_{ab}\delta'(x-y). \tag{18}$$

We consider these currents acting, as Wightman fields, on the vacuum Hilbert space $\mathcal{H}$ so that the vacuum vector $\Omega \in \mathcal{H}$ is cyclic for the polynomial algebra that they generate.

Starting from these currents, one can define an energy-momentum tensor $\Theta(x)$ by the Sugawara formula

$$\Theta(x) = \frac{2\pi}{3}\sum_{a=1}^{3} : j^a(x)j^a(x) : \tag{19}$$

which satisfies the commutation relations

$$[\Theta(x), \Theta(y)] = i\delta'(x-y)\{\Theta(x) + \Theta(y)\} - i\frac{c}{24\pi}\delta'''(x-y), \tag{20}$$

with central charge $c = 1$.

For every $I \in \mathcal{K}_0$ we define the local von Neumann algebra

$$\mathcal{F}(I) = \{j^a(f) |\ f \in \mathrm{C}^\infty(\mathbb{R}) \text{ real},\ \mathrm{supp} f \subset I,\ a = 1,2,3\}''. \tag{21}$$

Using the results in [4] (cf. also [15]) one can verify that the family $\mathcal{F} = \{\mathcal{F}(I) |\ I \in \mathcal{K}_0\}$ is a chiral net on $\mathbb{R}$. The representation U of $\mathrm{SL}(2,\mathbb{R})$ is obtained by integrating the Lie algebra generated by the first three moments of the energy-momentum tensor. Similarly the charges

$$Q^a = \int j^a(x)\mathrm{d}x \tag{22}$$

can be integrated to a strongly continuous unitary representation V of the gauge automorphisms group SO(3) of $\mathcal{F}$.

An example of conformal subsystem of $\mathcal{F}$ is given by the family

$$\mathcal{A}_{vir}(I) = \{\Theta(f)|\ f \in \mathrm{C}^\infty(\mathbb{R}) \text{ real},\ \mathrm{supp} f \subset I\}''. \tag{23}$$

It has been proved by Rehren [26] that $\mathcal{A}_{vir} = \mathcal{F}^{\mathrm{SO}(3)}$. In [6] it is proved that $\mathcal{A}_{vir}$ is a minimal subsystem in the sense that it has no proper conformal subsystems. We will now prove that it is a minimum in the sense that every conformal subsystem $\mathcal{B}$ of $\mathcal{F}$ is an extension of $\mathcal{A}_{vir}$. The proof will follow essentially the ideas in [6] (cf. also [5]) and crucially depends on the properties of the representation U and on the analysis of the scaling behaviour of local operators given in [14].

Because of the positivity of L_0, U splits into a direct sum of irreducible representations τ acting on irreducible subspaces $\mathcal{H}_\tau \subset \mathcal{H}$ [22]. The spectrum of the restriction of L_0 to each $\mathcal{H}_\tau$, which consists only of natural numbers, is simple and its minimum $n(\tau)$ (the conformal dimension) completely determines the equivalence class of τ. The case $n(\tau) = 0$ corresponds to the trivial representation. In this case $\mathcal{H}_\tau$ is one dimensional. If $n(\tau) > 0$ then the spectrum of the restriction of L_0 to $\mathcal{H}_\tau$ is simple and coincides with the set $\{n \in \mathbf{N}|\ n \geq n(\tau)\}$.

From the well known character formula [10] we find

$$\mathrm{Tr}\ t^{L_0} = 1 + 3t + 4t^2 + O(t^3). \tag{24}$$

Hence the multiplicities of the eigenvalues 0, 1 and 2 are given by 1, 3 and 4 respectively. As a consequence, in the decomposition of U there appear one representation with $n(\tau) = 0$, three representations with $n(\tau) = 1$ and one representation with $n(\tau) = 2$. The corresponding irreducible subspaces can be identified with (cf. [5, 6, 14]): $\mathbb{C}\Omega$ for $n(\tau) = 0$; the closed subspaces

$\mathcal{H}_a$, $a = 1, 2, 3$, generated by the vectors of the form $j^a(f)\Omega$ for $f \in C^\infty(\mathbb{R})$ with compact support for $n(\tau) = 1$; the closed subspace $\mathcal{H}_\Theta$ generated by the vectors of the form $\Theta(f)\Omega$ for $f \in C^\infty(\mathbb{R})$ with compact support for $n(\tau) = 2$.

Given a vector $u = (u_1,\ u_2,\ u_3) \in \mathbb{R}^3$ we consider the current

$$j^{(u)}(x) = \sum_{a=1}^{3} u_a j^a(x). \tag{25}$$

For $u \neq 0$ the currents $j^{(u)}(x)$ generate subsystems $\mathcal{F}_{(u)}$ of $\mathcal{F}$ in the usual way. Thus, we have

$$\mathcal{F}_{(u)}(I) = \{j^{(u)}(f)|\ f \in C^\infty(\mathbb{R}) \text{ real},\ \text{supp} f \subset I\}'' \ \text{ for } I \in \mathcal{K}_0. \tag{26}$$

Consider now, for a fixed $u \neq 0$, the field $: j^{(u)}(x)j^{(u)}(x) :$. A straightforward calculation shows the well known fact that $\Theta(x)$ is proportional to $: j^{(u)}(x)j^{(u)}(x) :$. Therefore the energy-momentum is, for nonzero u a local function of $j^{(u)}(x)$ only. This implies that for each nonzero u and every $I \in \mathcal{K}_0$ we have (cf. [3])

$$\mathcal{A}_{vir}(I) \subset \mathcal{F}_{(u)}(I). \tag{27}$$

Let us now state the following

Proposition 3.1 *For every conformal subsystem $\mathcal{B}$ of $\mathcal{F}$ we have*

$$\mathcal{A}_{vir}(I) \subset \mathcal{B}(I) \ \ \forall I \in \mathcal{K}_0. \tag{28}$$

Proof. For every representation τ let P_τ be the orthogonal projection on $\mathcal{H}_\tau$. Moreover let P_Θ and P_a be the orthogonal projections on $\mathcal{H}_\Theta$ and $\mathcal{H}_a$ respectively. By the same arguments as in the proof of the lemma in [6] it can be shown that given a subsystem $\mathcal{B}$ of $\mathcal{F}$ there exist an interval $J \in \mathcal{K}_0$ and a self-adjoint operator $B \in \mathcal{B}(J)$ such that $P_\Theta B\Omega \neq 0$ and $(\Omega, B\Omega) = 0$. It can further be assumed that B is infinitely differentiable (in the norm topology) with respect to the action of $\mathrm{SL}(2,\mathbb{R})$. Now, given $\varphi \in \mathrm{C}^\infty(\mathbb{R})$ with compact support, let

$$B_\lambda(\varphi) = \int \varphi(x) T(x) D(\lambda) B D(\lambda)^{-1} T(x)^{-1} \mathrm{d}x. \tag{29}$$

We first suppose that for $a = 1, 2, 3$ we have $P_a B\Omega = 0$. Then , following the arguments in the proof of the proposition in [6] (cf. also [5]), it can be shown that if $\mathrm{supp}\varphi \subset I$, for a given $I \in K$, then

$$\lim_{\lambda \to 0} \lambda^{-2} B_\lambda(\varphi)\Omega = \eta \Theta(\varphi)\psi \tag{30}$$

for every $\psi \in \mathcal{F}(I)'\Omega$, in the weak topology of $\mathcal{H}$, where η is a real constant that can be assumed to be different from zero. From this fact it is not difficult to prove (see the proof of the theorem in [6]) that for every $I \in \mathcal{K}_0$

$$\mathcal{A}_{vir}(I) \subset \mathcal{B}(I). \tag{31}$$

On the contrary, if for some index a we have $P_a B\Omega \neq 0$ we find (cf. [5, 6, 14]), for $\mathrm{supp}\varphi \in I$,

$$\lim_{\lambda \to 0} \lambda^{-1} B_\lambda(\varphi)\psi = j^{(u)}(\varphi)\psi \tag{32}$$

for every $\psi \in \mathcal{F}(I)'\Omega$, in the weak topology of $\mathcal{H}$, for a given $u \in \mathbb{R}^3$ which can be assumed to be different from zero. Using this fact one gets

$$\mathcal{A}_{vir}(I) \subset \mathcal{F}_{(u)}(I) \subset \mathcal{B}(I) \tag{33}$$

for every $I \in \mathcal{K}_0$ *q.e.d.*

We now state our first classification result.

Theorem 3.2. *The map $H \to \mathcal{F}^H$ gives a one-to-one correspondence between the closed subgroups H of SO(3) and the conformal subsystems $\mathcal{B}$ of $\mathcal{F}$.*

Proof. Let $\mathcal{B}$ be a conformal subsystem of $\mathcal{F}$ and let $J \in \mathcal{K}_0$. By proposition 3.1 and the result in [26] we have

$$\mathcal{F}(J)^{\mathrm{SO}(3)} \subset \mathcal{B}(J) \subset \mathcal{F}(J). \tag{34}$$

$\mathcal{F}(J)$ is a type III_1 factor which, $\mathcal{H}$ being separable, has a separable predual. By proposition 2.1. the relative commutant of $\mathcal{F}(J)^{\mathrm{SO}(3)}$ in $\mathcal{F}(J)$ is trivial. Thus, by [20, theorem 3.15], there exists a unique closed subgroup H of SO(3) such that

$$\mathcal{B}(J) = \mathcal{F}(J)^H. \tag{35}$$

Finally given another interval $I \in \mathcal{K}_0$ by conformal covariance we find

$$\mathcal{B}(I) = \mathcal{F}(I)^H. \tag{36}$$

q.e.d.

The closed subgroups of SO(3) are well known. Up to conjugation there are only three infinite closed subgroups: SO(3) itself; the group H_1 of orientation preserving rotations around the third axis; the subgroup H_2 generated by H_1 and the element

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

All other closed subgroups are finite. Their classification can be found in [30]. The subsystems associated with the finite subgroups correspond to the orbifold models. Moreover, it is not difficult to see that the subsystem associated to H_1 is given by $\mathcal{F}_{(e_3)}$, where $e_3 = (0,\ 0,\ 1)$ and that the one associated to H_2 corresponds to the one obtained by taking the local operators in $\mathcal{F}_{(e_3)}$ invariant under the transformation $j^3(x) \to -j^3(x)$.

4 Classification for the $\hat{u}(1)$ Model

This model is generated by a current $j(x)$ (the $\hat{u}(1)$-current), $x \in \mathbb{R}$, satisfying the local commutation relations

$$[j(x), j(y)] = i\delta'(x-y). \tag{37}$$

Again we consider this current acting (as a Wightman field) on the vacuum Hilbert space $\mathcal{H}_1$ so that the vacuum vector $\Omega_1 \in \mathcal{H}_1$ is cyclic for the polynomial algebra that it generates. An energy-momentum tensor with $c = 1$ is defined by

$$\Theta_1(x) = \frac{1}{2} : j(x)j(x) : . \tag{38}$$

We define the chiral net $\mathcal{F}_1$ by

$$\mathcal{F}_1(I) = \{j(f) |\ f \in \mathrm{C}^\infty(\mathbb{R}) \text{ real},\ \mathrm{supp} f \subset I\}'', \ \text{ for } I \in \mathcal{K}_0 \tag{39}$$

and denote by U_1 the corresponding representation of $\mathrm{SL}(2, \mathbb{R})$. The conformal subsystem generated by $\Theta_1(x)$ is defined by

$$\mathcal{B}_{vir}(I) = \{\Theta(f)_1 |\ f \in \mathrm{C}^\infty(\mathbb{R}) \text{ real},\ \mathrm{supp} f \subset I\}'', \ \text{ for } I \in \mathcal{K}_0. \tag{40}$$

Another conformal subsystem is given by ${\mathcal{F}_1}^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts as an unbroken gauge group of symmetries implementing the transformation $j(x) \to -j(x)$. We then have

$$\mathcal{B}_{vir}(I) \subset {\mathcal{F}_1}^{\mathbb{Z}_2}(I) \subset \mathcal{F}_1(I), \ \text{ for } I \in \mathcal{K}_0. \tag{41}$$

The following argument allows us to classify the conformal subsystems of $\mathcal{F}_1$: the map Φ defined by

$$\Phi(e^{ij(f)}) = e^{i2\sqrt{\pi}j^3(f)}, \tag{42}$$

for $f \in C^\infty(\mathbb{R})$ with compact support, extends to an isomorphism of $\mathcal{F}_1$ into $\mathcal{F}$ with the following, easily verifiable, properties:

$$\Phi(\mathcal{F}_1(I)) = \mathcal{F}^{H_1}(I) \text{ for } I \in \mathcal{K}_0; \tag{43}$$

$$\Phi(\mathcal{B}_{vir}(I)) = \mathcal{A}_{vir}(I) \text{ for } I \in \mathcal{K}_0; \tag{44}$$

$$\Phi({\mathcal{F}_1}^{\mathbb{Z}_2}(I)) = \mathcal{F}^{H_2}(I) \text{ for } I \in \mathcal{K}_0. \tag{45}$$

Moreover, if A, $U_1(\alpha)AU_1(\alpha)^{-1} \in \mathcal{F}_1(I)$ for some $I \in \mathcal{K}_0$, we have

$$\Phi(U_1(\alpha)AU_1(\alpha)^{-1}) = U(\alpha)\Phi(A)U(\alpha)^{-1}. \tag{46}$$

We conclude this section with the following theorem.

Theorem 4.1. *The only proper conformal subsystems of $\mathcal{F}_1$ are $\mathcal{B}_{vir}$ and ${\mathcal{F}_1}^{\mathbb{Z}_2}$.*

Proof. If $\mathcal{B}$ is a proper conformal subsystem of $\mathcal{F}_1$ then the family

$$\Phi(\mathcal{B}) = \{\Phi(\mathcal{B}(I)) | I \in \mathcal{K}_0\} \tag{47}$$

is a conformal subsystem of $\mathcal{F}$ such that

$$\Phi(\mathcal{B})(I) \subset \mathcal{F}^{H_1}(I) \text{ for } I \in \mathcal{K}_0, \tag{48}$$

where the inclusion is proper. Thus, by theorem 3.3., there must be a closed subgroup H of SO(3) strictly containing H_1 such that $\Phi(\mathcal{B}) = \mathcal{F}^H$. The only possibilities are $H = H_2$ and $H = \mathrm{SO}(3)$. Using the fact that Φ is an isomorphism we then find $\mathcal{B} = {\mathcal{F}_1}^{\mathbb{Z}_2}$ in the first case or $\mathcal{B} = \mathcal{B}_{vir}$ in the second. *q.e.d.*

5 Discussion of Other Models

In this section we briefly discuss the limits of our approach in the application to other models of chiral conformal field theory.

We first consider the case of the current algebra $\hat{g}_k$ of the Lie algebra $\hat{g}$ of a compact connected simply connected simple Lie group G at level k [10]. We denote by $\mathcal{F}_{\hat{g}_k}$ the associated chiral net. Let $\mathcal{A}_{vir}$ be the net generated by the Sugawara energy-momentum tensor. It has been pointed out by Rehren [26] that the inclusion $\mathcal{A}_{vir} \subset \mathcal{F}^G_{\hat{g}_k}$ is proper unless G=SU(2) and $k = 1$. So one of the fundamental ingredients that we used in section 3 fails in all the chiral current algebras models with the exception of the one we considered. Moreover also the analogue of proposition 3.1. fails in the general case. In fact, except in the case of the $\hat{su}(2)_1$ model, the subsystem generated by a single current does not contains $\mathcal{A}_{vir}$. This is due to the fact that the energy-momentum tensor associated to a single current has central charge $c = 1$ while for the one associated to the full theory we have $c > 1$. The corresponding nonvanishing coset energy-momentum tensor, and hence the one of the full theory, cannot be affiliated to the net generated by a single current (cf. [27, 28]). More generally every Lie subalgebra of $\hat{g}$ giving rise to a "nonconformal" imbedding (see [10]) generates a conformal subsystem not containing $\mathcal{A}_{vir}$.

The situation is different for $c = 1$ models. Let us consider the case of the local extensions of the $\hat{u}(1)$ model. These have been classified by Buchholz, Mack and Todorov [3]. Let $\mathcal{A}_N$ ($N = 1, 2, ...$) be the corresponding chiral nets. The case $N = 1$ corresponds to the $\hat{su}(2)_1$ model. For almost all these models the analogue of proposition 3.1. holds. In fact, using the trace formulas given in [3] we find

$$\mathrm{Tr}\ t^{L_0} = 1 + t + 4t^2 + O(t^3) \ \text{ for } N = 2 \tag{49}$$

$$\mathrm{Tr}\ t^{L_0} = 1 + t + 2t^2 + O(t^3) \ \text{ for } N > 2. \tag{50}$$

Thus for $N > 2$ there appear only two nontrivial irreducible representations τ with conformal dimension $n(\tau) \leq 2$: the one associated to the $\hat{u}(1)$-current

$j(x)$ and the one associated with the energy-momentum tensor $\frac{1}{2} : j(x)j(x) :$. A proof similar to that of proposition 3.1. leads then to

Proposition 5.1. *For every conformal subsystem* $\mathcal{B}$ *of* $\mathcal{A}_N$, $N \neq 2$, *we have* $\mathcal{A}_{vir} \subset \mathcal{B}$.

This fact should be of interest for the classification of the subsystems of the nets $\mathcal{A}_N$ and relates this program to the problem of classification for the local extensions of the $c = 1$ Virasoro algebra.

Acknowledgements

It is a pleasure to thank Dr. Roberto Conti, Prof. Sergio Doplicher, Prof. Klaus Fredenhagen and Prof. Roberto Longo for useful discussions and hints.

References

[1] Araki H.: Symmetries in the theory of local observables and the choice of the net of local algebras, *Rev. Math. Phys.*, **Special Issue** (1992), 1.

[2] Böckenhauer J., Evans D.E.: Modular invariants graphs and α-induction for nets of subfactors, *Comm. Math. Phys.*, **197** (1998), 361.

[3] Buchholz D., Mack G., Todorov I.T.: The current algebra on the circle as a germ of local field theories, *Nucl. Phys. B* (Proc. Suppl.) **5B** (1988), 20.

[4] Buchholz D., Schulz-Mirbach H.: Haag duality in conformal quantum field theory, *Rev. Math. Phys.* **2** (1990), 105.

[5] Carpi S.: Quantum Noether's theorem and conformal field theory: study of some models, Preprint, Università di Roma "La Sapienza" (January 1998), to appear in *Rev. Math. Phys.*

[6] Carpi S.: Absence of subsystems for the Haag-Kastler net generated by the energy-momentum in two dimensional conformal field theory, *Lett. Math. Phys.* **45** (1998), 259.

[7] Conti R.: On the intrinsic definition of local observables, *Lett. Math. Phys.* **35** (1995), 237.

[8] D'Antoni C., Longo R., Radulescu F.: Conformal nets, maximal temperature and models from free probability, hep-th 98100003.

[9] Davidson D. R.: Classification of subsystems of local algebras, Ph.D. Thesis, University of California at Berkeley (1993).

[10] Di Francesco Ph., Mathieu P., Sénéchal D.: *Conformal field theory.* Berlin-Heielberg-New York: Springer-Verlag, 1996.

[11] Dong G., Griess L. Jr.: Rank one vertex operator algebras and their automorphism groups, q-alg/9710017, to appear in *J. Algebra.*

[12] Doplicher S., Haag R., Roberts J. E.: Fields, observable and gauge transformations I, *Comm. Math. Phys.* **13** (1969), 1.

[13] Doplicher S., Roberts J.E.: Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics, *Comm. Math. Phys.* **131** (1990), 51.

[14] Fredenhagen K., Jörß M.: Conformal Haag-Kastler nets, pointlike localized fields and the existence of operator product expansions, *Comm. Math. Phys.* **176** (1996), 541.

[15] Gabbiani F., Fröhlich J.: Operator algebras and conformal field theory, *Comm. Math. Phys.* **155** (1993), 569.

[16] Guido D., Longo R.: The conformal spin and statistic theorem, *Comm. Math. Phys.* **181** (1996), 11.

[17] Guido D., Longo R., Wiesbrock H.-W.: Extension of conformal nets and superselection structures, *Comm. Math. Phys.* **192** (1998), 217.

[18] Haag R.: *Local quantum physics.* Berlin-Heielberg-New York: Spriger-Verlag, 1996.

[19] Høegh-Krohn R., Landstad M., Størmer E.: Compact ergodic groups of automorphisms, *Ann. Math.* **114** (1981), 75.

[20] Izumi M., Longo R., Popa S.: A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras, *J. Funct. Anal.* **155** (1998), 25.

[21] Jörß M.: The construction of pointlike localized charged fields from conformal Haag-Kastler nets, *Lett. Math. Phys.* **38** (1996), 257.

[22] Lang S.: $\mathrm{SL}_2(R)$. Berlin-Heielberg-New York: Springer-Verlag, 1975.

[23] Langerholc J., Schroer B.: On the structure of the von Neumann algebras generated by local functions of the free Bose field, *Comm. Math. Phys.* **1** (1965), 215.

[24] Langerholc J., Schroer B.: Can current operators determine a complete theory?, *Comm. Math. Phys.* **4** (1967), 123.

[25] Longo R., Rehren K.-H.: Nets of subfactors, *Rev. Math. Phys.* **7** (1995), 567.

[26] Rehren K.-H.: A new view of the Virasoro algebra, *Lett. Math. Phys.* **30** (1994), 125.

[27] Rehren K.-H.: Subfactors and coset models, In: *Generalized Symmetries in Physics*, V. Dobrev et al. eds., Singapore: World Scientific, 1994, pp. 338-356.

[28] Rehren K.-H., Stanev Y. S. Todorov I.T.: Characterizing invariants for local extensions of current algebras, *Comm. Math. Phys.* **174** (1996), 605.

[29] Takesaki M.: Tomita' s theory of modular Hilbert algebras and applications. *Lect. Notes in Math.* **128**. Berlin-Heielberg-New York: Springer-Verlag, 1970.

[30] Weyl H.: *Symmetry.* Princeton: Princeton University Press, 1952.

[31] Wichmann E. H.: On systems of local operators and the duality condition, *J. Math. Phys.* **24** (1983), 1633.