Absence of Subsystems for the Haag-Kastler Net Generated by the Energy-Momentum Tensor in Two Dimensional Conformal Field Theory

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Abstract

We show that if $\mathcal{A}$ is the Haag-Kastler net generated by the energy-momentum tensor in a chiral quantum field theory, then every subsystem $\mathcal{B} \subset \mathcal{A}$ which is covariant under the action of $SL(2, \mathbf{R})$ given on $\mathcal{A}$ must coincide with $\mathcal{A}$. The result is valid for all the allowed values of the central charge and is obtained using scaling limit techniques.

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1 Introduction

Let $\Theta(x)$ be the energy-momentum tensor of a chiral quantum field theory. The Luscher-Mack theorem (see [6]) restricts its commutation relations to being given by

$$[\Theta(x), \Theta(y)] = i\delta'(x-y)\{\Theta(x) + \Theta(y)\} - i\frac{c}{24\pi}\delta'''(x-y), \tag{1}$$

where the central charge c is a positive constant whose allowed values are $c \geq 1$ and $c = 1 - \frac{6}{(m+2)(m+3)}$ for $m = 1, 2, ...$ Moreover using the Cayley transformation one can extend $\Theta(x)$ to a field on the circle whose Fourier coefficients L_n satisfy the following commutation rules

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} \tag{2}$$

which define the famous Virasoro algebra. If we consider only Wightman functions involving $\Theta(x)$ then the Hilbert space $\mathcal{H}$ obtained by the Wightman reconstruction theorem (see [16]) carries an irreducible representation (of lowest weight 0) of the Virasoro algebra (see [6]).

For every bounded open interval $I \subset \mathbf{R}$ one can define the local von Neumann algebra

$$\mathcal{A}(I) = \{\Theta(f) |\ f \in C^\infty(\mathbf{R}) \text{ real, } \operatorname{supp} f \subset I\}''. \tag{3}$$

We now denote by $\mathcal{K}$ the family of all bounded open intervals of the real line. It has been proved in [2] that the family $\mathcal{A} = \{\mathcal{A}(I) |\ I \in \mathcal{K}\}$, that we assume is represented in the vacuum Hilbert space $\mathcal{H}$ defined above, satisfies standard assumptions for a chiral Haag-Kastler net (see for example [1], [5] and [7]). In particular there is a strongly continuous representation U of $SL(2, \mathbf{R})$ leaving the vacuum vector Ω invariant and such that

$$U(-1) = 1 \tag{4}$$

$$U(\alpha)\mathcal{A}(I)U(\alpha)^{-1} = \mathcal{A}(\alpha I) \ \text{ for } I,\ \alpha I \in \mathcal{K}, \tag{5}$$

where

$$SL(2, \mathbf{R}) \ni \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on $\mathbf{R}$ by

$$x \to \alpha x = \frac{ax + b}{cx + d}. \tag{6}$$

The representation U is obtained by integrating the selfadjoint part of the complex Lie algebra spanned by the Fourier coefficients $L_{-1},\ L_0$, and L_{+1} and is a positive energy representation since the conformal Hamiltonian L_0 must be positive.

We now define a conformal subsystem of $\mathcal{A}$ to be a family of (non trivial) von Neumann algebras $\mathcal{B} = \{\mathcal{B}(I)|\ I \in \mathcal{K}\}$ such that for every $I \in \mathcal{K}$, $\mathcal{B}(I) \subset \mathcal{A}(I)$ and

$$U(\alpha)\mathcal{B}(I)U(\alpha)^{-1} = \mathcal{B}(\alpha I) \ \text{ for } I,\ \alpha I \in \mathcal{K}. \tag{7}$$

Moreover the family $\mathcal{B}$ is assumed to satisfy isotony i.e.

$$\mathcal{B}(I) \subset \mathcal{B}(J) \ \text{ if } I \subset J. \tag{8}$$

We will prove in this Letter that each conformal subsystem of $\mathcal{A}$ must coincide with $\mathcal{A}$ for all the allowed values of the central charge. In the proof we will use scaling limit procedures and arguments very similar to those developed in [3] with a different motivation and inspired by the work of Fredenhagen and Jörß [5].

Other results limiting the number of the possible subsystems of a given model have been obtained in previous works by Langerholc and Schroer [10],

[11], Davidson [4] and by Rehren [13]. They however do not cover the above situation.

In the approach to chiral field theory based on Wightman fields (see [6] for an introduction to the subject) it is generally assumed that to every model there is associated an energy-momentum tensor with the properties discussed above. This assumption is crucial for many aspects; it is for example one of the starting points for classification. However in the formulation of chiral field theory in therms of Haag-Kasteler nets the energy-momentum tensor is no more assumed to exists. In this contest in fact one starts from local von Neumann algebras covariant under the action of $SL(2,\ \mathbf{R})$. Actually there exists chiral Haag-Kastler nets without an enegy-momentum tensor (see [8]). Thus from this point of view a proper conformal subsystem $\mathcal{B}$ of the algebra $\mathcal{A}$ should be considered as an admissible model. The fact that such subsystems does not exists is in agreement with the idea that the energy-momentum is a fundamental object.

2 The Result

We consider the local algebras $\mathcal{A}(I)$ generated by the energy-momentum tensor in the vacuum representation. The field $\Theta(x)$ is covariant under the action of $SL(2,\mathbf{R})$ with conformal dimension equal to two, i.e.

$$U(\alpha)\Theta(x)U(\alpha)^{-1} = (\frac{d\alpha x}{dx})^2\Theta(\alpha x) \;\text{ for } \alpha \in SL(2,\mathbf{R}). \tag{9}$$

Thus the closed subspace $\mathcal{H}_\Theta$ generated by the vectors of the form $\Theta(f)\Omega$ for $f \in C^\infty(\mathbf{R})$ with compact support, is U-invariant. Actually it is U-irreducible. In fact the two point function of the energy-momentum tensor is given by (see [6])

$$(\Omega,\ \Theta(x)\Theta(y)\Omega) = \frac{c}{8\pi^2(x-y-i0)^4} \tag{10}$$

and thus

$$(\Theta(f)\Omega,\ \Theta(g)\Omega) = \frac{c}{24\pi}\int_0^\infty \bar{\hat{f}}(p)\hat{g}(p)p^3 dp. \tag{11}$$

The last equation implies that we can define an unitary operator V from $\mathcal{H}_\Theta$ to $L^2(\mathbf{R}_+,\ p^3dp)$ such that

$$(V\Theta(f)\Omega)(p) = (\frac{c}{24\pi})^{\frac{1}{2}}\hat{f}(p) \tag{12}$$

and intertwining the restriction of U to $\mathcal{H}_\Theta$ with the irreducible representation of $SL(2,\mathbf{R})$ in $L^2(\mathbf{R}_+,\ p^3dp)$ described in [5]. Moreover we recall that if P is the (positive) selfadjoint generator of the group of translations $T(a) = U\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ then we have

$$P = \int \Theta(x)dx \tag{13}$$

so that, if for a given $I \in \mathcal{K}$, $h \in C^\infty(\mathbf{R})$ is a positive function with compact support such that $h(x) = 1$ if $x \in I$, we have (cf. [2])

$$T(a)AT(a)^{-1} = e^{ia\Theta(h)}Ae^{-ia\Theta(h)} \ \text{ for } A,\ T(a)AT(a)^{-1} \in \mathcal{A}(I). \tag{14}$$

The first step in the proof of our main result is the following lemma.

Lemma. *Let P_Θ be the orthogonal projection on $\mathcal{H}_\Theta$ and let $\mathcal{B}$ be a conformal subsystem of $\mathcal{A}$. There exists an interval $I \in \mathcal{K}$ and a selfadjoint operator $B \in \mathcal{B}(I)$ such that $P_\Theta B\Omega \neq 0$.*

Proof. Let $J \in \mathcal{K}$ be an arbitrary open interval. We can find a non-zero selfadjoint operator $C \in \mathcal{B}(J)$ such that $(\Omega,\ C\Omega) = 0$. If $P_\Theta C\Omega \neq 0$ we take

$B = C$ and $I = J$. If $P_\Theta C\Omega = 0$ we consider the operator

$$C_x C = T(x) C T(x)^{-1} C \tag{15}$$

which, because of locality, is selfadjoint for $|x| > \mathrm{diam} J$. For every x with $|x| > \mathrm{diam} J$ we can find a real function $f_x \in C^\infty(\mathbf{R})$ such that, for ϵ small enough

$$C_{x+\epsilon} C = e^{i\epsilon\Theta(f_x)} C_x C e^{-i\epsilon\Theta(f_x)}. \tag{16}$$

We now suppose that $|x| > \mathrm{diam} J$ implies

$$P_\Theta C_x C\Omega = 0. \tag{17}$$

Then $|x| > \mathrm{diam} J$ also implies

$$\begin{aligned} \frac{d}{dx}(\Omega,\ C_x C\Omega) &= i(\Theta(f_x)\Omega,\ C_x C\Omega) - i(\Omega,\ C_x C\Theta(f_x)\Omega) \\ &= 0. \end{aligned} \tag{18}$$

The last equation together with the conformal cluster theorem [5] implies that the function $x \to (\Omega,\ CT(-x)C\Omega)$ has compact support and therefore, because of the positivity of P, it must be identically zero. In particular $(\Omega,\ C^2\Omega) = 0$, so the Reeh-Schlieder property implies $C = 0$ which is a contradiction. Now let $x_0 \in \mathbf{R}$ such that $|x_0| > \mathrm{diam} J$ and $P_\Theta C_{x_0} C\Omega \neq 0$; we can then take $B = C_{x_0} C$ and I to a bounded open interval containing J and $J + x_0$. *q.e.d.*

The next step crucially depends on the properties of the representation U and on the analysis of the scaling behaviour of the local operators given in [5]. Because of the positivity of L_0 the representation U splits in to a direct sum of irreducible representations τ acting on closed subspaces $\mathcal{H}_\tau \subset \mathcal{H}$ (see [9] and [5]). The spectrum of the restriction of L_0 to each $\mathcal{H}_\tau$, which consists only of natural numbers, is simple and its minimum $n(\tau)$ (the conformal dimension) completely determine the equivalence class of τ. The case $n(\tau) = 0$ corresponds to the trivial representation. In this case $\mathcal{H}_\tau$ is one dimensional. If $n(\tau) > 0$ then the spectrum of the restriction of L_0 to $\mathcal{H}_\tau$ coincides with the set $\{n \in \mathbf{N} | \, n \geq n(\tau)\}$.

In our case the spectrum of L_0 in $\mathcal{H}$ can be completely determined and therefore one can give a description of the decomposition of U. In fact one finds that the eigenspace of L_0 corresponding to the eigenvalue n is spanned by vectors of the form

$$L_{-n_1}L_{-n_2}...L_{-n_k}\Omega, \tag{19}$$

with $n_1 \geq n_2 ... \geq n_k \geq 2$ and $n_1 + n_2 ... + n_k = n$ (see [6]). These vectors are not necessarily linearly independent but one can find by direct computations an orthogonal base for each eigenspace. In particular one finds that $\mathbf{C}\Omega$ is the only eigenspace corresponding to the eigenvalue 0, that 1 is not in the spectrum of L_0 and that the eigenspace corresponding to the eigenvalue 2 is one dimensional and is spanned by $L_{-2}\Omega$. Thus in the decomposition of U, $\mathbf{C}\Omega$ is the only irreducible subspace with conformal dimension equal to 0, there are no irreducible subspaces with conformal dimension equal to 1 and there is only one subspace with conformal dimension equal to 2. Actually this last subspace is $\mathcal{H}_\Theta$ (cf. [5]).

For every bounded linear operator $A \in \mathbf{B}(\mathcal{H})$ and every $\varphi \in C^\infty(\mathbf{R})$ with compact support we define

$$A(\varphi) = \int \varphi(x) T(x) A T(x)^{-1} dx. \tag{20}$$

The next proposition is the main step in the proof of the claimed result.

Proposition. *For every conformal subsystem $\mathcal{B} \subset \mathcal{A}$ there is a sequence*

B_n of bounded selfadjoint operators which is eventually in $\mathcal{B}(I)$ for every $I \in \mathcal{K}$ containing the origin and such that, for every $\varphi \in C^\infty(\mathbf{R})$ with support contained in some interval $J \in \mathcal{K}$

$$\lim_{n\to\infty} B_n(\varphi)\psi = \Theta(\varphi)\psi \tag{21}$$

for every $\psi \in \mathcal{A}(J)'\Omega$, in the weak topology of $\mathcal{H}$.

Proof. If for every $I \in \mathcal{K}$ containing the origin B_n is eventually in $\mathcal{B}(I)$ then if supp$\varphi \subset J,\ J \in \mathcal{K}\ B_n(\varphi)$ is eventually in $\mathcal{B}(J)$. Because of locality it is thus enough to prove the proposition in the particular case in which $\psi = \Omega$.

By the Lemma we can chose an interval $\hat{I} \in \mathcal{K}$ and a selfadjoint operator $B \in \mathcal{B}(\hat{I})$ such that $(\Omega,\ B\Omega) = 0$ and $P_\Theta B\Omega \neq 0$. Moreover we can suppose, by possibly smearing B with a smooth function on $SL(2,\mathbf{R})$ with support sufficiently close to the identity (cf. [5] and [3]), that $B \in C^\infty(SL(2,\mathbf{R}))$ with respect to the norm topology. Now let $D(\lambda) = U\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix}$ be the group of dilations and let be $B_\lambda = D(\lambda)BD(\lambda)^{-1}$. It has been proved in [5], in a more general situation, that if $A \in C^\infty(SL(2,\mathbf{R}))$ is a local operator and P_τ is the orthogonal projection on $\mathcal{H}_\tau$, then the limit for $\lambda \to 0$ of $\lambda^{-n(\tau)}P_\tau A_\lambda(\varphi)\Omega$ exists for every irreducible representation τ appearing in the decomposition of U and for every infinitely differentiable function with compact support φ. In our case this implies that

$$\lim_{\lambda\to 0} \lambda^{-2} P_\tau B_\lambda(\varphi)\Omega = 0 \tag{22}$$

if P_τ is orthogonal to P_Θ. If now $g(p) = (VP_\Theta B\Omega)(p)$ then $(V\lambda^{-2}P_\Theta B_\lambda(\varphi)\Omega)(p)$ $= \hat{\varphi}(p)g(\lambda p)$. Since $g(p)$ is continuous and bounded (see [5]) it follows that

$$\lim_{\lambda\to 0} \lambda^{-2} P_\Theta B_\lambda(\varphi)\Omega = \eta\Theta(\varphi)\Omega, \tag{23}$$

where $\eta = (\frac{c}{24\pi})^{-\frac{1}{2}} g(0)$. It has been shown in [5] that an accidental vanishing of $g(0)$ can be avoided with an arbitrary small conformal transformation of B so that we can assume that η is different from zero.

We can now conclude that if ψ is in the linear span of a finite set of U-irreducible subspaces of $\mathcal{H}$

$$\lim_{\lambda\to 0}(\psi,\ \lambda^{-2}B_\lambda(\varphi)\Omega) = \eta(\psi,\ \Theta(\varphi)\Omega). \tag{24}$$

Since the set of such vectors ψ is dense in $\mathcal{H}$ to prove the weak convergence it is enough to show that $||\lambda^{-2}B_\lambda(\varphi)\Omega||$ is bounded in λ. By the conformal cluster theorem [5] the Fourier transform of $(\Omega,\ BT(x)B\Omega)$ can be written as $\vartheta(p)p^3F(p)$,where $\vartheta(p)$ is the Heaviside step function and $F(p)$ is an analytic function of rapid decrease (see [5]). A direct computation then shows that

$$||\lambda^{-2}B_\lambda(\varphi)\Omega||^2 = \int_0^\infty |\hat{\varphi}(p)|^2 F(\lambda p)p^3 dp \le$$
$$\le \max F \int_0^\infty |\hat{\varphi}(p)|^2 p^3 dp. \tag{25}$$

By the existence of the limit it easily follows that η is a real constant and thus the sequence $B_n = \frac{n^2}{\eta} D(\frac{1}{n}) B D(\frac{1}{n})^{-1}$ has all the claimed poperties. *q.e.d.*

The last step is based on the fact that if $\varphi \in C^\infty(\mathbf{R})$ is a real function with supp$\varphi \subset I \in \mathcal{K}$, the domain $\mathcal{A}(I)'\Omega$, which is dense in $\mathcal{H}$ because of the Reeh-Schlieder property, is a core for $\Theta(\varphi)$. This is a consequence of the energy bounds proved in [2] together with the fact that $\mathcal{A}(I)'\Omega$ contains a core for L_0 (see the appendix of [3]). We now state the claimed theorem.

Theorem. *For every conformal subsystem $\mathcal{B}$ of $\mathcal{A}$ one has $\mathcal{B} = \mathcal{A}$.*

Proof. Let $\varphi \in C^\infty(\mathbf{R})$ be a real function with supp$\varphi \subset I \in \mathcal{K}$ and let $C \in \mathcal{B}(I)'$. It follows from the previous proposition that if $\psi_1,\ \psi_2 \in \mathcal{A}(I)'\Omega$

then

$$(\psi_1,\ C\Theta(\varphi)\psi_2) = \lim_{n\to\infty}(\psi_1,\ CB_n(\varphi)\psi_2) =$$

$$\lim_{n\to\infty}(B_n(\varphi)\psi_1,\ C\psi_2) = (\Theta(\varphi)\psi_1,\ C\psi_2) \tag{26}$$

and thus, since $\mathcal{A}(I)'\Omega$ is a core for $\Theta(\varphi)$ we have

$$(\psi_1,\ C\Theta(\varphi)\psi_2) = (\Theta(\varphi)\psi_1,\ C\psi_2) \tag{27}$$

for every $\psi_1,\ \psi_2$ in the domain of $\Theta(\varphi)$. This implies that for every real function $\varphi \in C^\infty(\mathbf{R})$ with support contained in I

$$C\Theta(\varphi) \subset \Theta(\varphi)C \tag{28}$$

and so that $C \in \mathcal{A}(I)'$. Since $C \in \mathcal{B}(I)'$ is arbitrary we have

$$\mathcal{B}(I)' \subset \mathcal{A}(I)' \tag{29}$$

and thus

$$\mathcal{A}(I) \subset \mathcal{B}(I) \tag{30}$$

for every $I \in \mathcal{K}$. *q.e.d.*

3 Concluding Remarks

In the study of models and of general features of conformal quantum field theory one is often led to consider the situation in which the local net $\mathcal{A}$ generated by the energy-momentum tensor is imbedded in a larger local net $\mathcal{M}$. This means that $\mathcal{A}$ is a conformal subsystem of $\mathcal{M}$ or, using the terminology commonly adopted in the literature, that $\mathcal{M}$ is a local (conformal) extension of $\mathcal{A}$.

If we consider the set of conformal subsystems of $\mathcal{M}$ ordered by inclusion, the theorem of the previous section is equivalent to the assertion that $\mathcal{A}$ is a minimal element of this ordered set. In general $\mathcal{A}$ will not be a minimum but it is in some cases. For example if $\mathcal{M}$ is the local net generated by a chiral current $j(x)$ (the free scalar field) and the energy momentum tensor is given by $\Theta(x) = \frac{1}{2} : j^2 : (x)$, or in the case in which $\mathcal{M}$ is the gauge invariant part of the local net generated by the chiral current algebra of a compact simple Lie group and $\Theta(x)$ is the Sugawara energy-momentum tensor (in this case the equality $\mathcal{M} = \mathcal{A}$ is an exception [13]), it can be shown, by similar methods to those used in this paper (cf. [3]) that $\mathcal{A}$ is a minimum. So in these last cases every conformal subsystems of $\mathcal{M}$ must be a local extension of $\mathcal{A}$. This fact, together with some recent works on local extensions (see for example [12], [14] and [15]) should be useful for the classification of the conformal subsystems of $\mathcal{M}$.

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