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Locally finite theories with model companion

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Logica matematica. — *Locally finite theories with model companion.* (*) Nota di PAOLO LIPPARINI (**), presentata (***) dal Socio G. ZAPPA.

RIASSUNTO. — Si dà, mediante condizioni per l'amalgamabilità di tre modelli, una caratterizzazione completa delle teorie localmente finite con model-compagno. Si introduce inoltre un concetto intermedio tra quelli di model-compagno e model-completamento.

It is known that a locally finite theory with the amalgamation property has model companion (and hence model completion), if the language is finite (see [7], [8], [1], [6] and perhaps others). In § 1 I give a direct proof of a little more general fact, and then compare this result with Corollary 1 of [8], showing a deep resemblance. The proof of Theorem 1 is simple, and is given only in order to make clear the subsequent ideas. Indeed, § 2 and § 3 are formally independent of the first paragraph.

The method doesn't work so well when the language is infinite, and, as a matter of fact, the same difficulties are encountered in the case when the amalgamation property doesn't hold. In § 2 companionable, universal locally finite theories are characterized by a sort of diagramatic property: then I show how to use this in order to decide if a general theory is companionable, though I believe the methods presented in this paper are more interesting and useful in studying locally finite theories, or ones easily reducible to this case.

When I was on a wrong way for generalizing Theorem 1, I introduced a new concept intermediate between model companion and model completion, which nevertheless may be of some interest. It is given in § 3, together with some motivations. I thank prof. G. Cherlin for some useful conversations.

The reader is supposed to be familiar with a little model theory (see for example [2]) and to know the main facts about model companions and the like (see [4]). Notations are quite standard; $\bar{x}$ represents an n-tuple of x_i's (n is clear from the context), and sometimes simply the set of those elements; $\langle\ \rangle$ means 'generated by'. I use the (ugly) abbreviations e.c. for existentially complete, m.c. for model companion, a.p. for amalgamation property. Locally finite will always mean universal and locally finite.

I introduce the following condition for a universal theory T (note that it is satisfied if T is locally finite in a finite language):

($'$) *for any* n *and any model* $\mathbf{A}$ *of* T *generated by* $a_0, \cdots, a_{n-1}$, *the diagram of* $\mathbf{A}$, D ($\mathbf{A}$), *is finitely axiomatizable relative to* T. If this is the case, $D_{\mathbf{A}}(\bar{x})$ *or* $D_{\bar{a}}(\bar{x})$ will be a formula such that $D_{\mathbf{A}}(\bar{a})$ is one of these finite axiomatizations.

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§ 1. THE MAIN IDEA

THEOREM 1. *A universal theory* T *satisfying* (′) *and with the amalgamation property has a model companion.*

Proof. The class of e.c. structures for T is axiomatized by $T' = T + \{\forall \bar{x}\,(D_{\mathbf{A}}(\bar{x}) \Rightarrow \varphi(\bar{x})) \mid \varphi$ is an existential formula, and there are two models $\mathbf{B} \supseteq \mathbf{A} = \langle \bar{a} \rangle$ with $\mathbf{B} \models T + \varphi(\bar{a})\}$. Indeed, let $\mathbf{C} \models T'$, and $\varphi(\bar{c})$ be existential with constants from C (I can assume that there is at least one constant!). If $\mathbf{D} \supseteq \mathbf{C}, \mathbf{D} \models T + \varphi(\bar{c})$, from $\mathbf{C} \models \forall x\,(D_{\bar{c}}(\bar{x}) \Rightarrow \varphi(\bar{x}))$ & $D_{\bar{c}}(\bar{c})$ it follows $\mathbf{C} \models \varphi(\bar{c})$. This means that **C** is e.c.

Vice versa, let **C** be e.c., $\varphi(\bar{x})$ an existential formula, $\mathbf{B} \supseteq \mathbf{A} = \langle \bar{a} \rangle$, $\mathbf{B} \models T + \varphi(\bar{a})$. For any $\bar{c}$ in C, if $D_{\bar{a}}(\bar{c})$, then $\langle \bar{c} \rangle$ is isomorphic to $\langle \bar{a} \rangle$ (since they have the same diagram), so there exists a **D** making the following diagram commute:

$$\begin{array}{ccc} & \mathbf{D} & \\ \subseteq & & \supseteq \\ \mathbf{C} & & \mathbf{B} \\ \cup\!| & & |\!\cup \\ & \langle \bar{c} \rangle = \mathbf{A}\,; & \end{array}$$

and then $\mathbf{D} \models \varphi(\bar{c})$ and $\mathbf{C} \models \varphi(\bar{c})$, since it is e.c.

I proved $\mathbf{C} \models \forall \bar{x}\,(D_{\bar{a}}(\bar{x}) \Rightarrow \varphi(\bar{x}))$; so $\mathbf{C} \models T'$.

LEMMA 1. *If* T *is a universal theory, the following are equivalent*:

i) *condition* (′) *is satisfied* (*only for n*);

ii) *there is a finite number of non-isomorphic models of* T *generated by n elements*;

iii) (′) *is satisfied for n, with the additional hypothesis that the axiomatization uses only sentences of* D (**A**);

iv) *there is a finite number of basic* (*i.e. atomic or negated atomic*) *formulas such that any other basic formula containing only the first n variables is* (T-*equivalent to*) *a Boolean combination of those.*

Proof. The only non-trivial part is i) ⇒ ii) (since ii) & iii) ⇒ iv)). So assume there are infinite models of T generated by *n* elements, and arrange all them in a sequence $\mathbf{A}_1, \cdots, \mathbf{A}_\beta, \cdots$ $(\beta \in \alpha)$.

In $\mathbf{L}(T) \cup \bar{a}$, $\{\neg D_{\mathbf{A}_\beta}(\bar{a}) \mid \beta \in \alpha\} + T$ is finitely consistent, so it has a model; but the submodel generated by $\bar{a}$ is a model of T different from any one of the $\mathbf{A}_\beta$'s, absurd.

Note that if T satisfies iv), and all symbols of **L** (T) have a-rity less or equal than *n*, then T is equivalent to a theory in a finite language, with the other symbols thought as definitions. This is false if T satisfies iv) for each n and the a-rity of the symbols of **L** (T) is not bounded (the counterexample is easy).

COROLLARY 1. *If* T *is a universal, locally finite theory in a language with only a finite number of relations, then the following are equivalent*:

i) T *has* finite presentation decomposition (*see* [8] *for the definition*);

ii) *anyone of the conditions of Lemma 1 is true for every n.*

In particular, in this case, Corollary 1 of [8] *and my Theorem 1 coincide.*

Proof. i) $\Rightarrow$ ii) Take $\psi \equiv x = x$ in the definition of finite presentation decomposition, and ii) of Lemma 1 is verified, since a finite model with finitely many relations has only a finite number of homomorphic images. The converse is easy.

§ 2. THE GENERAL CASE

The proof of Theorem 1 suggests that what we really need is not the a.p. for models, but the one for 'finitely definable properties', instead. So the treatment for the case when the a.p. fails, or the language is 'really' infinite is almost the same (the best way to see this is to state and prove the next theorem separately for the two cases).

A *pseudo submodel* of **A** is a substructure of a reduct of **A**; a *pseudo model* of T is a pseudo submodel of a model of T; it is *finitely generated* iff it has finite language $\mathbf{L}_0$ and is finitely generated as a model in $\mathbf{L}_0$.

Condition (′) will be replaced by:

(″) *for every finitely generated pseudo model* X *of* T, D (X) *is finitely axiomatizable relative to* T.

(A locally finite theory clearly satisfies (″)).

For the remainder of the paragraph $\mathbf{A}, \mathbf{B}, \cdots$ will be models, $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \cdots$ finitely generated pseudo models of a fixed theory T. '$\subseteq$' will mean pseudo submodel.

THEOREM 2. *Let* T *be a universal theory satisfying* (″). *Then* T *has a model companion* T′ *iff*

$\forall \mathrm{X}, \forall \mathrm{Y} \supseteq \mathrm{X}$, *there is a finite number of models* $\mathrm{Y}_i \supseteq \mathrm{X}$ *such that one of the following equivalent conditions holds*:

i) *for any e.c. structure* $\mathbf{B} \supseteq \mathrm{X}$:

Y *can be embedded in* **B** (*over* X) *iff none of the* Y_i*'s can be embedded in* **B** (*over* X).

ii) Y *cannot be amalgamated with anyone of the* Y_i*'s and for any* $\mathbf{A} \supseteq X$, *if* **A** *cannot be amalgamated with anyone of the* Y_i*'s, then it can be amalgamated with* Y.

iii) *same as* ii) *but with* Z *in place of* **A**.

Proof. It must be clear from the proof of Theorem 1 that a model **B** is e.c. iff for any X′, Y′, **C**:

$$\begin{array}{ccc} & \mathbf{C} & \\ \subseteq & & \supseteq \\ \mathbf{B} & & Y' \\ \cup\!| & & |\!\cup \\ & X' & \end{array} \qquad \text{commutes implies } Y' \text{ can be embedded in } \mathbf{B}.$$

Indeed, an existential formula $E\bar{y}\,(\varphi\,(\bar{x}\,,\bar{y}))$ is equivalent to $\bigvee_{Z} E\bar{y}\,(D_Z(\bar{x}\,,\bar{y}))$, where Z varies over the pseudo models generated by a number of elements equal to the number of variables of φ, in the language of φ and which satisfy φ. (There is a finite number of such X's by a reasoning similar to the proof of Lemma 1).

So assume T is companionable, then in T′ any existential formula is equivalent to a universal one; in particular:

$$(1) \qquad E\bar{y}\,(D_Y\,(\bar{x}\,,\bar{y})) \Longleftrightarrow \forall\bar{z}\varphi\,(\bar{x}\,,\bar{z}) \Longleftrightarrow \bigwedge_i \forall\bar{z}\,(\sim D_{Y_i}\,(\bar{x}\,,\bar{z}))\,,$$

for certain Y_i's. That is exactly the meaning of i).

Conversely, the equivalence of the edge formulas, for any expression like (1), axiomatizes the model-companion of T, if i) holds.

i) $\Rightarrow$ iii): if an amalgam between Y and Y_i exists, it can be extended to an e.c. structure, absurd. The other condition is obtained extending Z to an e.c. model.

ii) $\Rightarrow$ i): use the first observation to relate amalgamability and embeddability for e.c. structures.

iii) $\Rightarrow$ ii): by compactness (two models are not amalgamable iff T plus the theory of their diagrams is inconsistent).

If we think of any n-ary function symbol as an $n+1$-ary relation, we can transform a theory T to another 'equivalent' theory T^{Rel} whose language has only relation symbols. It's easy to see that T is companionable iff T^{Rel} is, or iff $(T^{Rel})_\forall$ is; and this last theory clearly satisfies (″). From this point of view, pseudo models of $(T^{Rel})_\forall$ are *relative partial subalgebras* of models of T (cf. [3]), and all above can be translated for a general theory using this concept.

Similar methods (which I don't think at all to be new) may be of practical utility, too. For example, in [5], what I really proved is that the theory T of Boolean algebras with a relation $R(x, y)$ meaning 'y is the closure of x' has not m.c. Even if I didn't know, Proposition 3 gives a set of axioms for $T_\forall$. Moreover, problem v) can be solved in this way considering " $'$ " as a binary relation.

§ 3. THE LOCAL MODEL COMPLETION OF A THEORY

$\mathbf{A}, \mathbf{B}, \cdots$ will be models of T; $\mathbf{A}', \mathbf{B}', \cdots$ models of T$'$.

THEOREM 3. *Let* T *be a universal theory satisfying* ($'$), *and such that there are sentences* φ_i $(i \in I)$ *with*:

a) $A, B \models \varphi_i$ and $\begin{array}{ccc} \mathbf{A} & & \mathbf{B} \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array}$

imply that there is a $\mathbf{D}$ such that:

$$\begin{array}{ccc} & \mathbf{D} & \\ \subseteq & & \supseteq \\ \mathbf{A} & & \mathbf{B} \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array};$$

b) If $\begin{array}{ccc} & \mathbf{D} & \\ \subseteq & & \supseteq \\ \mathbf{A} & & \mathbf{B} \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array}$, then there is an $i \in I$ such that $\mathbf{A}, \mathbf{B} \models \varphi_i$.

Then T *has model companion.*

Proof. The class of e.c. structures is axiomatized by: $T + \{\varphi_i \Rightarrow \Rightarrow \forall \bar{x} (D_A(\bar{x}) \Rightarrow \varphi(\bar{x})) \mid i \in I,\ \varphi$ is an existential formula, and there are two models $\mathbf{B} \supseteq \mathbf{A} = \langle \bar{a} \rangle$ with $\mathbf{B} \models T + \varphi_i + \varphi(\bar{a})\}$ (use the method of Theorem 1).

If the hypothesis of Theorem 3 are satisfied, T has more than m.c.:

DEFINITION. *A theory* T *has* local model completion *iff it has a model companion* T$'$ *and one of the following conditions (equivalent under the hypothesis) holds*:

a) $\begin{array}{ccc} \mathbf{A}' & & \mathbf{B}' \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array}$ and $\mathbf{A}' \equiv_{L(T)} \mathbf{B}'$ imply $\mathbf{A}' \equiv_{L(C)} \mathbf{B}'$;

b) $\begin{array}{ccc} \mathbf{A} & & \mathbf{B} \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array}$ and $\begin{array}{ccc} & \mathbf{D} & \\ \subseteq & & \supseteq \\ \mathbf{A} & & \mathbf{B} \end{array}$ imply that there is an $\mathbf{E}$ with $\begin{array}{ccc} & \mathbf{E} & \\ \subseteq & & \supseteq \\ \mathbf{A} & & \mathbf{B} \\ |\!\cup & & \cup\!| \\ & \mathbf{C} & \end{array}$.

(The equivalence is shown as in the proofs of Propositions 2.6 and 2.8 of [4]).

If, in the definition, we require $T' \supseteq T$, the local model completion is a concept strictly between m.c. and model completion. Indeed, if T' is complete and model complete, but not substructurally complete, it is the model companion of $T'_\forall$, but not the local model completion; the other counterexample can be the theory of non unitary commutative rings satisfying "$yx = 0 \Rightarrow x = 0 \vee y = 0$".

Problems (half of them are probably crazy, or off point, or both!):

i) If T satisfies the hypothesis of Theorem 3, then T has local model completion. Is the converse true?

ii) Characterize locally finite theories such that $Z \cap F = \varnothing$, or $Z = F$, etc. etc. (cf. Diagram 1 of [7]).

iii) Try to see what the results of [8] become when finite presentation decomposition is not assumed (perhaps the observation in Corollary 1 may help).

iv) What does all this become in infinitary languages, soft model theory, or categories etc. etc.?

v) A universal theory has model completion iff it is mutually model consistent with a theory in which every formula is equivalent to a quantifier free one; it has model companion iff it is mutually model consistent with a theory in which every formula is equivalent to an existential one. Would be interesting to extend this to Σ_n formulas?

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