Research Article

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Radon measure-valued solutions of first order scalar conservation laws

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Abstract: We study nonnegative solutions of the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x[\varphi(u)] = 0 & \text{in } \mathbb{R}\times(0,T), \\ u = u_0 \geq 0 & \text{in } \mathbb{R}\times\{0\}, \end{cases}$$

where u_0 is a Radon measure and $\varphi\colon [0,\infty) \mapsto \mathbb{R}$ is a globally Lipschitz continuous function. We construct suitably defined entropy solutions in the space of Radon measures. Under some additional conditions on φ, we prove their uniqueness if the singular part of u_0 is a finite superposition of Dirac masses. Regarding the behavior of φ at infinity, we give criteria to distinguish two cases: either all solutions are function-valued for positive times (an instantaneous regularizing effect), or the singular parts of certain solutions persist until some positive *waiting time* (in the linear case $\varphi(u) = u$ this happens for all times). In the latter case, we describe the evolution of the singular parts.

Keywords: First order hyperbolic conservation laws, Radon measure-valued solutions, entropy inequalities, uniqueness

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1 Introduction

In this paper we consider the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x[\varphi(u)] = 0 & \text{in } \mathbb{R}\times(0,T) =: S, \\ u = u_0 & \text{in } \mathbb{R}\times\{0\}, \end{cases} \tag{P}$$

where $T > 0$, u_0 is a nonnegative finite Radon measure on $\mathbb{R}$, and $\varphi\colon [0,\infty) \mapsto \mathbb{R}$, $\varphi(0) = 0$, is a Lipschitz continuous function (see assumption (H1)). Therefore, φ grows at most linearly.

Problem (P) with a *superlinear* φ of the type $\varphi(u) = u^p$, $p > 1$, was studied in [19], proving existence and uniqueness of nonnegative entropy solutions (see also [8]). By definition, in that paper the solution for positive times takes values in $L^1(\mathbb{R})$, although the initial data u_0 is a finite Radon measure. Interesting, albeit sparse results concerning (P) with *φ at most linear at infinity* can be found in the pioneering paper [10], in which the same definition of Radon measure-valued solutions used below (see equality (3.8)) was proposed.

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When $\varphi(u) = Cu$ ($C \in \mathbb{R}$), problem (P) is the Cauchy problem for the *linear transport equation*

$$\begin{cases} \partial_t u + C\partial_x u = 0 & \text{in } S, \\ u = u_0 & \text{in } \mathbb{R}\times\{0\}, \end{cases}$$

whose solution is trivially the translated of u_0 along the lines $x = Ct + x_0$ ($x_0 \in \mathbb{R}$). In particular, the singular part $u_s(\cdot, t)$ of the solution is nonzero for $t > 0$ if and only if the same holds for $t = 0$.

It is natural to ask what happens if φ is *sublinear*. To address this case we must consider *solutions of problem* (P) *which, for* $t > 0$, *possibly are finite Radon measures on* $\mathbb{R}$ *as the initial data* u_0. Therefore, throughout the paper we consider solutions of problem (P) as maps from $[0, T]$ to the cone of nonnegative finite Radon measures on $\mathbb{R}$, which satisfy (P) in the following sense: for a suitable class of test functions ζ, we have

$$\iint_S [u_r\partial_t\zeta + \varphi(u_r)\partial_x\zeta]\,dx\,dt + \int_0^T \langle u_s(\cdot, t), \partial_v\zeta(\cdot, t)\rangle_{\mathbb{R}}\,dt = -\langle u_0, \zeta(\cdot, 0)\rangle_{\mathbb{R}}$$

(see Definition 3.3). Here the measure $u(t)$ is defined for a.e. $t \in (0, T)$, $u_r \in L^1(S)$ is the density of its absolutely continuous part, $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ denotes the duality map, and

$$\partial_v\zeta := \partial_t\zeta + C_\varphi\partial_x\zeta, \quad C_\varphi := \lim_{u\to\infty}\frac{\varphi(u)}{u}.$$

Measure-valued entropy solutions are defined similarly (see Definition 3.3).

We use an approximation procedure to construct measure-valued entropy solutions of problem (P) (see Theorem 3.7). In addition, we prove that the singular part u_s of an entropy solution of problem (P) does not increase along the lines $x = x_0 + C_\varphi t$ (see Proposition 3.8). In particular, if $C_\varphi = 0$, the map $t \mapsto u_s(\cdot, t)$ is nonincreasing.

Concerning the case when φ is sublinear, the following example is particularly instructive:

$$\begin{cases} \partial_t u + \partial_x[\varphi(u)] = 0 & \text{in } S, \\ u = \delta_0 & \text{in } \mathbb{R}\times\{0\}, \end{cases} \tag{1.1}$$

with $S := \mathbb{R}\times(0, T)$, $T > 1$ and

$$\varphi(u) = \operatorname{sgn} p[(1+u)^p - 1] \quad (p < 1, p \neq 0). \tag{1.2}$$

The function in (1.2) is increasing and concave, with $C_\varphi = 0$, and belongs to a class for which the constructed entropy solution of problem (1.1)–(1.2) is unique (see Theorem 3.22). Hence, the following holds.

Proposition 1.1. (i) *Let* $p < 0$. *Let* $\xi(t)$ *be defined by*

$$\xi' = -\frac{(|p|t\xi^{-1})^{\frac{p}{1-p}} - 1}{(|p|t\xi^{-1})^{\frac{1}{1-p}} - 1} \quad \textit{in } (1, T), \qquad \xi(1) = 0.$$

Let

$$A := \{(x, t) \in S \mid 0 < x \le |p|t,\ 0 \le t \le 1\} \cup \{(x, t) \in S \mid \xi(t) \le x \le |p|t,\ 1 < t \le T\}$$

and

$$u_s(t) := \max\{1 - t, 0\}\delta_0, \qquad u_r(x, t) := [(|p|tx^{-1})^{\frac{1}{1-p}} - 1]\chi_A(x, t) \quad ((x, t) \in S). \tag{1.3}$$

Then $u = u_r + u_s$ *is the unique constructed entropy solution of problem* (1.1)–(1.2).

(ii) *Let* $0 < p < 1$. *Let* $\xi(t)$ *be defined by*

$$\xi' = \frac{(|p|t\xi^{-1})^{\frac{p}{1-p}} - 1}{(|p|t\xi^{-1})^{\frac{1}{1-p}} - 1} \quad \textit{in } (0, T), \qquad \xi(0) = 0.$$

If $B := \{(x, t) \in S \mid \xi(t) \le x \le |p|t,\ 0 < t \le T\}$, *then*

$$u(x, t) = u_r(x, t) := [(|p|tx^{-1})^{\frac{1}{1-p}} - 1]\chi_B(x, t) \quad ((x, t) \in S) \tag{1.4}$$

is the unique constructed entropy solution of problem (1.1)–(1.2).

Let us define the *waiting time* $t_0 \in [0, T]$ for solutions u of (P):

$$t_0 := \inf\{\tau \in (0, T] \mid u_s(\cdot, t) = 0,\ u_r(\cdot, t) \in L^\infty(\mathbb{R}) \text{ for a.e. } t \in (\tau, T)\} \tag{1.5}$$

(by abuse of language, we call t_0 "waiting time" even if $t_0 = T$). Then, by Proposition 1.1,
($*$) positive waiting times occur in problem (1.1)–(1.2) if and only if $p < 0$.

More precisely, if $p < 0$, the singular part $u_s(\cdot, t)$ persists until the waiting time $t_0 = 1$ at which it disappears, whereas for $0 < p < 1$, the singular part vanishes for all $t > 0$, thus $t_0 = 0$ – an instantaneous regularizing effect. Instantaneous regularization also occurs if $p > 1$ (see [19] and Remark 3.24), whereas, as already remarked, in the linear case $p = 1$, we have $t_0 = T$ if $u_{0s} \neq 0$.

Since $\varphi(u) = \operatorname{sgn} p[(1 + u)^p - 1]$ ($p < 1, p \neq 0$) is bounded if and only if $p < 0$, and $C_\varphi = 0$, statement ($*$) could be rephrased as follows.

Proposition 1.2. *Positive waiting times occur in problem* (1.1) *if and only if the map* $u \mapsto \varphi(u) - C_\varphi u$*, with* φ *as in* (1.2)*, is bounded in* $[0, \infty)$.

The above result is generalized to problem (P), by Theorem 3.18, for functions φ which satisfy for u large a condition implying either concavity or convexity (see assumption (H4) and Remark 3.13). The proof of Theorem 3.18 makes use of estimates of the density u_r of the solution of (P), which are strongly reminiscent of the *Aronson–Bénilan inequality* for the porous medium equation (see Proposition 6.2). The main results on the waiting time and the regularity of solutions of (P) are collected in Section 3.3. The existence and an upper bound, in terms of φ and u_0, of a waiting time was already pointed out in [10, Proposition 2.1] (see also Theorem 3.8 (ii)).

Another interesting feature of the solution of (1.1)–(1.2), with $p < 0$, is that for $t \in (0, 1)$, i.e., as long as $u_s(\cdot, t) > 0$, we have

$$\lim_{x \to 0^+} u_r(x, t) = \infty.$$

Namely, the regular part $u_r(\cdot, t)$ diverges when approaching from the right the point $x_0 = 0$, where $u_s(\cdot, t)$ is concentrated. As we shall see below (see (3.24)–(3.25)), this property can be generalized to entropy solutions of a larger class of problems, characterized by the concavity/convexity property on φ mentioned before. In this class a generalized form of this property will also be used as a uniqueness criterion, provided that $\varphi(u) - C_\varphi u$ is bounded in $[0, \infty)$ and u_{0s} is a finite superposition of Dirac masses (see Proposition 3.17 and Theorem 3.22). In [10] it was already observed that Kruzkov's entropy inequalities do not guarantee the uniqueness of solutions (see also Remark 3.23 below), and the formulation of an additional uniqueness criterion was left as an open problem. This problem is addressed in a forthcoming paper, where more general *compatibility conditions* are given, which ensure uniqueness also for non-convex or non-concave functions φ (see [3]).

Apart from the intrinsic mathematical interest of problem (P), it is worth pointing out its connection with a class of relevant models. Ion etching is a common technique for the fabrication of semiconductor devices, also relevant in other fields of metallurgy, in which the material to be etched is bombarded with an ion beam (see [16, 24, 25]). Mathematical modelling of the process leads to the Hamilton–Jacobi equation in one space dimension

$$\begin{cases} \partial_t U + \varphi(\partial_x U) = 0 & \text{in } \mathbb{R} \times (0, T), \\ U = U_0 & \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{HJ}$$

where $U = U(x, t)$ denotes the thickness of the material and φ is bounded, non-convex and vanishing at infinity. Formal differentiation with respect to x suggests to describe the problem in terms of the unknown $u := \partial_x U$, which formally solves (P) with $u_0 = U_0'$. In this way, discontinuous solutions of (HJ) correspond to Radon measure-valued solutions of (P) having a Dirac mass δ_{x_0} concentrated at any point x_0, where $U(\cdot, t)$ is discontinuous ($t \in (0, T)$). A rigorous justification of the above argument, relating *discontinuous viscosity solutions* of (HJ) to *Radon measure-valued entropy solutions* of (P), is to our knowledge an open problem (in this connection, see [7, 14]).

In the context of conservation laws, the term "measure-valued solution" usually refers to solutions in the sense of Young measures, after DiPerna's seminal paper [11]. We stress that this concept of "statistical solutions" is completely different from that of Radon measure-valued solutions, introduced by Demengel and Serre [10], and discussed in the present paper. On the other hand, we do use Young measures in this paper, since they are an important ingredient in the construction of Radon measure valued solutions (see Section 3 and, in particular, Section 5).

A number of ideas used in the present paper go back to papers dealing with Radon measure-valued solutions of quasilinear parabolic problems, also of forward-backward type (in particular, see [4–6, 21, 23, 27]).

The results presented in this paper naturally lead to some open problems. Among them we mention a general statement about an instantaneous regularizing effect for fluxes with superlinear growth (singular parts should disappear instantaneously for $t > 0$), and an appropriate generalization of our results to the case of solutions with changing signs, when additional nonuniqueness phenomena (such as N-waves, see [19]) may occur; in this regard, the general case of an initial signed Radon measure u_0 in problem (P) will be considered in a forthcoming paper. Another open problem is whether new phenomena occur if φ is uniformly Lipschitz continuous on $[0, \infty)$ but the limit $\varphi(s)/s$ as $s \to \infty$ does not exist.

The paper is organized as follows. In Section 2 we recall several known results used in the sequel and introduce some notation. In Section 3 we present the main results of the paper. In Section 4 we introduce the approximation procedure needed for the construction of solutions. Sections 5–7 are devoted to the proofs of existence, qualitative properties and uniqueness of solutions.

2 Preliminaries

2.1 Function spaces and Radon measures

We denote by $\mathcal{M}(\mathbb{R})$ the Banach space of finite Radon measures on $\mathbb{R}$, with norm $\|\mu\|_{\mathcal{M}(\mathbb{R})} := |\mu|(\mathbb{R})$. By $\mathcal{M}^+(\mathbb{R})$, we denote the cone of nonnegative finite Radon measures; if $\mu_1, \mu_2 \in \mathcal{M}(\mathbb{R})$, then we write $\mu_1 \leq \mu_2$ if $\mu_2 - \mu_1 \in \mathcal{M}^+(\mathbb{R})$. We denote the convex set of probability measures on $\mathbb{R}$ by $\mathcal{P}(\mathbb{R}) \subset \mathcal{M}^+(\mathbb{R})$. We have $\|\tau\|_{\mathcal{M}(\mathbb{R})} = \tau(\mathbb{R}) = 1$ for $\tau \in \mathcal{P}(\mathbb{R})$.

We denote by $C_c(\mathbb{R})$ the space of continuous real functions with compact support in $\mathbb{R}$. The space of the functions of bounded variation in $\mathbb{R}$ is denoted by $\mathrm{BV}(\mathbb{R}) := \{u \in L^1(\mathbb{R}) \mid u' \in \mathcal{M}(\mathbb{R})\}$, where u' is the distributional derivative of u. It is endowed with the norm $\|u\|_{\mathrm{BV}(\mathbb{R})} := \|u\|_{L^1(\mathbb{R})} + \|u'\|_{\mathcal{M}(\mathbb{R})}$. We say that $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ if $u \in \mathrm{BV}(\Omega)$ for every open bounded subset $\Omega \subset \mathbb{R}$.

The Lebesgue measure, either on $\mathbb{R}$ or $S := \mathbb{R} \times (0, T)$, is denoted by $|\cdot|$. Integration with respect to the Lebesgue measure on $\mathbb{R}$ or on S will be denoted by the usual symbols dx, respectively $dx\,dt$. A Borel set E is *null* if $|E| = 0$. The expression "almost everywhere", or shortly "a.e.", means "up to null sets". For every measurable function f defined on $\mathbb{R}$ and $x_0 \in \mathbb{R}$, we write $\operatorname{ess\,lim}_{x \to x_0} f(x) = l \in \mathbb{R}$ if there is a null set $E^* \subseteq \mathbb{R}$ such that $f(x_n) \to l$ for any sequence $\{x_n\} \subseteq \mathbb{R} \setminus (E^* \cup \{x_0\})$, $x_n \to x_0$. We set $f^\pm := \max\{\pm f, 0\}$ for every measurable function f on $\mathbb{R}$.

We denote the duality map between $\mathcal{M}(\mathbb{R})$ and $C_c(\mathbb{R})$ by $\langle \mu, \rho \rangle_{\mathbb{R}} := \int_{\mathbb{R}} \rho \, d\mu$. By abuse of notation, we extend $\langle \mu, \rho \rangle_{\mathbb{R}}$ to any μ-integrable function ρ. A sequence $\{\mu_n\}$ converges strongly to μ in $\mathcal{M}(\mathbb{R})$ if $\|\mu_n - \mu\|_{\mathcal{M}(\mathbb{R})} \to 0$ as $n \to \infty$. A sequence $\{\mu_n\}$ of (possibly not finite) Radon measures on $\mathbb{R}$ converges weakly$*$ to a (possibly not finite) Radon measure μ, i.e., $\mu_n \overset{*}{\rightharpoonup} \mu$, if $\langle \mu_n, \rho \rangle_{\mathbb{R}} \to \langle \mu, \rho \rangle_{\mathbb{R}}$ for all $\rho \in C_c(\mathbb{R})$. Similar definitions are used for (possibly not finite) Radon measures on $\Omega \times (0, T)$, with $\Omega \subseteq \mathbb{R}$.

Every $\mu \in \mathcal{M}(\mathbb{R})$ has a unique decomposition $\mu = \mu_{\mathrm{ac}} + \mu_s$, with $\mu_{\mathrm{ac}} \in \mathcal{M}(\mathbb{R})$ absolutely continuous and $\mu_s \in \mathcal{M}(\mathbb{R})$ singular with respect to the Lebesgue measure. We denote by $\mu_r \in L^1(\mathbb{R})$ the density of μ_{ac}. Every function $f \in L^1(\mathbb{R})$ can be identified to a finite absolutely continuous Radon measure on $\mathbb{R}$; we shall denote this measure by the same symbol f used for the function.

The restriction $\mu \llcorner E$ of $\mu \in \mathcal{M}(\mathbb{R})$ to a Borel set $E \subseteq \mathbb{R}$ is defined by $(\mu \llcorner E)(A) := \mu(E \cap A)$ for any Borel set $A \subseteq \mathbb{R}$. Similar notations are used for the spaces of finite Radon measures $\mathcal{M}(\Omega)$, with $\Omega \subseteq \mathbb{R}$, $\mathcal{M}(S)$ and $\mathcal{M}(S \times \mathbb{R})$, where $S := \mathbb{R} \times (0, T)$.

We shall use measures $u \in \mathcal{M}(S)$ which, roughly speaking, admit a parametrization with respect to the time variable.

Definition 2.1. We denote by $L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ the set of finite nonnegative Radon measures $u \in \mathcal{M}^+(S)$ such that for a.e. $t \in (0, T)$, there is a measure $u(\cdot, t) \in \mathcal{M}^+(\mathbb{R})$ with the following properties:
(i) if $\zeta \in C([0, T]; C_c(\mathbb{R}))$, the map $t \mapsto \langle u(\cdot, t), \zeta(\cdot, t)\rangle_{\mathbb{R}}$ belongs to $L^1(0, T)$ and

$$\langle u, \zeta\rangle_S = \int_0^T \langle u(\cdot, t), \zeta(\cdot, t)\rangle_{\mathbb{R}}\, dt, \tag{2.1}$$

(ii) the map $t \mapsto \|u(\cdot, t)\|_{\mathcal{M}(\mathbb{R})}$ belongs to $L^\infty(0, T)$.
Accordingly, we set

$$\|u\|_{L^\infty(0,T;\mathcal{M}(\mathbb{R}))} := \operatorname*{ess\,sup}_{t\in(0,T)} \|u(\cdot, t)\|_{\mathcal{M}(\mathbb{R})} \quad \text{for } u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R})).$$

Remark 2.2. The definition implies that for all $\rho \in C_c(\mathbb{R})$, the map $t \mapsto \langle u(\cdot, t), \rho\rangle_{\mathbb{R}}$ is measurable, thus the map $u\colon (0, T) \to \mathcal{M}(\mathbb{R})$ is weakly$*$ measurable (e.g., see [22, Section 6.7]). For simplicity, we prefer the notation $L^\infty(0, T; \mathcal{M}(\mathbb{R}))$ to the more correct one $L^\infty_{w*}(0, T; \mathcal{M}(\mathbb{R}))$, which is used in [22].

If $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$, then also $u_{ac}, u_s \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ and, by (2.1),

$$\langle u_{ac}, \zeta\rangle_S = \iint_S u_r\zeta\, dx\, dt, \quad \langle u_s, \zeta\rangle_S = \int_0^T \langle u_s(\cdot, t), \zeta(\cdot, t)\rangle_{\mathbb{R}}\, dt \tag{2.2}$$

for $\zeta \in C([0, T]; C_c(\mathbb{R}))$. One can easily check that for a.e. $t \in (0, T)$,

$$u_{ac}(\cdot, t) = [u(\cdot, t)]_{ac}, \quad u_s(\cdot, t) = [u(\cdot, t)]_s, \quad u_r(\cdot, t) = [u(\cdot, t)]_r, \tag{2.3}$$

where $[u(\cdot, t)]_r$ denotes the density of the measure $[u(\cdot, t)]_{ac}$. For $\rho \in C_c(\mathbb{R})$, we have

$$\langle [u(\cdot, t)]_{ac}, \rho\rangle_{\mathbb{R}} = \int_{\mathbb{R}} [u(\cdot, t)]_r\rho\, dx = \int_{\mathbb{R}} u_r(\cdot, t)\rho\, dx \quad \text{for a.e. } t \in (0, T).$$

In view of (2.2)–(2.3), we shall always identify the quantities which appear on either side of equalities (2.3).

For any $\mu \in \mathcal{M}(\mathbb{R})$ and $a \in \mathbb{R}$, the *translated measure* $\mathcal{T}_a(\mu)$ is defined by

$$\langle \mathcal{T}_a(\mu), \rho\rangle_{\mathbb{R}} := \langle \mu, \rho_{-a}\rangle_{\mathbb{R}}$$

for any $\rho \in C_c(\mathbb{R})$, where $\rho_{-a}(x) := \rho(x + a)$ $(x \in \mathbb{R})$. Clearly, $\mathcal{T}_a(\mu) \in \mathcal{M}(\mathbb{R})$ and

$$[\mathcal{T}_a(\mu)]_{ac} = \mathcal{T}_a(\mu_{ac}), \quad [\mathcal{T}_a(\mu)]_s = \mathcal{T}_a(\mu_s).$$

2.2 Young measures

We recall the following result [2].

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be Lebesgue measurable, let $K \subseteq \mathbb{R}$ be closed, and let $u_n\colon \Omega \mapsto \mathbb{R}$ be a sequence of Lebesgue measurable functions such that*

$$\lim_{n\to\infty} \big|\{x \in \Omega \mid u_n(x) \notin U\}\big| = 0$$

for any open neighborhood U of K in $\mathbb{R}$. Then there exist a subsequence $\{u_j\} \equiv \{u_{n_j}\} \subseteq \{u_n\}$ and a family $\{\tau_x\}$ of nonnegative measures on $\mathbb{R}$, depending measurably on $x \in \Omega$, such that
(i) $\|\tau_x\|_{\mathcal{M}(\mathbb{R})} := \int_{\mathbb{R}} d\tau_x \leq 1$ *for a.e. $x \in \Omega$,*
(ii) $\operatorname{supp} \tau_x \subseteq K$ *for a.e. $x \in \Omega$,*

(iii) *for every continuous function* $f\colon \mathbb{R} \mapsto \mathbb{R}$ *satisfying* $\lim_{|\xi|\to\infty} f(\xi) = 0$, *we have*

$$f(u_j) \overset{*}{\rightharpoonup} f^* \quad \text{in } L^\infty(\Omega),$$

where

$$f^*(x) := \langle \tau_x, f\rangle_{\mathbb{R}} = \int_{\mathbb{R}} f(\xi)\, d\tau_x(\xi) \quad \text{for a.e. } x \in \Omega. \tag{2.4}$$

Suppose further that $\{u_j\}$ *satisfies the boundedness condition*

$$\lim_{k\to\infty} \sup_j \big|\{x \in \Omega \cap B_R \mid |u_j(x)| \geq k\}\big| = 0 \tag{2.5}$$

for every $R > 0$, *where* $B_R := \{x \in \mathbb{R}^N \mid |x| < R\}$. *Then*

(iv) τ_x *is a probability measure for a.e.* $x \in \Omega$,

(v) *given any measurable subset* $A \subseteq \Omega$, *we have*

$$f(u_j) \rightharpoonup f^* \quad \text{in } L^1(A) \tag{2.6}$$

for all continuous functions $f\colon \mathbb{R} \mapsto \mathbb{R}$ *such that* $\{f(u_j)\}$ *is sequentially weakly compact in* $L^1(A)$.

Below we shall always refer to the family $\{\tau_x\}$ of probability measures given by the previous theorem as the *disintegration of the Young measure* τ (or briefly Young measure) associated to the sequence $\{u_j\}$. We denote the set of Young measures on $\Omega \times \mathbb{R}$ by $\mathcal{Y}(\Omega; \mathbb{R})$; in particular, $\mathcal{Y}(S; \mathbb{R})$ denotes the set of Young measures on $S \times \mathbb{R}$, with $S := \mathbb{R} \times (0, T)$.

Remark 2.4. (i) The argument used in the proof of Theorem 2.3 shows that, under hypothesis (2.5), the convergence in (2.6) holds true for Carathéodory functions $f\colon A \times \mathbb{R} \mapsto \mathbb{R}$ if $\{f(\cdot\,, u_j)\}$ is sequentially weakly relatively compact in $L^1(A)$.

(ii) Condition (2.5) is very weak. It is equivalent to the statement that for any $R > 0$, there is a continuous nondecreasing function $g_R\colon [0, \infty) \mapsto \mathbb{R}$ such that

$$\lim_{\xi\to\infty} g_R(\xi) = \infty, \qquad \sup_j \int_{\Omega\cap B_R} g_R(|u_j(x)|)\, dx < \infty.$$

Therefore, Theorem 2.3 applies to bounded sequences $\{u_j\}$ in $L^1(\Omega)$ (in which case $g_R(\xi) = \xi$).

If $\Omega \subset \mathbb{R}^N$ is bounded and $\{u_j\}$ is a bounded but not uniformly integrable sequence in $L^1(\Omega)$, it is possible to extract a uniformly integrable subsequence "by removing sets of small measure". This is the content of the following "Biting lemma" (e.g., see [17, 28] and references therein).

Theorem 2.5. *Let* $\{u_n\}$ *be a bounded sequence in* $L^1(\Omega)$, *where* $\Omega \subset \mathbb{R}^N$ *is a bounded open set. Moreover, let* $\{u_j\} \subseteq \{u_n\}$ *and* $\{\tau_x\}$ *be the subsequence and the Young measure given in Theorem 2.3, respectively. Then there exist a subsequence* $\{u_k\} \equiv \{u_{j_k}\} \subseteq \{u_j\}$ *and a decreasing sequence of measurable sets* $E_k \subseteq \Omega$ *of Lebesgue measure* $|E_k| \to 0$ *such that the sequence* $\{u_k \chi_{\Omega\setminus E_k}\}$ *is uniformly integrable and*

$$u_k \chi_{\Omega\setminus E_k} \rightharpoonup Z := \int_{\mathbb{R}} \xi\, d\tau(\xi) \quad \text{in } L^1(\Omega),$$

where $Z \in L^1(\Omega)$ *is called the* barycenter *of the disintegration* $\{\tau_x\}$.

3 Main results

Throughout the paper we assume that $u_0 \in \mathcal{M}^+(\mathbb{R})$. Concerning φ, we always suppose that

(H1) $\varphi \in C([0, \infty))$, $\varphi(0) = 0$, $\varphi' \in L^\infty(0, \infty)$, and $\lim_{u\to\infty} \frac{\varphi(u)}{u} =: C_\varphi$ exists.

Hence, there exists $M > 0$ such that

$$|\varphi'(u)| \leq M, \quad |\varphi(u)| \leq Mu \quad \text{for a.e. } u > 0. \tag{3.1}$$

3.1 Definition of solution

In the following definitions, we denote by

$$\partial_\nu \zeta := \partial_t \zeta + C_\varphi \partial_x \zeta \tag{3.2}$$

the derivative of any $\zeta \in C^1(S)$ along the vector $\underline{\tau} \equiv (C_\varphi, 1)$.

Definition 3.1. By a *solution* of problem (P) *in the sense of Young measures*, we mean a pair (u, τ) such that

(i) $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$, $\tau \in \mathcal{Y}(S; \mathbb{R})$,

(ii) $\operatorname{supp} \tau_{(x,t)} \subseteq [0, \infty)$ for a.e. $(x, t) \in S$, and

$$u_r(x, t) = \int_{[0,\infty)} \xi \, d\tau_{(x,t)}(\xi), \tag{3.3}$$

where $\tau_{(x,t)} \in \mathcal{P}(\mathbb{R})$ is the disintegration of τ,

(iii) for all $\zeta \in C^1([0, T]; C_c^1(\mathbb{R}))$, with $\zeta(\cdot, T) = 0$ in $\mathbb{R}$, we have

$$\iint_S [u_r \partial_t \zeta + \varphi^* \partial_x \zeta] \, dx \, dt + \int_0^T \langle u_s(\cdot, t), \partial_\nu \zeta(\cdot, t)\rangle_{\mathbb{R}} \, dt = -\langle u_0, \zeta(\cdot, 0)\rangle_{\mathbb{R}}, \tag{3.4}$$

where $\partial_\nu \zeta$ is defined by (3.2) and

$$\varphi^*(x, t) := \int_{[0,\infty)} \varphi(\xi) \, d\tau_{(x,t)}(\xi) \quad \text{for a.e. } (x, t) \in S. \tag{3.5}$$

By an *entropy solution* of problem (P) *in the sense of Young measures*, we mean a solution such that

$$\iint_S [E^* \partial_t \zeta + F^* \partial_x \zeta] \, dx \, dt + C_E \int_0^T \langle u_s(\cdot, t), \partial_t \zeta(\cdot, t)\rangle_{\mathbb{R}} \, dt + C_F \int_0^T \langle u_s(\cdot, t), \partial_x \zeta(\cdot, t)\rangle_{\mathbb{R}} \, dt$$
$$\geq - \int_{\mathbb{R}} E(u_{0r}) \zeta(x, 0) \, dx - C_E \langle u_{0s}, \zeta(\cdot, 0)\rangle_{\mathbb{R}} \tag{3.6}$$

for all ζ as above, $\zeta \geq 0$, and for every pair (E, F), $E, F : [0, \infty) \mapsto \mathbb{R}$, such that

(C1) E is convex, $E', F' \in L^\infty(0, \infty)$, $F' = E' \varphi'$ in $(0, \infty)$, and $\lim_{u \to \infty} \frac{E(u)}{u} =: C_E$, $\lim_{u \to \infty} \frac{F(u)}{u} =: C_F$ exist.

In (3.6), for a.e. $(x, t) \in S$, we set

$$E^*(x, t) := \int_{[0,\infty)} E(\xi) \, d\tau_{(x,t)}(\xi), \quad F^*(x, t) := \int_{[0,\infty)} F(\xi) \, d\tau_{(x,t)}(\xi).$$

Entropy subsolutions (respectively *supersolutions*) of problem (P) *in the sense of Young measures* are defined by requiring that inequality (3.6) be satisfied for all ζ and (E, F) as above, with E nondecreasing (respectively nonincreasing).

Observe that choosing $E(u) = \pm u$ in the entropy inequality (3.6) plainly gives the weak formulation (3.4).

Remark 3.2. (i) By (3.1), (3.3) and (3.5),

$$|\varphi^*(x, t)| \leq M \int_{[0,\infty)} \xi \, d\tau_{(x,t)}(\xi) = M u_r(x, t) \quad \text{for a.e. } (x, t) \in S. \tag{3.7}$$

Since $u_r \in L^\infty(0, T; L^1(\mathbb{R}))$, by (3.7), we have that $\varphi^* \in L^\infty(0, T; L^1(\mathbb{R}))$.

(ii) By (C1), the functions E, F have at most linear growth. Arguing as in (i), it follows that E^* and F^* belong to $L^\infty(0, T; L^1_{\mathrm{loc}}(\mathbb{R}))$ and $L^\infty(0, T; L^1(\mathbb{R}))$, respectively, if $E(0) = F(0) = 0$.

Definition 3.3. A measure $u \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$ is called a *solution* of problem (P) if for all $\zeta \in C^1([0,T];C_c^1(\mathbb{R}))$, $\zeta(\cdot,T)=0$ in $\mathbb{R}$, we have

$$\iint_S [u_r\partial_t\zeta + \varphi(u_r)\partial_x\zeta]\,dx\,dt + \int_0^T \langle u_s(\cdot,t), \partial_\nu\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt = -\langle u_0, \zeta(\cdot,0)\rangle_{\mathbb{R}}, \tag{3.8}$$

where $\partial_\nu\zeta$ is defined by (3.2). A solution of problem (P) is called an *entropy solution* if for all $\zeta \geq 0$ as above and for all (E,F) as in (C1), it satisfies the *entropy inequality*

$$\begin{aligned}\iint_S [E(u_r)\partial_t\zeta + F(u_r)\partial_x\zeta]\,dx\,dt + C_E\int_0^T \langle u_s(\cdot,t), \partial_t\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt + C_F\int_0^T \langle u_s(\cdot,t), \partial_x\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt \\ \geq -\int_{\mathbb{R}} E(u_{0r})\zeta(x,0)\,dx - C_E\langle u_{0s}, \zeta(\cdot,0)\rangle_{\mathbb{R}}.\end{aligned} \tag{3.9}$$

Entropy subsolutions (respectively *supersolutions*) of problem (P) are defined by requiring (3.9) to be satisfied for all ζ and (E,F) as before, with E nondecreasing (respectively nonincreasing).

A solution of problem (P) is also a solution in the sense of Young measures. Moreover, it follows from (3.1) that $\varphi(u_r) \in L^\infty(0,T;L^1(\mathbb{R}))$. Similar remarks hold for entropy solutions, subsolutions and supersolutions.

Remark 3.4. (i) If $C_\varphi = 0$, equality (3.8) reads

$$\iint_S [u\partial_t\zeta + \varphi(u_r)\partial_x\zeta]\,dx\,dt = -\langle u_0, \zeta(\cdot,0)\rangle_{\mathbb{R}},$$

whence $\partial_t u = -\partial_x[\varphi(u_r)]$ in $\mathcal{D}'(S)$.

(ii) For the Kružkov entropies $E(u) = |u-k|$, $F(u) = \operatorname{sgn}(u-k)[\varphi(u)-\varphi(k)]$ ($k \in [0,\infty)$), we have $C_E = 1$, $C_F = C_\varphi$. Then inequality (3.9), for all $k \in [0,\infty)$, reads

$$\begin{aligned}\iint_S \{|u_r-k|\partial_t\zeta + \operatorname{sgn}(u_r-k)[\varphi(u_r)-\varphi(k)]\partial_x\zeta\}\,dx\,dt + \int_0^T \langle u_s(\cdot,t), \partial_\nu\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt \\ \geq -\int_{\mathbb{R}} |u_{0r}-k|\zeta(x,0)\,dx - \langle u_{0s}, \zeta(\cdot,0)\rangle_{\mathbb{R}}.\end{aligned} \tag{3.10}$$

The following proposition states that for any solution of (P) in the sense of Young measures, the map $t \mapsto u(t)$, possibly redefined in a null set, is continuous up to $t=0$ with respect to the weak$*$ topology of $\mathcal{M}^+(\mathbb{R})$. In particular, it explains in which sense the initial condition is satisfied.

Proposition 3.5. *Let* (H1) *be satisfied, let* (u,τ) *be a solution of problem* (P) *in the sense of Young measures, and let* $\rho \in C_c(\mathbb{R})$. *Then*

$$\operatorname*{ess\,lim}_{t\to 0^+} \langle u(\cdot,t), \rho\rangle_{\mathbb{R}} = \langle u_0, \rho\rangle_{\mathbb{R}}, \tag{3.11}$$

$$\operatorname*{ess\,lim}_{t\to t_0} \langle u(\cdot,t), \rho\rangle_{\mathbb{R}} = \langle u(\cdot,t_0), \rho\rangle_{\mathbb{R}} \quad \textit{for a.e. } t_0 \in (0,T). \tag{3.12}$$

The map $t \mapsto u(t)$ *has a representative, defined for all* $t \in [0,T]$, *such that*

$$\lim_{t\to t_0} \langle u(\cdot,t), \rho\rangle_{\mathbb{R}} = \langle u(\cdot,t_0), \rho\rangle_{\mathbb{R}} \quad \textit{for all } t_0 \in [0,T]. \tag{3.13}$$

3.2 Existence and monotonicity

The existence of solutions is proven by an approximation procedure. If $u_0 \in \mathcal{M}^+(\mathbb{R})$, then there exist $u_{0n} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$u_{0n} \geq 0 \quad \text{in } \mathbb{R}, \qquad \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}, \tag{3.14}$$

$$u_{0n} \overset{*}{\rightharpoonup} u_0, \qquad u_{0n} \to u_{0r} \quad \text{a.e. in } \mathbb{R}, \qquad \|u_{0n} - u_{0r}\|_{L^1_{\mathrm{loc}}(\mathbb{R}\setminus \operatorname{supp} u_{0s})} \to 0 \tag{3.15}$$

(e.g., see [23, Lemma 4.1]). Consider the approximating problem

$$\begin{cases} \partial_t u_n + \partial_x[\varphi(u_n)] = 0 & \text{in } S, \\ u_n = u_{0n} & \text{in } \mathbb{R} \times \{0\}\ (n \in \mathbb{N}). \end{cases} \tag{Pn}$$

Let us recall the definition of entropy solution of problem (Pn) (e.g., see [9]).

Definition 3.6. A function $u_n \in L^\infty(0, T; L^1(\mathbb{R})) \cap L^\infty(S)$ is called an *entropy solution* of problem (Pn) if for every $\zeta \in C^1([0, T]; C^1_c(\mathbb{R}))$, with $\zeta(\cdot, T) = 0$ in $\mathbb{R}$ and $\zeta \geq 0$, and for any couple (E, F), with E convex and $F' = E'\varphi'$, we have

$$\iint_S [E(u_n)\partial_t\zeta + F(u_n)\partial_x\zeta]\, dx\, dt \geq -\int_{\mathbb{R}} E(u_{0n})\zeta(x, 0)\, dx. \tag{3.16}$$

Entropy solutions are weak solutions if $\zeta \in C^1([0, T]; C^1_c(\mathbb{R}))$, $\zeta(\cdot, T) = 0$ in $\mathbb{R}$ and

$$\iint_S [u_n\partial_t\zeta + \varphi(u_n)\partial_x\zeta]\, dx\, dt + \int_{\mathbb{R}} u_{0n}\zeta(x, 0)\, dx = 0. \tag{3.17}$$

By studying the limiting points of the sequence $\{u_n\}$, we shall prove the following result.

Theorem 3.7. (i) *Let* (H1) *be satisfied. Then problem* (P) *has a solution u, which is obtained as a limiting point of the sequence $\{u_n\}$ of entropy solutions to problems* (Pn). *In addition, u is an entropy solution of problem* (P) *in the sense of Young measures.*
(ii) *Let* (H1) *and the following assumption be satisfied:*
(C2) $\varphi \in C^1([0, \infty))$, *and for every $\bar{u} > 0$, there exist $a, b \geq 0$, $a + b > 0$, such that φ' is strictly monotone in $[\bar{u} - a, \bar{u} + b]$.*
Then u is an entropy solution of problem (P).

Hypothesis (C2) fails if for example φ is affine in an interval $(a, b) \subset (0, \infty)$. In that case, Proposition 5.9 (iii), which characterizes the limiting Young measure, gives some additional information.

The following proposition shows that the singular part of an entropy subsolution of (P) does not increase along the lines $x = C_\varphi t + x_0$.

Proposition 3.8. *Let* (H1) *be satisfied.*
(i) *Let u be an entropy subsolution of problem* (P) *in the sense of Young measures. Then*

$$u_s(\cdot, t_2) \leq \mathcal{T}_{C_\varphi(t_2-t_1)}(u_s(\cdot, t_1)) \quad \text{in } \mathcal{M}^+(\mathbb{R}), \text{ for a.e. } 0 \leq t_1 \leq t_2 \leq T. \tag{3.18}$$

In particular,

$$u_s(\cdot, t) \leq \mathcal{T}_{C_\varphi t}(u_{0s}) \quad \text{in } \mathcal{M}^+(\mathbb{R}), \text{ for a.e. } t \in (0, T), \tag{3.19}$$

whence $\|u_s(\cdot, t)\|_{\mathcal{M}(\mathbb{R})} \leq \|u_{0s}\|_{\mathcal{M}(\mathbb{R})}$ *for a.e.* $t \in (0, T)$.
(ii) *Let u be a solution of problem* (P). *Then there is conservation of mass, i.e.,*

$$\|u(\cdot, t)\|_{\mathcal{M}(\mathbb{R})} = \|u_0\|_{\mathcal{M}(\mathbb{R})} \quad \text{for a.e. } t \in (0, T).$$

The linear case $\varphi(u) = u$ shows that equality may hold in (3.18). Moreover, if $C_\varphi = 0$, it follows from (3.18) that the map $t \mapsto u_s(\cdot, t)$ is nonincreasing.

3.3 Waiting time and regularity

It is convenient to distinguish two cases: $C_\varphi = 0$ (sublinear growth at infinity) and $C_\varphi \neq 0$ (linear growth at infinity), with C_φ defined by (H1).

3.3.1 Sublinear growth

Beside (H1), we will use the following assumption:

(H2) $\varphi \in C^\infty([0,\infty))$, $C_\varphi = 0$, there exist $H \geq -1$, $K \in \mathbb{R}$ such that $\varphi''(u)\,[H\varphi(u)+K] \leq -[\varphi'(u)]^2 < 0$ for all $u \in [0,\infty)$.

By (H2) the map $u \mapsto \varphi''(u)\,[H\varphi(u)+K]$ is strictly negative and continuous in $[0,\infty)$, hence two cases are possible: either (a) $H\varphi + K > 0$, $\varphi'' < 0$, or (b) $H\varphi + K < 0$, $\varphi'' > 0$ in $[0,\infty)$. In case (a), we have $\varphi' > 0$ in $[0,\infty)$, since $\varphi'' < 0$ and $\lim_{u\to\infty}\varphi'(u) = C_\varphi = 0$. Similarly, in case (b), we have plainly $\varphi' < 0$ in $[0,\infty)$. In particular, in both cases (H2) implies (C2). Moreover, if also (H1) holds, thus $\varphi(0)=0$, we have $H\varphi + K > 0$ in $[0,\infty)$ if and only if $K > 0$.

Remark 3.9. The following examples show that all values of $H \geq -1$ may occur in (H2):

$$\varphi(u) = \operatorname{sgn} p[(1+u)^p - 1]\ (p<1, p\neq 0) \ \Rightarrow\ H = \frac{p}{1-p} \in (-1,0)\cup(0,\infty),\ K = |H|,$$
$$\varphi(u) = 1 - e^{-\alpha u}\ (\alpha > 0) \ \Rightarrow\ H=-1,\ K=1,$$
$$\varphi(u) = \log(1+u) \text{ or } \varphi(u) = 1 - \frac{1}{\log(e+u)} \ \Rightarrow\ H=0,\ K=1.$$

The following property of constructed entropy solutions plays an important role as a uniqueness criterion (see its generalized form given by Proposition 3.17 and Theorem 3.22 below).

Proposition 3.10. *Let* (H1)–(H2) *be satisfied, and let φ be bounded in* $[0,\infty)$. *Then every entropy solution u of problem* (P) *given by Theorem 3.7 satisfies, for a.e. $t \in (0,T)$ and all $x_0 \in \operatorname{supp} u_s(\cdot,t)$,*

$$\operatorname*{ess\,lim}_{x\to x_0^+} u_r(x,t) = \infty \quad \text{if } \varphi' > 0 \text{ in } [0,\infty), \tag{3.20}$$
$$\operatorname*{ess\,lim}_{x\to x_0^-} u_r(x,t) = \infty \quad \text{if } \varphi' < 0 \text{ in } [0,\infty).$$

Theorem 3.11. (i) *Let* (H1) *be satisfied, let $u_{0s}(\{x_0\}) > 0$ for some $x_0 \in \mathbb{R}$ and let u be a solution of problem* (P). *If φ is bounded in $(0,\infty)$ (in particular, $C_\varphi = 0$), then the waiting time t_0 defined by* (1.5) *satisfies*

$$t_0 \geq \min\left\{T, \frac{u_{0s}(\{x_0\})}{\|\varphi\|_{L^\infty(0,\infty)}}\right\} > 0. \tag{3.21}$$

(ii) *Let* (H1)–(H2) *be satisfied, and let u be the entropy solution of problem* (P) *given by Theorem 3.7.*

(a) *If φ is bounded in $(0,\infty)$ and, moreover, $H > -1$, $|K| < \lim_{u\to\infty}|\varphi(u)| =: \gamma$, then*

$$t_0 \leq \min\left\{T, \frac{(H+1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - |K|}\right\}. \tag{3.22}$$

(b) *If φ is unbounded in $(0,\infty)$, then $t_0 = 0$.*

Remark 3.12. Concerning estimates (3.21) and (3.22), it is worth considering the case in which $u_0 = \delta_0$ and $\varphi(u) = 1-(1+u)^p$, $p<0$. By explicit calculations, in Proposition 1.1, we show that in this case the waiting time defined in (1.5) is $t_0 = 1$. Hence, in this case, estimates (3.21)–(3.22) are sharp, since

$$\frac{\delta_0(\{0\})}{\|\varphi\|_{L^\infty(0,\infty)}} = 1 \quad\text{and}\quad \frac{(H+1)\|\delta_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - |K|} = \frac{(p/(1-p)+1)\|\delta_0\|_{\mathcal{M}(\mathbb{R})}}{1+p/(1-p)} = 1.$$

Remark 3.13. In part (ii) of Theorem 3.11, it is enough to require condition (H2) for large values of u. More precisely (see Remark 6.10), Theorem 3.11 (ii) remains valid if instead of (H2), for some $k > 0$, the following holds:

(H3) the function $\varphi_k\colon [0,\infty)\to\mathbb{R}$, $\varphi_k(u) := \varphi(u+k) - \varphi(k)$, satisfies (H2).

In this connection, observe that the conditions $H > -1$ and $|K| < \lim_{u\to\infty}|\varphi(u)|$ exclude the function $\varphi(u) = 1 - e^{-u}$. The same conditions also exclude the function $\varphi(u) = 1 - \frac{1}{\log(e+u)}$, where $K = 1 = \gamma$. However, in this case, we can use hypothesis (H3) for $k > 0$, which is satisfied with $H = 0$ and $K = \log^{-2}(e+k) < \gamma_k = \log^{-1}(e+k)$.

Let us finally mention the following regularization result.

Proposition 3.14. *Let* (H1)–(H2) *be satisfied, and let φ be bounded in* $[0,\infty)$ *(in particular, $C_\varphi = 0$). Then, for a.e. $t \in (0,T)$,* $\operatorname{supp} u_s(t)$ *is a null set.*

Remark 3.15. It suffices to prove Proposition 3.10, Theorem 3.11 and Proposition 3.14 by assuming $\varphi'' < 0$ in (H2) (hence, $K > 0$, by (H2) and the assumption $\varphi(0) = 0$). Otherwise, it can be easily seen that if $u \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$ is a solution of problem (P), the map $\tilde{u}$ defined by setting

$$\langle \tilde{u}, \zeta\rangle_S := \int_0^T \langle u(\cdot,t), \zeta(-\cdot,t)\rangle_{\mathbb{R}}\, dt$$

for every $\zeta \in C([0,T]; C_c(\mathbb{R}))$ is a solution of the problem

$$\begin{cases} \partial_t \tilde{u} + \partial_x[\tilde{\varphi}(\tilde{u})] = 0 & \text{in } S,\\ \tilde{u} = \tilde{u}_0 & \text{in } \mathbb{R}\times\{0\}. \end{cases} \tag{3.23}$$

Here $\langle \tilde{u}_0, \rho\rangle_{\mathbb{R}} := \langle u_0, \rho(-\cdot)\rangle_{\mathbb{R}}$ for all $\rho \in C_c(\mathbb{R})$, and the function $\tilde{\varphi} := -\varphi$ satisfies (H2) with $\tilde{K} := -K$. The same holds for entropy solutions.

3.3.2 Linear growth

Let φ satisfy the following assumption:
(H4) $\varphi \in C^\infty([0,\infty))$ and there exist $H \geq -1$, $K \in \mathbb{R}$ such that

$$\varphi''(u)\{H[\varphi(u) - C_\varphi u] + K\} \leq -[\varphi'(u) - C_\varphi]^2 < 0 \quad \text{for all } u \in [0,\infty)$$

(observe that (H4) reduces to (H2) if $C_\varphi = 0$). If (H4) holds, the function $\tilde{\varphi} := \varphi(u) - C_\varphi u$ satisfies (H2) since $C_{\tilde{\varphi}} = 0$.

Remark 3.16. It is easily seen that if u is a solution (respectively an entropy solution) of problem (P), then $v \in L^\infty(0,T;\mathcal{M}^+(\Omega))$, defined by

$$v(\cdot,t) = \mathcal{T}_{-h}(u(\cdot,t)) \quad \text{in } \mathcal{M}(\mathbb{R})$$

for any $h \in \mathbb{R}$, is a solution (respectively an entropy solution) of (P) with u_0 replaced by $v_0 := \mathcal{T}_{-h}(u_0)$. Similarly, $\tilde{u}(\cdot,t) := \mathcal{T}_{-C_\varphi t}(u(\cdot,t))$ is a solution (respectively an entropy solution) of problem (3.23), with $\tilde{u}_0 = u_0$ and $\tilde{\varphi}(u) = \varphi(u) - C_\varphi u$.

By Remark 3.16, the above results for the case $C_\varphi = 0$ can be generalized as follows.

Proposition 3.17. *Let* (H1) *and* (H4) *be satisfied, and let $u \mapsto \varphi(u) - C_\varphi u$ be bounded in* $(0,\infty)$*. Then every entropy solution u of problem* (P) *given by Theorem 3.7 satisfies, for a.e. $t \in (0,T)$ and all $x_0 \in \operatorname{supp} u_s(\cdot,t)$,*

$$\operatorname*{ess\,lim}_{x\to x_0^+} u_r(x + C_\varphi t, t) = \infty \quad \text{if } \varphi' > C_\varphi \text{ in } [0,\infty), \tag{3.24}$$

$$\operatorname*{ess\,lim}_{x\to x_0^-} u_r(x + C_\varphi t, t) = \infty \quad \text{if } \varphi' < C_\varphi \text{ in } [0,\infty). \tag{3.25}$$

Theorem 3.18. (i) *Let* (H1) *be satisfied, let $u_{0s}(\{x_0\}) > 0$ for some $x_0 \in \mathbb{R}$, and let u be a solution of problem* (P)*. If $u \mapsto \varphi(u) - C_\varphi u$ is bounded in* $(0,\infty)$*, then*

$$t_0 \geq \min\Big\{T, \frac{u_{0s}(\{x_0\})}{\|\varphi - C_\varphi u\|_{L^\infty(0,\infty)}}\Big\} > 0.$$

(ii) *Let* (H1) *and* (H4) *be satisfied, and let u be the entropy solution of problem* (P) *given by Theorem 3.7.*

(a) *Let $u \mapsto \varphi(u) - C_\varphi u$ be bounded in* $(0,\infty)$*. If $H > -1$ and $|K| < \lim_{u\to\infty}|\varphi(u) - C_\varphi u| =: \bar{\gamma}$, then*

$$t_0 \leq \min\Big\{T, \frac{(H+1)\,\|u_0\|_{\mathcal{M}(\mathbb{R})}}{\bar{\gamma} - |K|}\Big\}.$$

(b) *Let $u \mapsto \varphi(u) - C_\varphi u$ be unbounded in* $(0,\infty)$*. Then $t_0 = 0$.*

Again, Theorem 3.18 (ii) remains valid if, for some $k > 0$, the function φ_k defined in Remark 3.13 satisfies (H4).

Proposition 3.19. *Let* (H1) *and* (H4) *be satisfied, and let $u \mapsto \varphi(u) - C_\varphi u$ be bounded in* $(0, \infty)$. *Then for a.e. $t \in (0, T)$, $\operatorname{supp} u_s(t)$ is a null set.*

3.4 Uniqueness

In connection with equality (3.11), observe that if $u_{0s} \neq 0$ and the waiting time t_0 is equal to 0, then the map $t \mapsto u(\cdot, t)$ is not continuous at $t = 0$ in the strong topology of $\mathcal{M}(\mathbb{R})$ (otherwise we would have $\lim_{t\to 0^+} \|u_s(\cdot, t)\|_{\mathcal{M}(\mathbb{R})} = 0 = \|u_{0s}\|_{\mathcal{M}(\mathbb{R})}$, a contradiction). Instead, continuity along the lines $x = x_0 + C_\varphi t$ may occur if the waiting time t_0 is positive.

Proposition 3.20. *Let* (H1) *be satisfied. Let $u \mapsto \varphi(u) - C_\varphi u$ be bounded in* $(0, \infty)$, *and let u_0 satisfy*

$$u_{0s} = \sum_{l=1}^{N} c_l \delta_{x_l}, \quad \text{with } c_l \in [0, \infty),\ l = 1, \dots, N \text{ for some } N \in \mathbb{N}. \tag{3.26}$$

(i) *If condition* (C2) *holds, then every entropy solution u of problem* (P) *given by Theorem 3.7* (ii) *satisfies*

$$\operatorname*{ess\,lim}_{t\to 0^+} \|\mathcal{T}_{-C_\varphi t}(u(\cdot, t)) - u_0\|_{\mathcal{M}(\mathbb{R})} = 0. \tag{3.27}$$

(ii) *All entropy solutions u of problem* (P) *satisfy* $\mathcal{T}_{-C_\varphi t}(u(\cdot, t)) \in C((0, T]; \mathcal{M}(\mathbb{R}))$.

Let us mention that the above statement (ii) holds for any $u_0 \in \mathcal{M}^+(\mathbb{R})$ if φ satisfies (H1) and (H4) (see Proposition 6.2).

The following uniqueness result will be proven in Section 7.

Theorem 3.21. *Let* (H1) *be satisfied and let $u \mapsto \varphi(u) - C_\varphi u$ be bounded and monotonic in* $(0, \infty)$. *Let u_0 satisfy* (3.26). *Then there exists at most one entropy solution u of problem* (P) *which satisfies either* (3.24) *or* (3.25), *and the condition*

$$\operatorname*{ess\,lim}_{t\to 0^+} \|u_r(\cdot, t) - u_{0r}\|_{L^1(\mathbb{R})} = 0. \tag{3.28}$$

By Propositions 3.17, 3.20 and Theorem 3.21, we have the following existence and uniqueness result (observe that (H4) implies (C2)).

Theorem 3.22. *Let* (H1) *and* (H4) *be satisfied, and let $u \mapsto \varphi(u) - C_\varphi u$ be bounded in* $(0, \infty)$. *Let u_0 satisfy* (3.26). *Then there exists a unique entropy solution of problem* (P) *which satisfies* (3.24)–(3.25).

Remark 3.23. Conditions (3.24)–(3.25) in Theorem 3.22 cannot be omitted. In fact, there exist entropy solutions of problem (P) which do not satisfy either (3.24) or (3.25), depending on φ. Therefore, by Proposition 3.17, they are different from those given by Theorem 3.7, thus uniqueness fails.

For example, let $u_{0s} \neq 0$ and $u_{0r} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ be defined by

$$u(\cdot, t) := u_r(\cdot, t) + \mathcal{T}_{C_\varphi t}(u_{0s}) \quad \text{for a.e. } t \in (0, T),$$

where $u_r \in C([0, T]; L^1(\mathbb{R})) \cap L^\infty(S)$ is the unique entropy solution of problem (P) with u_0 replaced by u_{0r}. Since $u(\cdot, 0) = u_r(\cdot, 0) + u_{0s} = u_{0r} + u_{0s} = u_0$, one easily checks that (3.8)–(3.9) are satisfied, thus u is an entropy solution of (P). On the other hand, $u_r \in L^\infty(S)$, so $u_r(\cdot, t) \in L^\infty(\mathbb{R})$ for a.e. $t \in (0, T)$, and (3.24)–(3.25) fails.

Remark 3.24. If $u \mapsto \varphi(u) - C_\varphi u$ is unbounded and satisfies assumptions (H1) and (H4), by [19, Theorem 1.1] and Theorem 3.18, for every $u_0 \in \mathcal{M}^+(\mathbb{R})$ there exists a unique entropy solution of problem (P) with waiting time t_0 equal to 0. In fact, every entropy solution u given by Theorem 3.18 is a solution according to [19]. This follows if we show that

$$u = u_r \in L^\infty(\mathbb{R} \times (\tau, T)) \quad \text{for every } \tau > 0 \tag{3.29}$$

and $\operatorname{ess\,lim}_{t\to 0} u(\cdot,t) = u_0$ narrowly in $\mathcal{M}(\mathbb{R})$, i.e. $\operatorname{ess\,lim}_{t\to 0}\langle u(\cdot,t),\rho\rangle = \langle u_0,\rho\rangle$ for all bounded $\rho \in C(\mathbb{R})$. The latter follows from (3.11) and Proposition 3.8 (ii) (see [17, Proposition 2, p. 38]).

To prove (3.29) we fix $\tau > 0$. By (1.5) we may assume that $u_r(\cdot,\tau) \in L^\infty(\mathbb{R})$ and $u(\cdot,t) = u_r(\cdot,t)$ for all $t \geq \tau$. By standard approximation arguments, we may substitute in the entropy inequality (3.9) $E(u) = [s - k_\tau]_+$, with $k_\tau = \|u_r(\cdot,\tau)\|_{L^\infty(\mathbb{R})}$, and $\zeta(x,t) \equiv \chi_{[\tau,t]}(t)$. Hence, $\int_{\mathbb{R}}[u_r(\cdot,t) - k_\tau]_+\,dx \leq 0$ for a.e. $t \geq \tau$ and (3.29) follows.

4 Approximating problems

In this section we consider problem (Pn). Let $u_{0n} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy (3.14) and let $\{u^\varepsilon_{0n}\} \subseteq C^\infty_c(\mathbb{R})$, $u^\varepsilon_{0n} \geq 0$ be any sequence such that

$$\|u^\varepsilon_{0n}\|_{L^1(\mathbb{R})} \leq \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}, \qquad \|u^\varepsilon_{0n}\|_{L^\infty(\mathbb{R})} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})}, \tag{4.1}$$

$$u^\varepsilon_{0n} \to u_{0n} \quad \text{in } L^1(\mathbb{R}), \qquad u^\varepsilon_{0n} \overset{*}{\rightharpoonup} u_{0n} \quad \text{in } L^\infty(\mathbb{R}). \tag{4.2}$$

Let $\eta \in C^\infty_c(\mathbb{R})$ be a standard mollifier, let $\eta_\varepsilon(u) := \frac{1}{\varepsilon}\eta(\frac{u}{\varepsilon})$ for $\varepsilon > 0$, and set

$$\varphi_\varepsilon(u) := (\eta_\varepsilon * \overline{\varphi})(u) - (\eta_\varepsilon * \overline{\varphi})(0) = \int_{\mathbb{R}} \eta_\varepsilon(u-v)\overline{\varphi}(v)dv - \int_{\mathbb{R}} \eta_\varepsilon(-v)\overline{\varphi}(v)\,dv, \quad u \in \mathbb{R}$$

(here $\overline{\varphi}(u) = \varphi(u)$ for $u \geq 0$ and $\overline{\varphi}(u) = 0$ for $u < 0$). The regularized problem associated with (Pn) is

$$\begin{cases} \partial_t u^\varepsilon_n + \partial_x[\varphi_\varepsilon(u^\varepsilon_n)] = \varepsilon\partial^2_x u^\varepsilon_n & \text{in } S,\\ u^\varepsilon_n = u^\varepsilon_{0n} & \text{in } \mathbb{R}\times\{0\} \end{cases} \tag{4.3}$$

(where $\varepsilon > 0$, $n \in \mathbb{N}$), has a unique strong solution $u^\varepsilon_n \in C([0,T];H^2(\mathbb{R})) \cap L^\infty(S)$, $\partial_t u^\varepsilon_n \in L^2(S)$ (e.g., see [20]). Some properties of the family $\{u^\varepsilon_n\}$ are collected in the following lemmata. Up to minor changes, the proof is standard (e.g., see [9]), thus is omitted.

Lemma 4.1. *Let u^ε_n be the solution of problem* (4.3). *Then, for every $n \in \mathbb{N}$ and $\varepsilon > 0$,*

$$u^\varepsilon_n \geq 0 \quad \text{in } S, \qquad \|u^\varepsilon_n\|_{L^\infty(S)} \leq \|u_{0n}\|_{L^\infty(\mathbb{R})}, \tag{4.4}$$

$$\int_{\mathbb{R}} u^\varepsilon_n(x,t)\,dx = \int_{\mathbb{R}} u^\varepsilon_{0n}(x)\,dx \quad (t \in (0,T)),$$

$$\sup_{t\in(0,T)} \|u^\varepsilon_n(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_{0n}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}, \tag{4.5}$$

$$\sup_{t\in(0,T)} \|u^\varepsilon_n(\cdot+h,t) - u^\varepsilon_n(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u^\varepsilon_{0n}(\cdot+h) - u^\varepsilon_{0n}\|_{L^1(\mathbb{R})} \quad \text{for any } h \in \mathbb{R}. \tag{4.6}$$

Lemma 4.2. *Let φ satisfy* (3.1). *Then there exists $C > 0$, which only depends on $\|u_0\|_{\mathcal{M}(\mathbb{R})}$, such that for all $n \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $p \in (0,1)$,*

$$\varepsilon \iint_S (1+u^\varepsilon_n)^{p-2}(\partial_x u^\varepsilon_n)^2\,dx\,dt \leq \frac{C}{p(1-p)}. \tag{4.7}$$

Proof. Let $U \in C^2([0,\infty))$, $U' \geq 0$ in $(0,\infty)$, and set

$$\Theta_{U,\varepsilon}(u) := \int_0^u U'(s)\varphi'_\varepsilon(s)\,ds + \theta_U \quad (\theta_U \in \mathbb{R}). \tag{4.8}$$

Multiplying the first equation in (4.3) by $U'(u^\varepsilon_n)$ gives

$$\partial_t[U(u^\varepsilon_n)] + \partial_x[\Theta_{U,\varepsilon}(u^\varepsilon_n)] = \varepsilon\partial^2_x[U(u^\varepsilon_n)] - \varepsilon U''(u^\varepsilon_n)(\partial_x u^\varepsilon_n)^2 \quad \text{in } S. \tag{4.9}$$

Hence, for all $\zeta \in C^1([0,T]; C_c^2(\mathbb{R}))$,

$$\varepsilon \iint_S U''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2 \zeta \, dx\, dt + \int_{\mathbb{R}} U(u_n^\varepsilon(x,T))\zeta(x,T)\, dx = \int_{\mathbb{R}} U(u_{0n}^\varepsilon)\zeta(x,0)\, dx + \iint_S \{U(u_n^\varepsilon)\partial_t \zeta + \Theta_{U,\varepsilon}(u_n^\varepsilon)\partial_x \zeta + \varepsilon U(u_n^\varepsilon)\partial_x^2 \zeta\}\, dx\, dt. \tag{4.10}$$

By (3.1) and the definition of the function φ_ε, for all $u \geq 0$,

$$|\Theta_{U,\varepsilon}(u)| \leq \int_0^u U'(s)|\varphi_\varepsilon'(s)|\, ds + |\theta_U| \leq M[U(u) - U(0)] + |\theta_U|. \tag{4.11}$$

Choose $\theta_U = 0$, $U(u) = (1+u)^p - 1$, with $p \in (0,1)$, and

$$\zeta = \rho_k := \chi_{\{|x|\leq k\}} + \rho(\cdot - k)\chi_{\{k \leq x < k+1\}} + \rho(\cdot + k)\chi_{\{-(k+1)<x\leq -k\}} \quad (k \in \mathbb{N}),$$

with any $\rho \in C_c^2((-1,1))$ such that $\rho(0) = 1$, $0 \leq \rho \leq 1$, and the derivatives ρ', ρ'' vanish at 0. Then $0 \leq U(u) \leq u$ for $u \geq 0$ and, by (4.5), (4.10) and (4.11),

$$\varepsilon p(1-p) \iint_S (1+u_n^\varepsilon)^{p-2}(\partial_x u_n^\varepsilon)^2 \rho_k \, dx\, dt \leq \int_{\mathbb{R}} u_{0n}^\varepsilon(x)\, dx + \iint_S \{M u_n^\varepsilon |\rho_k'| + \varepsilon u_n^\varepsilon |\rho_k''|\}\, dx\, dt \leq \{1 + (M+1)T\|\rho\|_{C^2([-1,1])}\}\|u_0\|_{\mathcal{M}(\mathbb{R})} =: C$$

for all $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$. Passing to the limit as $k \to \infty$, we obtain (4.7). □

Lemma 4.3. *Let φ satisfy* (3.1) *and let $U \in C^2([0,\infty))$ be such that*

$$|U''(u)| \leq K(1+u)^{p-2} \quad \text{for all } u \in [0,\infty), \text{ for some } K \geq 0 \text{ and } p \in (0,1). \tag{4.12}$$

Then there exists $C_p > 0$ such that for all $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\varepsilon \iint_S |U''(u_n^\varepsilon)|(\partial_x u_n^\varepsilon)^2\, dx\, dt \leq C_p. \tag{4.13}$$

If, moreover, $U' \in L^\infty(0,\infty)$, then the family $\{U_{n,\rho}^\varepsilon\}$, where

$$U_{n,\rho}^\varepsilon(t) := \int_{\mathbb{R}} U(u_n^\varepsilon)(x,t)\rho(x)\, dx \quad (t \in (0,T)) \tag{4.14}$$

and $\rho \in C_c^2(\mathbb{R})$, is bounded in $\mathrm{BV}(0,T)$.

Proof. Inequality (4.13) follows immediately from (4.7) and (4.12). To prove that $\{U_{n,\rho}^\varepsilon\}$ is bounded in $\mathrm{BV}(0,T)$, observe that, by (4.9),

$$(U_{n,\rho}^\varepsilon)'(t) = \int_{\mathbb{R}} [\Theta_{U,\varepsilon}(u_n^\varepsilon)\rho' + \varepsilon U(u_n^\varepsilon)\rho'' - \varepsilon U''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2 \rho](x,t)\, dx. \tag{4.15}$$

Since $U' \in L^\infty(0,\infty)$, there exists $N > 0$ such that $|U(u)| \leq N(1+u)$ for $u \geq 0$. Hence, $|U(u_n^\varepsilon)| \leq N(1+u_n^\varepsilon)$ and, by (4.8), (3.1) and the definition of φ_ε, we have

$$|\Theta_{U,\varepsilon}(u_n^\varepsilon)| \leq \|\varphi_\varepsilon' U'\|_{L^\infty((0,\infty))}|u_n^\varepsilon| + |\theta_U| =: \tilde{M} u_n^\varepsilon + |\theta_U|.$$

Then it follows from (4.15) that

$$|(U_{n,\rho}^\varepsilon)'|(t) \leq \|\rho\|_{C^2(\mathbb{R})} \int_{\operatorname{supp}\rho} \{(\tilde{M} + \varepsilon N)u_n^\varepsilon(x,t) + \varepsilon N + |\theta_U|\}\, dx + \varepsilon\|\rho\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} [|U''(u_n^\varepsilon)|\,(\partial_x u_n^\varepsilon)^2](x,t)\, dx,$$

and, by (4.5) and (4.13), there exists a constant $C_{p,\rho} > 0$ such that

$$\int_0^T |(U^\varepsilon_{n,\rho})'|(t)\,dt \le \|\rho\|_{C^2(\mathbb{R})}\{(\tilde{M}+N)T\|u_0\|_{\mathcal{M}(\mathbb{R})} + C_{p,\rho}\}. \tag{4.16}$$

On the other hand, by (4.5) and since $|U(u^\varepsilon_n)| \le N(1+u^\varepsilon_n)$, we have

$$\int_0^T |U^\varepsilon_{n,\rho}|(t)\,dt \le NT\|\rho\|_{L^\infty(\mathbb{R})}(\|u_0\|_{\mathcal{M}(\mathbb{R})} + |\operatorname{supp}\rho|), \tag{4.17}$$

whence the result follows. □

From the above lemmata, we get the following convergence results.

Lemma 4.4. (i) *If $\varphi \in C([0,\infty))$, there exist a subsequence $\{u^{\varepsilon_m}_n\} \subseteq \{u^\varepsilon_n\}$ and $u_n \in L^\infty(S)\cap L^\infty(0,T;L^1(\mathbb{R}))$ such that, as $\varepsilon_m \to 0$,*

$$u^{\varepsilon_m}_n \overset{*}{\rightharpoonup} u_n \quad \text{in } L^\infty(S), \qquad u^{\varepsilon_m}_n \to u_n \qquad \text{and} \qquad \varphi_{\varepsilon_m}(u^{\varepsilon_m}_n) \to \varphi(u_n) \quad \text{a.e. in } S, \tag{4.18}$$

$$u^{\varepsilon_m}_n \to u_n \quad \text{in } L^1((-L,L)\times(0,T)), \text{ for all } L>0. \tag{4.19}$$

Moreover, $u_n \ge 0$ a.e. in S, $\|u_n\|_{L^\infty(S)} \le \|u_{0n}\|_{L^\infty(\mathbb{R})}$ and

$$\sup_{t\in(0,T)} \|u_n(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_{0n}\|_{L^1(\mathbb{R})} \le \|u_0\|_{\mathcal{M}(\mathbb{R})}. \tag{4.20}$$

(ii) *Let φ satisfy (3.1), let $\rho \in C^2_c(\mathbb{R})$, and let $U \in C^2([0,\infty))$, with $U' \in L^\infty(0,\infty)$, satisfy (4.12). Let $U^{\varepsilon_m}_{n,\rho}$ be defined by (4.14) and set*

$$U_{n,\rho}(t) := \int_{\mathbb{R}} U(u_n)(x,t)\rho(x)\,dx \quad (t \in (0,T)). \tag{4.21}$$

Then

$$U^{\varepsilon_m}_{n,\rho} \to U_{n,\rho} \quad \text{in } L^1(0,T) \text{ and a.e. in } (0,T). \tag{4.22}$$

Proof. By (4.4), $u^{\varepsilon_m}_n \overset{*}{\rightharpoonup} u_n$ in $L^\infty(S)$, where $u_n \in L^\infty(S)$, $\|u_n\|_{L^\infty(S)} \le \|u_{0n}\|_{L^\infty(\mathbb{R})}$ and $u_n \ge 0$ a.e. in S. The a.e.-convergence of $u^{\varepsilon_m}_n$ and part (ii) follow from (4.19), and since φ_ε converges uniformly to the continuous function φ on compact subsets of $\mathbb{R}$, we also obtain the a.e.-convergence of $\varphi_{\varepsilon_m}(u^{\varepsilon_m}_n)$.

It remains to prove (4.19) and (4.20). We claim that for a.e. $t \in (0,T)$,

$$u^{\varepsilon_m}_n(\cdot,t) \overset{*}{\rightharpoonup} u_n(\cdot,t) \quad \text{in } L^\infty(\mathbb{R}) \text{ as } \varepsilon_m \to 0. \tag{4.23}$$

Set $I^{\varepsilon_m}_{n,\rho}(t) := \int_{\mathbb{R}} u^{\varepsilon_m}_n(x,t)\rho(x)\,dx$ for $t \in (0,T)$) and let $\rho \in C^2_c(\mathbb{R})$. By Lemma 4.3, with $U(u)=u$, the sequence $\{I^{\varepsilon_m}_{n,\rho}\}$ is bounded in $\mathrm{BV}(0,T)$ and has a subsequence (not relabeled) $\{I^{\varepsilon_m}_{n,\rho}\}$ such that

$$I^{\varepsilon_m}_{n,\rho} \to I_{n,\rho} \quad \text{in } L^1(0,T) \text{ as } \varepsilon_m \to 0 \tag{4.24}$$

for some $I_{n,\rho} \in \mathrm{BV}(0,T)$. Since $u^{\varepsilon_m}_n \overset{*}{\rightharpoonup} u_n$ in $L^\infty(S)$,

$$\lim_{m\to\infty} \int_0^T I^{\varepsilon_m}_{n,\rho}(t)\,dt = \iint_S u_n(x,t)\rho(x)\,dx\,dt = \int_0^T \Big(\int_{\mathbb{R}} u_n(x,t)\rho(x)\,dx\Big)\,dt,$$

whence $I_{n,\rho} = \int_{\mathbb{R}} u_n(x,t)\rho(x)\,dx$ for a.e. $t \in (0,T)$), and the convergence in (4.24) is satisfied along the whole sequence $\{I^{\varepsilon_m}_{n,\rho}\}$. Hence, for all $\rho \in C^2_c(\mathbb{R})$, there exists a null set $N \subset (0,T)$ such that

$$\lim_{\varepsilon_m\to 0} \int_{\mathbb{R}} u^{\varepsilon_m}_n(x,t)\rho(x)\,dx = \int_{\mathbb{R}} u_n(x,t)\rho(x)\,dx \quad \text{for all } t \in (0,T)\setminus N.$$

Since $C_c^2(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ and $L^1(\mathbb{R})$ is separable, the choice of the set N can be made independent of ρ. Hence, we have proven (4.23).

By (4.2), (4.5), (4.6), and the Fréchet–Kolmogorov theorem, $\{u_n^{\varepsilon_m}(\cdot,t)\}$ is relatively compact in $L^1((-L,L))$ for all $t \in (0,T)$ and $L > 0$. Hence, by (4.23),

$$u_n^{\varepsilon_m}(\cdot,t) \to u_n(\cdot,t) \quad \text{in } L^1((-L,L)) \text{ as } \varepsilon_m \to 0, \text{ for } L > 0 \text{ and a.e. } t \in (0,T), \tag{4.25}$$

and (4.20) follows from (4.5). Finally, (4.19) follows from (4.5), (4.25) and the dominated convergence theorem. □

Proposition 4.5. *Let $\varphi \in C([0,\infty))$. For all $n \in \mathbb{N}$, problem* (Pn) *has an entropy solution u_n, which is unique if φ is locally Lipschitz continuous. For a.e. $t \in (0,T)$, we have*

$$\|u_n(\cdot+h,t) - u_n(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_{0n}(\cdot+h) - u_{0n}\|_{L^1(\mathbb{R})} \quad \textit{for any } h \in \mathbb{R}, \tag{4.26}$$

$$\int_{\mathbb{R}} u_n(x,t)\,dx = \int_{\mathbb{R}} u_{0n}(x)\,dx. \tag{4.27}$$

Moreover, given $\rho \in C_c^2(\mathbb{R})$ and $U \in C^2([0,\infty))$, with $U' \in L^\infty(0,\infty)$, satisfying (4.12), *the sequence $\{U_{n,\rho}\}$ defined by* (4.21) *is bounded in* BV$(0,T)$.

Proof. Let ζ and E be as in Definition 3.6, and $F_\varepsilon' = E'\varphi_\varepsilon'$. Then

$$\iint_S \{E(u_n^\varepsilon)(\partial_t\zeta + \varepsilon\partial_x^2\zeta) + F_\varepsilon(u_n^\varepsilon)\partial_x\zeta\}\,dx\,dt + \int_{\mathbb{R}} E(u_{0n}^\varepsilon)\zeta(x,0)\,dx \ge 0, \tag{4.28}$$

where $u_n^{\varepsilon_m}$ is defined by Lemma 4.4. By (4.4), it is not restrictive to assume that $E(u) = |u-k|$ and $F_\varepsilon(u) = \operatorname{sgn}(u-k)[\varphi_\varepsilon(u) - \varphi_\varepsilon(k)]$ ($k \in [0,\infty)$). By (4.4),

$$\|\varphi_{\varepsilon_m}(u_n^{\varepsilon_m})\|_{L^\infty(S)} \le \sup_{|v| \le \|u_{0n}\|_{L^\infty(\mathbb{R})}} |\varphi_{\varepsilon_m}(v)| \le \sup_{|v| \le \|u_{0n}\|_{L^\infty(\mathbb{R})}+1} |\varphi(v)|.$$

Since $\varphi_{\varepsilon_m}(u_n^{\varepsilon_m}) \to \varphi(u_n)$ a.e. in S (see (4.18)), it follows from (4.19) and the dominated convergence theorem that

$$\iint_S F_{\varepsilon_m}(u_n^{\varepsilon_m})\partial_x\zeta\,dx\,dt \to \iint_S F(u_n)\partial_x\zeta\,dx\,dt \quad \text{as } \varepsilon_m \to 0.$$

The remaining terms in (4.28) (with $\varepsilon = \varepsilon_m$) are dealt with similarly. Letting $\varepsilon_m \to 0$, we obtain (3.16), so u_n is an entropy solution of problem (Pn). Its uniqueness follows from Kružkov's theorem [26].

Inequality (4.26) follows from (4.6) and (4.25). Concerning (4.27), it follows from (3.17) that for all $\rho \in C_c^1(\mathbb{R})$ and a.e. $t \in (0,T)$,

$$\int_{\mathbb{R}} u_n(x,t)\rho(x)\,dx - \int_{\mathbb{R}} u_{0n}(x)\rho(x)\,dx = \int_0^t \int_{\mathbb{R}} \varphi(u_n)(x,s)\rho'(x)\,dx\,ds. \tag{4.29}$$

Let $\{\rho_k\} \subseteq C_c^1(\mathbb{R})$ be such that $\rho_k(x) = 1$ for $x \in [-k,k]$, $\rho_k(x) = 0$ if $|x| \ge k+1$, and $\|\rho_k'\|_{L^\infty(\mathbb{R})} \le 2$. Setting $\rho = \rho_k$ in (4.29) and letting $k \to \infty$, we get

$$\left|\int_0^t \int_{\mathbb{R}} \varphi(u_n)(x,s)\rho_k'(x)\,dx\,ds\right| \le 2M \int_0^t \int_{\{x\in\mathbb{R} \mid k \le |x| \le k+1\}} |u_n(x,s)|\,dx\,ds \to 0,$$

since $u_n \in L^1(S)$. On the other hand, by the monotone convergence theorem,

$$\int_{\mathbb{R}} u_n(x,t)\rho_k(x)\,dx \to \int_{\mathbb{R}} u_n(x,t)\,dx, \qquad \int_{\mathbb{R}} u_{0n}(x)\rho_k(x)\,dx \to \int_{\mathbb{R}} u_{0n}(x)\,dx,$$

and (4.27) follows from (4.29).

Finally, let us show that $\{U_{n,\rho}\}$ is bounded in BV$(0,T)$. By (4.17) and (4.22),

$$\int_0^T |U_{n,\rho}(t)|\,dt = \lim_{\varepsilon_m\to 0}\int_0^T |U^{\varepsilon_m}_{n,\rho}(t)|\,dt \le N\|\rho\|_{L^\infty(\mathbb{R})}(T\|u_0\|_{\mathcal{M}(\mathbb{R})} + |\operatorname{supp}\rho|),$$

and, by (4.16) and the lower semicontinuity of the total variation in $L^1(0,T)$ ([15, Theorem 1, Section 5.2.1]), we get

$$\|U'_{n,\rho}\|_{\mathcal{M}(0,T)} \le \|\rho\|_{C^2(\mathbb{R})}\{(\tilde{M}+N)T\|u_0\|_{\mathcal{M}(\mathbb{R})} + C_{p,\rho}\},$$

with $C_{p,\rho} > 0$ as in (4.16). This completes the proof. □

5 Existence and monotonicity: Proofs

We proceed with the proof of Theorem 3.7.

Proposition 5.1. *Let* (H1) *hold and let u_n be the entropy solution of problem* (Pn). *Then there exist a sequence $\{u_{n_j}\}$ and $u \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$ such that*

$$u_{n_j} \overset{*}{\rightharpoonup} u \quad \text{in } \mathcal{M}(S). \tag{5.1}$$

For all $L > 0$, there exists a decreasing sequence $\{E_j\} \subset (-L,L)\times(0,T)$ of Lebesgue measurable sets, with $|E_j| \to 0$ as $j\to\infty$, such that

$$u_{n_j}\chi_{((-L,L)\times(0,T))\setminus E_j} \rightharpoonup u_b := \int_{[0,\infty)} \xi\, d\tau(\xi) \quad \text{in } L^1((-L,L)\times(0,T)), \tag{5.2}$$

where $\tau \in \mathcal{Y}(S;\mathbb{R})$ is the Young measure associated with $\{u_{n_j}\}$, and

$$u_{n_j}\chi_{E_j} \overset{*}{\rightharpoonup} \mu := u - u_b \quad \text{in } \mathcal{M}((-L,L)\times(0,T)). \tag{5.3}$$

Proof. By (4.20), there exist $u \in \mathcal{M}^+(S)$ and a sequence $\{u_{n_j}\}$ such that $u_{n_j} \overset{*}{\rightharpoonup} u$ in $\mathcal{M}(S)$. Arguing as in [27, Proposition 4.2], we obtain that $u \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$.

Since by (4.20) the sequence $\{u_{n_j}\}$ is bounded in $L^1(S)$, by Theorem 2.3 there exist a subsequence of $\{u_{n_j}\}$ (not relabeled) and a Young measure $\tau \in \mathcal{Y}(S;\mathbb{R})$ such that

(i) for every measurable set $A \subseteq S$, (2.4)–(2.6) are valid for any $f \in C(\mathbb{R})$ such that the sequence $\{f(u_{n_j})\}$ is sequentially weakly relatively compact in $L^1(A)$,
(ii) supp $\tau_{(x,t)} \subseteq [0,\infty)$ for a.e. $(x,t)\in S$ (here $\tau_{(x,t)}$ is the disintegration of τ).

Then the result follows by Theorem 2.5 and a standard diagonal procedure. □

Remark 5.2. The function u_b in (5.2) is defined for a.e. in $(x,t)\in S$, since τ is globally defined in S. In addition, by (4.20) and the arbitrariness of L in Proposition 5.1, a routine proof shows that $u_b \in L^\infty(0,T;L^1(\mathbb{R}))$ and $u_b \ge 0$ a.e. in S. Therefore, the Radon measure $\mu \ge 0$ (see (5.3)) is defined on S, $\mu \in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$, and

$$\mu = u - u_b \;\Rightarrow\; u = u_b + \mu \quad \text{in } \mathcal{M}(S). \tag{5.4}$$

Proposition 5.3. *Let* (H1) *hold, let μ be as in* (5.4) *and let $U \in C([0,\infty))$. If*

$$\lim_{u\to\infty}\frac{U(u)}{u} =: C_U \in [0,\infty), \tag{5.5}$$

then, for all $L > 0$,

$$U(u_{n_j}) \overset{*}{\rightharpoonup} U^* + C_U\mu \quad \text{in } \mathcal{M}((-L,L)\times(0,T)), \tag{5.6}$$

where $U^* \in L^\infty(0,T;L^1_{\mathrm{loc}}(\mathbb{R}))$ is defined by

$$U^*(x,t) := \int_{[0,\infty)} U(\xi)\,d\tau_{(x,t)}(\xi) \quad \text{for a.e. } (x,t)\in S.$$

Remark 5.4. If $U \in C([0,\infty))$ satisfies (5.5), there exists $N > 0$ such that

$$|U(u)| \le N(1+u) \quad \text{for } u \ge 0. \tag{5.7}$$

Moreover, $U^* \in L^\infty(0,T;L^1(\mathbb{R}))$ if $|U(u)| \le Nu$, since $u_b \in L^\infty(0,T;L^1(\mathbb{R}))$ and

$$|U^*(x,t)| \le \int_{[0,\infty)} |U(\xi)|\,d\tau_{(x,t)}(\xi) \le N \int_{[0,\infty)} \xi\,d\tau_{(x,t)}(\xi) = Nu_b(x,t) \quad \text{for a.e. } (x,t) \in S.$$

Proof of Proposition 5.3. For all $\varepsilon > 0$, there exist $m_\varepsilon > 0$ such that

$$-\varepsilon u < U(u) - C_U u < \varepsilon u \quad \text{if } u > m_\varepsilon. \tag{5.8}$$

For any $m \in \mathbb{N}$, $m > m_\varepsilon$, let $l_{1m}, l_{2m} \in C([0,\infty))$ be such that $0 \le l_{1m} \le 1$, $0 \le l_{2m} \le 1$, $l_{1m} + l_{2m} = 1$ in $[0,\infty)$, supp $l_{1m} \subseteq [0, m+1]$ and supp $l_{2m} \subseteq [m,\infty)$. Then, by (5.8),

$$\big|U(u_{n_j}) - [U(u_{n_j})l_{1m}(u_{n_j}) + C_U u_{n_j} l_{2m}(u_{n_j})]\big| < \varepsilon u_{n_j} l_{2m}(u_{n_j}) \quad \text{for } j \in \mathbb{N}. \tag{5.9}$$

Since $\sup_S[|U(u_{n_j})|l_{1m}(u_{n_j})] \le \sup_{u\in[0,m+1]}|U(u)| < \infty$, it follows that $\{U(u_{n_j})l_{1m}(u_{n_j})\}$ is uniformly integrable in $(-L,L)\times(0,T)$. Hence, by Theorem 2.3, for all $L > 0$,

$$U(u_{n_j})l_{1m}(u_{n_j}) \rightharpoonup U^*_{1m} := \int_{[0,\infty)} U(\xi)l_{1m}(\xi)\,d\tau(\xi) \tag{5.10}$$

in $L^1((-L,L)\times(0,T))$. Here U^*_{1m} belongs to $L^\infty(0,T;L^1_{\text{loc}}(\mathbb{R}))$, since, by (5.7),

$$|U^*_{1m}| \le \int_{[0,\infty)} |U(\xi)|l_{1m}(\xi)\,d\tau(\xi) \le N \int_{[0,\infty)} (1+\xi)\,d\tau(\xi) \le N(1+u_b). \tag{5.11}$$

Similarly, by (5.1), (5.2), (5.4) and (5.10), with $U(u) = u$,

$$\begin{aligned} u_{n_j} l_{2m}(u_{n_j}) = u_{n_j} - u_{n_j} l_{1m}(u_{n_j}) &\overset{*}{\rightharpoonup} u - \int_{[0,\infty)} \xi l_{1m}(\xi)\,d\tau(\xi) \\ &= u_b - \int_{[0,\infty)} \xi l_{1m}(\xi)\,d\tau(\xi) + \mu \\ &= \int_{[0,\infty)} \xi[1 - l_{1m}(\xi)]\,d\tau(\xi) + \mu \\ &= \int_{[0,\infty)} \xi l_{2m}(\xi)\,d\tau(\xi) + \mu \\ &=: l^*_{2m} + \mu \quad \text{in } \mathcal{M}((-L,L)\times(0,T)). \end{aligned} \tag{5.12}$$

From (5.9)–(5.12), for any $\zeta \in C_c((-L,L)\times(0,T))$, $\zeta \ge 0$, and m as above, we get

$$\begin{aligned} &\iint_{(-L,L)\times(0,T)} [U^*_{1m} + (C_U - \varepsilon)l^*_{2m}]\zeta\,dx\,dt + (C_U - \varepsilon)\langle\mu,\zeta\rangle_{(-L,L)\times(0,T)} \\ &\quad \le \liminf_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta\,dx\,dt \\ &\quad \le \limsup_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta\,dx\,dt \\ &\quad \le \iint_{(-L,L)\times(0,T)} [U^*_{1m} + (C_U + \varepsilon)l^*_{2m}]\zeta\,dx\,dt + (C_U + \varepsilon)\langle\mu,\zeta\rangle_{(-L,L)\times(0,T)}. \end{aligned} \tag{5.13}$$

Since $U^*_{1m} \in L^\infty(0,T;L^1_{\mathrm{loc}}(\mathbb{R}))$,

$$0 \le l^*_{2m} \le \int_{[m,\infty)} \xi \, d\tau(\xi) \le u_b \in L^\infty(0,T;L^1(\mathbb{R}))$$

and

$$\lim_{\varepsilon_m\to 0} l^*_{2m}(x,t) = 0, \quad \lim_{\varepsilon_m\to 0} U^*_{1m}(x,t) = U^*(x,t) \quad \text{for a.e. } (x,t) \in S,$$

by letting $m \to \infty$ in (5.13), we get plainly

$$\begin{aligned}\iint_{(-L,L)\times(0,T)} U^*\zeta \, dx\, dt + (C_U - \varepsilon)\langle \mu, \zeta\rangle_{(-L,L)\times(0,T)} &\le \liminf_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta \, dx\, dt \\ &\le \limsup_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta\, dx\, dt \\ &\le \iint_{(-L,L)\times(0,T)} U^*\zeta\, dx\, dt + (C_U+\varepsilon)\langle\mu,\zeta\rangle_{(-L,L)\times(0,T)},\end{aligned}$$

whence

$$0 \le \limsup_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta\, dx\, dt - \liminf_{n_j\to\infty} \iint_{(-L,L)\times(0,T)} U(u_{n_j})\zeta\, dx\, dt \le 2\varepsilon\langle\mu,\zeta\rangle_{(-L,L)\times(0,T)}.$$

From the above inequalities, the conclusion follows. □

Proposition 5.5. *Let* (H1) *hold. Let* μ, U *and* U^* *be as in Proposition 5.3. Then*

$$\int_0^T \left| \int_{\mathbb{R}} U(u_{n_j})(x,t)\rho(x)\, dx - \int_{\mathbb{R}} U^*(x,t)\rho(x)\, dx - C_U\langle \mu(\cdot,t),\rho\rangle_{\mathbb{R}} \right| dt \to 0 \tag{5.14}$$

as $j\to\infty$ *for* $\rho \in C_c(\mathbb{R})$. *Moreover, for all* $L > 0$, *there exist a null set* $N \subset (0,T)$ *and a subsequence of* $\{u_{n_j}\}$ *(not relabeled), such that for all* $t \in (0,T)\setminus N$,

$$U(u_{n_j})(\cdot,t) \overset{*}{\rightharpoonup} U^*(\cdot,t) + C_U\mu(\cdot,t) \quad \textit{in } \mathcal{M}((-L,L)). \tag{5.15}$$

Remark 5.6. Choosing $U(u) = u$ in (5.15), we obtain that

$$u_{n_j}(\cdot,t) \overset{*}{\rightharpoonup} u(\cdot,t) \quad \text{in } \mathcal{M}((-L,L)) \text{ for a.e. } t\in(0,T) \text{ and } L>0. \tag{5.16}$$

If $U \in C([0,\infty))$ satisfies (5.5), $U^* \in L^\infty(0,T;L^1_{\mathrm{loc}}(\mathbb{R}))$ and $\{U(u_n)\}$ is bounded in $L^\infty(0,T;L^1_{\mathrm{loc}}(\mathbb{R}))$ (see (4.20) and (5.7)). Since every $\zeta \in C(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ can be uniformly approximated in bounded sets by finite sums $\sum_{i=1}^p f^{i,p}(x)g^{i,p}(t)$, with $f^{i,p}$, $g^{i,p}$ bounded and continuous functions ($1\le i\le p$; e.g., see [12, Théorème D.1.1]), it follows from (5.14) that, as $j\to\infty$, for all $\zeta \in C([0,T];C_c(\mathbb{R}))$,

$$\int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j})\zeta](x,t)\, dx - \int_{\mathbb{R}} [U^*\zeta](x,t)\, dx - C_U\langle \mu(\cdot,t),\zeta(\cdot,t)\rangle_{\mathbb{R}} \right| dt \to 0. \tag{5.17}$$

Proof of Proposition 5.5. (i) Let us first prove (5.14) for $U \in C^2([0,\infty))$, with $U' \in L^\infty(0,\infty)$, satisfying (4.12) and (5.5). Let $\rho \in C_c(\mathbb{R})$, $h \in C_c(0,T)$, and fix any $L>0$ such that $\operatorname{supp}\rho \subset (-L,L)$. Then, by (5.6),

$$\int_0^T U_{n_j,\rho}(t)h(t)\, dt \to \int_0^T U^*_\rho(t)h(t)\, dt + C_U\int_0^T h(t)\langle\mu(\cdot,t),\rho\rangle_{\mathbb{R}}\, dt, \tag{5.18}$$

where $U_{n_j,\rho}$ is defined by (4.21) and $U^*_\rho(t) := \int_{\mathbb{R}} U^*(x,t)\rho(x)\, dx$. Since, by Proposition 4.5, $\{U_{n_j,\rho}\}$ is bounded in $\mathrm{BV}(0,T)$ if $\rho \in C^2_c(\mathbb{R})$, there exists a subsequence which converges in $L^1(0,T)$. Combined with (5.18), this yields that $U_{n_j,\rho} \to U^*_\rho + C_U\langle\mu(\cdot,\cdot),\rho\rangle_{\mathbb{R}}$ in $\mathcal{D}(0,T)$ and in $L^1(0,T)$ for all $\rho \in C_c^2(\mathbb{R})$. Since the sequence $\{U(u_{n_j})\}$ is bounded in $L^\infty(0,T;L^1((-L,L)))$ and $U^* \in L^\infty(0,T;L^1((-L,L)))$, the condition $\rho \in C_c^2(\mathbb{R})$ may be relaxed to $\rho \in C_c(\mathbb{R})$, and we have found (5.14).

(ii) Next we prove (5.14) for all $U \in C([0,\infty)) \cap L^\infty((0,\infty))$ (in this case $C_U = 0$). To this end, let $U_k(u) := (U\chi_{[0,k]} * \theta_k)(u)$ for any $u \geq 0$, where $\theta_k \geq 0$ is a sequence of standard mollifiers ($k \in \mathbb{N}$). Then $\{U_k\} \subseteq C_c^2([0,\infty))$, $U_k \to U$ uniformly on compact subsets of $[0,\infty)$ and $\|U_k\|_{L^\infty(\mathbb{R})} \leq \|U\|_{L^\infty(\mathbb{R})}$. By part (i) and (4.20), for all $\rho \in C_c(\mathbb{R})$ and $k \in \mathbb{N}$, $M > 0$,

$$
\begin{aligned}
&\limsup_{j\to\infty}\int_0^T dt\Big|\int_{\mathbb{R}} U(u_{n_j})\rho(x)\,dx - \int_{\mathbb{R}} U^*(x,t)\rho(x)\,dx\Big| \\
&\quad\leq \limsup_{j\to\infty}\iint_{\{0\leq u_{n_j}\leq M\}} |U(u_{n_j}) - U_k(u_{n_j})||\rho|\,dx\,dt + \limsup_{j\to\infty}\iint_{\{u_{n_j}>M\}} |U(u_{n_j}) - U_k(u_{n_j})||\rho|\,dx\,dt \\
&\qquad + \iint_S |U^* - U_k^*||\rho|\,dx\,dt \\
&\quad\leq \|\rho\|_\infty|\operatorname{supp}\rho|T\|U - U_k\|_{L^\infty(0,M)} + \|\rho\|_\infty\Big\{\frac{2T}{M}\|u_0\|_{\mathcal{M}(\mathbb{R})}\|U\|_{L^\infty(\mathbb{R})} \\
&\qquad + \iint_{\operatorname{supp}\rho\times(0,T)} dx\,dt\int_{[0,\infty)} |U_k(\xi) - U(\xi)|\,d\tau_{(x,t)}(\xi)\Big\} \\
&\quad\leq 2\|\rho\|_\infty|\operatorname{supp}\rho|T\|U - U_k\|_{L^\infty(0,M)} + 2\|\rho\|_\infty\|U\|_{L^\infty(\mathbb{R})}\Big\{\frac{T\|u_0\|_{\mathcal{M}(\mathbb{R})}}{M} \\
&\qquad + \iint_{\operatorname{supp}\rho\times(0,T)} dx\,dt\int_{\{\xi>M\}} d\tau_{(x,t)}(\xi)\Big\},
\end{aligned}
$$

where we have used Chebychev's inequality and the inequality

$$
\int_{\{0\leq\xi\leq M\}} |U_k(\xi) - U(\xi)|\,d\tau_{(x,t)}(\xi) + \int_{\{\xi>M\}} |U_k(\xi) - U(\xi)|\,d\tau_{(x,t)}(\xi) \leq \|U_k - U\|_{L^\infty(0,M)} + 2\|U\|_{L^\infty(\mathbb{R})}\int_{\{\xi>M\}} d\tau_{(x,t)}(\xi).
$$

Letting $k \to \infty$, since $U_k \to U$ uniformly on compact sets in $[0,\infty)$, we obtain

$$
\begin{aligned}
&\limsup_{j\to\infty}\int_0^T dt\Big|\int_{\mathbb{R}} U(u_{n_j})\rho(x)\,dx - \int_{\mathbb{R}} U^*(x,t)\rho(x)\,dx\Big| \\
&\quad\leq 2\|\rho\|_{C(\overline{\mathbb{R}})}\|U\|_{L^\infty(\mathbb{R})}\Big\{\frac{T\|u_0\|_{\mathcal{M}(\mathbb{R})}}{M} + \iint_{\operatorname{supp}\rho\times(0,T)} dx\,dt\int_{\{\xi>M\}} d\tau_{(x,t)}(\xi)\Big\}.
\end{aligned}
\tag{5.19}
$$

Since $\tau_{(x,t)}$ is a probability measure, we have $\int_{\{\xi>M\}} d\tau_{(x,t)}(\xi) \to 0$ as $M \to \infty$ for a.e. $(x,t) \in S$, thus, by the dominated convergence theorem,

$$
\iint_{\operatorname{supp}\rho\times(0,T)} dx\,dt\int_{\{\xi>M\}} d\tau_{(x,t)}(\xi) \to 0 \quad \text{as } M \to \infty.
$$

Then, letting $M \to \infty$ in (5.19), we obtain (5.14).

(iii) Now let $U \in C([0,\infty))$ be any function satisfying (5.5). Arguing as in the proof of Proposition 5.3, let $l_{1m}, l_{2m} \in C^2([0,\infty))$ ($m \in \mathbb{N}$) satisfy $l_{1m}, l_{2m} \geq 0$ and $l_{1m} + l_{2m} = 1$ in $[0,\infty)$, $\operatorname{supp} l_{1m} \subseteq [0, m+1]$, and $\operatorname{supp} l_{2m} \subseteq [m,\infty)$. Then

$$
U(u_{n_j}) = U(u_{n_j})l_{1m}(u_{n_j}) + U(u_{n_j})l_{2,m}(u_{n_j}) \tag{5.20}
$$

and, by (5.8), for all $\varepsilon > 0$ and $m > m_\varepsilon$,

$$
(C_U - \varepsilon)u_{n_j}l_{2m}(u_{n_j}) \leq U(u_{n_j})l_{2m}(u_{n_j}) \leq (C_U + \varepsilon)u_{n_j}l_{2m}(u_{n_j}). \tag{5.21}
$$

Since $\|Ul_{1m}\|_{L^\infty(\mathbb{R})} \le \|U\|_{C([0,m+1])} < \infty$, the function Ul_{1m} belongs to $C([0,\infty)) \cap L^\infty(\mathbb{R})$. Then, by part (ii),

$$\int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j})l_{1m}(u_{n_j})](x,t)\rho(x)\,dx - \int_{\mathbb{R}} U_{1m}^*(x,t)\rho(x)\,dx \right| dt \to 0 \tag{5.22}$$

as $j \to \infty$, where $\rho \in C_c(\mathbb{R})$ and U_{1m}^* is defined by (5.10). By (5.21) and (4.20),

$$\int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j})l_{2m}(u_{n_j}) - C_U u_{n_j} l_{2m}(u_{n_j})](x,t)\rho(x)\,dx \right| dt \le \varepsilon \iint_S |u_{n_j}||\rho(x)|\,dx \le \varepsilon T\|u_0\|_{\mathcal{M}(\mathbb{R})}\|\rho\|_\infty.$$

Then we obtain that

$$\begin{aligned}
&\int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j})l_{2m}(u_{n_j}) - C_U l_{2m}^*](x,t)\rho(x)\,dx - C_U\langle \mu(\,\cdot\,,t),\rho\rangle_{\mathbb{R}} \right| dt \\
&\qquad \le \varepsilon T\|u_0\|_{\mathcal{M}(\mathbb{R})}\|\rho\|_\infty + C_U \int_0^T \left| \int_{\mathbb{R}} [u_{n_j} l_{2m}(u_{n_j}) - l_{2m}^*](x,t)\rho(x)\,dx - \langle \mu(\,\cdot\,,t),\rho\rangle_{\mathbb{R}} \right| dt,
\end{aligned}$$

with l_{2m}^* defined as in (5.12). The map $u \mapsto ul_{2m}(u)$ belongs to $C^2([0,\infty))$, has bounded derivative and satisfies (4.12) and (5.5), with $C_U = 1$. Then, by part (i), (5.20) and (5.22),

$$\limsup_{j\to\infty} \int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j}) - U_{1m}^* - C_U l_{2m}^*](x,t)\rho(x)\,dx - C_U\langle \mu(\,\cdot\,,t),\rho\rangle_{\mathbb{R}} \right| dt \le \varepsilon T\,\|u_0\|_{\mathcal{M}(\mathbb{R})}\|\rho\|_\infty \quad \text{if } m > m_\varepsilon. \tag{5.23}$$

To complete the proof of (5.14), we show that

$$\lim_{m\to\infty} \iint_S |U^* - U_{1m}^* - C_U l_{2m}^*|(x,t)|\rho(x)|\,dx\,dt = 0. \tag{5.24}$$

By (5.21),

$$\begin{aligned}
|U^* - U_{1m}^* - C_U l_{2m}^*|(x,t) &\le \int_{[0,\infty)} \big|U(\xi) - U(\xi)l_{1m}(\xi) - C_U\xi l_{2m}(\xi)\big|\,d\tau_{(x,t)}(\xi) \\
&= \int_{[0,\infty)} \big|U(\xi)l_{2m}(\xi) - C_U\xi l_{2m}(\xi)\big|\,d\tau_{(x,t)}(\xi) \\
&\le \varepsilon \int_{[m,\infty)} \xi\,d\tau_{(x,t)}(\xi) \le \varepsilon u_b(x,t)
\end{aligned}$$

for a.e. $(x,t) \in S$. Since $u_b \in L^\infty(0,T;L^1(\mathbb{R}))$ and $\int_{[m,\infty)} \xi\,d\tau_{(x,t)}(\xi) \to 0$ as $m \to \infty$ for a.e. $(x,t) \in S$, (5.24) follows from the dominated convergence theorem.

Letting $m \to \infty$ in (5.23), it follows from (5.24) that

$$\begin{aligned}
&\limsup_{j\to\infty} \int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j}) - U^*](x,t)\rho(x)\,dx - C_U\langle \mu(\,\cdot\,,t),\rho\rangle_{\mathbb{R}} \right| dt \\
&\qquad \le \limsup_{m\to\infty}\left(\limsup_{j\to\infty} \int_0^T \left| \int_{\mathbb{R}} [U(u_{n_j}) - U_{1m}^* - C_U l_{2m}^*]\rho\,dx - C_U\langle \mu(\,\cdot\,,t),\rho\rangle_{\mathbb{R}} \right| dt \right) \\
&\qquad \le \varepsilon T\|u_0\|_{\mathcal{M}(\mathbb{R})}\|\rho\|_\infty,
\end{aligned}$$

and (5.14) follows from the arbitrariness of ε.

Finally, (5.15) follows from (5.14), the separability of $C_c(\mathbb{R})$ and a diagonal argument; we leave the details to the reader. □

Proposition 5.7. *Let* (H1) *hold. Then* (5.4) *is the Lebesgue decomposition of u, i.e.,*

$$u_b = u_r \quad \text{a.e. in } S, \qquad \mu = u_s \quad \text{in } \mathcal{M}(S). \tag{5.25}$$

Proof. Let U be a convex function with $U(0) = 0$ and $U' \in L^\infty(0, \infty)$. By (3.16),

$$\int_{\mathbb{R}} U(u_{n_j})(x, \bar{t})\zeta(x, \bar{t})\,dx - \int_{\mathbb{R}} U(u_{0n_j})(x)\zeta(x, 0)\,dx \le \iint_{\mathbb{R}\times(0,\bar{t})} \{U(u_{n_j})\partial_t\zeta + \Theta_U(u_{n_j})\partial_x\zeta\}dx\,dt \tag{5.26}$$

for all $\zeta \in C^1([0, T]; C^1_c(\mathbb{R}))$ and a.e. $\bar{t} \in (0, t)$, where

$$\Theta_U(u) := \int_0^u U'(s)\varphi'(s)\,ds + \theta_U \quad (\theta_U \in \mathbb{R}). \tag{5.27}$$

Let $U_m(u) = (u - m)\chi_{[m,\infty)}(u)$ and $\theta_{U_m} = 0$ ($m \in \mathbb{N}$). Since $U_m(u)/u \to C_{U_m} = 1$ and $\Theta_{U_m}(u)/u \to C_\varphi$ as $u \to \infty$ (with C_φ as in (H1)), it follows from (5.17) that

$$\int_0^{\bar{t}} \left| \int_{\mathbb{R}} [U_m(u_{n_j})\partial_t\zeta](x, t)\,dx - \int_{\mathbb{R}} [U^*_m\partial_t\zeta](x, t)\,dx - \langle \mu(\cdot, t), \partial_t\zeta(\cdot, t)\rangle_{\mathbb{R}} \right| dt \to 0$$

and

$$\int_0^{\bar{t}} \left| \int_{\mathbb{R}} [\Theta_{U_m}(u_{n_j})\partial_x\zeta](x, t)\,dx - \int_{\mathbb{R}} [\Theta^*_{U_m}\partial_x\zeta](x, t)\,dx - C_\varphi\langle \mu(\cdot, t), \partial_x\zeta(\cdot, t)\rangle_{\mathbb{R}} \right| dt \to 0$$

as $j \to \infty$, where

$$U^*_m(x, t) := \int_{[0,\infty)} U_m(\xi)\,d\tau_{(x,t)}(\xi), \quad \Theta^*_{U_m}(x, t) := \int_{[0,\infty)} \Theta_{U_m}(\xi)\,d\tau_{(x,t)}(\xi)$$

belong to $L^\infty(0, T; L^1_{\mathrm{loc}}(\mathbb{R}))$. In particular, setting $\partial_\nu\zeta := \partial_t\zeta + C_\varphi\partial_x\zeta$, we have that

$$\iint_{\mathbb{R}\times(0,\bar{t})} \{U_m(u_{n_j})\partial_t\zeta + \Theta_{U_m}(u_{n_j})\partial_x\zeta\}\,dx\,dt \to \iint_{\mathbb{R}\times(0,\bar{t})} \{U^*_m\partial_t\zeta + \Theta^*_{U_m}\partial_x\zeta\}\,dx\,dt + \int_0^{\bar{t}} \langle \mu(\cdot, t), \partial_\nu\zeta(\cdot, t)\rangle_{\mathbb{R}}\,dt. \tag{5.28}$$

By (5.15) and a diagonal argument, there exist a null set $N \subset (0, T)$ and a subsequence, denoted again by $\{u_{n_j}\}$, such that for all $\bar{t} \in (0, T) \setminus N$ and $m \in \mathbb{N}$,

$$\lim_{n_j\to\infty} \int_{\mathbb{R}} U_m(u_{n_j})(x, \bar{t})\zeta(x, \bar{t})\,dx = \int_{\mathbb{R}} U^*_m(x, \bar{t})\zeta(x, \bar{t})\,dx + \langle \mu(\cdot, \bar{t}), \zeta(x, \bar{t})\rangle_{\mathbb{R}}. \tag{5.29}$$

Since $\{U_m(u_{0n_j}) - u_{0n_j}\}$ is bounded in $L^\infty(\mathbb{R})$ and converges a.e. to $U_m(u_{0r}) - u_{0r}$, it follows from (3.15) that

$$\lim_{n_j\to\infty} \int_{\mathbb{R}} U_m(u_{0n_j})(x)\zeta(x, 0)\,dx = \int_{\mathbb{R}} U_m(u_{0r})(x)\zeta(x, 0)\,dx + \langle u_{0s}, \zeta(\cdot, 0)\rangle_{\mathbb{R}}. \tag{5.30}$$

Setting $U = U_m$ in (5.26) and letting $j \to \infty$, we obtain from (5.28)–(5.30) that

$$\begin{aligned}\int_{\mathbb{R}} U^*_m(x, \bar{t})\zeta(x, \bar{t})\,dx + \langle \mu(\cdot, \bar{t}), \zeta(\cdot, \bar{t})\rangle_{\mathbb{R}} &\le \iint_{\mathbb{R}\times(0,\bar{t})} \{U^*_m\partial_t\zeta + \Theta^*_{U_m}\partial_x\zeta\}\,dx\,dt + \int_0^{\bar{t}} \langle \mu(\cdot, t), \partial_\nu\zeta(\cdot, t)\rangle_{\mathbb{R}}\,dt \\ &\quad + \int_{\mathbb{R}} U_m(u_{0r})(x)\zeta(x, 0)\,dx + \langle u_{0s}, \zeta(\cdot, 0)\rangle_{\mathbb{R}}\end{aligned} \tag{5.31}$$

for all $\bar{t} \in (0, T) \setminus N$ and $m \in \mathbb{N}$. Since for all $u \ge 0$ (see (3.1)),

$$0 \le U_m(u) \le u\chi_{[m,\infty)}(u), \quad |\Theta_{U_m}(u)| = |\varphi(u) - \varphi(m)|\chi_{[m,u)}(u) \le Mu\chi_{[m,\infty)}(u),$$

we have that $|U_m^*| \le u_b$, $|\Theta^*_{U_m}| \le Mu_b$, $U_m^* \to 0$ and $\Theta^*_{U_m}(x,t) \to 0$ (as $m \to \infty$) a.e. in S. Thus, by the dominated convergence theorem and (5.31), for all $\bar t \in (0,T) \setminus N$,

$$\langle \mu(\cdot,\bar t), \zeta(\cdot,\bar t)\rangle_{\mathbb{R}} \le \int_0^{\bar t} \langle \mu(\cdot,t), \partial_v \zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt + \langle u_{0s}, \zeta(\cdot,0)\rangle_{\mathbb{R}}. \tag{5.32}$$

Let $\rho \in C^1_c(\mathbb{R})$ and $\zeta(x,t) = \rho(x - C_\varphi t)$, so $\zeta_v \equiv 0$. By (5.32), $\langle \mu(\cdot,\bar t), \rho(\cdot - C_\varphi \bar t)\rangle_{\mathbb{R}} \le \langle u_{0s}, \rho\rangle_{\mathbb{R}}$. Hence, $\mu(\cdot,\bar t)$ is singular with respect to the Lebesgue measure and, since $\mu(\cdot,\bar t) = [\mu(\cdot,\bar t)]_s = \mu_s(\cdot,\bar t)$ for a.e. $\bar t \in (0,T)$ (see (2.3)), (5.25) follows from the uniqueness of the Lebesgue decomposition. □

The following result is based on the concept of compensated compactness (e.g., see [13]).

Proposition 5.8. *Let* (H1) *hold. Then* $\varphi(u_r) = \int_{[0,\infty)} \varphi(\xi)\,d\tau(\xi)$ *a.e. in* S.

Proof. Let $U, V \in C^2([0,\infty)) \cap L^\infty((0,\infty))$ satisfy (4.12), and assume that Θ_U, Θ_V, defined by (5.27), belong to $L^\infty((0,\infty))$. By (4.13), we have

$$\varepsilon \|U''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2\|_{L^1(S)} \le C_p \quad \text{and} \quad \varepsilon \|V''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2\|_{L^1(S)} \le C_p$$

for all $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$, and up to a subsequence,

$$\varepsilon U''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2 \overset{*}{\rightharpoonup} \lambda_n, \quad \varepsilon V''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2 \overset{*}{\rightharpoonup} \mu_n \quad \text{in } \mathcal{M}(S) \text{ as } \varepsilon \to 0, \tag{5.33}$$

for some $\lambda_n, \mu_n \in \mathcal{M}(S)$. By the lower semicontinuity of the norm,

$$\|\lambda_n\|_{\mathcal{M}(S)} \le C_p, \quad \|\mu_n\|_{\mathcal{M}(S)} \le C_p \quad \text{for } n \in \mathbb{N}. \tag{5.34}$$

Let $\zeta \in C^2_c(S)$. Then (see (4.9))

$$\varepsilon \iint_S U''(u_n^\varepsilon)(\partial_x u_n^\varepsilon)^2 \zeta\,dx\,dt = \iint_S \{U(u_n^\varepsilon)\partial_t \zeta + \Theta_{U,\varepsilon}(u_n^\varepsilon)\partial_x \zeta + \varepsilon U(u_n^\varepsilon)\partial_x^2 \zeta\}\,dx\,dt, \tag{5.35}$$

where $\Theta_{U,\varepsilon}(u) = \int_0^u U'(s)\varphi'_\varepsilon(s)\,ds + \theta_U$, $\theta_U \in \mathbb{R}$. By (3.1) and (4.4), for all $n \in \mathbb{N}$,

$$|\Theta_{U,\varepsilon}(u_n^\varepsilon)| \le \int_0^{\|u_{0n}\|_\infty} |U'(s)\varphi'_\varepsilon(s)|\,ds + |\theta_U| \le M \int_0^{\|u_{0n}\|_\infty} |U'(s)|\,ds + |\theta_U| \le \gamma_{n,U}$$

for some $\gamma_{n,U} \ge 0$, so for fixed $n \in \mathbb{N}$, the family $\{\Theta_{U,\varepsilon}(u_n^\varepsilon)\}_\varepsilon$ is uniformly bounded in $L^\infty(S)$. Similar results hold for V and $\Theta_{V,\varepsilon}(u) = \int_0^u V'(s)\varphi'_\varepsilon(s)\,ds + \theta_V$, and letting $\varepsilon \to 0$ in (5.35) along some subsequence $\{\varepsilon_m\}$ (see the proof of Proposition 4.5), it follows from by (5.33) that for all $n \in \mathbb{N}$ and $\zeta \in C^1_c(S)$,

$$\iint_S \{U(u_n)\partial_t \zeta + \Theta_U(u_n)\partial_x \zeta\}\,dx\,dt = \langle \lambda_n, \zeta\rangle_S, \quad \iint_S \{V(u_n)\partial_t \zeta + \Theta_V(u_n)\partial_x \zeta\}\,dx\,dt = \langle \mu_n, \zeta\rangle_S, \tag{5.36}$$

where u_n is the entropy solution of the approximating problem (Pn) (see (4.18)).

Let $A \subset\subset S$ be a bounded open set and let $Y_n, Z_n : A \to \mathbb{R}^2$ be defined by

$$Y_n := (\Theta_U(u_n), U(u_n)), \quad Z_n := (V(u_n), -\Theta_V(u_n)).$$

By (5.36),

$$\operatorname{div} Y_n = -\lambda_n, \quad \operatorname{curl} Z_n = -\mu_n \quad \text{in } \mathcal{D}'(A). \tag{5.37}$$

Since U, Θ_U, V, Θ_V are bounded in $(0,\infty)$, the sequences $U(u_n)$, $\Theta_U(u_n)$, $V(u_n)$ and $\Theta_V(u_n)$ are bounded in $L^1(A)$ and uniformly integrable, and, by Theorem 2.3,

$$U(u_n) \rightharpoonup U^* := \int_{[0,\infty)} U(\xi)\,d\tau_{(\cdot,\cdot)}(\xi), \quad \Theta_U(u_n) \rightharpoonup \Theta_U^* := \int_{[0,\infty)} \Theta_U(\xi)\,d\tau_{(\cdot,\cdot)}(\xi),$$

$$V(u_n) \rightharpoonup V^* := \int_{[0,\infty)} V(\xi)\,d\tau_{(\cdot,\cdot)}(\xi), \quad \Theta_V(u_n) \rightharpoonup \Theta_V^* := \int_{[0,\infty)} \Theta_V(\xi)\,d\tau_{(\cdot,\cdot)}(\xi)$$

in $L^1(A)$, where $\tau_{(\cdot,\cdot)}$ denotes the disintegration of the Young measure τ associated with $\{u_n\}$. Since the sequences $U(u_n)$, $\Theta_U(u_n)$, $V(u_n)$ and $\Theta_V(u_n)$ are bounded in $L^\infty(A) \subset L^2(A)$, they also converge weakly in $L^2(A)$, so

$$Y_n \rightharpoonup Y^* := (\Theta_U^*, U^*), \quad Z_n \rightharpoonup Z^* := (V^*, -\Theta_V^*) \quad \text{in } [L^2(A)]^2.$$

By a similar argument,

$$Y_n \cdot Z_n := \Theta_U(u_n)V(u_n) - \Theta_V(u_n)U(u_n) \rightharpoonup \int_{[0,\infty)} [\Theta_U(\xi)V(\xi) - \Theta_V(\xi)U(\xi)]\, d\tau_{(\cdot,\cdot)}(\xi) \quad \text{in } L^2(A). \tag{5.38}$$

By (5.34) and (5.37), $\{\operatorname{div} Y_n\}$ and $\{\operatorname{curl} Z_n\}$ are precompact in $W^{-1,2}(A)$ (see [13, Chapter 1, Corollary 1]) and, by the div-curl lemma,

$$Y_n \cdot Z_n \to Y^* \cdot Z^* = \Theta_U^* V^* - \Theta_V^* U^* \quad \text{in } \mathcal{D}'(A). \tag{5.39}$$

By (5.38) and (5.39),

$$\int_{[0,\infty)} [\Theta_U(\xi) - \Theta_U^*]V(\xi)\, d\tau(\xi) = \int_{[0,\infty)} [U(\xi) - U^*]\Theta_V(\xi)\, d\tau(\xi) \quad \text{a.e. in } A. \tag{5.40}$$

For every U as above with $U' > 0$ in $(0,\infty)$, by a standard approximation argument, we may choose $V(u) = |U^* - U(u)|$, so $\Theta_V(u) = \operatorname{sgn}(U(u) - U^*)[\Theta_U(u) - \Theta_U(U^{-1}(U^*))]$ and, by (5.40),

$$[\Theta_U^* - \Theta_U(U^{-1}(U^*))] \int_{[0,\infty)} |U^* - U(\xi)|\, d\tau(\xi) = 0. \tag{5.41}$$

Let $U_k \in C^2([0,\infty)) \cap L^\infty((0,\infty))$ satisfy (4.12) and

$$U_k(0) = 0, \qquad 0 < U_k' \le U_{k+1}' \le 1 \quad \text{in } [0,\infty), \qquad U_k'(u) \to 1 \quad \text{for } u \ge 0 \text{ as } k \to \infty. \tag{5.42}$$

By (3.1),

$$|\Theta_{U_k}(u)| \le \int_0^u U_k'(s)|\varphi'(s)|\, ds + |\theta_{U_k}| \le M U_k(u) + |\theta_{U_k}|,$$

thus Θ_{U_k} is bounded in $(0,\infty))$ for every $k \in \mathbb{N}$. We claim that, as $k \to \infty$,

$$U_k^* := \int_{[0,\infty)} U_k(\xi)\, d\tau(\xi) \to u_r \qquad \text{a.e. in } A, \tag{5.43}$$

$$\Theta_{U_k}^* - \Theta_{U_k}(U_k{}^{-1}(U_k^*)) \to \int_{[0,\infty)} \varphi(\xi)\, d\tau(\xi) - \varphi(u_r) \quad \text{a.e. in } A, \tag{5.44}$$

where $\Theta_{U_k}^* := \int_{[0,\infty)} \Theta_{U_k}(\xi)\, d\tau(\xi)$ (recall that $\varphi \in L^1([0,\infty); d\tau_{(x,t)})$, see Remark 3.2). By (5.43) and the dominated convergence theorem, for a.e. $(x,t) \in A$,

$$\int_{[0,\infty)} |U_k^*(x,t) - U_k(\xi)|\, d\tau_{(x,t)}(\xi) \to \int_{[0,\infty)} |u_r(x,t) - \xi|\, d\tau_{(x,t)}(\xi) \quad \text{as } k \to \infty,$$

since $0 \le U_k(\xi) \le \xi$ for all $k \in \mathbb{N}$ and $I(\xi) := \xi$ belongs to $L^1([0,\infty), d\tau_{(x,t)})$ (recall that, by (5.25) and the definition of u_b in (5.2), $u_r(x,t) = \int_{[0,\infty)} \xi\, d\tau_{(x,t)}(\xi) < \infty$ for a.e. $(x,t) \in S$). Letting $k \to \infty$ in (5.41), with $U = U_k$, we obtain that for a.e. $(x,t) \in A$,

$$\Big[\int_{[0,\infty)} \varphi(\xi)\, d\tau_{(x,t)}(\xi) - \varphi(u_r)(x,t)\Big] \int_{[0,\infty)} |u_r(x,t) - \xi|\, d\tau_{(x,t)}(\xi) = 0,$$

and Proposition 5.8 follows from the arbitrariness of A.

It remains to prove (5.43) and (5.44). By (5.42) and the monotone convergence theorem, $U_k(\xi) \to \xi$ for any $\xi \in [0,\infty)$, and (5.43) follows (recall that $I(\xi) = \xi \in L^1([0,\infty), d\tau)$). Concerning (5.44), we observe that

$$\Theta^*_{U_k} - \Theta_{U_k}({U_k}^{-1}(U_k^*)) = \int_{[0,\infty)} \Big(\int_0^{\xi} U_k'(s)\varphi'(s)\,ds \Big)\, d\tau(\xi) - \int_0^{{U_k}^{-1}({U_k}^*)} U_k'(s)\varphi'(s)\,ds. \tag{5.45}$$

Since $U_k'(\xi) \to 1$ and $|U_k'(\xi)\varphi'(\xi)| \le M$ for $\xi \ge 0$ (see (5.42) and (3.1)), it follows from the dominated convergence theorem that

$$\int_{[0,\infty)} \Big(\int_0^{\xi} U_k'(s)\varphi'(s)\,ds \Big)\, d\tau_{(x,t)}(\xi) \to \int_{[0,\infty)} \varphi(\xi)\, d\tau_{(x,t)}(\xi). \tag{5.46}$$

On the other hand,

$$\int_0^{{U_k}^{-1}(U_k^*(x,t))} U_k'(s)\varphi'(s)\,ds - \varphi(u_r)(x,t) = \int_0^{u_r(x,t)} [U_k'(s) - 1]\varphi'(s)\,ds + \int_{u_r(x,t)}^{{U_k}^{-1}(U_k^*(x,t))} U_k'(s)\varphi'(s)\,ds. \tag{5.47}$$

Arguing as before, one can show that the first term in the right-hand side of (5.47) vanishes as $k \to \infty$. As for the second term, we observe, by (5.42) and (5.43), that

$$\begin{aligned}\Big| \int_{u_r(x,t)}^{{U_k}^{-1}(U_k^*(x,t))} U_k'(s)\varphi'(s)\,ds \Big| &\le M|u_r(x,t) - {U_k}^{-1}(U_k^*(x,t))| \\ &\le M\Big(|u_r(x,t) - U_k^{-1}(u_r(x,t))| + \sup_{s\in I_\delta(u_r(x,t))} \frac{1}{U_1'(s)} |u_r(x,t) - U_k^*(x,t)| \Big)\end{aligned}$$

for some $\delta > 0$ and $k \in \mathbb{N}$ sufficiently large, where $I_\delta(q) \equiv (q-\delta, q+\delta)$. Hence,

$$\int_0^{{U_k}^{-1}(U_k^*(x,t))} U_k'(s)\varphi'(s)\,ds \to \varphi(u_r)(x,t) \quad \text{for a.e. } (x,t) \in A, \tag{5.48}$$

and we obtain (5.44) from (5.45), (5.46) and (5.48). □

To prove the second part of Theorem 3.7, we need the following result which characterizes the disintegration of the Young measure τ.

Proposition 5.9. *Let* (H1) *hold and* $\varphi \in C^1([0,\infty))$ *satisfy for all* $\bar u > 0$ *either* (C2) *or the following:*
(C3) *there exist* $a > 0$, $b \in (0,\infty]$ *such that* φ' *is constant in* $I_{a,b} = [\bar u - a, \bar u + b]$ *and, if* $b < \infty$, *then* φ' *is strictly monotone in* $[\bar u + b, \bar u + \tilde b]$ *and* $[\bar u - \tilde a, \bar u - a]$ *for some* $\tilde b > b$ *and* $\tilde a \in (a, \bar u)$.
Then, for a.e. $(x,t) \in S$, *the following hold:*
(i) *If* $u_r(x,t) = 0$, *then* $\tau_{(x,t)} = \delta_0$.
(ii) *If* φ' *is strictly monotone in* $I_{a,b} = [u_r(x,t) - a, u_r(x,t) + b]$, *with* $a, b \ge 0$, $a + b > 0$, *then*

$$\tau_{(x,t)} = \delta_{u_r(x,t)}. \tag{5.49}$$

(iii) *If* φ' *is constant in the above interval* $I_{a,b}$ *for some* $a > 0$, $b > 0$, *then*

$$\operatorname{supp} \tau_{(x,t)} \subseteq I_{(x,t)} \quad \textit{for a.e. } (x,t) \in S, \tag{5.50}$$

where $I_{(x,t)} \supseteq I_{a,b}$ *is the maximal interval where* $\varphi'(\cdot) \equiv \varphi'(u_r(x,t))$.

Proof. Let $(x,t) \in S$ be fixed. If $u_r(x,t) = 0$, it follows from (5.25) and the definition of u_b in (5.2) that $\int_{[0,\infty)} \xi\, d\tau_{(x,t)}(\xi) = 0$, which implies part (i): $\tau_{(x,t)} = \delta_0$.

So let $u_r(x,t) > 0$. Let $l_1 := u_r(x,t)$, $l_2 > l_1$ and

$$V_k(u) := k(u - l_1)\chi_{(l_1, l_1+\frac{1}{k})}(u) + \chi_{[l_1+\frac{1}{k}, l_2)}(u) + k\Big(l_2 + \frac{1}{k} - u\Big)\chi_{[l_2, l_2+\frac{1}{k})}(u)$$

for $u \geq 0$ and sufficiently large $k \in \mathbb{N}$. Then $V_k(u) \to \chi_{(l_1,l_2]}(u)$ as $k \to \infty$, and

$$\Theta_{V_k}(u) = \int_0^u V_k'(s)\varphi'(s)\,ds \to \varphi'(l_1)\chi_{(l_1,l_2]}(u) + [\varphi'(l_1) - \varphi'(l_2)]\chi_{(l_2,\infty)}(u) \quad (u \geq 0).$$

By standard approximation arguments, (5.40) is satisfied with $U = U_k$ and $V = V_k$, where $\{U_k\}$ is the sequence in the proof of Proposition 5.8 (see (5.42)), i.e.,

$$\int_{[0,\infty)} [\Theta_{U_k}(\xi) - \Theta^*_{U_k}(x,t)]V_k(\xi)\,d\tau_{(x,t)}(\xi) = \int_{[0,\infty)} [U_k(\xi) - U_k^*(x,t)]\Theta_{V_k}(\xi)\,d\tau_{(x,t)}(\xi).$$

Letting $k \to \infty$ and arguing as in the proof of Proposition 5.8, we obtain that

$$U_k(\xi) - U_k^*(x,t) \to \xi - \int_{[0,\infty)} \xi\,d\tau_{(x,t)}(\xi) = \xi - u_r(x,t) = \xi - l_1,$$

$$\Theta^*_{U_k}(x,t) - \Theta_{U_k}(\xi) \to \int_{[0,\infty)} \varphi(\xi)d\tau_{(x,t)}(\xi) - \varphi(\xi) = \varphi(u_r)(x,t) - \varphi(\xi) = \varphi(l_1) - \varphi(\xi)$$

for all $\xi \geq 0$ (see (5.25) and Proposition 5.8). This implies that

$$\int_{[0,\infty)} [\Theta_{U_k}(\xi) - \Theta^*_{U_k}(x,t)]V_k(\xi)\,d\tau_{(x,t)}(\xi) \to \int_{(l_1,l_2]} [\varphi(\xi) - \varphi(l_1)]\,d\tau_{(x,t)}(\xi),$$

$$\int_{[0,\infty)} [U_k(\xi) - U_k^*(x,t)]\Theta_{V_k}(\xi)\,d\tau_{(x,t)}(\xi) \to \int_{(l_1,l_2]} \varphi'(l_1)(\xi - l_1)\,d\tau_{(x,t)}(\xi) + [\varphi'(l_1) - \varphi'(l_2)] \int_{(l_2,\infty)} (\xi - l_1)\,d\tau_{(x,t)}(\xi),$$

whence

$$\int_{(l_1,l_2]} [\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)]\,d\tau_{(x,t)}(\xi) = [\varphi'(l_1) - \varphi'(l_2)] \int_{(l_2,\infty)} (\xi - l_1)\,d\tau_{(x,t)}(\xi). \tag{5.51}$$

Similarly, let $l_0 \in (0, l_1)$ and set

$$\tilde{V}_k(u) := k(u - l_0)\chi_{[l_0,l_0+\frac{1}{k}]}(u) + \chi_{(l_0+\frac{1}{k},l_1-\frac{1}{k})}(u) + k(l_1 - u)\chi_{[l_1-\frac{1}{k},l_1]}(u).$$

Then $\tilde{V}_k(u) \to \chi_{(l_0,l_1)}(u)$ and

$$\Theta_{\tilde{V}_k}(u) = \int_{l_1}^u \tilde{V}_k'(s)\varphi'(s)\,ds \to \varphi'(l_1)\chi_{(l_0,l_1)}(u) + [\varphi'(l_1) - \varphi'(l_0)]\chi_{[0,l_0]}(u) \quad (u \geq 0).$$

Letting $k \to \infty$ in (5.40), with $U = U_k$ as above and $V = \tilde{V}_k$, we obtain that

$$\int_{(l_0,l_1)} [\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)]\,d\tau_{(x,t)}(\xi) = [\varphi'(l_1) - \varphi'(l_0)] \int_{[0,l_0]} (\xi - l_1)\,d\tau_{(x,t)}(\xi). \tag{5.52}$$

By (C2) and (C3), we can distinguish two cases.

(a) If φ is strictly convex or strictly concave in $[l_1, l_2]$, it follows from (5.51) that

$$\int_{(l_1,l_2]} |\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)|\,d\tau_{(x,t)}(\xi) + |\varphi'(l_1) - \varphi'(l_2)| \int_{(l_2,\infty)} |l_1 - \xi|\,d\tau_{(x,t)}(\xi) = 0,$$

where

$$\chi_{(l_1,l_2]}(\xi)|\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)| > 0 \quad \text{and} \quad |\varphi'(l_1) - \varphi'(l_2)| > 0.$$

This implies that supp $\tau_{(x,t)} \subseteq [0, l_1]$. Since $\tau_{(x,t)}$ is a probability measure and $l_1 := u_r(x,t)$,

$$u_r(x,t) = \int_{[0,u_r(x,t)]} \xi \, d\tau_{(x,t)}(\xi) = \int_{[0,u_r(x,t)]} [\xi - u_r(x,t)] \, d\tau_{(x,t)}(\xi) + u_r(x,t)$$

(see (5.2) and (5.25)), thus

$$\int_{[0,u_r(x,t)]} |\xi - u_r(x,t)| \, d\tau_{(x,t)}(\xi) \;\Rightarrow\; \tau_{(x,t)}([0, u_r(x,t))) = 0.$$

Hence, supp $\tau_{(x,t)} = \{u_r(x,t)\}$ and (5.49) follows since $\tau_{(x,t)}$ is a probability measure.

Similarly, if φ is strictly convex or strictly concave in (l_0, l_1), it follows from (5.52) that $\tau_{(x,t)}([0, l_1)) = 0$ (we omit the details). Thus, supp $\tau_{(x,t)} \subseteq [l_1, \infty)$, and arguing as above we obtain (5.49).

(b) If φ is affine in $[l_1 - c, l_1 + c]$ for some $c > 0$, let $I = [\bar{l}_0, \bar{l}_2]$ be the maximal interval containing l_1, where $\varphi'(\xi) = \varphi'(l_1)$. If $I = [0, \infty)$, (5.50) is satisfied. If $\bar{l}_2 < \infty$, by (C3) and the maximality of I, φ is strictly convex (or concave) in $[\bar{l}_2, \bar{l}_2 + b]$ for some $b > 0$ (and affine in $[l_1, \bar{l}_2]$). By (5.51), with $l_2 \in (\bar{l}_2, \bar{l}_2 + b)$, we obtain that

$$\int_{(\bar{l}_2, l_2]} \big|\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)\big| \, d\tau_{(x,t)}(\xi) + |\varphi'(l_1) - \varphi'(l_2)| \int_{(l_2,\infty)} |l_1 - \xi| \, d\tau_{(x,t)}(\xi) = 0,$$

where

$$\chi_{(\bar{l}_2, l_2]}(\xi)\big|\varphi(\xi) - \varphi(l_1) - \varphi'(l_1)(\xi - l_1)\big| > 0 \quad \text{and} \quad |\varphi'(l_1) - \varphi'(l_2)| > 0.$$

It follows that $\tau_{(x,t)}((\bar{l}_2, \infty)) = 0$, whence supp $\tau_{(x,t)} \subseteq [0, \bar{l}_2]$. Similarly, if $\bar{l}_0 > 0$, by (C3) and the maximality of I, φ is strictly convex (or concave) in $[\bar{l}_0 - a, \bar{l}_0]$ for some $a > 0$ (and affine in $[\bar{l}_0, l_1]$). Arguing as before, we obtain from (5.52), with $l_0 \in (\bar{l}_0 - a, \bar{l}_0)$, that supp $\tau_{(x,t)} \subseteq [\bar{l}_0, \infty)$ (we omit the details). Summing up, we obtain (5.50): supp $\tau_{(x,t)} \subseteq [0, \bar{l}_2] \cap [\bar{l}_0, \infty) = I$. □

Remark 5.10. If (C2) is satisfied for all $\bar{u} > 0$, it follows from (5.49) and standard properties of narrow convergence of Young measures (see [28]) that $u_{n_j} \to u_r$ in measure, where $\{u_{n_j}\}$ is the subsequence in Proposition 5.1. Therefore, up to a subsequence, $u_{n_j} \to u_r$ a.e. in S. Hence, if φ is bounded, it follows from the dominated convergence theorem that $\varphi(u_{n_j}) \to \varphi(u_r)$ in $L^1((-L, L) \times (0, T))$ for all $L > 0$.

Now we can prove Theorem 3.7.

Proof of Theorem 3.7. Let $\zeta \in C^1([0,T]; C_c^1(\mathbb{R}))$, with $\zeta(\cdot, T) = 0$ in $\mathbb{R}$, and let $L > 0$ be such that supp $\zeta \subset (-L, L) \times [0, T]$. By (5.17), with $U(u) = u$ and $U(u) = \varphi(u)$,

$$\iint_S u_{n_j} \partial_t \zeta \, dx \, dt \to \iint_S u_r \partial_t \zeta \, dx \, dt + \int_0^T \langle u_s(\cdot, t), \partial_t \zeta(\cdot, t) \rangle_{\mathbb{R}} \, dt,$$

$$\iint_S \varphi(u_{n_j}) \partial_x \zeta \, dx \, dt \to \iint_S \varphi^* \partial_x \zeta \, dx \, dt + C_\varphi \int_0^T \langle u_s(\cdot, t), \partial_x \zeta(\cdot, t) \rangle_{\mathbb{R}} \, dt$$

(see also (5.25)). Letting $j \to \infty$ in (3.17), with $n = n_j$, we obtain (3.4). Inequality (3.6) is proven similarly, since by arguing as in Proposition 5.3, we get

$$E(u_{0n_j}) \overset{*}{\rightharpoonup} E(u_{0r}) + C_E u_{0s} \quad \text{in } \mathcal{M}(\mathbb{R})$$

(in this regard, see also (3.15)). Thus, the function $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ given by Proposition 5.1 is an entropy solution of problem (P) in the sense of Young measures. By Proposition 5.8, it is also a solution in the sense of Definition 3.3. This proves the first part of the theorem. The second part is an immediate consequence of Proposition 5.9; in fact, (3.9) follows from (3.6) and (5.49). □

Let us end this section by proving Proposition 3.8.

Proof of Proposition 3.8. For every $\tilde\zeta \in C^1([0,T];C_c^1(\mathbb{R}))$, $\tilde\zeta(\cdot,T)=0$, we set $E(u)=U_m(u)=(u-m)\chi_{\{u>m\}}(u)$ and $F(u)=F_m(u)=\int_0^u U_m'(\xi)\varphi'(\xi)\,d\xi=(\varphi(u)-\varphi(m))\chi_{\{u>m\}}(u)$ in the entropy inequalities (3.6) ($m\in\mathbb{N}$). Then we get

$$\iint_S \{U_m^*\partial_t\tilde\zeta+F_m^*\partial_x\tilde\zeta\}\,dx\,dt+\int_0^T\langle u_s(\cdot,t),\partial_t\tilde\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt+C_\varphi\int_0^T\langle u_s(\cdot,t),\partial_x\tilde\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt$$
$$\geq -\int_{\mathbb{R}} U_m(u_{0r})\tilde\zeta(x,0)\,dx-\langle u_{0s},\tilde\zeta(\cdot,0)\rangle_{\mathbb{R}},$$

where, for a.e. $(x,t)\in S$,

$$U_m^*(x,t):=\int_{[0,\infty)}U_m(\xi)\,d\tau_{(x,t)}(\xi),\quad F_m^*(x,t):=\int_{[0,\infty)}F_m(\xi)\,d\tau_{(x,t)}(\xi).$$

As in the proof of Proposition 5.7, we have $\iint_S\{U_m^*\partial_t\tilde\zeta+F_m^*\partial_x\tilde\zeta\}\,dx\,dt\to 0$ and $\int_{\mathbb{R}}U_m(u_{0r})\tilde\zeta(x,0)\,dx\to 0$ as $m\to\infty$, whence

$$\int_0^T\langle u_s(\cdot,t),\partial_t\tilde\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt+C_\varphi\int_0^T\langle u_s(\cdot,t),\partial_x\tilde\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt\geq-\langle u_{0s},\tilde\zeta(\cdot,0)\rangle_{\mathbb{R}}. \tag{5.53}$$

Let $\zeta\in C([0,T];C_c(\mathbb{R}))$. By Definition 2.1 (for $L^\infty(0,T;\mathcal{M}(\mathbb{R}))$), the map $t\mapsto\langle u_s(\cdot,t),\zeta(\cdot,t)\rangle_{\mathbb{R}}$ belongs to $L^\infty(0,T)$. Hence,

$$\lim_{h\to0}\frac1h\int_{\bar t}^{\bar t+h}\langle u_s(\cdot,t),\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt=\langle u_s(\cdot,\bar t),\zeta(\cdot,\bar t)\rangle_{\mathbb{R}}\quad\text{for every }\bar t\in(0,T)\setminus N, \tag{5.54}$$

for some null set $N\subset(0,T)$ (by separability arguments, we have that N is independent of ζ; see the proof of [23, Lemma 3.1]). Let $t_1,t_2\in(0,T)\setminus N$, $0<t_1<t_2<T$. By standard approximation arguments, we can choose $\tilde\zeta(x,t)=g_h(t)\zeta(x,t)$ in (5.53), where

$$g_h(t):=\frac1h(t-t_1)\chi_{\{t_1\le t\le t_1+h\}}(t)+\chi_{\{t_1+h<t<t_2\}}(t)+\frac1h(t_2+h-t)\chi_{\{t_2\le t\le t_2+h\}}(t) \tag{5.55}$$

and $h\in(0,\min\{t_2-t_1,T-t_2\})$. Letting $h\to0$ in (5.53), we obtain that

$$\langle u_s(\cdot,t_2),\zeta(\cdot,t_2)\rangle_{\mathbb{R}}\le\int_{t_1}^{t_2}\langle u_s(\cdot,t),\zeta_\nu(\cdot,t)\rangle_{\mathbb{R}}\,dt+\langle u_s(\cdot,t_1),\zeta(\cdot,t_1)\rangle_{\mathbb{R}}. \tag{5.56}$$

Similarly, let $f_h(t):=\chi_{\{0\le t<t_2\}}(t)+\frac1h(t_2+h-t)\chi_{\{t_2\le t\le t_2+h\}}(t)$. Setting $\tilde\zeta(x,t)=f_h(t)\zeta(x,t)$ in (5.53) and letting $h\to0^+$, we obtain that

$$\langle u_s(\cdot,t),\zeta(\cdot,t)\rangle_{\mathbb{R}}\le\int_0^t\langle u_s(\cdot,\tau),\zeta_\nu(\cdot,\tau)\rangle_{\mathbb{R}}\,d\tau+\langle u_{0s},\zeta(\cdot,0)\rangle_{\mathbb{R}}. \tag{5.57}$$

Arguing as in the last part of the proof of Proposition 5.7, we obtain (3.18) and (3.19) from, respectively, (5.56) and (5.57) (we omit the details).

(ii) It follows from (3.8) that for a.e. $\tau\in(0,T)$ and $m\in\mathbb{N}$,

$$\langle u(\cdot,\tau),\rho_m\rangle_{\mathbb{R}}-\langle u_0,\rho_m\rangle_{\mathbb{R}}=\int_0^\tau\Big\{\int_{\Omega_m}\varphi(u_r)(x,t)\rho_m'\,dx+C_\varphi\langle u_s(\cdot,t)\llcorner\Omega_m,\rho_m'\rangle_{\mathbb{R}}\Big\}\,dt, \tag{5.58}$$

where $\{\rho_m\}\subset C_c^1(\mathbb{R})$ is such that $\rho_m=1$ in $[-m,m]$, $\operatorname{supp}\rho_m\subseteq[-m-1,m+1]$, $0\le\rho_m\le1$ and $|\rho_m'|\le2$ in $\mathbb{R}$, and $\Omega_m:=[-m-1,-m]\cup[m,m+1]$. Since $u_s\in L^\infty(0,T;\mathcal{M}^+(\mathbb{R}))$ and $\varphi(u_r)\in L^\infty(0,T;L^1(\mathbb{R}))$, a routine proof shows that

$$\lim_{m\to\infty}\int_0^\tau\int_{\mathbb{R}}\varphi(u_r)(x,t)\rho_m'(x)\,dx\,dt=\lim_{m\to\infty}\int_0^\tau\langle u_s(\cdot,t)\llcorner\Omega_m,\rho_m'\rangle_{\mathbb{R}}\,dt=0.$$

Since $\rho_m(x) \to 1$ for all $x \in \mathbb{R}$, we also get that $\langle u(\cdot,\tau), \rho_m\rangle_{\mathbb{R}} \to \|u(\cdot,\tau)\|_{\mathcal{M}(\mathbb{R})}$ and $\langle u_0, \rho_m\rangle_{\mathbb{R}} \to \|u_0\|_{\mathcal{M}(\mathbb{R})}$ as $m \to \infty$. Letting $m \to \infty$ in (5.58), we obtain claim (ii). □

6 Regularity: Proofs

The first regularity result which we prove is Proposition 3.5. Hence, we need the following lemma.

Lemma 6.1. *Let* (H1) *be satisfied. Let* (u, τ) *be a Young measure solution of problem* (P). *Then there exists a null set* $F^* \subset (0, T)$ *such that for every* $t_0, t_1 \in (0, T) \setminus F^*$, $t_0 < t_1$, *and any* $\rho \in C^1_c(\mathbb{R})$, *we have*

$$\langle u(\cdot, t_1), \rho\rangle_{\mathbb{R}} - \langle u_0, \rho\rangle_{\mathbb{R}} = \int_0^{t_1} \Big\{ \int_{\mathbb{R}} \varphi^*(x,t)\rho'(x)\,dx + C_\varphi \langle u_s(\cdot,t), \rho'\rangle_{\mathbb{R}} \Big\}\,dt, \tag{6.1}$$

$$\langle u(\cdot, t_1), \rho\rangle_{\mathbb{R}} - \langle u(\cdot, t_0), \rho\rangle_{\mathbb{R}} = \int_{t_0}^{t_1} \Big\{ \int_{\mathbb{R}} \varphi^*(x,t)\rho'(x)\,dx + C_\varphi \langle u_s(\cdot,t), \rho'\rangle_{\mathbb{R}} \Big\}\,dt. \tag{6.2}$$

Proof. Since $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$, there exists a null set $F_0 \subseteq (0, T)$ such that the spatial disintegration $u(\cdot, t) \in \mathcal{M}^+(\mathbb{R})$ is defined for every $t \in (0, T) \setminus F_0$. Arguing as in the proof of [23, Lemma 3.1], we can show that there exists a null set $F^* \subset (0, T)$, $F_0 \subseteq F^*$, such that for every $\rho \in C_c(\mathbb{R})$ and $t \in (0, T) \setminus F^*$,

$$\lim_{q\to\infty} \Big\{ 2q \int_{t-\frac{1}{q}}^{t+\frac{1}{q}} \big|\langle u(\cdot, s), \rho\rangle_{\mathbb{R}} - \langle u(\cdot, t), \rho\rangle_{\mathbb{R}}\big|\,ds \Big\} = 0. \tag{6.3}$$

The proof of (6.1) is based on (3.4) and (6.3). Let $\rho \in C^1_c(\mathbb{R})$ and $t_1 \in (0, T) \setminus F^*$. By standard regularization arguments, we can set $\zeta = \rho(x)k_q(t)$ in (3.4), with $q \geq \frac{1}{T-t_1} + 1$ ($q \in \mathbb{N}$) and

$$k_q(t) := \min\Big\{1, q\Big(t_1 + \frac{1}{q} - t\Big)_+\Big\} \to \chi_{(0,t_1]} \quad \text{in } (0, T) \text{ as } q \to \infty$$

to get

$$q \int_{t_1}^{t_1+\frac{1}{q}} \langle u(\cdot, t), \rho\rangle_{\mathbb{R}}\,dt - \langle u_0, \rho\rangle_{\mathbb{R}} = \int_0^T \Big\{ \int_{\mathbb{R}} \varphi^*(x,t)\rho'(x)\,dx + C_\varphi \langle u_s(\cdot,t), \rho'\rangle_{\mathbb{R}} \Big\} k_q(t)\,dt.$$

Letting $q \to \infty$, we obtain (6.1) from (3.7) and (6.3). Subtracting from (6.1) the same inequality with t_1 replaced by t_0, we obtain (6.2). □

Proof of Proposition 3.5. Let $F^* \subset (0, T)$ be the null set given by Lemma 6.1. Let $\{\tau_n\} \subseteq (0, T) \setminus F^*$, with $\tau_n \to 0^+$ as $n \to \infty$. Since, by (3.7), $u \in L^\infty(0, T; \mathcal{M}^+(\mathbb{R}))$ and $\varphi^* \in L^\infty(0, T; L^1(\mathbb{R}))$, it follows from (6.1) that $\langle u(\cdot, \tau_n), \rho\rangle_{\mathbb{R}} \to \langle u_0, \rho\rangle_{\mathbb{R}}$ for all $\rho \in C^1_c(\mathbb{R})$. Since, by Definition 2.1 (ii), $\sup_n \|u(\cdot, \tau_n)\|_{\mathcal{M}(\mathbb{R})} \leq C$, there exist $\mu_0 \in \mathcal{M}^+(\mathbb{R})$ and a subsequence $\{\tau_{n_k}\}$ such that $u(\cdot, \tau_{n_k}) \overset{*}{\rightharpoonup} \mu_0$ in $\mathcal{M}(\mathbb{R})$ as $k \to \infty$. By standard density arguments, this implies that $\mu_0 = u_0$. Hence, $u(\cdot, \tau_n) \overset{*}{\rightharpoonup} u_0$ along the whole sequence $\{\tau_n\}$, and (3.11) follows from (6.1) and the arbitrariness of $\{\tau_n\}$.

Similarly, it follows from (6.2) that $\langle u(\cdot, \tau_n), \rho\rangle_{\mathbb{R}} \to \langle u(\cdot, t_0), \rho\rangle_{\mathbb{R}}$ for all $\rho \in C^1_c(\mathbb{R})$ as $\tau_n \to t_0$ if $t_0, \tau_n \in (0, T) \setminus F^*$, and we obtain (3.12).

To prove (3.13), we observe that, given $t_0 \in [0, T]$ and two sequences τ^1_n and τ^2_n contained in $(0, T) \setminus F^*$ and converging to t_0, we have $\langle u(\cdot, \tau^1_n) - u(\cdot, \tau^2_n), \rho\rangle_{\mathbb{R}} \to 0$ for all $\rho \in C_c(\mathbb{R})$. Hence, if $t_0 \notin F^*$, the continuous extension of $u(\cdot, t)$ from $(0, T) \setminus F^*$ with respect to the weak$*$ topology is well-defined. □

Let us now prove the results of Section 3.3. As explained there, replacing x by $x - C_\varphi t$ we may assume, without loss of generality, that $C_\varphi = 0$; namely, it suffices to prove Proposition 3.10, Theorem 3.11 and Proposition 3.14. Moreover, replacing x by $-x$ and φ by $-\varphi$, it suffices to do so by assuming that (H2) is satisfied with $\varphi'' < 0$, $\varphi' > 0$ in $(0, \infty)$ (see Remark 3.15). Therefore, we make use of the following assumption:

(H5) $\varphi \in C^\infty([0,\infty))$, $C_\varphi = 0$, $\varphi''(u) < 0$, and there exist $H \geq -1$, $K > 0$ such that

$$\varphi''(u)[H\varphi(u) + K] \leq -[\varphi'(u)]^2 < 0 \quad \text{for all } u \in [0,\infty).$$

(Recall that in this case $\varphi' > 0$ and $H\varphi(u) + K > 0$ in $[0,\infty)$.)

First we prove some estimates of the constructed entropy solutions. As already said, these estimates are analogous to the Aronson–Bénilan inequality for the convex case u^p, $p > 1$ (see [1]).

Proposition 6.2. *Let* (H1) *and* (H5) *be satisfied, and let u be an entropy solution of problem* (P) *given by Theorem 3.7. Then, for a.e.* $0 < t_1 < t_2 \leq T$,

$$\varphi(u_r)(\cdot, t_2) + \frac{K}{H} \leq \left(\frac{t_2}{t_1}\right)^H \left[\varphi(u_r)(\cdot, t_1) + \frac{K}{H}\right] \quad \text{a.e. in } \mathbb{R} \text{ if } H \neq 0, \tag{6.4}$$

$$\varphi(u_r)(\cdot, t_2) - K\log(t_2) \leq \varphi(u_r)(\cdot, t_1) - K\log t_1 \quad \text{a.e. in } \mathbb{R} \text{ if } H = 0. \tag{6.5}$$

Moreover, if

(C4) *there exists* $L > 0$ *such that*

$$H\varphi(u) + K \leq L(1+u)\varphi'(u) \quad \text{for } u \geq 0,$$

then $\partial_t u \in \mathcal{M}(\Omega \times (\tau, T))$, $\partial_t[\varphi(u_r)] \in \mathcal{M}(\Omega \times (\tau, T))$, *and* $u \in C((0,T]; \mathcal{M}(\Omega))$ *for every bounded open set* $\Omega \subset \mathbb{R}$ *and* $\tau > 0$.

Remark 6.3. If $\varphi(u) = \operatorname{sgn} p[(1+u)^p - 1]$ ($p < 1$, $p \neq 0$), (6.4) becomes

$$u_r(\cdot, t_2) \leq \left(\frac{t_2}{t_1}\right)^{\frac{1}{1-p}} [1 + u_r(\cdot, t_1)] - 1 \quad \text{a.e. in } \mathbb{R}, \text{ for a.e. } 0 < t_1 \leq t_2 \leq T$$

(see Remark 3.9). Similarly, if $\varphi(u) = \log(1+u)$, (6.5) becomes

$$u_r(\cdot, t_2) \leq \left(\frac{t_2}{t_1}\right)[1 + u_r(\cdot, t_1)] - 1 \quad \text{a.e. in } \mathbb{R}, \text{ for a.e. } 0 < t_1 \leq t_2 \leq T.$$

Let (H5) hold. To prove Proposition 6.2, we use a different regularization of (Pn), that is,

$$\begin{cases} \partial_t y_n^\varepsilon + \partial_x[\varphi(y_n^\varepsilon)] = \varepsilon \partial_x^2[\varphi(y_n^\varepsilon)] & \text{in } S, \\ y_n^\varepsilon = u_{0n}^\varepsilon & \text{in } \mathbb{R} \times \{0\}, \end{cases} \tag{6.6}$$

where $\{u_{0n}^\varepsilon\}$ satisfies (4.1)–(4.2). The existence, uniqueness and regularity results recalled in Section 4 for problem (4.3), as well as the a priori estimates in Lemma 4.1 and the convergence results in Lemma 4.4 (i), continue to hold for solutions of (6.6) (see [18]). In particular, there exist a sequence $\{y_n^{\varepsilon_m}\}$ and $y_n \in L^\infty(S) \cap L^\infty(0,T; L^1(\mathbb{R}))$ such that $y_n^{\varepsilon_m} \overset{*}{\rightharpoonup} y_n$ in $L^\infty(S)$ and for all $L > 0$,

$$y_n^{\varepsilon_m} \to y_n \quad \text{in } L^1((-L,L) \times (0,T)) \text{ as } \varepsilon_m \to 0. \tag{6.7}$$

From (6.6), for every E convex, $F' = E'\varphi'$, and ζ as in Definition 3.6, we get

$$\iint_S \{E(y_n^{\varepsilon_m})\partial_t\zeta + F(y_n^{\varepsilon_m})\partial_x\zeta\}\,dx\,dt + \int_{\mathbb{R}} E(u_{0n}^\varepsilon)\zeta(x,0)\,dx \geq \varepsilon_m \iint_S F'(y_n^{\varepsilon_m})\partial_x y_n^{\varepsilon_m}\partial_x\zeta\,dx\,dt.$$

Arguing as in the proof of Proposition 4.5 and letting $\varepsilon_m \to 0$, we obtain that

$$\iint_S [E(y_n)\partial_t\zeta + F(y_n)\partial_x\zeta]\,dx\,dt \geq -\int_{\mathbb{R}} E(u_{0n})\zeta(x,0)\,dx.$$

So y_n satisfies (3.16) and, by Kružkov's uniqueness theorem, $y_n = u_n$. Hence, we have shown the following lemma.

Lemma 6.4. *Let* (H1) *and* (H5) *be satisfied, and let* u_n *be the unique entropy solution of problem* (Pn) *given by Proposition 4.5. Then there exists a subsequence* $\{y_n^{\varepsilon_m}\}$ *of solutions of* (6.6) *such that* $y_n^{\varepsilon_m} \overset{*}{\rightharpoonup} u_n$ *in* $L^\infty(S)$ *and satisfies* (6.7).

Lemma 6.5. *Let* (H1) *and* (H5) *be satisfied. Then*

$$\partial_t\Big[\frac{H\varphi(y_n^\varepsilon)(\cdot,t)+K}{t^H}\Big]\begin{cases}\le 0 & \text{in } \mathbb{R} \text{ if } H>0,\\ \ge 0 & \text{in } \mathbb{R} \text{ if } H<0,\end{cases} \tag{6.8}$$

$$\partial_t[\varphi(y_n^\varepsilon)(\cdot,t)-K\log t]\le 0 \quad \text{in } \mathbb{R} \text{ if } H=0, \tag{6.9}$$

for all $t\in(0,T)$, $\varepsilon>0$ *and* $n\in\mathbb{N}$. *Moreover, if* (C4) *is satisfied, then*

$$t\partial_t y_n^\varepsilon \le L(1+y_n^\varepsilon) \quad \text{in } S. \tag{6.10}$$

Proof. For convenience, we set $A\equiv\varepsilon\partial_x^2-\partial_x$, thus $\partial_t y_n^\varepsilon = A[\varphi(y_n^\varepsilon)]$ in S. Let

$$z_n^\varepsilon := t\partial_t y_n^\varepsilon - g(y_n^\varepsilon), \quad \text{where } g(y_n^\varepsilon) := \frac{H\varphi(y_n^\varepsilon)+K}{\varphi'(y_n^\varepsilon)} \; (n\in\mathbb{N}).$$

It follows from (H5) and a straightforward calculation that

$$\partial_t z_n^\varepsilon = A[\varphi'(y_n^\varepsilon)z_n^\varepsilon] + \underbrace{[H+1-g'(y_n^\varepsilon)]}_{\le 0}\frac{z_n^\varepsilon+g(y_n^\varepsilon)}{t} \le A[\varphi'(y_n^\varepsilon)z_n^\varepsilon] + [H+1-g'(y_n^\varepsilon)]\frac{z_n^\varepsilon}{t}$$

in S. Since $z_n^\varepsilon = -g(u_{0n}^\varepsilon)\le 0$ in $\mathbb{R}\times\{0\}$, it follows from the comparison principle for parabolic equations that $z_n^\varepsilon\le 0$ in S for all $n\in\mathbb{N}$. Hence, $t\partial_t y_n^\varepsilon(\cdot,t)\le g(y_n^\varepsilon)(\cdot,t)$ in $\mathbb{R}$ for all $t\in(0,T)$, which implies (6.8), (6.9) and, if (C4) is satisfied, (6.10). □

Proof of Proposition 6.2. Let $\{y_n^{\varepsilon_m}\}$ be as in the proof of Lemma 6.4. By (6.8)–(6.9),

$$\varphi(y_n^{\varepsilon_m})(\cdot,t_2)+\frac{K}{H}\le\Big(\frac{t_2}{t_1}\Big)^H\Big[\varphi(y_n^{\varepsilon_m})(\cdot,t_1)+\frac{K}{H}\Big] \quad \text{in } \mathbb{R} \text{ if } H\ne 0,$$
$$\varphi(y_n^\varepsilon)(x,t_2)-K\log(t_2)\le\varphi(y_n^\varepsilon)(x,t_1)-K\log t_1 \quad \text{in } \mathbb{R} \text{ if } H=0,$$

for all $0<t_1\le t_2\le T$ and $n\in\mathbb{N}$. Hence, by Lemma 6.4,

$$\varphi(u_n)(\cdot,t_2)+\frac{K}{H}\le\Big(\frac{t_2}{t_1}\Big)^H\Big[\varphi(u_n)(\cdot,t_1)+\frac{K}{H}\Big] \quad \text{a.e. in } \mathbb{R} \text{ if } H\ne 0, \tag{6.11}$$
$$\varphi(u_n)(\cdot,t_2)-K\log(t_2)\le\varphi(u_n)(\cdot,t_1)-K\log t_1 \quad \text{a.e. in } \mathbb{R} \text{ if } H=0, \tag{6.12}$$

for a.e. $0<t_1\le t_2\le T$. Since φ' is strictly decreasing in $[0,\infty)$ (recall that φ is concave by assumption (H5)), possibly extracting another subsequence (denoted again by $\{n_j\}$), $\varphi(u_{n_j})\to\varphi(u_r)$ a.e. in S (see Remark 5.10). Letting $j\to\infty$ in (6.11)–(6.12) (with $n=n_j$), we obtain (6.4)–(6.5).

Let $\Omega=(-L,L)$. If (C4) is satisfied, it follows from (6.10) and (4.5) that

$$t\int_\Omega[\partial_t y_n^\varepsilon]^+(x,t)\,dx\le L|\Omega|+\|u_0\|_{\mathcal{M}(\mathbb{R})} \quad \text{for all } t\in(0,T]. \tag{6.13}$$

Since $|\partial_t y_n^\varepsilon| = 2[\partial_t y_n^\varepsilon]^+-\partial_t y_n^\varepsilon$ a.e. in S, there exists $C_\Omega>0$ such that

$$\int_\tau^T\int_\Omega|\partial_t y_n^\varepsilon|(x,t)\,dx\,dt\le 2(T-\tau)\frac{L|\Omega|+\|u_0\|_{\mathcal{M}(\mathbb{R})}}{\tau}+\int_\Omega\{y_n^\varepsilon(x,\tau)-y_n^\varepsilon(x,T)\}\le\frac{C_\Omega}{\tau}$$

for all $\tau>0$, $\varepsilon>0$ and $n\in\mathbb{N}$, and, by (3.1),

$$\int_\tau^T\int_\Omega|\partial_t y_n^\varepsilon|\,dx\,dt\le\frac{C_\Omega}{\tau}, \quad \int_\tau^T\int_\Omega|\partial_t[\varphi(y_n^\varepsilon)]|dx\,dt\le\frac{MC_\Omega}{\tau}. \tag{6.14}$$

Let $\{\varepsilon_m\}$ and $\{n_j\}$ be as in Lemma 6.4 and (5.1). Then

$$\lim_{n_j\to\infty}\lim_{\varepsilon_m\to 0}\langle y_{n_j}^{\varepsilon_m},\partial_t\zeta\rangle_{\Omega\times(\tau,T)}=\langle u,\partial_t\zeta\rangle_{\Omega\times(\tau,T)} \quad \text{for all } \zeta\in C_c^1(\Omega\times(\tau,T)),$$

whence, by (6.14) and the lower semicontinuity of the total variation,

$$\|\partial_t u\|_{\mathcal{M}(\Omega\times(\tau,T))} \le \frac{C_\Omega}{\tau}.$$

Similarly, by (5.6), (5.25) and Proposition 5.8,

$$\lim_{n_j\to\infty}\lim_{\varepsilon_m\to 0}\langle \varphi(y_{n_j}^{\varepsilon_m}), \partial_t\zeta\rangle_{\Omega\times(\tau,T)} = \int_\tau^T\int_\Omega \varphi(u_r)\partial_t\zeta\,dx\,dt \quad \text{for all } \zeta\in C_c^1(\Omega\times(\tau,T)),$$

and, by (6.14) and the lower semicontinuity of the total variation,

$$\|\partial_t[\varphi(u_r)]\|_{\mathcal{M}(\Omega\times(\tau,T))} \le \frac{MC_\Omega}{\tau}.$$

It remains to prove that $u\in C((0,T];\mathcal{M}(\Omega))$. Observe that for all $t_1, t_2\in(0,T]$, $0<\tau<t_1<t_2$, and $\rho\in C_c^2(\mathbb{R})$, $0\le\rho\le 1$ in $\mathbb{R}$, $\rho=1$ in Ω,

$$\begin{aligned}
\int_\Omega |y_n^\varepsilon(x,t_2)-y_n^\varepsilon(x,t_1)|\,dx &\le \int_{\mathbb{R}} |y_n^\varepsilon(x,t_2)-y_n^\varepsilon(x,t_1)|\rho(x)\,dx\\
&\le \int_{t_1}^{t_2}\int_{\mathbb{R}} |\partial_t y_n^\varepsilon|\rho\,dx\,dt\\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}} (2[\partial_t y_n^\varepsilon]^+ - \partial_t y_n^\varepsilon)\rho\,dx\,dt\\
&= 2\int_{t_1}^{t_2}\int_{\mathbb{R}} [\partial_t y_n^\varepsilon]^+\rho\,dx\,dt - \int_{t_1}^{t_2}\int_{\mathbb{R}} \{\varphi(y_n^\varepsilon)\rho'(x)+\varepsilon\varphi(y_n^\varepsilon)\rho''(x)\}\,dx\,dt\\
&\le 2\frac{L|\operatorname{supp}\rho| + \|u_0\|_{\mathcal{M}(\mathbb{R})}}{\tau}(t_2-t_1) - \int_{t_1}^{t_2}\int_{\mathbb{R}} \{\varphi(y_n^\varepsilon)\rho' + \varepsilon\varphi(y_n^\varepsilon)\rho''\}\,dx\,dt,
\end{aligned}$$

where we have used (6.13). We let $\varepsilon=\varepsilon_m\to 0$ and use (3.1) and (4.20) to obtain

$$\begin{aligned}
\int_\Omega |u_n(x,t_2)-u_n(x,t_1)|\,dx &\le 2\frac{L|\operatorname{supp}\rho| + \|u_0\|_{\mathcal{M}(\mathbb{R})}}{\tau}(t_2-t_1) - \int_{t_1}^{t_2}\int_{\mathbb{R}} \varphi(u_n)\rho'(x)\,dx\,dt\\
&\le \Big(2\frac{L|\operatorname{supp}\rho| + \|u_0\|_{\mathcal{M}(\mathbb{R})}}{\tau} + M\|u_0\|_{\mathcal{M}(\mathbb{R})}\|\rho'\|_{L^\infty(\mathbb{R})}\Big)(t_2-t_1) =: \frac{\tilde C}{\tau}(t_2-t_1).
\end{aligned}$$

By (5.16) and the lower semicontinuity of the total variation,

$$\|u(\cdot,t_2)-u(\cdot,t_1)\|_{\mathcal{M}(\Omega)} \le \frac{\tilde C}{\tau}|t_1-t_2| \quad \text{for a.e. } 0<\tau<t_1<t_2\le T.$$

So we may define $u(\cdot,t)$ for all $t\in[\tau,T]$ such that $u\in C([\tau,T];\mathcal{M}(\Omega))$. Since $\tau>0$ is arbitrary, the proof is complete. □

To prove Proposition 3.10, we need the following lemma.

Lemma 6.6. *Let* (H1) *be satisfied, and let u be the solution of problem* (P) *given by Theorem 3.7. Let* $\{u_{n_j}\}$ *be as in the proof of Theorem 3.7. Then, for a.e.* $t\in(0,T)$ *and all* $x_0\in\operatorname{supp}u_s(\cdot,t)$*, there exist a sequence* $\{x_{0k}\}\subset\mathbb{R}$ *and a subsequence* $\{u_{n_k}\}$ *of* $\{u_{n_j}\}$ *such that* $x_{0k}\to x_0$ *and* $u_{n_k}(x_{0k},t)\to\infty$ *as* $k\to\infty$.

Proof. Let $x_0\in\operatorname{supp}u_s(\cdot,t)$. We may assume that the convergence in (5.16) is satisfied for this t. Since $x_0\in\operatorname{supp}u_s(\cdot,t)$, there is no neighborhood $I_\delta(x_0)$ such that the sequence $\{u_{n_j}(\cdot,t)\}$ lies in a bounded subset of $L^\infty(I_\delta(x_0))$. Otherwise, up to a subsequence, $u_{n_j}(\cdot,t)\overset{*}{\rightharpoonup} f_t$ in $L^\infty(I_\delta(x_0))$ for some $f_t\in L^\infty(I_\delta(x_0))$, $f_t\ge 0$. However, this would imply that $u_s(\cdot,t)=0$ in $I_\delta(x_0)$, a contradiction.

Setting $\delta=1/k$, we obtain that $\sup_{n_j\in\mathbb{N}}\|u_{n_j}(\cdot,t)\|_{L^\infty(I_{1/k}(x_0))}=\infty$ for all $k\in\mathbb{N}$. Hence, for all $k\in\mathbb{N}$, there exists $x_{0k}\in I_{1/k}(x_0)$ such that $u_{n_k}(x_{0k},t)\ge k$. □

Proof of Proposition 3.10. As pointed out above, it suffices to prove equality (3.20) by assuming (H5). Let $\{u_{n_j}\}$ be as in the proof of Lemma 6.6. By Lemma 6.4, for every $n_j \in \mathbb{N}$, there exists $\varepsilon_m \to 0$ such that

$$y_{n_j}^{\varepsilon_m}(\cdot, t) \to u_{n_j}(\cdot, t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}) \text{ as } \varepsilon_m \to 0 \text{ for a.e. } t \in (0, T). \tag{6.15}$$

By the proof of Lemma 6.5, for all $t \in (0, T)$,

$$\varepsilon_m \partial_x^2[\varphi(y_{n_j}^{\varepsilon_m})(\cdot, t)] - \partial_x[\varphi(y_{n_j}^{\varepsilon_m})(\cdot, t)] = \partial_t(y_{n_j}^{\varepsilon_m}) \le \frac{g(y_{n_j}^{\varepsilon_m})(\cdot, t)}{t} \quad \text{in } \mathbb{R}, \tag{6.16}$$

where $g(u) = \frac{H\varphi(u)+K}{\varphi'(u)} > 0$. For every $\underline{x} < \bar{x}$, let $\rho \in C_c^1((\underline{x}, \bar{x}))$, $\rho \ge 0$. Multiplying (6.16) by $\rho/g(y_{n_j}^{\varepsilon_m}(\cdot, t))$, integrating by parts and setting $\Psi(y) := \int_0^y \frac{\varphi'(u)}{g(u)}\, du$, we find that

$$\int_{\underline{x}}^{\bar{x}} \Psi(y_{n_j}^{\varepsilon_m})(x, t)[\varepsilon_m \rho''(x) + \rho'(x)]\, dx \le \frac{1}{t}\int_{\underline{x}}^{\bar{x}} \rho(x)\, dx - \varepsilon_m \int_{\underline{x}}^{\bar{x}} \frac{\varphi'(y_{n_j}^{\varepsilon_m}) g'(y_{n_j}^{\varepsilon_m})[(y_{n_j}^{\varepsilon_m})_x]^2}{[g(y_{n_j}^{\varepsilon_m})]^2}(x, t)\rho(x)\, dx \le \frac{1}{t}\int_{\underline{x}}^{\bar{x}} \rho(x)\, dx$$

(observe that by (H5) we have $g'(u) \ge H + 1 \ge 0$ and Ψ is bounded). Hence, by (6.15),

$$\int_{\underline{x}}^{\bar{x}} \Psi(u_{n_j})(x, t)\rho'(x)\, dx \le \frac{1}{t}\int_{\underline{x}}^{\bar{x}} \rho(x)\, dx. \tag{6.17}$$

Let $x_0 \in \operatorname{supp} u_s(\cdot, t)$, and let $\{x_{0k}\} \subset \mathbb{R}$, $\{u_{n_k}\}$ be as in Lemma 6.6, for a.e. $t \in (0, T)$. Let $\bar{x} > x_0$ be fixed. Since $x_{0k} \to x_0$, there exists $\bar{k} \in \mathbb{N}$ such that $\bar{x} > x_{0k}$ for all $k > \bar{k}$. Consider any sequence $\{\rho_m\} \subset C_c^1((x_{0k}, \bar{x}))$, $0 \le \rho_m \le 1$, $\rho_m \to \chi_{(x_{0k}, \bar{x})}$ in $\mathbb{R}$. Without loss of generality, we may assume that both x_{0k} and $\bar{x}$ are Lebesgue points of $u_{n_k}(\cdot, t)$ for all $k \in \mathbb{N}$. Setting $\rho = \rho_m$ and $\underline{x} = x_{0k}$ in (6.17), and letting $m \to \infty$, we find that

$$\Psi(u_{n_k})(x_{0k}, t) \le \Psi(u_{n_k})(\bar{x}, t) + \frac{1}{t}(\bar{x} - x_{0k}) \quad \text{for all } n_k.$$

Since Ψ is continuous, by Lemma 6.6 and Remark 5.10 (recall that φ satisfies (C2), since φ is strictly concave by assumption (H5)), letting $n_k \to \infty$ gives

$$\Psi(u_r)(\bar{x}, t) + \frac{1}{t}(\bar{x} - x_0) \ge \Psi(\infty) \quad \text{for a.e. } \bar{x} > x_0,$$

whence, by the invertibility of Ψ,

$$u_r(\bar{x}, t) \ge \Psi^{-1}\Big(\Psi(\infty) - \frac{1}{t}(\bar{x} - x_0)\Big) \quad \text{for a.e. } \bar{x} > x_0. \tag{6.18}$$

Letting $\bar{x} \to x_0^+$ in the previous inequality, we obtain (3.20). □

To prove Theorem 3.11, we need the following result.

Proposition 6.7. *Let* (H1) *be satisfied. Let* $C_\varphi = 0$, *and let* u *be a solution of problem* (P). *Then, for a.e.* $0 \le t_1 \le t_2 \le T$,
(i) *the map* $x \mapsto \Phi(x, t_1, t_2) := \int_{t_1}^{t_2} \varphi(u_r)(x, t)\, dt$ *belongs to* BV($\mathbb{R}$),
(ii) *for all* $x_0, x_1 \in \mathbb{R}$, $x_0 \le x_1$,

$$u(\cdot, t_2)([x_0, x_1]) - u(\cdot, t_1)([x_0, x_1]) = \Phi(x_0^-, t_1, t_2) - \Phi(x_1^+, t_1, t_2), \tag{6.19}$$
$$u(\cdot, t_2)([x_0, x_1]) - u_0([x_0, x_1]) = \Phi(x_0^-, 0, t_2) - \Phi(x_1^+, 0, t_2). \tag{6.20}$$

Remark 6.8. It is easily seen that for $C_\varphi \ne 0$, equalities (6.19)–(6.20) are replaced by

$$u(\cdot, t_2)([x_0, x_1]) - \mathcal{T}_{C_\varphi(t_2 - t_1)}(u(\cdot, t_1))([x_0, x_1]) = \Phi(x_0^-, t_1, t_2) - \Phi(x_1^+, t_1, t_2), \tag{6.21}$$
$$u(\cdot, t_2)([x_0, x_1]) - \mathcal{T}_{C_\varphi t_2}(u_0)([x_0, x_1]) = \Phi(x_0^-, 0, t_2) - \Phi(x_1^+, 0, t_2),$$

where now

$$\Phi(x, t_1, t_2) := \int_{t_1}^{t_2} [\varphi(u_r) - C_\varphi u_r](x + C_\varphi(t - t_1), t)\, dt. \tag{6.22}$$

Proof of Proposition 6.7. (i) By (3.1), $|\int_{t_1}^{t_2} \varphi(u_r)(x,t)\,dt| \le M \int_{t_1}^{t_2} u_r(x,t)\,dt \in L^1(\mathbb{R})$. We argue as in the proof of Proposition 3.8 (see (5.54)). There exists a null set $N \subset (0,T)$ such that

$$\lim_{h\to 0} \frac{1}{h} \int_{\bar t}^{\bar t+h} \langle u(\cdot,t),\rho\rangle_{\mathbb{R}}\,dt = \langle u(\cdot,\bar t),\rho\rangle_{\mathbb{R}} \quad \text{for all } \rho \in C_c(\mathbb{R}) \text{ and } \bar t \in (0,T)\setminus N. \tag{6.23}$$

Let $t_1, t_2 \in (0,T)\setminus N$, $0 < t_1 < t_2 < T$, $\rho \in C_c^1(\mathbb{R})$, and $\zeta(x,t) = g_h(t)\rho(x)$, with g_h as in (5.55). Since $C_\varphi = 0$, we obtain from (3.8) that

$$\frac{1}{h}\int_{t_1}^{t_1+h} \langle u(\cdot,t),\rho\rangle_{\mathbb{R}}\,dt - \frac{1}{h}\int_{t_2}^{t_2+h} \langle u(\cdot,t),\rho\rangle_{\mathbb{R}}\,dt + \int_0^T\int_{\mathbb{R}} g_h(t)\rho'(x)\varphi(u_r)(x,t)\,dx\,dt = 0.$$

Letting $h \to 0$, it follows from (6.23) that

$$\langle u(\cdot,t_2),\rho\rangle_{\mathbb{R}} - \langle u(\cdot,t_1),\rho\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \Phi(x,t_1,t_2)\rho'(x)\,dx. \tag{6.24}$$

Hence, the distributional derivative $\Phi_x(x,t_1,t_2)$ belongs to $\mathcal{M}(\mathbb{R})$.

(ii) We set, for $m \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\rho_m(x) := m\Big(x - x_0 + \frac{1}{m}\Big)\chi_{[x_0-\frac{1}{m},x_0]} + \chi_{(x_0,x_1)}(x) + m\Big(-x + x_1 + \frac{1}{m}\Big)\chi_{[x_1,x_1+\frac{1}{m}]}.$$

By standard regularization arguments, we can choose $\rho = \rho_m$ in (6.24) to obtain

$$\langle u(\cdot,t_2),\rho_m\rangle_{\mathbb{R}} - \langle u(\cdot,t_1),\rho_m\rangle_{\mathbb{R}} = m\int_{x_0-\frac{1}{m}}^{x_0} \Phi(x,t_1,t_2)\,dx - m\int_{x_1}^{x_1+\frac{1}{m}} \Phi(x,t_1,t_2)\,dx. \tag{6.25}$$

By the dominated convergence theorem, $\langle u(\cdot,t_i),\rho_m\rangle_{\mathbb{R}} \to u(\cdot,t_i)([x_0,x_1])$ as $m \to \infty$ $(i = 1,2)$, whereas, by part (i),

$$m\int_{x_0-\frac{1}{m}}^{x_0} \Phi(x,t_1,t_2)\,dx \to \Phi(x_0^-,t_1,t_2), \quad m\int_{x_1}^{x_1+\frac{1}{m}} \Phi(x,t_1,t_2)\,dx \to \Phi(x_1^+,t_1,t_2).$$

Hence, (6.19) follows from (6.25). The proof of (6.20) is similar. □

Remark 6.9. Observe that, by (3.18) and (6.21) with $x_0 = x_1 = x$, all entropy solutions of problem (P) satisfy, for a.e. $0 \le t_1 \le t_2 \le T$,

$$\Phi(x^-,t_1,t_2) \le \Phi(x^+,t_1,t_2) \quad \text{for all } x \in \mathbb{R},$$

with Φ defined by (6.22).

Now we are ready to prove Theorem 3.11 and Proposition 3.14. As pointed out at the beginning of this section, in doing so it is not restrictive to assume that (H5) holds.

Proof of Theorem 3.11. (i) By (6.20), for a.e. $0 \le t \le T$,

$$u_s(t)(\{x_0\}) = u_{0s}(\{x_0\}) + \Phi(x_0^-,0,t) - \Phi(x_0^+,0,t) \ge u_{0s}(\{x_0\}) - \|\varphi\|_{L^\infty(0,\infty)}t,$$

whence $u_s(t)(\{x_0\}) > 0$ if $t \in (0, \frac{u_{0s}(\{x_0\})}{\|\varphi\|_{L^\infty(0,\infty)}})$. Hence, (3.21) follows.

(ii) Let u_n be the entropy solution of problem (Pn) given by Proposition 4.5. We argue as in the proof of Proposition 6.7. For all $n \in \mathbb{N}$, the map $x \mapsto \Phi_n(x,t_1,t_2) := \int_{t_1}^{t_2} \varphi(u_n)(x,t)\,dt$ belongs to $\mathrm{BV}(\mathbb{R})$ and, for a.e. $0 \le t_1 \le t_2 \le T$ and a.e. $x_0 \le x_1 \in \mathbb{R}$,

$$\int_{x_0}^{x_1} u_n(x,t_2)\,dx - \int_{x_0}^{x_1} u_n(x,t_1)\,dx = \Phi_n(x_0^-,t_1,t_2) - \Phi_n(x_1^+,t_1,t_2).$$

Letting $x_1 \to \infty$, it follows from (4.27) and (3.14) that

$$\int_{t_1}^{t_2} \varphi(u_n)(x,t)\,dt \le \|u_0\|_{\mathcal{M}(\mathbb{R})} \quad \text{for } n \in \mathbb{N} \text{ and a.e. } x \in \mathbb{R}. \tag{6.26}$$

Let $\{y_n^{\varepsilon_m}\}$ be the subsequence used in the proof of Lemma 6.4. By (6.8) and (6.9), for every $0 < t_1 \le t \le T$ and $x \in \mathbb{R}$,

$$\begin{aligned}\int_{t_1}^{t} \varphi(y_n^{\varepsilon_m})(x,s)\,ds &= \frac{1}{H}\int_{t_1}^{t} \frac{H\varphi(y_n^{\varepsilon_m})(x,s)+K}{s^H} s^H\,ds - \frac{K}{H}(t-t_1)\\ &\ge \frac{H\varphi(y_n^{\varepsilon_m})(x,t)+K}{Ht^H}\frac{t^{H+1}-t_1^{H+1}}{H+1} - \frac{K}{H}(t-t_1) \quad \text{if } H \ne 0,\\ \int_{t_1}^{t} \varphi(y_n^{\varepsilon_m})(x,s)\,ds &= \int_{t_1}^{t} [\varphi(y_n^{\varepsilon_m})(x,s) - K\log s]\,ds + K\int_{t_1}^{t} \log s\,ds\\ &\ge [\varphi(y_n^{\varepsilon_m})(x,t) - K\log t](t-t_1) + K[t\log t - t] - K[t_1\log t_1 - t_1] \quad \text{if } H = 0.\end{aligned}$$

Letting $\varepsilon_m \to 0$, by (6.26), we obtain, for a.e. $t \in (t_1, T)$ and a.e. $x \in \mathbb{R}$,

$$\|u_0\|_{\mathcal{M}(\mathbb{R})} \ge \Phi_n(x,t_1,t) \ge \begin{cases} \dfrac{H\varphi(u_n)(x,t)+K}{Ht^H}\dfrac{t^{H+1}-t_1^{H+1}}{H+1} - \dfrac{K}{H}(t-t_1) & \text{if } H \ne 0,\\[2ex] [\varphi(u_n)(x,t) - K](t-t_1) + Kt_1\log\dfrac{t}{t_1} & \text{if } H = 0.\end{cases}$$

Letting $t_1 \to 0^+$, we find in both cases that

$$\varphi(u_n)(x,t) \le \frac{(H+1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{t} + K \quad \text{for a.e. } t \in (t_1, T) \text{ and a.e. } x \in \mathbb{R} \tag{6.27}$$

(recall that we have assumed $H > -1$ if φ is bounded; otherwise, if φ is unbounded, we have $H \ge 0$, since $\varphi' > 0$ and $H\varphi + K > 0$ in $[0,\infty)$ by (H5)). If $\lim_{u\to\infty}\varphi(u) =: \gamma < \infty$, $K < \gamma$ and $H > -1$, the sequence $\{u_n(\cdot\,,t)\}$ lies in a bounded subset of $L^\infty(\mathbb{R})$ (thus, by (5.16) $u_s(\cdot\,,t) = 0$ and $u_r(\cdot\,,t) \in L^\infty(\mathbb{R})$) for a.e. $t \in (0,T)$ such that

$$\frac{(H+1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{t} + K < \gamma \iff t > \frac{(H+1)\|u_0\|_{\mathcal{M}(\mathbb{R})}}{\gamma - K}.$$

This proves claim (ii) (a).

If $\gamma = \infty$, we have $H \ge 0$, since $H\varphi + K > 0$ in $[0,\infty)$ (see (H5)). Then, by (6.27), the sequence $\{u_n(\cdot\,,t)\}$ lies in a bounded subset of $L^\infty(\mathbb{R})$ for a.e. $t \in (0,T)$, hence, by (5.16) as $n \to \infty$, we obtain that $t_0 = 0$. Thus, claim (ii) (b) follows. This completes the proof. □

Remark 6.10. As we claimed in Remark 3.13, in Theorem 3.11 (ii), we may relax hypothesis (H2) to (H3), with $k > 0$. To prove this, for every $u_0 \in \mathcal{M}^+(\Omega)$, let $\{u_{0n}\}$ be any sequence as in (3.14)–(3.15), and let u_n be the entropy solution of problem (Pn). Set $v_{0n} := G_k(u_{0n})$, where $G_k(u) := (u-k)^+$ for every $u \ge 0$, and let v_n be the entropy solution of the following problem:

$$\begin{cases} \partial_t v_n + \partial_x[\varphi_k(v_n)] = 0 & \text{in } S,\\ v_n = v_{0n} & \text{in } \mathbb{R}\times\{0\}\end{cases}$$

($\varphi_k(u) = \varphi(u+k) - \varphi(k)$). A standard calculation shows that $G_k(u_n)$ is an entropy subsolution of the above problem, whence

$$G_k(u_n) \le v_n \quad \text{a.e. in } S. \tag{6.28}$$

Following the proof of Theorem 3.7, the sequence $\{v_n\}$ converges to an entropy solution v of problem (P) with initial datum $v_0 = u_{0s} + G_k(u_{0r})$. Moreover, by assumption (H3), φ_k satisfies (H2) and we may apply Theorem 3.11 (ii) to v. Therefore, the conclusion follows from (6.28).

Proof of Proposition 3.14. By the proof of Proposition 3.10, inequality (6.18) is satisfied for a.e. $t \in (0, T)$ and all $x_0 \in \operatorname{supp} u_s(\cdot, t)$. We fix such t. Let $x_1 \in \operatorname{supp} u_s(\cdot, t)$ and set $\mathcal{I}_1 := (x_1 - \varepsilon, x_1 + \varepsilon)$ with $\varepsilon > 0$. By (6.18),

$$\int_{\mathcal{I}_1} u_r(x,t)\,dx \geq \int_{x_1}^{x_1+\varepsilon} \Psi^{-1}\Big(\Psi(\infty) - \frac{1}{t}(x - x_1)\Big)\,dx = \int_0^{\varepsilon} \Psi^{-1}\Big(\Psi(\infty) - \frac{y}{t}\Big)\,dy =: B_\varepsilon.$$

If $\operatorname{supp} u_s(\cdot, t) \not\subset \mathcal{I}_1$, let $x_2 \in \operatorname{supp} u_s(\cdot, t) \setminus \mathcal{I}_1$ and set $\mathcal{I}_2 := (x_2 - \varepsilon, x_2 + \varepsilon)$. Since $(x_1, x_1 + \varepsilon) \cap (x_2, x_2 + \varepsilon) = \emptyset$, we have that

$$\int_{\mathcal{I}_1 \cup \mathcal{I}_2} u_r(x,t)\,dx \geq \int_{x_1}^{x_1+\varepsilon} u_r(x,t)\,dx + \int_{x_2}^{x_2+\varepsilon} u_r(x,t)\,dx \geq 2B_\varepsilon.$$

We continue this construction recursively as long as $\operatorname{supp} u_s(\cdot, t) \not\subset \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{n-1}$, with $\mathcal{I}_{n-1} := (x_{n-1} - \varepsilon, x_{n-1} + \varepsilon)$: there exists $x_n \in \operatorname{supp} u_s(\cdot, t) \setminus \{\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{n-1}\}$ such that, setting $\mathcal{I}_n := (x_n - \varepsilon, x_n + \varepsilon)$,

$$nB_\varepsilon \leq \int_{\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n} u_r(x,t)\,dx \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}.$$

Hence, this construction stops at some $n = n_\varepsilon$, and $n_\varepsilon B_\varepsilon \leq \|u_0\|_{\mathcal{M}(\mathbb{R})}$. Therefore,

$$\operatorname{supp} u_s(\cdot, t) \subset \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{n_\varepsilon}, \quad |\operatorname{supp}_s(\cdot, t)| \leq |\mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{n_\varepsilon}| \leq 2n_\varepsilon \varepsilon \leq \frac{2\varepsilon}{B_\varepsilon}\|u_0\|_{\mathcal{M}(\mathbb{R})}.$$

Since $B_\varepsilon/\varepsilon \to \infty$ as $\varepsilon \to 0$, the claim follows. □

7 Uniqueness: Proofs

Again, without loss of generality, we may assume that $C_\varphi = 0$ in the following proofs (see Remark 3.16).

Proof of Proposition 3.20. (i) The first step of the proof consists in showing that

$$\operatorname*{ess\,lim}_{t \to 0^+} \|u_r(\cdot, t) - u_{0r}\|_{L^1(\mathbb{R})} = 0. \tag{7.1}$$

Let $\{u_n^\varepsilon\}$ be the sequence of solutions to problems (4.3) considered in Section 4, and let $\{x_l\}$ $(l = 1, \ldots, N)$ be as in (3.26). We set $I_l := (x_l, x_{l+1})$, $Q_l := I_l \times (0, \tau)$ $(l = 1, \ldots, N-1)$, $I_- := (-\infty, x_1)$, $I_+ := (x_N, \infty)$, and $Q_\pm := I_\pm \times (0, \tau)$.

Let $1 \leq l \leq N - 1$ and $\rho \in C_c^2(I_l)$, $\rho \geq 0$. Let $h_0 > 0$ be such that $x + h \in I_l$ if $x \in \operatorname{supp} \rho$ and $|h| < h_0$. Let $\delta > 0$. Setting $v_n^\varepsilon(x,t) := u_n^\varepsilon(x + h, t)$ and $z := (u_n^\varepsilon - v_n^\varepsilon)(\rho + \delta)$, we apply the L^1-contraction property to the parabolic equation

$$\partial_t z + \partial_x\Big[\Big(R + \frac{2\varepsilon\rho'}{\rho + \delta}\Big)z\Big] - \varepsilon\partial_x^2 z = \Big(\frac{R\rho'}{\rho + \delta} + \frac{\varepsilon\rho''}{\rho + \delta}\Big)z = (\varphi_\varepsilon(u_n^\varepsilon) - \varphi_\varepsilon(v_n^\varepsilon))\rho' + \varepsilon[u_n^\varepsilon - v_n^\varepsilon]\rho'',$$

where

$$R := \begin{cases} \dfrac{\varphi_\varepsilon(u_n^\varepsilon) - \varphi_\varepsilon(v_n^\varepsilon)}{u_n^\varepsilon - v_n^\varepsilon} & \text{if } u_n^\varepsilon \neq v_n^\varepsilon, \\ \varphi_\varepsilon'(u_n^\varepsilon) & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned}\int_{I_l} |z(x,\tau)|\,dx \leq \int_{I_l} |z(x,0)|\,dx + \int_0^\tau \int_{I_l} \big|\varphi_\varepsilon(u_n^\varepsilon(x,t)) - \varphi_\varepsilon(u_n^\varepsilon(x+h,t))\big||\rho'(x)|\,dx\,dt \\ + \varepsilon \int_0^\tau \int_{I_l} \big|u_n^\varepsilon(x,t) - u_n^\varepsilon(x+h,t)\big||\rho''(x)|\,dx\,dt \quad \text{for } \tau \in (0,T).\end{aligned}$$

First we let $\delta \to 0$ and then $\varepsilon = \varepsilon_m \to 0$, where $\{\varepsilon_m\}$ is as in Lemma 4.4. Hence,

$$\int_{I_l} \big|u_n(x,\tau) - u_n(x+h,\tau)\big|\rho(x)\,dx \le \int_{I_l} |u_{0n}(x) - u_{0n}(x+h)|\rho(x)\,dx + \int_0^\tau \int_{I_l} \big|\varphi(u_n(x,t)) - \varphi(u_n(x+h,t))\big||\rho'(x)|\,dx\,dt \tag{7.2}$$

for a.e. $\tau \in (0,T)$, where u_n is the entropy solution of problem (Pn) ($n \in \mathbb{N}$). Since, by (5.16), $u_n(\cdot,t) \overset{*}{\rightharpoonup} u(\cdot,t)$ in $\mathcal{M}(I_l)$ for a.e. $t \in (0,T)$ and, by (3.19) and (3.26), $u_s(\cdot,t) \llcorner I_l \le u_{0s} \llcorner I_l = 0$, the lower semicontinuity of the total variation implies that for a.e. $\tau \in (0,T)$,

$$\int_{I_l} |u_r(x,\tau) - u_r(x+h,\tau)|\rho(x)\,dx \le \liminf_{n\to\infty} \int_{I_l} |u_n(x,\tau) - u_n(x+h,\tau)|\rho(x)\,dx.$$

By (3.15), $\int_{I_l} |u_{0n}(x) - u_{0n}(x+h)|\rho(x)\,dx \to \int_{I_l} |u_{0r}(x) - u_{0r}(x+h)|\rho(x)\,dx$. In addition, $\varphi(u_{n_j}) \to \varphi(u_r)$ in $L^1(Q_l)$ for a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ (see Remark 5.10). Letting $n = n_j \to \infty$ in (7.2), we obtain, for a.e. $\tau \in (0,T)$,

$$\int_{I_l} |u_r(x,\tau) - u_r(x+h,\tau)|\rho(x)\,dx \le \int_{I_l} |u_{0r}(x) - u_{0r}(x+h)|\rho(x)\,dx + \int_0^\tau \int_{I_l} \big|\varphi(u_r(x,t)) - \varphi(u_r(x+h,t))\big||\rho'(x)|\,dx\,dt. \tag{7.3}$$

Let $\{\tau_n\} \subset (0,T)$ be any sequence such that $\tau_n \to 0^+$ and (7.3) is satisfied with $\tau = \tau_n$. Since $u_{0r} \in L^1(\mathbb{R})$ and $\varphi(u_r) \in L^1(S)$, it follows from (7.3) and the Fréchet–Kolmogorov theorem that the sequence $\{u_r(\cdot,\tau_n)\rho\}$ is relatively compact in $L^1(\mathbb{R})$. Then, by (3.11) and a standard argument,

$$u_r(\cdot,\tau_n)\rho \to u_{0r}\rho \quad \text{in } L^1(\mathbb{R}). \tag{7.4}$$

It follows from (3.8) and (3.13) that for each $n \in \mathbb{N}$,

$$\int_{I_l} [u_r(x,\tau_n) - u_{0r}(x)]\rho(x)\,dx = \int_0^{\tau_n} \int_{I_l} \varphi(u_r)(x,t)\rho'(x)\,dx\,dt. \tag{7.5}$$

For sufficiently small $\delta > 0$, the characteristic function $\chi_{(x_l,x_l+\delta)\cup(x_{l+1}-\delta,x_{l+1})}$ can be approximated by functions $\rho_k \in C_c^2(I_l)$, $\rho_k \ge 0$ such that $\int_{I_l} |\rho_k'(x)|\,dx \le 4$ for all $k \in \mathbb{N}$. Setting $\rho = \rho_k$ in (7.5) and letting $k \to \infty$, we find that

$$\int_{x_l}^{x_l+\delta} u_r(x,\tau_n)\,dx + \int_{x_{l+1}-\delta}^{x_{l+1}} u_r(x,\tau_n)\,dx \le \int_{x_l}^{x_l+\delta} u_{0r}(x)\,dx + \int_{x_{l+1}-\delta}^{x_{l+1}} u_{0r}(x)\,dx + 4\|\varphi\|_{L^\infty(0,\infty)}\tau_n. \tag{7.6}$$

Since $u_{0r} \in L^1(\mathbb{R})$, for every $\sigma > 0$, there exists $\delta > 0$ such that

$$\int_{x_l}^{x_l+\delta} u_{0r}(x)\,dx + \int_{x_{l+1}-\delta}^{x_{l+1}} u_{0r}(x)\,dx \le \sigma. \tag{7.7}$$

If $\rho \in C_c(I_l)$ is such that $0 \le \rho \le 1$ in I_l, $\rho = 1$ in $[x_l+\delta, x_{l+1}-\delta]$, then

$$|u_r(\cdot,\tau_n) - u_{0r}| = |u_r(\cdot,\tau_n) - u_{0r}|\rho + |u_r(\cdot,\tau_n) - u_{0r}|(1-\rho)\chi_{(x_l,x_l+\delta)\cup(x_{l+1}-\delta,x_{l+1})} \quad \text{in } I_l.$$

Hence, by (7.6) and (7.7),

$$\begin{aligned}\int_{I_l} |u_r(\cdot,\tau_n) - u_{0r}|\,dx &\le 2\Big\{\int_{x_l}^{x_l+\delta} u_{0r}\,dx + \int_{x_{l+1}-\delta}^{x_{l+1}} u_{0r}\,dx\Big\} + 4\|\varphi\|_{L^\infty(0,\infty)}\tau_n + \int_{I_l} |u_r(\cdot,\tau_n) - u_{0r}|\rho\,dx \\ &\le \int_{I_l} |u_r(\cdot,\tau_n) - u_{0r}|\rho\,dx + 4\|\varphi\|_{L^\infty(0,\infty)}\tau_n + 2\sigma.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by (7.4), we obtain that

$$\limsup_{n\to\infty} \int_{I_l} |u_r(\cdot\,,\tau_n) - u_{0r}|\,dx \le 2\sigma,$$

whence, by the arbitrariness of σ,

$$\lim_{n\to\infty} \int_{I_l} |u_r(x,\tau_n) - u_{0r}(x)|\,dx = 0 \quad (l = 1,\dots,N-1).$$

A similar argument shows that $\int_{I_\pm} |u_r(x,\tau_n) - u_{0r}(x)|\,dx \to 0$ as $n \to \infty$, thus (7.1) follows.

To complete the proof of (3.27), observe that by (3.19) we have $u_s(\cdot\,,t) \le u_{0s}$ in $\mathcal{M}(\mathbb{R})$ (recall that $C_\varphi = 0$, by assumption). Hence,

$$\langle u_{0s} - u_s(\cdot\,,t), \rho\rangle_{\mathbb{R}} \ge \|u_s(\cdot\,,t) - u_{0s}\|_{\mathcal{M}(\mathbb{R})}$$

for all $\rho \in C_c(\mathbb{R})$ such that $\rho(x) = 1$ for every $x \in \operatorname{supp} u_{0s}$. From the previous inequality, (3.11) and (7.1), we get

$$\begin{aligned}\operatorname*{ess\,lim}_{t\to0^+} \|u_s(\cdot\,,t) - u_{0s}\|_{\mathcal{M}(\mathbb{R})} &\le \operatorname*{ess\,lim}_{t\to0^+} \langle u_{0s} - u_s(\cdot\,,t), \rho\rangle_{\mathbb{R}} \\ &= \operatorname*{ess\,lim}_{t\to0^+} \Big\{\langle u_0 - u(\cdot\,,t), \rho\rangle_{\mathbb{R}} - \int_{\mathbb{R}} (u_r(x,t) - u_{0r})\rho(x)\,dx\Big\} = 0. \end{aligned} \tag{7.8}$$

Then (3.27) follows.

(ii) Let $\zeta^\pm \in C_c^1(Q_\pm)$, $\zeta^\pm \ge 0$, and for every $1 \le l \le N-1$ let $\zeta_l \in C_c^1(Q_l)$, $\zeta_l \ge 0$. Let $h_0 > 0$ be such that $(x+h,t) \in Q_l$ (respectively $(x+h,t) \in Q_\pm$) if $(x,t) \in \operatorname{supp}\zeta_l$ (respectively if $(x,t) \in \operatorname{supp}\zeta_\pm$) and $|h| < h_0$.

Let u be an entropy solution of problem (P), thus $v(\cdot\,,t) = \mathcal{T}_{-h}(u(\cdot\,,t))$ is an entropy solution of problem (P) with u_0 replaced by $v_0 := \mathcal{T}_{-h}(u_0)$ (see Remark 3.16). We shall prove, for all $l = 1,\dots,N-1$ and ζ_l as above, that

$$\iint_{Q_l} \{|v_r - u_r|\partial_t\zeta_l + \operatorname{sgn}(v_r - u_r)[\varphi(v_r) - \varphi(u_r)]\partial_x\zeta_l\}\,dx\,dt \ge 0 \tag{7.9}$$

and, for all $\zeta^\pm$ as above,

$$\iint_{Q_\pm} \{|v_r - u_r|\partial_t\zeta^\pm + \operatorname{sgn}(v_r - u_r)[\varphi(v_r) - \varphi(u_r)]\partial_x\zeta^\pm\}\,dx\,dt \ge 0. \tag{7.10}$$

Relying on (7.9)–(7.10) we can conclude the proof by an argument similar to that used in (i). Let $\rho \in C_c^1(I_l)$, $0 \le \rho \le 1$, be such that $x + h \in I_l$ if $x \in \operatorname{supp}\rho$ and $|h| < h_0$. By a proper choice of the function ζ_l in (7.9), for a.e. $0 < t_0 < t_1 \le T$, we get

$$\int_{I_l} |u_r(x,t_1) - v_r(x,t_1)|\rho(x)\,dx \le \int_{I_l} |u_r(x,t_0) - v_r(x,t_0)|\rho(x)\,dx + \int_{t_0}^{t_1} |\varphi(u_r) - \varphi(v_r)||\rho'(x)|\,dx\,dt.$$

Let $t_0 > 0$ be fixed. Then, for every $\tau \in (t_0,T]$, there exists a sequence $\tau_n \to \tau$ such that $\tau_n \in (t_0,T]$, and the above inequality holds true with $t_1 = \tau_n$ for every n, that is,

$$\begin{aligned}\int_{I_l} \big|u_r(x+h,\tau_n) - u_r(x,\tau_n)\big|\rho(x)\,dx &\le \int_{I_l} \big|u_r(x+h,t_0) - u_r(x,t_0)\big|\rho(x)\,dx \\ &\quad + \|\rho'\|_\infty \int_0^T \big|\varphi(u_r(x+h,t)) - \varphi(u_r(x,t))\big|\,dx\,dt. \end{aligned} \tag{7.11}$$

Since $\varphi(u_r) \in L^1(S)$ and $u_r(\cdot\,,t_0) \in L^1(\mathbb{R})$, inequality (7.11) and the Fréchet–Kolmogorov theorem imply that the sequence $\{u_r(\cdot\,,\tau_n)\rho\}$ is relatively compact in $L^1(\mathbb{R})$, whence, by Proposition 3.5 and a standard argument,

$$u_r(\cdot\,,\tau_n)\rho \to u_r(\cdot\,,\tau)\rho \quad \text{in } L^1(\mathbb{R}). \tag{7.12}$$

Moreover, by arguing as in (7.6) and (7.7) with u_{0r} replaced by $u_r(\cdot,\tau)$, for every $\sigma > 0$, there exists $\delta > 0$ such that

$$\int_{x_l}^{x_l+\delta} u_r(x,\tau_n)\,dx + \int_{x_{l+1}-\delta}^{x_{l+1}} u_r(x,\tau_n)\,dx \le \sigma + 4\|\varphi\|_{L^\infty(0,\infty)}|\tau_n - \tau|. \tag{7.13}$$

As in the proof of claim (i), combining (7.12) and (7.13) gives

$$\lim_{n\to\infty} \int_{I_l} |u_r(x,\tau_n) - u_r(x,\tau)|\,dx = 0, \quad l = 1,\dots,N-1$$

(by a similar argument, $\int_{I_\pm} |u_r(x,\tau_n) - u_r(x,\tau)|\,dx \to 0$ as $n\to\infty$), whence

$$\operatorname*{ess\,lim}_{t\to\tau} \|u_r(\cdot,t) - u_r(\cdot,\tau)\|_{L^1(\mathbb{R})} = 0.$$

Since $C_\varphi = 0$, it follows from (3.18) that $u_s(\cdot,t_2) \le u_s(\cdot,t_1)$ in $\mathcal{M}(\mathbb{R})$ if $t_2 > t_1$, whence, by arguing as in (7.8), we also obtain

$$\operatorname*{ess\,lim}_{t\to\tau^+} \|u_s(\cdot,t) - u_s(\cdot,\tau)\|_{\mathcal{M}(\mathbb{R})} = \operatorname*{ess\,lim}_{t\to\tau^-} \|u_s(\cdot,t) - u_s(\cdot,\tau)\|_{\mathcal{M}(\mathbb{R})} = 0,$$

and claim (ii) follows.

Finally, it remains to prove (7.9) (the proof of (7.10) is analogous). Let $1 \le l \le N-1$ and $\zeta_l \in C_c^1(Q_l)$, $\zeta_l \ge 0$, be fixed as above. Since $C_\varphi = 0$, it follows from (3.19) and (3.26) that $u_s(\cdot,t) = v_s(\cdot,t) = 0$ on $\operatorname{supp}\zeta_l(\cdot,t)$ for a.e. $t\in(0,T)$, and from (3.10) that, for $k \in [0,\infty)$,

$$\iint_{Q_l} \{|u_r - k|\partial_t\zeta_l + \operatorname{sgn}(u_r - k)[\varphi(u_r) - \varphi(k)]\partial_x\zeta_l\}\,dx\,dt \ge 0, \tag{7.14}$$

$$\iint_{Q_l} \{|v_r - k|\partial_t\zeta_l + \operatorname{sgn}(v_r - k)[\varphi(v_r) - \varphi(k)]\partial_x\zeta_l\}\,dx\,dt \ge 0. \tag{7.15}$$

We apply Kružkov's method of doubling variables. Let $Z_l = Z_l(x,t,y,s) \in C_c^1(Q_l\times Q_l)$, $Z_l \ge 0$. It follows from (7.14)–(7.15) that

$$\iint_{Q_l} \Big\{|u_r(x,t) - v_r(y,s)|\partial_t Z_l(x,t,y,s) + \operatorname{sgn}(u_r(x,t) - v_r(y,s))[\varphi(u_r(x,t)) - \varphi(v_r(y,s))]\partial_x Z_l(x,t,y,s)\Big\}\,dx\,dt \ge 0$$

and

$$\iint_{Q_l} \Big\{|v_r(y,s) - u_r(x,t)|\partial_s Z_l(x,t,y,s) + \operatorname{sgn}(v_r(y,s) - u_r(x,t))[\varphi(v_r(y,s)) - \varphi(u_r(x,t))]\partial_y Z_l(x,t,y,s)\Big\}\,dy\,ds \ge 0,$$

whence

$$\iiiint_{Q_l\times Q_l} \Big\{|u_r(x,t) - v_r(y,s)|(\partial_t Z_l + \partial_s Z_l)(x,t,y,s) + \operatorname{sgn}(u_r(x,t) - v_r(y,s))[\varphi(u_r)(x,t) - \varphi(v_r)(y,s)](\partial_x Z_l + \partial_y Z_l)(x,t,y,s)\Big\}\,dx\,dt\,dy\,ds \ge 0.$$

We choose

$$Z_l(x,t,y,s) = Z_l^\varepsilon(x,t,y,s) := \zeta_l(x,t)\zeta_\varepsilon(x-y,t-s) \quad (\varepsilon > 0),$$

where ζ_ε is a smooth approximation of the Dirac mass $\delta_{(0,0)}$,

$$\zeta_\varepsilon(x,y) = \frac{1}{\varepsilon^2}\theta\Big(\frac{x}{\varepsilon}\Big)\eta\Big(\frac{t}{\varepsilon}\Big) \ge 0, \quad \text{with } \operatorname{supp}\theta \subseteq (-1,1),\ \operatorname{supp}\eta \subseteq (-1,1).$$

Then $\partial_t Z_l + \partial_s Z_l = \partial_t \zeta_l \zeta_\varepsilon$ and $\partial_x Z_l + \partial_y Z_l = \partial_x \zeta_l \zeta_\varepsilon$, whence, for sufficiently small ε,

$$\iiiint\limits_{Q_l\times Q_l} \Big\{|u_r(x,t) - v_r(y,s)|\partial_t\zeta_l(x,t) + \operatorname{sgn}(u_r(x,t) - v_r(y,s))[\varphi(u_r)(x,t) - \varphi(v_r)(y,s)]\partial_x\zeta_l(x,t)\Big\}\zeta_\varepsilon(x-y,t-s)\,dx\,dt\,dy\,ds \ge 0.$$

Now (7.9) follows by letting $\varepsilon \to 0^+$, and we claim that

$$\lim_{\varepsilon\to 0^+} \iiiint\limits_{Q_l\times Q_l} |u_r(x,t) - v_r(y,s)|\partial_t\zeta_l(x,t)\zeta_{\varepsilon_n}(x-y,t-s)\,dx\,dt\,dy\,ds = \iint\limits_{Q_l} |u_r(x,t) - v_r(x,t)|\partial_t\zeta_l(x,t)\,dx\,dt. \tag{7.16}$$

Analogously, it can be proven that, as $\varepsilon_n \to 0^+$,

$$\iiiint\limits_{Q_l^2} \operatorname{sgn}[u_r(x,t) - v_r(y,s)][\varphi(u_r)(x,t) - \varphi(v_r)(y,s)]\partial_x\zeta_l(x,t)\zeta_{\varepsilon_n}(x-y,t-s)dx\,dt\,dy\,ds$$
$$\to \iint\limits_{Q_l} \operatorname{sgn}(u_r(x,t) - v_r(y,s))[\varphi(u_r)(x,t) - \varphi(v_r)(x,t)]\partial_x\zeta_l(x,t)\,dx\,dt.$$

In order to prove (7.16), for every sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0$, we set

$$F_n(x,t) := \iint\limits_{Q_l} |u_r(x,t) - v_r(y,s)|\zeta_{\varepsilon_n}(x-y,t-s)\,dy\,ds \quad \text{for } (x,t) \in K_l := \operatorname{supp}\zeta_l,$$

and observe that $F_n \to |u_r - v_r|$ a.e. in $(x,t) \in K_l$ and

$$\begin{aligned}|F_n(x,t)| &\le |u_r(x,t)| + \iint\limits_{Q_l} |v_r(y,s)|\zeta_{\varepsilon_n}(x-y,t-s)\,dy\,ds\\ &= |u_r(x,t)| + (\zeta_{\varepsilon_n} * |v_r|)(x,t) \to |u_r(x,t)| + |v_r(x,t)| \quad \text{in } L^1(K_l).\end{aligned}$$

Thus, by a variant of the dominated convergence theorem (e.g., see [15, Theorem 4, Section 1.3]), we have $F_n \to |u_r - v_r|$ in $L^1(K_l)$, and we obtain (7.16). This completes the proof of (7.9), thus the result follows. □

Proof of Theorem 3.21. Without loss of generality, we may assume that φ is nondecreasing, see Remark 3.15. By Theorem 3.11 (i),

$$\tau := \sup\{t \in [0,T) \mid u_{is}(\cdot,t)(\{x_l\}) > 0 \text{ for all } l = 1,\dots,N,\ i = 1,2\} > 0.$$

Let us first prove that

$$u_{1r} = u_{2r} \quad \text{a.e. in } \mathbb{R}\times(0,\tau). \tag{7.17}$$

To this end, let $x_1,\dots,x_N$ be the points in (3.26). Set $I_l := (x_l, x_{l+1})$, $Q_l := I_l \times (0,\tau)$ $(l = 1,\dots,N-1)$, $I_- := (-\infty, x_1)$, $I_+ := (x_N, \infty)$, and $Q_\pm := I_\pm \times (0,\tau)$. By arguing as in the last part of the proof of Proposition 3.20 (ii) (in particular, see the proof of (7.9)–(7.10)), it follows that, for all $l = 1,\dots,N-1$ and $\zeta_l \in C_c^1(Q_l)$, $\zeta_l \ge 0$,

$$\iint\limits_{Q_l} \{|u_{1r} - u_{2r}|\partial_t\zeta_l + |\varphi(u_{1r}) - \varphi(u_{2r})(x,t)|\partial_x\zeta_l\}\,dx\,dt \ge 0 \tag{7.18}$$

and, for all $\zeta^\pm \in C_c^1(Q_\pm)$, $\zeta^\pm \ge 0$,

$$\iint\limits_{Q_\pm} \{|u_{1r} - u_{2r}|\partial_t\zeta^\pm + |\varphi(u_{1r}) - \varphi(u_{2r})|\partial_x\zeta^\pm(x,t)\}\,dx\,dt \ge 0 \tag{7.19}$$

(recall that φ, by assumption, is increasing). We must show that (7.18) and (7.19) imply (7.17). For this purpose, let $h \in C_c^1(0,\tau_1)$, $h \ge 0$, and

$$\begin{aligned}\rho_{l,p}(x) &= p\Big(x - x_l - \frac{1}{p}\Big)\chi_{[x_l+1/p,\,x_l+2/p)} + \chi_{[x_l+2/p,\,x_{l+1}-2/p)}(x)\\ &\quad - p\Big(x - x_{l+1} + \frac{1}{p}\Big)\chi_{[x_{l+1}-2/p,\,x_{l+1}-1/p)}(x) \quad (l = 1,\dots,N-1),\end{aligned}$$

with $p \in \mathbb{N}$ sufficiently large. By standard approximation arguments, we may choose $\zeta_l = \zeta_{l,p} := \rho_{l,p}(x)h(t)$ in (7.18) to get

$$0 \le \iint_{Q_l} \{|u_{1r} - u_{2r}|\rho_{l,p}(x)h'(t) + |\varphi(u_{1r}) - \varphi(u_{2r})|\rho'_{l,p}(x)h(t)\}\,dx\,dt. \tag{7.20}$$

By the dominated convergence theorem, as $p \to \infty$,

$$\iint_{Q_l} \{|u_{1r} - u_{2r}|\rho_{l,p}(x)h'(t)\,dx\,dt \to \int_0^\tau dt\,h'(t) \int_{x_l}^{x_{l+1}} |u_{1r} - u_{2r}|\,dx.$$

Since $\rho'_{l,p}(x) = p\chi_{(x_l+1/p,x_l+2/p)}(x) - p\chi_{(x_{l+1}-2/p,x_{l+1}-1/p)}(x)$ and φ is bounded, it follows from (3.20) and the dominated convergence theorem that

$$\limsup_{p\to\infty} \iint_{Q_l} |\varphi(u_{1r}) - \varphi(u_{2r})|\rho'_{l,p}(x)h(t)\,dx\,dt \le \int_0^\tau h(t)\Big(\lim_{p\to\infty} p \int_{x_l+\frac{1}{p}}^{x_l+\frac{2}{p}} |\varphi(u_{1r}) - \varphi(u_{2r})|\,dx\Big)\,dt = 0.$$

Hence, by (7.20),

$$\int_0^\tau dt\,h'(t) \int_{x_l}^{x_{l+1}} |u_{1r}(x,t) - u_{2r}(x,t)|\,dx \ge 0$$

and, by a proper choice of h,

$$\|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(I_l)} \le \|u_{1r}(\cdot,t_1) - u_{2r}(\cdot,t_1)\|_{L^1(I_l)} \quad \text{for every } 0 < t_1 \le t \le \tau \tag{7.21}$$

(recall that $u_{ir} \in C((0,T];\mathcal{M}(\mathbb{R}))$, $i = 1,2$, by Proposition 3.20 (ii)). Letting $t_1 \to 0^+$, it follows from (3.28) that $\|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(I_l)} = 0$ for a.e. $t \in (0,\tau)$ and all $l = 1,\dots,N-1$. Similarly, we can prove that $\|u_{1r}(\cdot,t) - u_{2r}(\cdot,t)\|_{L^1(I_-\cup I_+)} = 0$ for a.e. $t \in (0,\tau)$, so we have proven (7.17).

Next let us prove that

$$u_1 = u_2 \quad \text{in } \mathcal{M}(\mathbb{R}\times(0,\tau)). \tag{7.22}$$

By (3.8) and (7.17), for every $\zeta \in C^1([0,\tau];C^1_c(\mathbb{R}))$, with $\zeta(\cdot,\tau) = 0$ in $\mathbb{R}$, we have

$$\int_0^T \langle u_{1s}(\cdot,t) - u_{2s}(\cdot,t), \partial_t\zeta(\cdot,t)\rangle_{\mathbb{R}}\,dt = \iint_S \{(u_{1r} - u_{2r})\partial_t\zeta + [\varphi(u_{1r}) - \varphi(u_{1r})]\partial_x\zeta\}\,dx\,dt = 0.$$

Arguing as in the proof of Lemma 6.1, there exists a null set $F_0 \subset (0,\tau)$ such that $\langle u_{1s}(\cdot,t) - u_{2s}(\cdot,t), \rho\rangle_{\mathbb{R}} = 0$ for all $t \in (0,\tau)\setminus F_0$ and $\rho \in C^1_c(\mathbb{R})$. Hence, $u_1 = u_2$ in $L^\infty(0,\tau;\mathcal{M}(\mathbb{R}))$ and, by (7.17), equality (7.22) follows.

If $\tau = T$, the proof is complete. Otherwise, there exist $N_1 < N$ different points $x_{l_k} \in \{x_1,\dots,x_N\}$ such that $u_{is}(\cdot,\tau)(\{x_{l_k}\}) > 0$ for each $k = 1,\dots,N_1$ and $i = 1,2$; moreover, for every point $x_l \in \{x_1,\dots,x_N\}$, $x_l \ne x_{l_k}$, it follows from (6.20), with $x_0 = x_1 = x_l$, that $u_{1s}(\cdot,\tau)(\{x_l\}) = u_{2s}(\cdot,\tau)(\{x_l\}) = 0$, since $\varphi(u_{1r}) = \varphi(u_{2r})$ in $\mathbb{R}\times(0,\tau)$ by (7.17). Then we set

$$\tau_1 := \sup\{t \in [\tau,T) \mid u_{is}(\cdot,t)(\{x_{l_k}\}) > 0 \text{ for all } k = 1,\dots,N_1,\ i = 1,2\}.$$

We can argue as in the proof of (7.17) to obtain that inequality (7.21) holds for every $\tau < t_1 \le t \le \tau_1$. Since $u_{ir} \in C((0,T];\mathcal{M}(\mathbb{R}))$, $i = 1,2$ (see Proposition 3.20 (ii)), and $u_{1r}(\cdot,\tau) = u_{2r}(\cdot,\tau)$, letting $t_1 \to \tau^+$, we get $u_{1r} = u_{2r}$ a.e. in $\mathbb{R}\times(\tau,\tau_1)$ (whence, also, $u_1 = u_2$ in $\mathcal{M}(\mathbb{R}\times(\tau,\tau_1))$) and the proof is completed in a finite number of steps. □

Let us finally prove Proposition 1.1.

Proof of Proposition 1.1. A calculation proves that the solution defined by (1.3) if $p < 0$, respectively, by (1.4) if $0 < p < 1$ is an entropy solution of problem (1.1)–(1.2). If $p < 0$, the solution also satisfies (3.20) for $0 < t < 1$ and (3.28), so claim (i) follows from the uniqueness result in Theorem 3.22. If $0 < p < 1$, uniqueness of entropy solutions such that $u_s(t) = 0$ for $t > 0$ and $u_r \in L^\infty(\mathbb{R}\times(\tau,T))$ for $\tau \in (0,T)$ can be used (the proof of this uniqueness result is very similar to that given in [19], thus we omit the details; see also Remark 3.24). Hence, claim (ii) follows. □

Remark 7.1. It is instructive to describe the approximation procedure which gives the solutions mentioned in Proposition 1.1. Consider the family of approximating problems

$$\begin{cases} \partial_t u_n + \partial_x[\varphi(u_n)] = 0 & \text{in } S, \\ u_n = \frac{n}{2}\chi_{(-\frac{1}{n},\frac{1}{n})} & \text{in } \mathbb{R}\times\{0\}, \end{cases} \tag{7.23}$$

with φ given by (1.2). It is easily seen that the entropy solution of (R_n) is

$$u_n(x,t) := \begin{cases} 0 & \text{if } x \ge |p|t + \frac{1}{n}, \\ \Big(\frac{n|p|t}{nx-1}\Big)^{\frac{1}{1-p}} - 1 & \text{if } |p|t + \frac{1}{n} > x \ge \Big(\frac{2}{n+2}\Big)^{1-p}|p|t + \frac{1}{n}, \\ \frac{n}{2} & \text{if } \Big(\frac{2}{n+2}\Big)^{1-p}|p|t + \frac{1}{n} > x \ge \frac{2(\operatorname{sgn} p)}{n}\Big[\Big(\frac{n+2}{2}\Big)^p - 1\Big]t - \frac{1}{n}, \\ 0 & \text{if } \frac{2(\operatorname{sgn} p)}{n}\Big[\Big(\frac{n+2}{2}\Big)^p - 1\Big]t - \frac{1}{n} > x, \end{cases}$$

for $0 \le t \le t_n := \frac{1}{\varphi(\frac{n}{2})-\frac{n}{2}\varphi'(\frac{n}{2})}$. At $t = t_n$ a shock $x = \xi(t)$ stems from $x = x_n := \frac{1}{n}\frac{\varphi(\frac{n}{2})+\frac{n}{2}\varphi'(\frac{n}{2})}{\varphi(\frac{n}{2})-\frac{n}{2}\varphi'(\frac{n}{2})}$, which solves the problem

$$\begin{cases} \xi_n'(t) = \dfrac{\varphi(u_n^{(1)}(\xi_n(t),t))}{u_n^{(1)}(\xi_n(t),t)} = \operatorname{sgn} p \dfrac{\big(\frac{n|p|t}{n\xi-1}\big)^{\frac{p}{1-p}} - 1}{\big(\frac{n|p|t}{n\xi-1}\big)^{\frac{1}{1-p}} - 1} & \text{if } t > t_n, \\ \xi_n(t_n) = x_n. \end{cases}$$

Hence, for $t > t_n$, the entropy solution of (7.23) is

$$u_n(x,t) := \begin{cases} 0 & \text{if } x \ge |p|t + \frac{1}{n}, \\ \Big(\frac{n|p|t}{nx-1}\Big)^{\frac{1}{1-p}} - 1 & \text{if } |p|t + \frac{1}{n} > x \ge \xi_n(t), \\ 0 & \text{if } \xi_n(t) \ge x. \end{cases}$$

Letting $n \to \infty$, we obtain the entropy solution defined in parts (i) (if $p < 0$) and (ii) (if $0 < p < 1$) of Proposition 1.1.

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