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Subfactors and Applications

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Abstract. Invariants of topological spaces of dimension three play a major role in many areas, in particular . . .

Introduction by the Organizers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds . . .

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Workshop: Subfactors and Applications

Table of Contents

Abstracts

Weak quasi-Hopf algebras, VOAs and conformal nets

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(joint work with Sergio Ciamprone, Claudia Pinzari)

Weak quasi-Hopf algebras, introduced by Mack and Schomerus in [13, 14], are a generalization of Drinfeld's quasi-Hopf algebras. Every fusion category is tensor equivalent to the representation category of a weak quasi-Hopf algebra [11]. After these early works there seems has not been no relevant progress in the theory until the recent work by Ciamprone and Pinzari [2] where some specific examples from quantum groups at roots of unity in the type A case where studied in detail.

In a subsequent work by Ciamprone and Pinzari and me [3] we develop various aspects of the theory and consider many examples and applications. Here, I will briefly report of some the results contained in the latter work with emphasis on the unitarity and conformal field theory aspects emerging in connection with vertex operator algebras (VOAs) and conformal nets. These results indicate that weak quasi-Hopf algebras give a useful and natural tool to study certain relevant properties of fusion categories and conformal field theory.

A a weak quasi-Hopf algebra is a quintuple $(A, \Delta, \varepsilon, S, \Phi)$ satisfying various assumptions. Here A is a unital associative algebra (over $\mathbb{C}$), the coproduct $\Delta : A \to A \otimes A$ is a homomorphism, the counit $\varepsilon : A \to \mathbb{C}$ is a nonzero homomorphism, the antipode $S : A \to A$ is an antiautomorphism, Φ is the associator.

In contrast with the quasi-Hopf algebra case the coproduct is not assumed to be unital so that $\Delta(1_A)$ is an idempotent in $A \otimes A$ commuting with $\Delta(A)$ which is in general different from $1_A \otimes 1_A$. This fact allows a much more flexibility.

The coproduct gives a tensor structure on the representation category $\mathrm{Rep}(A)$. The tensor product $\pi_1 \underline{\otimes} \pi_2$ on objects of $\mathrm{Rep}(A)$ is then given by the restriction of $\pi_1 \otimes \pi_2 \circ \Delta$ to the invariant subspace $\pi_1 \otimes \pi_2 \circ \Delta(1_A) V_{\pi_1} \otimes V_{\pi_2}$. If A is finite-dimensional and semisimple then $\mathrm{Rep}(A)$ is a fusion category.

Now, for a given (finite-dimensional and semisimple) A, the additive function $D : \mathrm{Gr}(\mathrm{Rep}(A)) \to \mathbb{Z}$ defined by $D([\pi]) := \dim(V_\pi)$ is a weak integral dimension function i.e. it satisfies $D([\pi_1 \underline{\otimes} \pi_2]) \leq D([\pi_1])D([\pi_2])$, $D([\iota]) = 1$ and $D([\overline{\pi}]) = D([\pi]) \geq 0$. All fusion categories have integral weak dimension functions.

The following result is due to Häring-Oldenburg [11].

Theorem ([11]). *Let $\mathcal{C}$ be a fusion category and $D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{Z}$ be an integral weak dimension. Then there exists a finite dimensional semisimple weak quasi-Hopf algebra $(A, \Delta, \varepsilon, S, \Phi)$ and a tensor equivalence $\mathcal{F} : \mathcal{C} \to \mathrm{Rep}(A)$ such that $D([X]) = \dim(V_{\mathcal{F}(X)})$ for all $X \in \mathrm{Obj}(\mathcal{C})$.*

Extra structures on $\mathcal{C}$ give extra structures on A (see [11] and [3]): braidings give R-matrices; C*-tensor structures on $\mathcal{C}$ give Ω-involutive structures on A (in particular the algebras A become a C*-algebras). The weak quasi-Hopf algebra associated to a fusion category $\mathcal{C}$ is highly non-unique. It depends on the choice of the integral weak dimension function D and, once D is fixed, is only defined up to a "twist".

Now let $\mathcal{C}^+$ be a linear C*-category, $\mathcal{C}$ be a fusion category and $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ be a linear equivalence. In the proof of the following theorem weak quasi-Hopf algebras plays a crucial role.

Theorem 1. ([3]). *If $\mathcal{C}$ is tensor equivalent to a unitary fusion category $\mathcal{D}^+$ then $\mathcal{C}^+$ can be upgraded to a unitary fusion category so that $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ becomes a tensor equivalence. This unitary tensor structure on C^+ is unique up to unitary equivalence and makes $\mathcal{C}^+$ unitary tensor equivalent to $\mathcal{D}^+$.*

In fact the result is still valid if $\mathcal{C}$ is only assumed to be rigid and semisimple provided that it has an integral weak dimension function. As a corollary we find a positive answer to a question by Cesar Galindo in [6]

Corollary 2. ([3]). *Two tensor equivalent unitary fusion categories must be unitary tensor equivalent.*

A different proof of the latter result was found independently by Reutter [15].

We now apply the previous theorem to the unitarizability of the representation categories of unitary affine VOAs. Let $\mathfrak{g}$ be a complex simple Lie algebra, let k be a positive integer and let $V_{\mathfrak{g}_k}$ be the corresponding simple level k affine VOA. It is known that $V_{\mathfrak{g}_k}$ is a unitary strongly rational VOA and that every $V_{\mathfrak{g}_k}$-module is unitarizable. We denote by $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ the linear C*-category of unitary $V_{\mathfrak{g}_k}$-modules. Because of the unitarizability of the $V_{\mathfrak{g}_k}$-modules the forgetful functor $\mathcal{F} : \mathrm{Rep}^u(V_{\mathfrak{g}_k}) \to \mathrm{Rep}(V_{\mathfrak{g}_k})$ is a linear equivalence. By a result of Finkelberg [4, 5] we know that $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is tensor equivalent to the "semisimplified" tensor category $\widetilde{\mathrm{Rep}}(G_q)$ associated to the representations of the quantum group G_q, with G the simply connected compact Lie group corresponding to $\mathfrak{g}$ and and the rooth of unity q is given by $q = e^{\frac{i\pi}{d(k+h^\vee)}}$. Here $h^\vee$ is the dual Coxeter number, $d = 1$ if $\mathfrak{g}$ is ADE, $d = 2$ if $\mathfrak{g}$ is BCF and $d = 3$ if $\mathfrak{g}$ is G_2.

It was shown by Wenzl and Xu [17, 18] that $\widetilde{\mathrm{Rep}}(G_q)$ is tensor equivalent to a unitary fusion category. As a consequence we have the following result.

Theorem 3. ([3]). $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ *has a structure of unitary fusion category which is unique up to unitary equivalence.*

Unitary tensor structures on $\mathrm{Rep}^u(V_{\mathfrak{g}_k})$ have been constructed directly in a series of papers [7, 7, 9] by Bin Gui for the Lie types A, B, C, D, and G_2 and more recently by James Tener in [16] for the remaining cases E_6, E_7, E_8 and F_4 by completely different methods. By our uniqueness result these structures agree with those we have found.

Our method works also for many other VOAs such as e.g. lattice VOAs and certain holomorphic orbifolds.

As another application of the theory of weak quasi-Hopf algebra we give a classification of pseudo-unitary type A ribbon fusion categories. The starting point is the work of Kazhdan and Wenzl [12] on the classification of type A tensor categories. As a consequence of our results we have in particular the following theorem.

Theorem 4. ([3])*Let $\mathcal{C}$ be a modular fusion category with modular matrices S, T coinciding with the Kac-Peterson matrices for the $\mathfrak{sl}(n)$ affine Lie algebra at positive integer level k. Then $\mathcal{C}$ is ribbon equivalent to* $\mathrm{Rep}(V_{\mathfrak{sl}(n)_k})$.

Now let $\mathcal{A}_{V_{\mathfrak{sl}(n)_k}}$ be the conformal net on S^1 associated to the strongly local unitary VOA $V_{\mathfrak{sl}(n)_k}$ [1]. As a first consequence of Theorem 4 we have

Corollary 5. *We have a unitary ribbon equivalence* $\mathcal{F} : \mathrm{Rep}^u(V_{\mathfrak{sl}(n)_k}) \to \mathrm{Rep}(\mathcal{A}_{V_{\mathfrak{sl}(n)_k}})$.

The same result has been independently obtained by Bin Gui [10] by different methods (direct analytic proof instead of classification). As a second consequence of Theorem 4 we obtain e new proof of Finkelberg's equivalence in the type A case.

Corollary 5. *If* $q = e^{\frac{i\pi}{(k+n)}}$ *there is a unitary ribbon equivalence* $\mathcal{F} : \mathrm{Rep}(V_{\mathfrak{sl}(n)_k}) \to \widetilde{\mathrm{Rep}}(\mathrm{SU}(n)_q)$.

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