Bollettino U.M.I.
Algebra e Geometria
Serie VI, Vol. I - D, N. 1 - 1982

EXISTENTIALLY COMPLETE CLOSURE ALGEBRAS

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SOMMARIO. *Si dà una caratterizzazione delle algebre di chiusura esistenzialmente complete, caratterizzazione che viene usata poi per dimostrare che la teoria delle algebre di chiusura non ha model-companion. Si mostra che anche altre teorie più deboli hanno la proprietà di amalgama ma non model-companion.*

§0 - INTRODUCTION.

I use +,·,',0,1 for the operations and the constants of a *Boolean algebra* (BA); axioms for this theory can be found in [1]. The needed properties of BA's are listed without proof in §1.

A *closure algebra* (CA) is a BA $\mathscr{A}$ together with an operator $K : A \to A$ satisfying all the formal properties of the closure in a topological space. The class of all CA's can be axiomatized in first order finitary logic, so one can talk of the theory of CA's. Weaker theories are those of F-, V-, V_α-, V_D-algebras, respectively (for exact definitions see [2]).

A CA $\mathscr{A}$ is said to be *existentially complete* (e.c.) if every system of equations and inequalities with parameters in A, which has a solution in an extension of $\mathscr{A}$, has already a solution in A (this is a particular case of the general definition of existential completeness. See [3]).

A theory is said to have a *model-companion* if the class of e.c. structures for this theory is axiomatizable. In §2 a characterization of e.c. CA's is given; it is used in §3 to prove that the theory of CA's has no model-companion; this is the central result of the paper.

A class of structures has the *amalgamation property* (a.p.) if whenever $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ belong to the class and $\mathscr{C}$ is a substructure of both $\mathscr{A}$ and $\mathscr{B}$, there exists a $\mathscr{D}$ in the class extending $\mathscr{A}$ and $\mathscr{B}$. In §4 the theories I consider are shown to have the a.p..

I use the following notations: $\bar{a}$ stays for the n-tuple $(a_0, \ldots a_{n-1})$.

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and also for the product $a_0a_1\ldots a_{n-1}$, if n is clear from the context; s, t, u and v vary over all the functions from n to 2 (n is identified with $\{0,1,\ldots,n-1\}$), J over all the subsets of n. I suppose a bijection is fixed between ${}^{n}2$ and 2^n, so that $(a_s)_{s:n\to 2}$ and $(a_0,\ldots,a_{2n-1})$ can be considered *exactly* the same thing.
a^1 is a and a^0 is a'; $\bar{a}^s=a_0^{s0}\ldots a_{n-1}^{s(n-1)}$, $K\bar{a}=(Ka_0)\ldots(Ka_{n-1})$ and $K^s\bar{a}=$ $=(Ka_0)^{s0}\ldots(Ka_{n-1})^{s(n-1)}$; Ia=(Ka')'. If $\mathscr{A}$ is a BA with an operator, $\hat{\mathscr{A}}$ denotes the BA obtained from $\mathscr{A}$ by not considering the operator.

I will use the following property of CA's: xy=0 implies IxKy=0.

§ 1 - FACTS ABOUT BOOLEAN ALGEBRAS.

PROPOSITION 1 - *The* BA's *generated by an algebra* $\mathscr{A}$ *and* n *elements* $\bar{b}$ *have the general form* $\mathop{X}_s \mathscr{A}/I_s$ *, where each* I_s *is a (possibly non proper) ideal of* $\mathscr{A}$ *, such that if* a_s *belongs to* I_s *for each* s*, then* $\bar{a}=0$*. Moreover each* b_i *is the equivalence class of* $(0^{si})_s$*; and* b_i *is an element of* A *iff* $\forall s\ Ea_s\varepsilon I_s \sim\Phi_i^n(\bar{a})$ *, where* $\Phi_i^n(x_0,\ldots,x_{2n-1})$ *is* $\prod_{si=1} x_s + \prod_{si=0} x_s \neq 1$*;* $b_i=b_j$ *iff* $\forall s\ Ea_s\varepsilon I_s \sim\Phi_i^n(\bar{a})$ *, where* $\Phi_{ij}^n(\bar{x})$ *is:* $\bigvee_s (si=sj\ \&\ x_s\neq 1)$.

Prop. 1 can be proved using isomorphism theorems (cf. [4]) ($\bar{a}=0$ makes sure that $X\mathscr{A}/I_s$ is an extension of $\mathscr{A}$).

PROPOSITION 2 - *If* $\mathscr{C}$ *is a Boolean subalgebra of both* $\mathscr{A}$ *and* $\mathscr{B}$*, the 'free product with amalgamated subalgebra'* $\mathscr{A}*_{\mathscr{C}}\mathscr{B}$ *exists and extends* $\mathscr{A}$ *and* $\mathscr{B}$*. If* a *belongs to* A *and* b *to* B*, then in* $\mathscr{A}*_{\mathscr{C}}\mathscr{B}$*:*
$a\leqslant b$ *iff* $Ec\varepsilon C\ \ a\leqslant c\leqslant b$.

§ 2 - EXISTENTIALLY COMPLETE CLOSURE ALGEBRAS.

The following proposition gives us the possibility of constructing a lot of CA's:

PROPOSITION 3 - *If* A *is a subset containing* 0 *of a complete* BA $\mathscr{B}$*, and* H *is a function from* A *to* B *such that* H0=0 *and for every* a *in* A $Ha\geqslant a$*, then* H *can be extended on* B *to a:*

a) F-operator *always;*
b) V-operator *iff* $a\leqslant b$ *implies* $Ha\leqslant Hb$*;*
c) V_α*-operator* " $a\leqslant Hb$ *implies* $Ha\leqslant Hb$*;*
d) V_D*-operator* *iff* $a\leqslant\Sigma b_i$ *implies* $Ha\leqslant\Sigma Hb_i$ *;*
e) closure operator iff $a\leqslant\Sigma Hb_i$ *implies* $Ha\leqslant\Sigma Hb_i$,

a, b, b_i *being any finite number of elements of* A. *In all the cases there exists the coarsest extension, in cases a), b), c) the finest, too (*K *is* finer *than* K° *iff for each* a $Ka \leq K°a$ *).*

Sketch of proof: Define in cases:

a), b): A°=A, H°=H;

c): A°={Ha| a∈A}, H° the identity on A°;

d): $A° = \{\Sigma a_i \mid a_i \in A,\ i \in n\}$, $H° \Sigma a_i = \Sigma H a_i$;

e): $A° = \{\Sigma H a_i \mid a_i \in A,\ i \in n\}$, H° the identity on A°.

This is a good definition. As in [2], each H° can be extended to an operator K with the requested regularity; and it is easy to see that K extends H, too.

I define:

$$w_{st}(0,\bar{a}) = a_{st};$$

$$w_{st}(m+1,\bar{a}) = w_{st}(1,\bar{w}(m,\bar{a})) = w_{st}(m,\bar{a}) + \sum_{\substack{\{i\in n \mid t_i=1\} \\ \{J \mid tJ=0\}}} \prod_{\substack{ui=1 \\ vJ=0}} I w_{uv}(m,\bar{a}) + \sum_{\{J \mid tJ=0\}} K \prod_{\substack{u \\ vJ=0}} w_{uv}(m,\bar{a}) + 1_{st},$$

1_{st} being 1 if Ei (si=1 & ti=0), and 0 otherwise.

I will consider only polynomials, with coefficients $\bar{a}$ in a CA, of the form $P = P(\bar{a},\bar{x}) = \Sigma_{s,t}\, a_{st}\, \bar{x}^s K^t \bar{x}$.

P_m is defined as $P_m(\bar{a},\bar{x}) = P(\bar{w}(m,\bar{a}),\bar{x})$.

LEMMA - *For any* CA *, any* P *and any* m, P=0 *iff* $P_m=0$.

Proof: an implication follows from the fact that $\bar{w}(p,\bar{a})$ is increasing as a function of p.

On the other side, let P=0; then, for every J:

$$0 = \sum_{\substack{s \\ tJ=0}} a_{st}\, \bar{x}^s K^t \bar{x} \geq \sum_{\substack{s \\ tJ=0}} \Big(\prod_{\substack{u \\ vJ=0}} a_{uv} \Big) \bar{x}^s K^t \bar{x} = \Big(\prod_{\substack{u \\ vJ=0}} a_{uv} \Big) \Big(\prod_{i\in J} K' x_i \Big) \sum_{\substack{s \\ tJ=0}} \bar{x}^s \prod_{h \in n-J} K^{th} x_h =$$

$$= \prod_{\substack{u \\ vJ=0}} a_{uv} \prod_{i \in J} K' x_i, \quad \text{and:}$$

$$(K \prod_{\substack{u \\ vJ=0}} a_{uv})(I \prod_{i\varepsilon J} K'x_i)=(K \prod_{\substack{u \\ vJ=0}} a_{uv}) \prod_{i\varepsilon J} K'x_i=0. \qquad (1_J)$$

Moreover, for each J and each $j\varepsilon n$:

$$0=\sum_{\substack{sj=1 \\ tJ=0}} a_{st}\bar{x}^s K^t \bar{x} \geq x_j (\prod_{\substack{uj=1 \\ vJ=0}} a_{uv})(\prod_{i\varepsilon J} K'x_i) \sum_{\substack{sj=1 \\ tJ=0}} (\prod_{i\neq j} x^s) \prod_{h\varepsilon n-J} K^{th} x_h = x_j (\prod_{uj=1} a_{uv})(\prod_{i\varepsilon J} K'x_i)$$

from which:

$$(\prod_{\substack{uj=1 \\ vJ=0}} Ia_{uv})(\prod_{i\varepsilon J} K'x_i)Kx_j=0. \qquad (2_{J,j})$$

Adding the terms given by (1_J) and $(2_{J,j})$ to P for each J and j, and since in any CA yK'y=0, I obtain $P_1=0$, and then, iterating, $P_m=0$.

THEOREM 1 - (Characterization of e.c. CA's) - *A* CA $\mathscr{A}$ *is e.c. iff it has the following property:*

For every $P(\bar{a},\bar{x})$, if for every $m\varepsilon\omega$, $i,j\varepsilon n$

$$\Phi_i^{2n}(\bar{w}(m,\bar{a})) \ \& \ \Phi_{ij}^{2n}(\bar{w}(m,\bar{a})) \ \& \ \bar{w}(m,\bar{a})=0 \qquad (3)$$

is true in $\mathscr{A}$, then

$$\begin{cases} P=0 \\ x_0\neq 0 \\ \cdots\cdot \\ x_{n-1}\neq 0 \end{cases} \qquad (4)$$

has a solution in A.

Proof: First, suppose (3) is true but (4) has no solution: I find an extension $\mathscr{B}$ of $\mathscr{A}$ in which (4) has a solution, thus contradicting the e.c. of $\mathscr{A}$.

Take $\mathscr{B}$ a complete extension of the BA $\underset{s,t}{X}\hat{\mathscr{A}}/I_{st}$, where I_{st} is the ideal of $\mathscr{A}$ generated by the elements $w_{st}(m,\bar{a})$ (such an extension exists for Stone's theorem, cf. [1]). Define H from A U $\{b_0,\ldots,b_{n-1}\}$ to B with:

$$\begin{cases} Ha=Ka & \text{if } a\varepsilon A \\ Hb_i=b_{n+i} \end{cases}$$

It is easy to see that:

i) (3) assures that $\mathscr{B}$ extends $\hat{\mathscr{A}}$ and H is well defined (cf. Prop 1);

ii) H can be extended to a closure operator K on B (verify the hypothesis of Prop 3);

iii) with this K, b is a solution of (4).

On the contrary, let $\mathscr{A}$ satisfy the property. To see that $\mathscr{A}$ is e.c. it is sufficient to verify that any system of equations and inequalities with parameters from A is equivalent either to a system of the form (4) satisfying (3) (and hence has a solution in $\mathscr{A}$), or to the false statement 0=1 (and cannot have solutions). Given a general system, eliminate the expressions of the form KX, where X is not an unknown, adding the equation X=y, y being a new unknown, and substituting Ky in place of KX; in a similar way, since $X\neq Y$ is equivalent to $XY'+X'Y\neq 0$, reduce all the inequalities to the form $x_i\neq 0$. X=0 & Y=0 is equivalent to X+Y=0, so you can have only one equation, which can be reduced to the form P=0 using properties of BA's. The system obtained is of the form (4); now, if there exists an m such that:

i) $\bar{w}(m,\bar{a})\neq 0$, then $P_m\geqslant\bar{w}\neq 0$, and the system is equivalent to 1=0;

ii) $\Phi_i^{2n}(\bar{w})$ or $\Phi_{ij}^{2n}(\bar{w})$ is false, then it can be proved that a solution $\bar{x}$ of P_m (and hence of P by the lemma) must satisfy, respectively, $x_i = \prod_{\substack{s i\neq 0\\ t}} w_{st}(m,\bar{a})$ or $x_i=x_j$, and you can get an equivalent system with a fewer number of variables performing one of those substitutions.

Applying i and ii a finite number of times, we prove the system is impossible, or we get a system satisfying (3).

Remarks: 1) It is easy to see that the above theorem yields an explicit axiomatization in $\mathscr{L}_{\omega_1\omega}$ of the class of e.c. CA's.

2) All above is true for V_α-algebras, too, putting:

$$w_{st}(1,\bar{a})=a_{st}+\sum_{\{i|ti=0\}} K \prod_{\substack{u\\ vi=0}} a_{uv}+\sum_{\{i|ti=1\}} I \prod_{\substack{ui=0\\ v}} a_{uv}+1_{st}+\sum_{\substack{\{i|ti=1\}\\ \{j|tj=0\}}} e_{ij} ,$$

where $e_{ij}=1$ if $\sum_{\substack{ui=1\\ vj=0}} a_{uv}=1$, and zero otherwise.

§ 3 - NON EXISTENCE OF MODEL COMPANIONS

A direct consequence of th. 1 is:

COROLLARY 1 - *If* a *is an element of a* CA $\mathscr{A}$, *then:*

$$\begin{cases} Q=ax+a'y+yK'x+xK'y=0 \\ x\neq 0 \\ y\neq 0 \end{cases}$$

has a solution in A *iff for every* m $a_m\neq 1$, *where*
$a_0=a$; $a_{1,0}=a'$; $a_{m+1}=a_m+Ia_{1,m}$; $a_{1,m+1}=a_{1,m}+Ia_m$.

THEOREM 2 - *The theory of* CA's *has no model companion.*

Proof: If the models of T^+ are exactly the e.c. CA's, then, by the corollary:
$T^+ + \{a_m\neq 1 | m\epsilon\omega\} \vdash Ex\neq 0 \quad Ey\neq 0 \quad Q(x,y)=0$,
and by compactness there exists a natural number p such that:
$T^+ + a_p\neq 1 \vdash Ex=0 \quad Ey\neq 0 \quad Q\neq 0$, then, again by Cor.1:
$T^+ + a_p\neq 1 \vdash a_{p+1}\neq 1$,
absurd, as one can find, for each p, a CA with an element a such that $a_p\neq 1$ and $a_{p+1}=1$ (take the topological space $\{-p,-p+1,\dots,p-1,p\}$ with all intervals containing 0 as open subsets, and let a be $\{1,3,5,\dots,-2,-4,\dots\}$).

THEOREM 3 - *The theories of* V_α-, V- *and* V_D-*algebras have no model companion.*

Proof: V_α-algebras: same as the proof of th.2, using remark 2 of §2.
V- and V_D-algebras: if a is an element of a V- or V_D-algebra, then $\forall m\ K^{(m)}a\neq 1$ iff in an extension there is a $c\geqslant a$ such that $c\neq 1$ and $Kc=c$.
The proof of the theorem now goes as the one of th.2.

§ 4 - OTHER RESULTS.

THEOREM 4 - *The theories of* F-, V-, V_D-, V_α *and closure algebras have the* a.p.

Proof: If $\mathscr{C}$ is a substructure of both $\mathscr{A}$ and $\mathscr{B}$, take a complete extension $\mathscr{D}$ of the BA $\mathscr{A} *_{\mathscr{C}} \mathscr{B}$ and put:

$$Ha=\begin{cases} K_{\mathscr{A}}a & \text{if } a\epsilon A \\ K_{\mathscr{B}}a & \text{if } a\epsilon B \end{cases}$$

then H is a function from A U B to D, and satisfies the hypothesis of prop.3 (use prop.2). Extend H to an operator of the desired regularity and the theorem is proved.

THEOREM 5 - *The theory of* F-*algebras has model-completion (i.e. model-companion + the a.p.).*

Proof: the following is a set of axioms for the class of e.c. F-algebras:

$$\forall\bar{y}\{[\bar{y}=0 \ \& \bigwedge_{ijst} (\Phi_i^{2n}(\bar{y}) \ \& \ \Phi_{ij}^{2n}(\bar{y}) \ \& \ (si=1 \ \& \ ti=0 \rightarrow y_{st}=1))] \rightarrow E\bar{x}(P(\bar{y},\bar{x})=0 \ \& \bigwedge_i x_i=0)\}$$

for each P. This is proved as in th.3, putting $w_{st}(1,\bar{a})=1_{st}$; the axiomatizability is due to the fact that here no infinite sequence of $\bar{w}$'s is needed.

Open problems:

i) Is it possible to reduce the number of polynomials to be considered in th.2?

ii) Is the class of algebraically closed CA's non-axiomatizable, too? (where algebraic closedness is defined similarly to e.c., admitting only equations)

iii) Which results are true if one considers only particular classes of CA's(e.g. connected CA's etc.etc.)?

iv) Study the finitely and infinitely generic CA's (cf. |3|).

v) Does all above hold for closure (distributive) lattices? (During the acceptation time of this paper I proved the answer is yes).

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Pervenuto il 29/10/80