

# Isospin-0 $\pi\pi$ scattering from twisted mass lattice QCD

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We present results for the isospin-0  $\pi\pi$  s-wave scattering length calculated in twisted mass lattice QCD. We use three  $N_f = 2$  ensembles with unitary pion mass at its physical value, 240 MeV and 330 MeV respectively. We also use a large set of  $N_f = 2 + 1 + 1$  ensembles with unitary pion masses varying in the range of 230 MeV - 510 MeV at three different values of the lattice spacing. A mixed action approach with the Osterwalder-Seiler action in the valence sector is adopted to circumvent the complications arising from isospin symmetry breaking of the twisted mass quark action. Due to the relatively large lattice artefacts in the  $N_f = 2 + 1 + 1$  ensembles, we do not present the scattering lengths for these ensembles. Instead, taking the advantage of the many different pion masses of these ensembles, we qualitatively discuss the pion mass dependence of the scattering properties of this channel based on the results from the  $N_f = 2 + 1 + 1$  ensembles. The scattering length is computed for the  $N_f = 2$  ensembles and the chiral extrapolation is performed. At the physical pion mass, our result  $M_\pi a_0^{I=0} = 0.198(9)(6)$  agrees reasonably well with various experimental measurements and theoretical predictions.

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## 1. Introduction

Elastic  $\pi\pi$  scattering is a fundamental QCD process at low energies. It provides an ideal testing ground for the mechanism of chiral symmetry breaking. The isospin-0  $\pi\pi$  scattering is particularly interesting because it accommodates the lowest resonance in QCD – the mysterious  $\sigma$  or  $f_0(500)$  scalar meson. Studying this channel in lattice QCD is difficult mainly due to the fermionic disconnected diagrams contributing to the isospin-0  $\pi\pi$  correlation function. To date there are only two full lattice QCD computations dedicated to this channel [1, 2]. In this work we compute the scattering length of the isospin-0  $\pi\pi$  channel in twisted mass lattice QCD [3]. We use a mixed action approach with the Osterwalder-Seiler (OS) action [4] in the valence sector to circumvent the complications arising from the isospin symmetry breaking of the twisted mass quark action.

## 2. Lattice setup

The results presented in this paper are based on the gauge configurations generated by the European Twisted Mass Collaboration (ETMC). We use three  $N_f = 2$  ensembles with Wilson clover twisted mass quark action at maximal twist [3]. The pion masses of the three ensembles are at the physical value, 240 MeV and 330 MeV, respectively. The lattice spacing is  $a = 0.0931(2)$  fm for all three ensembles. More details about these ensembles are presented in Ref. [5]. In addition, we use a set of  $N_f = 2 + 1 + 1$  ensembles with Wilson twisted mass quark action with pion masses varying in the range of 230 MeV - 510 MeV at three different values of the lattice spacing [6, 7]. We follow the notation in these references and denote the ensembles as A, B, and D ensembles with lattice spacing values  $a_A = 0.0863(4)$  fm,  $a_B = 0.0779(4)$  fm and  $a_D = 0.0607(2)$  fm, respectively.

In Table 1 we list all ensembles used in this study with the relevant input parameters, the lattice volume and the number of configurations.

In the valence sector we introduce quarks in the so-called Osterwalder-Seiler (OS) discretisation [4]. The OS *up* and *down* quarks have explicit SU(2) isospin symmetry if the proper parameters of the actions are chosen. The matching of OS to unitary actions is performed by matching the quark mass values. Masses computed with OS valence quarks differ from those computed with twisted mass valence quarks by lattice artefact of  $\mathcal{O}(a^2)$ , in particular  $(M_\pi^{\text{OS}})^2 - (M_\pi)^2 = \mathcal{O}(a^2)$ . For twisted clover fermions this difference is much reduced as compared to twisted mass fermions [5], however, the effect is still sizable. We use the OS pion mass in this paper, with the consequence that the pion mass values of all ensembles are higher than the values measured in the unitary theory.

As a smearing scheme we use the stochastic Laplacian Heavyside (sLapH) method [8, 9] for our computation. The details of the sLapH parameter choices for the  $N_f = 2 + 1 + 1$  Wilson twisted mass ensembles are given in Ref. [10]. The parameters for the  $N_f = 2$  ensembles are the same as those for  $N_f = 2 + 1 + 1$  ensembles with the corresponding lattice volume.

## 3. Lüscher's finite volume method

Lüscher showed that the infinite volume scattering parameters can be related to the discrete spectrum of the eigenstates in a finite-volume box [11, 12]. In the case of s-wave elastic scattering,

ensemble	$\beta$	$c_{\text{sw}}$	$a\mu_\ell$	$a\mu_\sigma$	$a\mu_\delta$	$(L/a)^3 \times T/a$	$N_{\text{conf}}$
<i>cA2.09.48</i>	2.10	1.57551	0.009	-	-	$48^3 \times 96$	615
<i>cA2.30.48</i>	2.10	1.57551	0.030	-	-	$48^3 \times 96$	352
<i>cA2.60.32</i>	2.10	1.57551	0.060	-	-	$32^3 \times 64$	337
<i>A30.32</i>	1.90	-	0.0030	0.150	0.190	$32^3 \times 64$	274
<i>A40.24</i>	1.90	-	0.0040	0.150	0.190	$24^3 \times 48$	1017
<i>A40.32</i>	1.90	-	0.0040	0.150	0.190	$32^3 \times 64$	251
<i>A60.24</i>	1.90	-	0.0060	0.150	0.190	$24^3 \times 48$	314
<i>A80.24</i>	1.90	-	0.0080	0.150	0.190	$24^3 \times 48$	307
<i>A100.24</i>	1.90	-	0.0100	0.150	0.190	$24^3 \times 48$	313
<i>B25.32</i>	1.95	-	0.0025	0.135	0.170	$32^3 \times 64$	201
<i>B55.32</i>	1.95	-	0.0055	0.135	0.170	$32^3 \times 64$	311
<i>B85.24</i>	1.95	-	0.0085	0.135	0.170	$32^3 \times 64$	296
<i>D15.48</i>	2.10	-	0.0015	0.120	0.1385	$48^3 \times 96$	313
<i>D30.48</i>	2.10	-	0.0030	0.120	0.1385	$48^3 \times 96$	198
<i>D45.32<sub>sc</sub></i>	2.10	-	0.0045	0.0937	0.1077	$32^3 \times 64$	301

**Table 1:** The gauge ensembles used in this study. The labelling of the ensembles follows the notations in Ref. [5, 6]. In addition to the relevant input parameters we give the lattice volume  $(L/a)^3 \times T/a$  and the number of evaluated configurations  $N_{\text{conf}}$ .

Lüscher's formula reads:  $q \cot \delta_0(k) = \mathcal{Z}_{00}(1; q^2)/\pi^{3/2}$ , where  $k$  is the scattering momentum and  $q$  is a dimensionless variable defined via  $q = kL/2\pi$ .  $\mathcal{Z}_{00}(1; q^2)$  is the Lüscher zeta-function which can be evaluated numerically given the value of  $q^2$ . Using the effective range expansion of s-wave elastic scattering near threshold, we have  $k \cot \delta_0(k) = \frac{1}{a_0} + \frac{1}{2}r_0k^2 + \mathcal{O}(k^4)$ , where  $a_0$  is the scattering length and  $r_0$  is the effective range parameter. Once the isospin-0  $\pi\pi$  interacting energy  $E_{\pi\pi}$  is determined from lattice QCD simulations, the scattering length  $a_0$  can be calculated from the following relation

$$\frac{2\pi}{L} \frac{\mathcal{Z}_{00}(1; q^2)}{\pi^{3/2}} = \frac{1}{a_0} + \frac{1}{2}r_0k^2 + \mathcal{O}(k^4). \quad (3.1)$$

#### 4. Finite volume spectrum

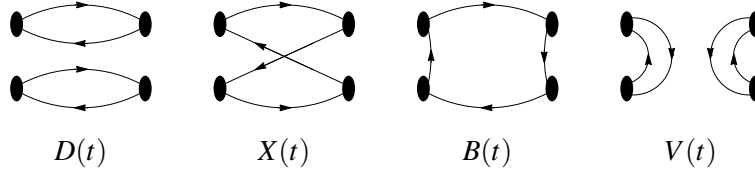
The discrete spectra of hadronic states are extracted from the correlation functions of the interpolating operators that resemble the states. We define the interpolating operator that represents the isospin-0  $\pi\pi$  state in terms of OS valence quarks

$$\mathcal{O}_{\pi\pi}^{I=0}(t) = \frac{1}{\sqrt{3}}(\pi^+\pi^-(t) + \pi^-\pi^+(t) + \pi^0\pi^0(t)), \quad (4.1)$$

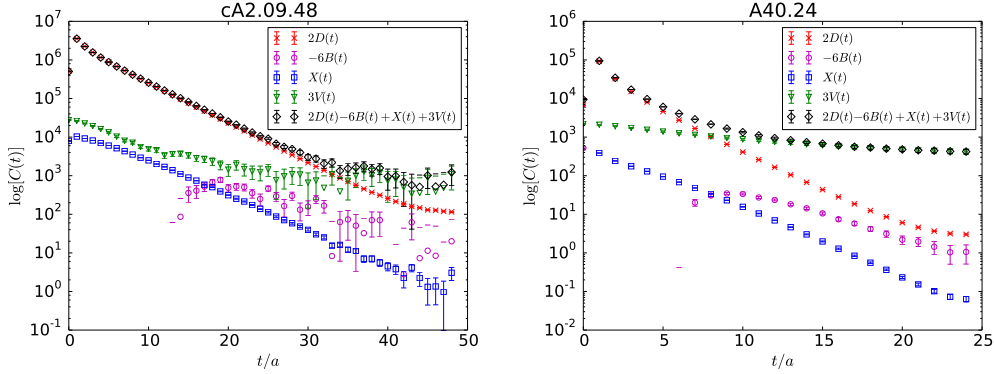
with single pion operators summed over spatial coordinates  $\mathbf{x}$  to project to zero momentum

$$\pi^+(t) = \sum_{\mathbf{x}} \bar{d}\gamma_5 u(\mathbf{x}, t), \pi^-(t) = \sum_{\mathbf{x}} \bar{u}\gamma_5 d(\mathbf{x}, t), \pi^0(t) = \sum_{\mathbf{x}} \frac{1}{\sqrt{2}}(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d)(\mathbf{x}, t). \quad (4.2)$$

Here  $u$  and  $d$  represent the OS *up* and *down* quarks, respectively. With OS valence quarks all three pions are mass degenerate and will be denoted as  $M_\pi^{\text{OS}}$ .



**Figure 1:** Diagrams contributing to the correlation function  $C_{\pi\pi}(t)$ .



**Figure 2:** Correlation functions of the operator  $\mathcal{O}_{\pi\pi}^{I=0}$  and the single diagrams  $D, X, B, V$  for the ensembles  $cA2.09.48$  and  $A40.24$ .

The energy of the isospin-0  $\pi\pi$  state can be computed from the exponential decay in time of the correlation function  $C_{\pi\pi}(t) = \frac{1}{T} \sum_{t_{src}=0}^{T-1} \langle \mathcal{O}_{\pi\pi}^{I=0}(t+t_{src}) (\mathcal{O}_{\pi\pi}^{I=0})^\dagger(t_{src}) \rangle$ , where  $T$  is the temporal lattice extend. The four diagrams contributing to this correlation function, namely the direct connected diagram  $D(t)$ , the cross diagram  $X(t)$ , the box diagram  $B(t)$  and the vacuum diagram  $V(t)$ , are depicted in Fig. 1. The correlation function can be expressed in terms of all relevant diagrams as  $C_{\pi\pi}(t) = 2D(t) + X(t) - 6B(t) + 3V(t)$ .  $C_{\pi\pi}$  and the contributions from individual diagrams  $D, X, B$  and  $V$  are plotted in Fig. 2 for the ensembles  $cA2.09.48$  and  $A40.24$  as examples.

Even though we have full  $SU(2)$  isospin symmetry in the valence sector when using OS valence quarks as described above, we have to consider effects of unitarity breaking. In particular, the  $n(n \geq 1)$  unitary neutral pion states can mix with the operator  $\mathcal{O}_{\pi\pi}^{I=0}$  via the vacuum diagram  $V$ . Since the neutral pion is the lightest meson in the spectrum with Wilson twisted mass fermions at finite lattice spacing, the appearance of such states with  $n = 1$  (and maybe  $n = 2$ ) will dominate the large Euclidean time behaviour of the correlation function  $C_{\pi\pi}$ . The effect of this mixing can be clearly seen in the plot of the correlation function  $C_{\pi\pi}(t)$  for the ensemble  $A40.24$  (the right panel of Fig. 2), in which the vacuum diagram starts to dominate  $C_{\pi\pi}(t)$  at around  $t = 10$ . While for the ensemble  $cA2.09.48$ , this effect is much less prominent. Please note that this mixing is purely a lattice artefact. Since the lattice artefacts of the twisted clover action are much smaller than of the twisted mass action [5], we expect that the effect of the unitary neutral pion mixing is much smaller for the  $N_f = 2$  ensembles compared to the  $N_f = 2 + 1 + 1$  ensembles. This is indeed what we observed generally for the  $N_f = 2$  ensembles and the  $N_f = 2 + 1 + 1$  ensembles used in this work. In order to resolve this mixing, we build a  $2 \times 2$  matrix of correlation functions  $C_{ij}(t) = \frac{1}{T} \sum_{t_{src}=0}^{T-1} \langle \mathcal{O}_i(t+t_{src}) \mathcal{O}_j^\dagger(t_{src}) \rangle$ , with  $i, j$  labeling the operator  $\mathcal{O}_{\pi\pi}^{I=0}$  and the unitary neu-

Ensemble	$aM_\pi^{\text{OS}}$	$aE_{\pi\pi}$	Ensemble	$aM_\pi^{\text{OS}}$	$aE_{\pi\pi}$
<i>cA2.09.48</i>	0.11985(15)	0.2356(4)	<i>B85.24</i>	0.2434(6)	0.4441(29)
<i>cA2.30.48</i>	0.15214(11)	0.3010(3)	<i>A30.32</i>	0.2143(10)	0.4043(36)
<i>cA2.60.32</i>	0.18844(24)	0.3647(5)	<i>A40.24</i>	0.2283(10)	0.4187(40)
<i>D15.48</i>	0.1082(3)	0.2067(13)	<i>A40.32</i>	0.2266(8)	0.4417(20)
<i>D30.48</i>	0.1299(2)	0.2526(8)	<i>A60.24</i>	0.2482(9)	0.4644(39)
<i>D45.32</i>	0.1466(6)	0.2686(17)	<i>A80.24</i>	0.2663(7)	0.5033(21)
<i>B25.32</i>	0.1843(13)	0.3496(31)	<i>A100.24</i>	0.2835(6)	0.5096(55)
<i>B55.32</i>	0.2105(4)	0.4051(23)			

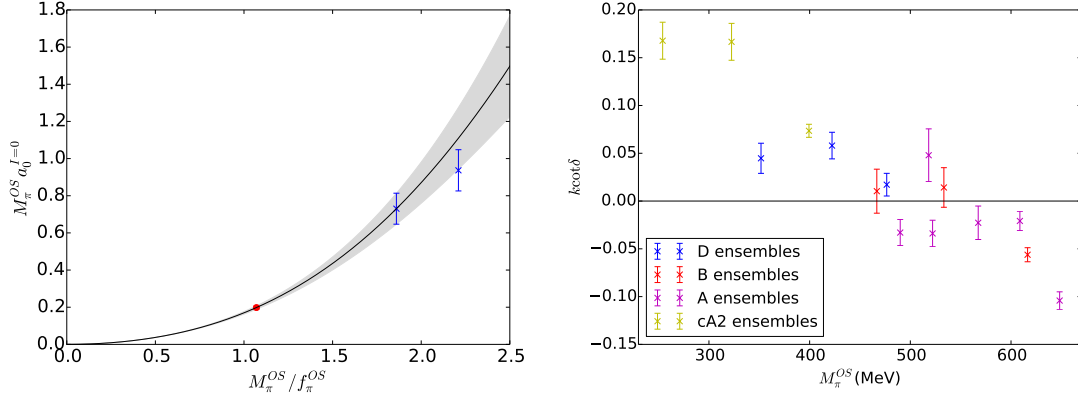
**Table 2:** OS pion masses and the  $\pi\pi$  interacting energies in lattice units for the three ensembles.

tral pion operator  $\pi^{0,uni}(t) = \sum_{\mathbf{x}} \frac{1}{\sqrt{2}} (\bar{u}\gamma_5 u - \bar{d}'\gamma_5 d')$ ( $\mathbf{x}, t$ ), where  $u$  and  $d'$  are the (unitary) Wilson (clover) twisted mass  $up$  and  $down$  quarks. We use  $d'$  to distinguish it from OS  $down$  quark in Eq. 4.2. The twisted mass  $up$  quark coincides with the OS  $up$  quark with our matching scheme of the OS to the unitary action.

We use a shifting procedure  $\tilde{C}_{ij}(t) = C_{ij}(t) - C_{ij}(t+1)$  to eliminate contaminations constant in time from so-called thermal states due to the finite time extension of the lattice. Solving the generalized eigenvalue problem(GEVP)  $\tilde{C}(t)v(t, t_0) = \lambda(t, t_0)\tilde{C}(t_0)v(t, t_0)$ , the desired energy of the  $\pi\pi$  isospin-0 system  $E_{\pi\pi}$  can be extracted from the exponential decay of the eigenvalues  $\lambda(t, t_0)$ . To further improve our results we adopt a method to remove excited state contaminations, which we have recently used successfully to study  $\eta$  and  $\eta'$  mesons [13, 14]. See Ref. [15] for more details about this method. In Table 2, we collect the values of  $E_{\pi\pi}$  obtained from the procedure described above. The OS pion masses  $M_\pi^{\text{OS}}$  are also given since they will be needed to compute the scattering length.

## 5. Results

The scattering momentum  $k^2$  is calculated from the energies  $E_{\pi\pi}$  and the OS pion masses listed in Table 2. Then the scattering length can be obtained from Eq. 3.1. Using the values of the effective range  $r_0$  determined from  $\chi$ PT [16, 15], we investigated the contribution of  $\mathcal{O}(k^2)$  term in the effective range expansion. For the ensembles *cA2.09.48* and *cA2.30.48*, the value of  $\frac{1}{2}r_0k^2$  is less than 3% of  $k \cot \delta(k)$ . So we can safely ignore the  $\mathcal{O}(k^4)$  term and compute the scattering length  $a_0^{\text{I}=0}$  using Eq. 3.1 for these two ensembles. The values of  $M_\pi^{\text{OS}}a_0^{\text{I}=0}$  are plotted in Fig. 3(a) as a function of  $M_\pi^{\text{OS}}/f_\pi^{\text{OS}}$ . For the ensemble *cA2.60.32*, the contribution of  $\frac{1}{2}r_0k^2$  is rather large – around 30% of  $k \cot \delta(k)$ . Since the contribution of  $\mathcal{O}(k^4)$  is unclear, we refrain from giving the scattering length for this ensemble. The reason for the invalidity of the effective range expansion is probably due to virtual or bound state poles appearing in the isospin-0  $\pi\pi$  scattering amplitude at the pion mass around 400 MeV, which is the OS pion mass of the ensemble *cA2.60.32*. Since the OS pion mass for the  $N_f = 2 + 1 + 1$  ensembles are generally above 400 MeV, we do not compute the scattering length for these ensembles either. However, the value of  $k \cot \delta(k)$  can be computed up to lattice artefacts. Fig 3(b) presents the values of  $k \cot \delta(k)$  for all ensembles as a function of  $M_\pi^{\text{OS}}$ . One can see that  $k \cot \delta(k)$  changes from positive to negative with increasing OS



**Figure 3:** (a): The values of  $M_\pi^{OS} a_0^{I=0}$  for the ensembles  $cA2.09.48$  and  $cA2.60.32$ . The black curve and the grey band represent the chiral fit using only the data point with lower pion mass. The red point indicates the extrapolated value at physical pion mass. (b): The values of  $k \cot \delta(k)$  for all ensembles as a function of  $M_\pi^{OS}$ .

pion mass. The pion mass range where the sign change happens is around 400 MeV - 600 MeV. Correspondingly, the scattering length will change from positive infinity to negative infinity in this range, which indicates the emergence of virtual or bound state poles in the scattering amplitude.

For the  $N_f = 2$  ensembles, chiral extrapolation is performed in order to obtain the scattering length at the physical pion mass. Since we only have two data points, we fit the NLO  $\chi$ PT formula, which contains one free parameter, to our data. The method we are applying here is valid only in the elastic region. Therefore, the pion mass values must be small enough to be below threshold where the  $\sigma$  meson becomes stable. Furthermore, the pion mass value should also be small enough to make the chiral expansion valid. To be safe, we perform the chiral extrapolation using only the data point with the lower pion mass (250 MeV). The fit results using the two data points are used to estimate the systematics arising from chiral extrapolation. This leads to our final result for the scattering length:

$$M_\pi a_0^{I=0} = 0.198(9)_{\text{stat}}(6)_{\text{sys}}. \quad (5.1)$$

## 6. Summary and discussions

The isospin-0  $\pi\pi$  scattering is studied with Lüscher's finite volume formalism in twisted mass lattice QCD using a mixed action approach with the OS action in the valence sector. The lowest energy level in the rest frame is extracted for three  $N_f = 2$  ensembles and a large set of  $N_f = 2 + 1 + 1$  ensembles with many different values of pion mass. The scattering length is computed for the two  $N_f = 2$  ensembles with the lowest pion mass values. After the chiral extrapolation, our result at the physical pion mass is  $M_\pi a_0^{I=0} = 0.198(9)(6)$ , which is compatible with the newer experimental and theoretical determinations available in the literature. The value of  $k \cot \delta(k)$  near threshold is computed for all ensembles. The pion mass dependence of the scattering properties of this channel is briefly discussed. We cannot exclude that our result is affected by residual systematic uncertainties stemming from unitarity breaking, which will vanish in the continuum limit. In order

to avoid isospin breaking and unitarity breaking effects, we will repeat this computation with an action without isospin breaking.

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