

Fragility and anomalous susceptibility of weakly interacting networks

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Percolation is a fundamental concept that brought new understanding on the robustness properties of complex systems. Here we consider percolation on weakly interacting networks, that is, network layers coupled together by much less interlinks than the connections within each layer. For these kinds of structures, both continuous and abrupt phase transition are observed in the size of the giant component. The continuous (second-order) transition corresponds to the formation of a giant cluster inside one layer, and has a well defined percolation threshold. The abrupt transition instead corresponds to the merger of coexisting giant clusters among different layers, and is characterised by a remarkable uncertainty in the percolation threshold, which in turns causes an anomalous trend in the observed susceptibility. We develop a simple mathematical model able to describe this phenomenon and to estimate the critical threshold for which the abrupt transition is more likely to occur. Remarkably, finite-size scaling analysis in the abrupt region supports the hypothesis of a genuine first-order phase transition.

Percolation theory is a very successful framework for understanding a broad range of critical phenomena taking place on networks, such as robustness to failures or attacks and spreading of diseases or information, and for unveiling the common principles underlying these processes [1, 2]. In this context, multilayer networks have been shown to exhibit critical percolation properties which are different from what is observed for a single isolated network—namely, a single continuous phase transition [3, 4] whose properties depend on the kind of process [5] and on the network features [6, 7]. Indeed, the presence of interconnections between the network layers can give rise to supercritical phenomena such as abrupt or multiple phase transitions. Discontinuous percolation transitions have been extensively reported in the case of interdependent networks—that is, two (or more) networks whose nodes are interconnected by dependency links, such that the removal of a node in a network causes the instantaneous removal of the dependent nodes in the other networks (see for instance [8–10]). Our focus here is instead on interacting networks (or network of networks), in which the connections between the network layers are ordinary links that thus take part in the percolation process. A system of this kind is therefore equivalent to a single modular network, characterised by a percolation threshold that is typically lower than in homogeneous networks—with a giant cluster appearing for a smaller total number of links [11]. A case of particular interest arises when the interaction between the network layers is weak, meaning that there is a sufficiently small number of interlinks between network layers, so that the removal of a few of them can easily separate the network layers into isolated modules [12]. This setup is common for neural systems, and therefore of major relevance to understand the resilience of neural processing [13]. Weakly interact-

ing networks are characterised by a mixed percolation phase, in which only one or some of the network layers do percolate [14, 15]. In particular, Colomer-de-Simón & Boguñá [16] identified multiple percolation transitions when the coupling between the different layers vanishes in the thermodynamic limit. In order to account for the emergence of coexisting percolating clusters, Faqeeh *et al.* [17] developed a modular message passing approach. In any event, the appearance of these coexisting clusters in weakly interacting networks is a fundamental source of error for percolation theory. In this work we develop a simple mathematical framework that allows estimating the most likely critical threshold at which the merging of coexisting clusters occurs in weakly interacting networks. Moreover, we characterise the percolation process in terms of a powder keg: due to the scarcity of the interlinks, the aggregation of the coexisting giant clusters is delayed, therefore giving rise to an abrupt percolation transition.

RESULTS

The intrinsic powder keg of weakly interacting networks

To illustrate the percolation properties of weakly interacting networks, we consider as in Figure 1 two layers A (with N_A nodes and average degree k_A) and B (with N_B nodes and average degree k_B), that are interconnected by a small number I of links ($I \ll \min\{N_A k_A, N_B k_B\}$). The bond percolation process consists in retaining each link of the system with occupation probability f and otherwise removing it. To simulate the process, we use the method proposed by Newman and Ziff [18]: for each realisation, we start from a system configuration with no

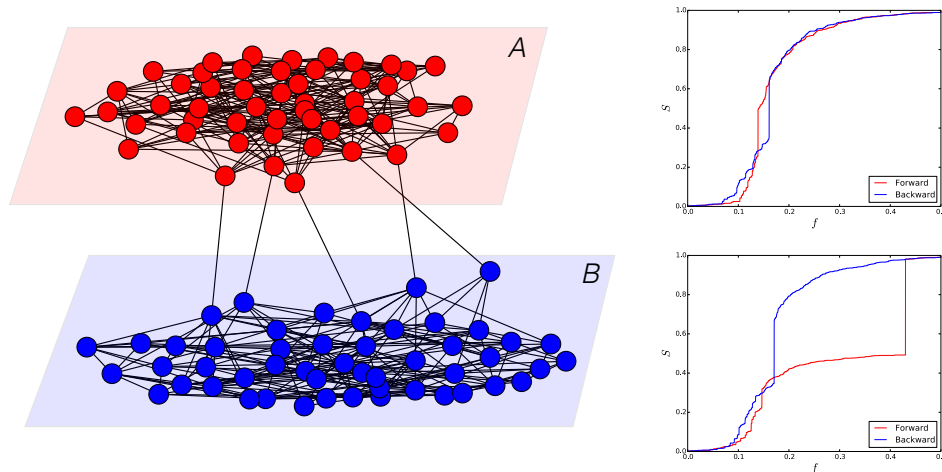


FIG. 1. **Single realisation of bond percolation on a weakly interacting network.** Left panel: pictorial representation of two weakly interacting networks A and B , in which the interconnection links I are much less than the intra-layer links. Right panel: two different instances of the percolation process on an interacting network composed by two Erdős-Rényi layers ($N = 500$ nodes and average degree $k = 10$ each) connected by $I = 5$ interconnection links. Each realisation is obtained as follows. Starting from an empty network, links are first randomly added (*forward*) up to half the total number of links, and then randomly removed (*backward*) until the network is empty again. The hysteresis cycle appearing in both cases are remarkably different, because of the large variability of the percolation threshold.

connections, and then sequentially add links in a random order. f is thus the fraction of links added to the system. In such a situation, we may observe large jumps for the order parameter S , that is, the size of the giant cluster spanning both layers. These jumps can be understood as resulting from the addition of one of the I interlinks *after* the formation of the two giant clusters S_A and S_B of layer A and B , respectively. Indeed, differently from what happens for standard percolation, when such interlink is about to be added the two giant clusters already contain a number of nodes that is proportional to the system size. According to the definition of Friedman *et al.* [19], this configuration corresponds to a *powder keg*, which is “ignited” as soon as that interconnection is added causing a discontinuous percolation transition. Note that if a system is initialised as a powder keg, then even a random link addition rule causes a discontinuous transition: as in our case, the formation of the giant cluster spanning both layers is not hindered by specific link selection rules [20–24], but is naturally delayed by the structure of the interconnections itself. However, the absence of any particular link selection criteria causes a large uncertainty for the percolation threshold.

Model for the anomalous susceptibility and percolation threshold

In order to gain a more quantitative insight on the described phenomenology, we start by defining the probability P_I that at least one of the I interconnections is added and actually connects the two giant clusters S_A

and S_B [17]:

$$P_I = 1 - \left[1 - \left(\frac{N}{N_A} S_A \right) \left(\frac{N}{N_B} S_B \right) f \right]^I = 1 - \left[1 - \frac{N}{\mu} S_A S_B f \right]^I, \quad (1)$$

where $N = N_A + N_B$ is the total number of nodes, the normalisation coefficients before S_A and S_B respectively denote their maximum size N_A/N and N_B/N , and $\mu = \frac{N_A N_B}{N_A + N_B}$ is the *reduced* number of nodes (equivalently to the concept of reduced mass for the classical two-body problem). Without loss of generality, we set the percolating thresholds f_A and f_B of the individual layers A and B respectively such that $f_A < f_B$ (the degenerate case $f_A = f_B$ is reported below and discussed in the Methods section). This implies that on average and for layers of the same nature we have $S_A > S_B$ for any given value of f such that both clusters exist. Hence, for $f > f_B$, the percolation cluster S of the whole system is either that of layer A if S_A and S_B are not connected, or abruptly jumps to $S_A + S_B$ provided that S_A and S_B are connected—which happens with probability P_I . In formulas,

$$S = \begin{cases} S_A + S_B & \text{with probability } P_I \\ S_A & \text{otherwise} \end{cases} \quad (2)$$

Overall, we have a first continuous transition at $f_1 = f_A$ (the standard percolation transition when layer A percolates), and a second discontinuous transition at f_2 when layer B percolates and at least one active interconnection is established between the two layers. Yet, because of the dichotomy characterising the outcome of the process for $f \simeq f_2$, the average value $\langle S \rangle = S_A + S_B P_I$

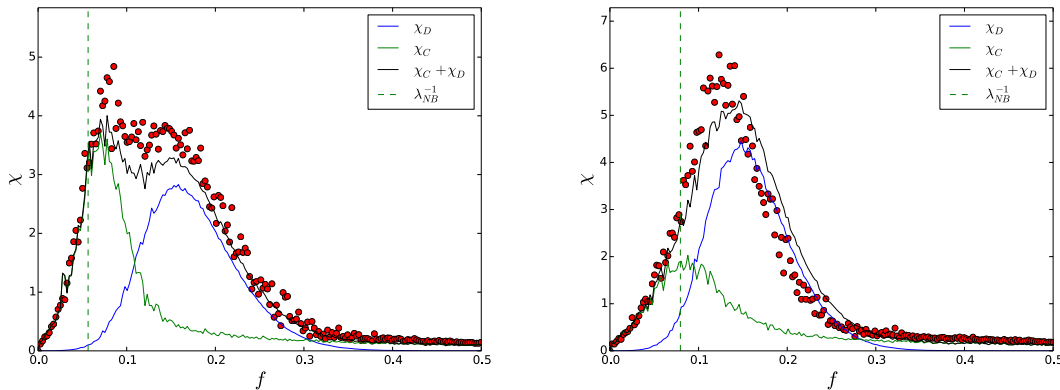


FIG. 2. **Susceptibility of air transportation networks.** We consider the duplex formed by the transportation network Lufthansa-Ryanair (left panel) and Lufthansa-Easyjet (right panel) [25], in which each layer is made up of airports (nodes) and flight routes (links) operated by a company. The layers are characterised by $N_{LH} = 106$ and $\langle k \rangle_{LH} = 4.604$, $N_{FR} = 128$ and $\langle k \rangle_{FR} = 9.391$, $N_{U2} = 99$ and $\langle k \rangle_{U2} = 6.202$. The interconnection links in each case are the airports in which both companies operate: we have $I_{LH-FR} = 36$ and $I_{LH-U2} = 51$. Red dots denote numerically computed values of χ from 400 realisations of the bond percolation process, χ_D is given by eq. (3) and χ_C is the susceptibility of the corresponding non-interacting system. λ_{NB} is the leading eigenvalue of the non-backtracking matrix of the network, whose inverse is a good approximation for the percolation threshold of sparse networks [26].

is not representative at all of what happens in the system. We thus study the behaviour of the susceptibility $\chi = N\text{Var}(S)/\langle S \rangle$ [16]. For $f > f_B$ each layer has its own percolating cluster, and thus the only contribution to χ comes from the Bernoulli trial described by eq. (2):

$$\chi_D = N \frac{S_B^2 P_I (1 - P_I)}{S_A + S_B P_I}. \quad (3)$$

Note that χ_D gives a non-vanishing contribution to the total susceptibility χ only in the weakly interacting regime, that is, when the Bernoulli trial of eq. (2) is not trivial. Indeed, eq. (1) tells us that for $I \rightarrow 0$ (as for the case of disconnected layers) we have $P_I \rightarrow 0$, and when I is very large (as is the case of strongly connected layers, see the Methods section) we have $P_I \rightarrow 1$. In both cases $\chi_D \rightarrow 0$. For fixed I , however, χ_D achieves its maximum for the value of f which maximises the uncertainty of the Bernoulli trial, at which the discontinuous jump of S is more likely to occur. We thus identify f_2 with the value for f which maximises χ_D .

Real and artificial networks

These simple mathematical arguments are indeed able to capture the behaviour of the susceptibility both in real and model networks. We first consider in Figure (2) the duplex (two-layer multiplex) formed by a pair of coupled air transportation networks, where each layer consists of the airports (nodes) and flight routes (links) operated by a given company, and the interlinks are the airports served by both companies. We see that the susceptibility of the two individual layers χ_C cannot capture the

observed behaviour of χ computed numerically. The difference between χ and χ_C is instead very well represented by χ_D .

A more precise assessment of our methodology is given by considering two Erdős-Rényi weakly interacting networks with the same number of nodes $N_A = N_B$ and average degrees k_A and k_B . In this case it is possible to derive an analytic approximation for f_2 , since S_A and S_B have a known analytic form in the thermodynamic limit. We get (see the Methods section for details):

$$f_2 = \frac{1 - 2^{-1/I}}{1 - \exp[-k_B(1 - 2^{-1/I})]}. \quad (4)$$

The specific case $k_A = k_B = k$ leads to a more accurate transcendental equation (see again Methods):

$$f_2 = \frac{1 - (2 - \sqrt{2})^{1/I}}{\left\{ 1 - \exp \left[-k \sqrt{f_2} (1 - (2 - \sqrt{2})^{1/I}) \right] \right\}^2}, \quad (5)$$

that can be easily solved numerically. As shown in Figure 3, in the case of two weakly interacting Erdős-Rényi networks with different average degrees, the numerical evaluation of χ_D by means of eq. (3) fits very well the observed anomalous susceptibility, and the numerical solution of eq. (4) gives with good approximation the position of the maximum of χ . In the degenerate case $k_A = k_B$, eq. (5) provides an even better approximation for the maximum of χ . Analysing single realisation of the percolation process, we confirm that f_2 marks the region in which S is subject to discontinuous jumps. However, this discontinuous behaviour is lost by averaging the outcomes of

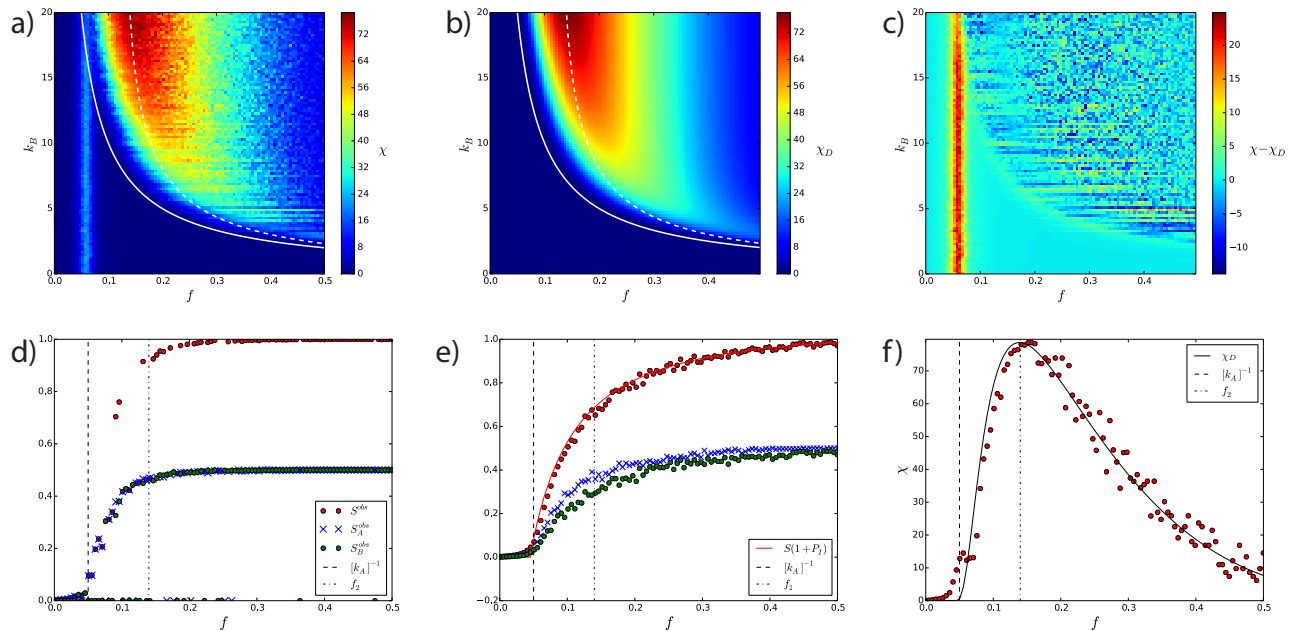


FIG. 3. **Susceptibility of weakly interacting Erdős-Rényi networks.** Panels (a-c) are heat maps of the susceptibility of two Erdős-Rényi coupled networks with $N_A = N_B = 500$ nodes, $k_A = 20$, $I = 5$ and k_B varying from 0 to k_A . Panel (a) reports χ obtained from numerical simulations of the percolation process, whereas, panel (b) reports χ_D from numerical solutions of eq. (3). In both cases, the continuous white line gives the relation $f = k_B^{-1}$ that marks the boundary for the region in which percolating clusters exist in both layers, and the dashed white line reports values of f_2 as given by eq. (4). Panel (c) reports the difference between the two values, which is high only in the vertical strip corresponding to χ_C . Panels (d-f) instead report S and χ for the degenerate case of two Erdős-Rényi interacting layers with $N_A = N_B = 500$ and $k_A = k_B = 20$, and $I = 5$. In all three cases, the dashed vertical line denotes the percolation threshold of the individual layers, whereas, the dashed-dotted line marks f_2 as derived from eq. (5). Panel (d) reports a single realisation of the process and Panel (e) reports averages of the same process over 300 realisations. In both cases, red dots are the observed values of S , blue crosses and green dots are the observed values of S_A and S_B respectively, and the red line gives the numerical estimate of $\langle S \rangle$ derived from eq. (2). Panel (f) finally reports the observed susceptibility (red dots) averaged over 300 realisations of the process, as well as the numerical value of χ_D (continuous black line).

the percolation process over many realisations, for which S becomes $\langle S \rangle = S_A + P_I S_B$ which fails to represent the outcome of the process.

Finite size scaling

We conclude our study with a finite-size scaling analysis carried out for the cases of two coupled Erdős-Rényi layers and two coupled Barabási-Albert layers with different average connectivities. For each of the two settings we considered networks made of two layers of size $N_{A,B}$ equal to 100, 500, 2500, 12500. According to standard percolation theory, the maximum of the susceptibility diverges around the critical value f_c according to the power law $\chi(f_c) \sim N^{1-\beta/\nu}$, while for the relative size of the giant component we have $S(f_c) \sim N^{-\beta/\nu}$ [5, 16]. Our analysis, reported in Figure 4, shows that the scaling properties around f_1 and f_2 are significantly different. While $S(f_1)$ and $\chi(f_1)$ exhibit the usual power law scaling typical of second-order phase transitions (with differ-

ent exponents according to the two different topologies of the network layers), $S(f_2)$ does not scale with N in both examples, which implies $\beta/\nu = 0$. This in turn implies $\chi(f_2) \sim N$. These particular scaling properties, that is the failure of finite size scaling relations and the extensive character of the susceptibility, are a clear trademark of a first order phase transition [27].

CONCLUSION

To sum up, in this work we have studied the bond percolation properties of weakly interacting networks. This class of systems encompass the important cases of multilayer/modular networks with very sparse connections within the layers/modules. We reported the existence of discontinuous jumps in the relative size of the giant component S , happening since the percolating cluster of the sparser layer can give either a full or zero contribution to the giant cluster of the whole system. Furthermore we observed that in this case the abrupt transition

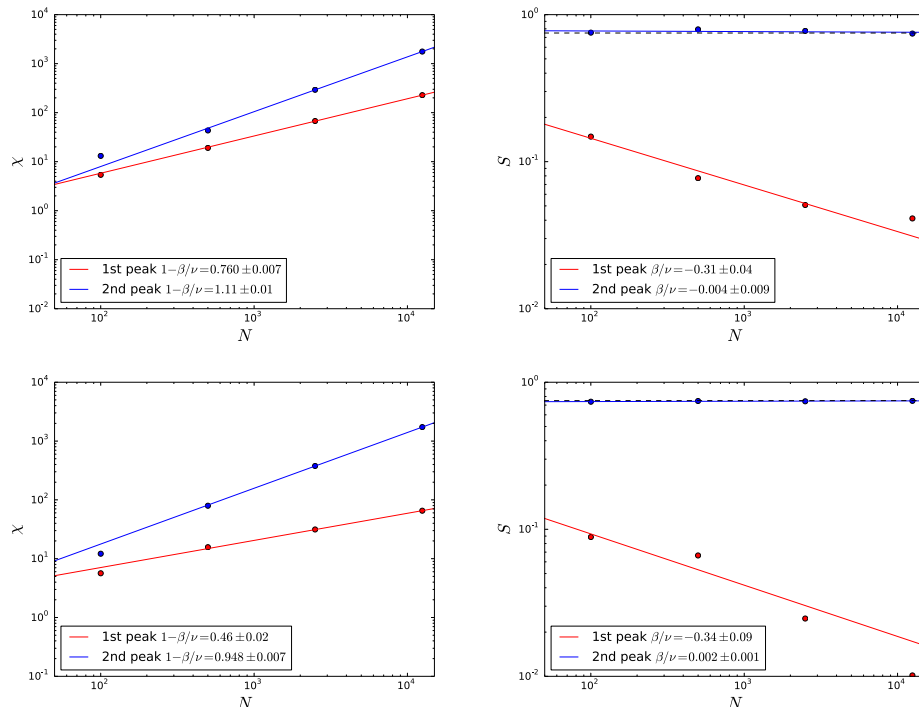


FIG. 4. **Finite size scaling analysis.** Top panels report the case of two weakly interacting Erdős-Rényi layers with $k_A = 20$, $k_B = 10$ and $I = 5$, for different size N . From (b) we see that while $S(f_1)$ shows a power law decay with exponent $\beta/\nu = 0.31 \pm 0.04$ (which is consistent with the mean-field values $\beta = 1$ and $\nu = 3$), $S(f_2)$ does not scale with N . Accordingly to those values, from (a) we can verify the different divergence rates for the two peaks of the susceptibility, and in particular we see that the divergence of $\chi(f_2)$ is almost linear. Bottom panels instead report the case of two weakly interacting Barabási-Albert layers with $m_A = 20$, $m_B = 10$ and $I = 5$, for different size N . Again we see that while $S(f_1)$ and $\chi(f_1)$ show a scaling behaviour ruled by the topology of the layers, $S(f_2)$ and $\chi(f_2)$ show the same behaviour of the Erdős-Rényi case: the one characteristic of first-order phase transitions.

does not have a definite threshold, but can occur for a wide range of values of the bond occupation probability. This causes an anomalous behaviour of the susceptibility, which we captured using simple probabilistic arguments. We successfully tested our predictions in both synthetic and real systems. Finally, from finite-size scaling analysis we showed that the critical behaviour of both S and χ in the abrupt region exhibits the features of a genuine first-order phase transition. Our work can have important applications in characterising the fragility of weakly interacting structures such as multiplex transportation networks, as well as in describing epidemic processes on networks with metapopulation structures [28–32].

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METHODS

In order to derive the analytic approximations presented in both eqs. (4) and (5) in the case of two Erdős-Rényi layers of the same size ($N_A = N_B = N/2$), we start from the implicit form of S_A and S_B in the thermodynamic limit:

$$S_X = \frac{1}{2} (1 - e^{-2fk_X S_X}) \quad (6)$$

with $X = \{A, B\}$. The above expression is obtained from the usual equation for a single Erdős-Rényi network, namely $S = 1 - e^{-fkS}$, using the substitution $S \rightarrow 2S$ (as S_X refers to only one layer with half of the N nodes). We thus obtain the same solution of the single network scaled by a factor $1/2$, as well as the same percolation threshold $f_X = k_X^{-1}$. The value of f which maximises χ_D of eq. (3) for fixed S_A and S_B is given by the following implicit equation

$$[1 - 4f_2 S_A S_B]^I = 1 + \frac{S_A - \sqrt{S_A^2 + S_A S_B}}{S_B}, \quad (7)$$

where both S_A and S_B are functions of f_2 according to eq. (6). Note that eq. (7) returns $f_2 = 1 - (2 - \sqrt{2})^{1/I}$

in the limit $S_{A,B} \rightarrow 1/2$. This regime corresponds to the case $I \ll k_{A,B}$, for which we can safely assume that both layers will fully percolate before the activation of at least one interconnection link as f increases, leading to a value of f_2 which does not depend on k_A nor k_B . Since eq. (7) is difficult to handle, we can approximate f_2 with the values that maximises $\text{Var}(S)$ instead of χ_D . For a Bernoulli trial we simply have $P_I(f_2) = 1/2$, implying $f_2 = [1 - 2^{-1/I}]/[4S_A S_B]$. With the further assumption $S_A \simeq 1/2$ (hence when layer A has already percolated) we have $f_2 = [1 - 2^{-1/I}]/[2S_B]$. Using eq. (6) we finally get the analytic solution presented in eq. (4):

$$f_2 = \frac{1 - 2^{-1/I}}{1 - \exp[-k_B(1 - 2^{-1/I})]}.$$

In the limit $S_B \rightarrow 1/2$ this expression simplifies to $f_2 = 1 - 2^{-1/I}$, which is very close to the value that maximises χ_D in the same regime.

In the degenerate case $k_A = k_B = k$, we have $S_A = S_B = S_X$ which leads to the simpler expressions:

$$\langle S \rangle = S_X(1 + P_I), \quad \chi_D = N \frac{S_X P_I (1 - P_I)}{1 + P_I}, \quad (8)$$

and thus the value of f which maximises χ_D at fixed S_X is simply given by the implicit expression $P_I(f_2) = \sqrt{2} - 1$, implying $f_2 = [1 - (2 - \sqrt{2})^{1/I}]/[4S^2]$. Plugging the latter in eq. (6) yields eq. (5)

$$f_2 = \frac{1 - (2 - \sqrt{2})^{1/I}}{\left\{ 1 - \exp \left[-k \sqrt{f_2} (1 - (2 - \sqrt{2})^{1/I}) \right] \right\}^2}.$$

We finally consider the case of strongly interacting Erdős-Rényi layers, that we define by $I \geq \max\{k_A, k_B\}$. As shown in Figure 5 (where $k_A > k_B$), as soon as $I > k_A$ the height of the second peak drastically decreases, while the corresponding value of f_2 approaches k_B^{-1} , that is, the percolation threshold of the weak layer. The fact that $f_2 \rightarrow k_B^{-1}$ is obtained by taking the limit $I^{-1} \rightarrow 0$ in eq. (4). Indeed, in this regime $P_I \simeq 1$ as soon as the percolating cluster appears in layer B : the process bears no uncertainty related to the interconnections, therefore contribution of χ_D vanishes and χ simply becomes that of the ordinary percolation process for the Erdős-Rényi layer B .

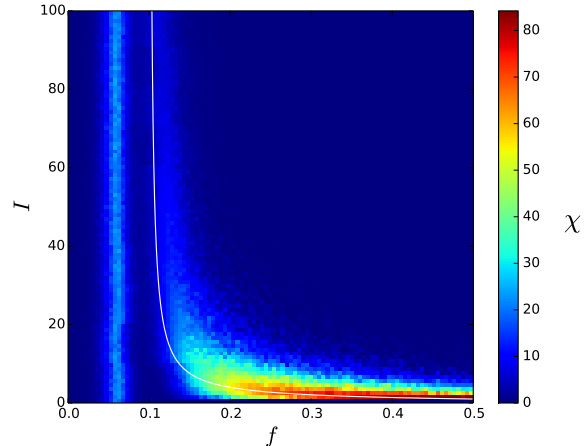


FIG. 5. **The case of strongly connected layers.** Heatmap of the susceptibility χ for two Erdős-Rényi layers of $N = 500$ with $k_A = 20$, $k_B = 10$ and $0 < I < 100$. For every fixed value of I , χ is averaged over 400 realisations of the bond percolation process. The continuous white line represents the theoretical prediction from eq. (4), which for large values of I converges to $k_B^{-1} = 0.1$.

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