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Evolution of convex entire graphs by curvature flows

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Abstract: We consider the evolution of an entire convex graph in euclidean space with speed given by a symmetric function of the principal curvatures. Under suitable assumptions on the speed and on the initial data, we prove that the solution exists for all times and it remains a graph. In addition, after appropriate rescaling, it converges to a homothetically expanding solution of the flow. In this way, we extend to a class of nonlinear speeds the well known results of Ecker and Huisken for the mean curvature flow.

MSC: 53C44

1 Introduction

Let $\mathbf{F}_0 : \mathcal{M} \longrightarrow \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional manifold. We consider the one-parameter family of immersions $\mathbf{F} : \mathcal{M} \times [0, T) \longrightarrow \mathbb{R}^{n+1}$ which satisfy

$$\frac{\partial \mathbf{F}}{\partial t}(\mathbf{p}, t) = -\mathcal{S}(\mathbf{p}, t)\, \nu(\mathbf{p}, t) \tag{1}$$

with initial data

$$\mathbf{F}(\mathbf{p}, 0) = \mathbf{F}_0(\mathbf{p}). \tag{2}$$

Here ν is the unit normal vector field to $M_t := \mathbf{F}_t(\mathcal{M})$, while the speed $\mathcal{S}$ is a smooth, positive symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$. The best known example is the mean curvature flow, corresponding to $\mathcal{S} = \lambda_1 + \cdots + \lambda_n$. In this paper we study the flow in the so-called graphical case, when the hypersurface M_t is an entire graph over $\mathbb{R}^n$ for all t. Our aim is to investigate the existence of solutions and their long time behaviour.

Problem (1)–(2) has been widely studied in the last decades under the assumption that M_0 is a compact convex hypersurface, see e.g. [2, 5, 14] and the references therein. In this case, the flow necessarily becomes singular in finite time and many researchers have studied the behaviour of the hypersurfaces as the maximal time is approached, giving conditions under which they converge to a spherical profile after rescaling.

On the other hand, much less results are known when $\mathcal{M}$ is a complete non compact hypersurface. The mean curvature flow in the graphical case was first considered by Ecker and Huisken. In [9], they assumed that the initial value is the graph of a globally Lipschitz function over $\mathbb{R}^n$ and they proved that, in contrast to the compact case, the solution exists for all times. In addition, assuming a suitable condition on the oscillation at infinity of the initial data, the solution converges for large times to an expanding selfsimilar solution of the flow, up to a rescaling. The same authors [10] showed that global time existence holds for any initial data which is an entire graph over $\mathbb{R}^n$, regardless of its growth. After this, Stavrou [19] proved the convergence to a selfsimilar profile of Lipschitz graphs having a unique cone at infinity, while Rasul [16] obtained a convergence result under a weaker oscillation condition than in [9]. The long time behaviour without rescaling

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has been investigated for special classes of solutions: Clutterbuck, Schnürer and Schulze [8] have proved stability of translating solutions of the graphical mean curvature flow, while Clutterbuck and Schnürer [7] have considered the stability of mean convex cones. We also recall that a result with some analogies to [9] has been obtained by Schulze and Simon [18] for complete solutions of the Ricci flow.

Only in the last years, some authors have investigated the evolution of graphs for some choices of speed $\mathcal{S}$ different from the mean curvature. Schnürer and Urbas [17] have proved long time existence of convex graphs evolving by powers of the Gauss curvature assuming that the initial graph is convex and satisfies a mild condition on the oscillation of the normal. A similar result has been obtained by Franzen [11] for the flow by powers of the mean curvature. Very recently, Holland [13] has considered speeds of the form $\mathcal{S} = E_k^{1/k}$, where E_k is the elementary symmetric polynomial of degree k, proving long time existence of the flow for k-convex initial data. He has also proved convergence to a selfsimilar profile after rescaling if the graphs are Lipschitz.

In this paper, we consider a whole class of speed functions $\mathcal{S}$, which is characterized by some requirements which are listed in Section 2. The main ones are that $\mathcal{S}$ is homogeneous of degree one, is monotone, is concave and inverse-concave on the positive cone. Such a class includes many examples which are obtained from the polynomials E_k, e.g. $\mathcal{S} = E_k/E_{k-1}$ or $\mathcal{S} = (E_k/E_l)^{\frac{1}{k-l}}$. The initial hypersurface is required to be the graph of a convex and globally Lipschitz entire function. Intuitively speaking, such a graph behaves like a cone at infinity. In addition, we assume that the initial graph satisfies a positive lower bound on the quotient $\mathcal{S}/H$, where H is the mean curvature. Under these hypotheses, we obtain results similar to those of Ecker and Huisken for the mean curvature flow. Namely, we show that the flow exists for all times, that M_t is a graph with Lipschitz constant nonincreasing in t, and that, under the same oscillation assumption as in [9], the rescaled flow converges to a selfsimilar expanding solution as time goes to infinity.

We point out that the existence part for a general speed needs different arguments from the case of the mean curvature flow, and even the small time existence does not follow directly from the standard parabolic theory. In fact, the problem usually exhibits some degeneracy or singularity when the curvatures are zero, due to the nonlinearity of $\mathcal{S}$: for instance, if $\mathcal{S}$ is homogeneous of degree one, its derivatives are discontinuous at zero. On an entire convex graph, however, the curvatures cannot be uniformly far from zero everywhere. Therefore, as in [11, 17], we prove existence by an approximating procedure where the initial graph is replaced by a closed uniformly convex hypersurface. We then can employ the existence theory of the compact case. The method yields directly the long time existence of the flow, because the lifespan of the approximating solutions becomes larger and larger. A crucial step in the proof is provided by a Harnack-type estimate due to Andrews [3] which we use to show that the curvature remains strictly positive for positive times on any compact subset of our solution.

The paper is organized as follows. In Section 2 we collect some introductory results, we describe our assumptions on the speed and on the initial data and give examples where they are satisfied. In Section 3 we prove the global existence of solutions to the flow by the method described above. In Section 4 we give estimates on the slope and on the speed of our solution by means of the maximum principle. Finally, in Section 5 we study the long time behaviour of the rescaled solution and prove convergence to a homothetically expanding profile.

2 Preliminaries

Let us first state the properties which we will require on the speed $\mathcal{S}$ appearing in (1). We recall that a function $\mathcal{S} = \mathcal{S}(\lambda_1, \dots, \lambda_n)$ is called inverse concave if

$$\mathcal{S}_\star(\lambda_1, \dots, \lambda_n) := \mathcal{S}^{-1}\left(\lambda_1^{-1}, \dots, \lambda_n^{-1}\right)$$

is concave. Let us denote by $\Gamma_+ \subset \mathbb{R}^n$ the positive cone. In most of the results of this paper we assume that $\mathcal{S}$ satisfies the following properties.

(H1) $\mathcal{S}$ is a C^∞ symmetric function, homogeneous of degree one, defined in an open cone Γ which contains Γ_+.
(H2) $\mathcal{S}$ is positive, concave and inverse concave on Γ_+, and the derivatives $\frac{\partial \mathcal{S}}{\partial \lambda_i}$ are positive at all points of Γ_+.
(H3) There exists $k \in \{1, \dots, n-1\}$ such that, for any $\bar{\lambda} \in \partial\Gamma_+$ we have

$$\bar{\lambda} \in \Gamma \iff \bar{\lambda} \text{ has at least } k \text{ nonzero entries.}$$

In addition, $\mathcal{S}$ and its derivatives $\frac{\partial \mathcal{S}}{\partial \lambda_i}$ are all positive at the points of $\partial\Gamma_+ \cap \Gamma$.
(H4) For all $\bar{\lambda} \in \partial\Gamma_+ \setminus \Gamma$, we have that $\lim_{\lambda \to \bar{\lambda}, \lambda \in \Gamma_+} \mathcal{S}(\lambda) = 0$.

In particular, condition (H3) implies that, if $\bar{\lambda} \in \partial\Gamma_+$ has exactly one zero entry, then $\bar{\lambda} \in \Gamma$ and $\mathcal{S}(\bar{\lambda}) > 0$. We remark that there is an alternative definition of inverse concavity, in terms of the convexity of $\mathcal{S}\left(\lambda_1^{-1}, \dots, \lambda_n^{-1}\right)$, see e.g. [4]; however, for a function which satisfies the other requirements of hypothesis (H2), this formulation is equivalent to the one used here, as it follows from [4, Theorem 2.1] and [5, Lemma 3].

Let us consider the elementary symmetric polynomials

$$E_k = \sum_{1 \le i_1 < \dots < i_k \le n} \lambda_{i_1} \cdot \dots \cdot \lambda_{i_k}$$

for $1 \le k \le n$. We also set $E_k \equiv 1$ if $k = 0$. Then many examples of speed functions satisfying our assumptions can be obtained from these polynomials, as the next result shows.

Proposition 1. *Any function of the form*

$$\mathcal{S} = \left(\frac{E_k}{E_l}\right)^{\frac{1}{k-l}} \quad \textit{for } 0 \le l < k < n,$$

satisfies (H1)–(H4). *In addition, any weighted geometric mean of functions* $\mathcal{S}_1, \dots, \mathcal{S}_N$ *satisfying* (H1)–(H4), *that is* $\mathcal{S} = \prod_{i=1}^N S_i^{\alpha_i}$ *with* $\alpha_i > 0$ *and* $\sum_i \alpha_i = 1$, *also satisfies the same assumptions.*

Proof. For $h = 1, \dots, n$, let us denote by Γ_h the cone where $E_1 > 0, \dots, E_h > 0$. Then $\left(\frac{E_k}{E_l}\right)^{\frac{1}{k-l}}$ is well defined on $\Gamma := \Gamma_k$. The properties in (H1) are obvious, while the ones in (H2) are well known, see e.g. Theorem 15.18 in [15]. Property (H3) also follows easily. In fact, a point $\bar{\lambda} \in \partial\Gamma_+$ satisfies $\mathcal{S}(\bar{\lambda}) > 0$ if and only if it has at least k nonzero entries. If $\mathcal{S}(\bar{\lambda}) > 0$, we can again use Theorem 15.18 in [15] to show that $\partial\mathcal{S}/\partial\lambda_i(\bar{\lambda}) > 0$ for all $i = 1, \dots, n$.

To check (H4), let us take a point $\bar{\lambda} \in \partial\Gamma_+ \setminus \Gamma$ Then $\bar{\lambda}$ has p positive entries, where $0 \le p < k$. We have to show that $\mathcal{S}(\lambda) \to 0$ as $\lambda \to \bar{\lambda}$ from inside the positive cone. This is not obvious in the case $p < l$, where both terms in the definition of $\mathcal{S}$ vanish. However, Lemma 1 in [5] ensures that the limit exists. Therefore, it is enough to compute

$$\lim_{\varepsilon \to 0^+} \mathcal{S}(\bar{\lambda} + \varepsilon(1, 1, \dots, 1)).$$

It is easily checked that $E_h(\bar{\lambda} + \varepsilon(1, 1, \dots, 1)) \sim \varepsilon^{h-p}$ for any $h > p$. Therefore $\mathcal{S}(\bar{\lambda} + \varepsilon(1, 1, \dots, 1)) = O(\varepsilon)$, and this concludes the proof of (H4).

Weighted geometric means of functions satisfying (H1)–(H4) clearly satisfy the same conditions on the intersection of the cones Γ^i considered for each function respectively. □

Observe that the above result includes in particular speeds of the form $E_k^{\frac{1}{k}}$ for $1 \le k < n$ and of the form E_k/E_{k-1} for $1 \le k < n$. On the other hand, speeds involving the Gauss curvature, such as $E_n^{1/n}$, are excluded from our setting because E_n vanishes everywhere on $\partial\Gamma_+$. In our analysis we shall also exclude the case of the mean curvature flow $\mathcal{S} = E_1$, since the methods of the paper would not improve the results known from [9].

For a given function $\mathcal{S} : \Gamma \to \mathbb{R}$ which satisfies (H1)–(H4) we define, for any m_0, the cone

$$\Gamma(m_0) = \{(\lambda_1, \dots, \lambda_n) \in \overline{\Gamma}_+ \cap \Gamma \ : \ \mathcal{S}(\lambda_1, \dots, \lambda_n) \geq m_0(\lambda_1 + \cdots + \lambda_n)\}. \tag{3}$$

Throughout the paper, we will assume that the initial value (2) of our problem satisfies the following assumptions.

(ID) M_0 is the graph of an entire convex function $w_0 \in C^2(\mathbb{R}^n)$ such that $w(0) = 0, Dw(0) = 0$, and such that $||Dw_0||_\infty$ and $||D^2 w_0||_\infty$ are both finite. In addition, at every point of M_0 the principal curvatures $\lambda_1, \dots, \lambda_n$ are positive and belong to the cone $\Gamma(m_0)$ for a fixed m_0.

In particular, w_0 satisfying (ID) is strictly convex, uniformly Lipschitz and has a graph with bounded curvature. The requirement that the curvatures belong to $\Gamma(m_0)$ is a kind of a pinching condition. For instance, in the example mentioned above with $\mathcal{S} = (E_k/E_l)^{\frac{1}{k-l}}$, it is equivalent to the pinching of the largest k curvatures. We recall that if M_0 is a complete noncompact convex hypersurface (in particular, the graph of an entire convex function) we cannot expect inequalities of the form $\lambda_1 \geq c$ or $\lambda_1 \geq cH > 0$ to hold everywhere on M_0 with a fixed constant $c > 0$, see [12]. Therefore, the pinching conditions on all curvatures usually considered in the compact case cannot hold on an entire graph. The condition described above, instead, defines a nontrivial class, as the next proposition shows.

Proposition 2. *For any speed $\mathcal{S}$ satisfying* (H1)—(H4)*, the class of functions w_0 satisfying* (ID) *is nonempty.*

Proof. Consider for instance

$$w_0(\mathbf{x}) = \sqrt{1 + |\mathbf{x}|^2} - 1.$$

Then it is easily checked that Dw_0 and $D^2 w_0$ are uniformly bounded and that, if we set $r = |\mathbf{x}|$, the principal curvatures of the graph of w_0 are

$$\lambda_1 = \frac{1}{\left(\sqrt{2r^2+1}\right)^3}, \qquad \lambda_k = \frac{1}{\sqrt{2r^2+1}}, \quad \text{for } k = 2, \dots, n.$$

We need to prove that the principal curvatures satisfy at every point $\mathcal{S} \geq m_0 H$ for some positive constant m_0.

Before showing this, observe that, using the symmetry, concavity and 1-homogeneity of $\mathcal{S}$, we have

$$\begin{aligned} \mathcal{S}(0, 1, \dots, 1) &= \frac{1}{n}\left(\mathcal{S}(0, 1, \dots, 1) + \mathcal{S}(1, 0, 1, \dots, 1) + \cdots + \mathcal{S}(1, \dots, 1, 0)\right) \\ &\leq \mathcal{S}\left(\frac{n-1}{n}, \frac{n-1}{n}, \dots, \frac{n-1}{n}\right) = \frac{n-1}{n}\mathcal{S}(1, 1, \dots, 1). \end{aligned}$$

We can now estimate $\mathcal{S}/(\lambda_1 + \cdots + \lambda_n)$ at a generic point of the graph. Setting for simplicity $\mu = \lambda_1/\lambda_2 = (2r^2+1)^{-1}$, we have

$$\begin{aligned} \frac{\mathcal{S}(\lambda_1, \dots, \lambda_n)}{\lambda_1 + \cdots + \lambda_n} &= \frac{\mathcal{S}(\mu, 1, \dots, 1)}{\mu + n - 1} \\ &\geq \frac{(1-\mu)\mathcal{S}(0, 1, \dots, 1) + \mu\mathcal{S}(1, \dots, 1)}{\mu + n - 1} \\ &\geq \frac{(1-\mu) + \mu\frac{n}{n-1}}{\mu + n - 1}\mathcal{S}(0, 1, \dots, 1) = \frac{\mathcal{S}(0, 1, \dots, 1)}{n-1} =: m_0 > 0, \end{aligned}$$

where the property that $\mathcal{S}(0, 1, \dots, 1) > 0$ follows from (H3). □

The next property will be used to show that the flow is uniformly parabolic along solutions which satisfy assumption (ID) for a fixed m_0.

Lemma 3. *Let $\mathcal{S}$ satisfy* (H1)–(H4)*. For any $m_0 > 0$ define*

$$\Gamma(m_0) = \{(\lambda_1, \dots, \lambda_n) \in \overline{\Gamma}_+ \cap \Gamma \ : \ \mathcal{S}(\lambda_1, \dots, \lambda_n) \geq m_0(\lambda_1 + \cdots + \lambda_n)\}.$$

Then there are $B \geq \beta > 0$ such that

$$\beta \leq \frac{\partial \mathcal{S}}{\partial \lambda_i} \leq B \tag{4}$$

everywhere in $\Gamma(m_0)$.

Proof. By homogeneity, it suffices to prove (4) on the set

$$\Gamma^\star(m_0) = \{\lambda \in \Gamma(m_0) \ : \ \lambda_1 + \cdots + \lambda_n = 1\}.$$

It is easy to see that such a set is a compact subset of Γ; in fact, by (H4), a point $\overline{\lambda} \in \partial\Gamma_+ \setminus \Gamma$ cannot be obtained as the limit of points in $\Gamma^\star(m_0)$. In addition, the derivatives of $\mathcal{S}$ are positive everywhere on $\Gamma^\star(m_0)$, by (H2) and (H3). By compactness, a bound of the form (4) holds. □

In this paper we consider solutions of the flow (1) which can be written as an entire graph for all times. Then there exists a function $w : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ which gives the graph representation of M_t, i.e., such that for all $(\mathbf{p}, t) \in \mathcal{M} \times [0, T)$ there is a unique $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{F}(\mathbf{p}, t) = (\mathbf{x}, w(\mathbf{x}, t))$. It is easy to see that w satisfies the partial differential equation

$$\frac{\partial w}{\partial t} = \sqrt{1 + |Dw|^2}\,\mathcal{S} \tag{5}$$

where $\mathcal{S}$ is evaluated in the curvatures of the graph of $w(\cdot, t)$.

On the hypersurfaces M_t, we denote the metric tensor by $g = g_{ij}$ and the second fundamental form by h_{ij}. In addition, we set $H = \lambda_1 + \cdots + \lambda_n$ and $|A|^2 = \lambda_1^2 + \ldots \lambda_n^2$ to denote the mean curvature and the squared norm of the second fundamental form. The signs are chosen in such a way that the normal vector ν is pointing downwards, and that M_t is convex if and only if the hessian of its graph representation $w(\cdot, t)$ is nonnegative. We recall the expressions of ν, g_{ij} and h_{ij} in terms of u

$$\nu(\mathbf{x}, t) = \frac{(Du(\mathbf{x}, t), -1)}{\sqrt{1 + |Du(\mathbf{x}, t)|^2}}, \tag{6}$$

$$g_{ij}(\mathbf{x}, t) = \delta_{ij} + D_i u(\mathbf{x}, t) D_j u(\mathbf{x}, t), \tag{7}$$

$$h_{ij}(\mathbf{x}, t) = \frac{D^2_{ij} u(\mathbf{x}, t)}{\sqrt{1 + |Du(\mathbf{x}, t)|^2}}. \tag{8}$$

Some useful evolution equations satisfied along solutions of the flow (1) are recalled below and are proved for instance in Section 3 of [2] or Proposition 1.9 of [1]. As in these references, we use the notation $\dot{\mathcal{S}}^i_j = \frac{\partial \mathcal{S}}{\partial h^j_i}$ for the first derivatives of the speed and $\dot{\mathcal{S}} g^\star Hess_\nabla = \dot{\mathcal{S}}^{ij} \nabla_i \nabla_j$ for the second order operator associated to $\mathcal{S}$, where ∇ denotes the covariant derivative associated to the metric on M_t.

Proposition 4. *The metric g and the unit normal vector ν evolve according to the equations*

$$\frac{\partial g_{ij}}{\partial t} = -2\mathcal{S} h_{ij}, \qquad \frac{\partial \nu}{\partial t} = \nabla \mathcal{S}.$$

In addition, the Weingarten map and the speed function satisfy respectively

$$\frac{\partial h^i_j}{\partial t} = \dot{\mathcal{S}} g^\star \left(Hess_\nabla h^i_j\right) + g^{pq} \ddot{\mathcal{S}}^{rl}_{sk} \left(\nabla_p h^s_r, \nabla_q h^k_l\right) + \dot{\mathcal{S}}^r_s \left(h^s_k h^k_r\right) h^i_j, \tag{9}$$

$$\frac{\partial \mathcal{S}}{\partial t} = \dot{\mathcal{S}} g^\star (Hess_\nabla \mathcal{S}) + \dot{\mathcal{S}}\left(\mathcal{W}^2\right) \mathcal{S}. \tag{10}$$

3 Existence

In this section we prove the existence of solutions of the flow (1)–(2) under the assumption that the speed $\mathcal{S}$ satisfies (H1)–(H4) and that the image of the initial immersion $M_0 = F_0(\mathcal{M})$ can be written as the graph of a function $w_0 : \mathbb{R}^n \to \mathbb{R}$ satisfying (ID). Equivalently, we prove the existence of the solutions to equation (5) with w_0 as initial data. The idea is to approximate M_0 with compact surfaces on larger and larger regions and use the theorem of short time existence for compact surfaces. Since the lifespan of the approximating surfaces becomes larger as they become larger, we obtain in the limit a solution to our original problem which is defined for all times. We recall that a similar approximation procedure for noncompact manifolds has been recently used by other authors in the field of geometric flows, see e.g. [6, 11, 17].

Although by assumption the principal curvatures are strictly positive at every point of M_0, their infimum over all M_0 is zero. A crucial part of the existence proof consists of showing that the curvature satisfies a positive lower bound on any compact set. To this purpose we will use the following Harnack-type estimate due to B. Andrews, see Theorem 5.21, part (2), in [3].

Theorem 5. *Let $F : \mathcal{N}\times[0, T) \to \mathbb{R}^{n+1}$ be a solution of the flow* (1) *defined on a compact n-dimensional manifold $\mathcal{N}$. Suppose that the speed $\mathcal{S}$ satisfies* (H1)-(H2) *and that the hypersurfaces $N_t = \mathbf{F}(\mathcal{N}, t)$ are strictly convex for all t. Suppose also that $\mathcal{S} \geq mH > 0$ on every N_t for some $m > 0$. Then we have, for any $\bar{\mathbf{p}}, \mathbf{p} \in \mathcal{N}$ and $t_2 > t_1 > 0$*

$$\frac{\mathcal{S}(\mathbf{p}, t_2)}{\mathcal{S}(\bar{\mathbf{p}}, t_1)} \geq \sqrt{\frac{t_1}{t_2}}\ \exp\left(-\frac{d^2}{4m(t_2 - t_1)}\right) \tag{11}$$

where d is the distance from $\bar{\mathbf{p}}$ to $\mathbf{p}$ with respect to the metric at time t_1.

In the following we denote by $\mathbf{x}$ the points in $\mathbb{R}^n$ and by $(\mathbf{x}, x_{n+1})$ the points in $\mathbb{R}^n \times \mathbb{R}$. We use the symbol $B_R(\mathbf{x})$ to denote balls in $\mathbb{R}^n$, while the balls in $\mathbb{R}^n \times \mathbb{R}$ will be described as “ball of center $(\mathbf{x}, x_{n+1})$ and radius R”, to avoid ambiguities.

The next lemma describes the approximation of our initial graph by compact surfaces which will be used in the existence theorem.

Lemma 6. *Let $M_0 \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of a function $w_0 \in C^2(\mathbb{R}^n)$ satisfying* (ID) *for some $m_0 > 0$. Then, for any $R > 1$, there exists a smooth closed convex hypersurface $N_R \subset \mathbb{R}^n \times \mathbb{R}$ such that*

(A1) $N_R \cap (B_R(0) \times [0, c_0R]) = M_0 \cap (B_R(0) \times [0, c_0R])$ *for some $c_0 > 1$ independent of R.*
(A2) *N_R encloses the ball of center $(0, c_0R)$ and radius R.*
(A3) *N_R has diameter less than c_1R for some $c_1 > 1$ independent of R.*
(A4) $\max_{N_R} \mathcal{S} \leq c_2 \sup_{M_0} \mathcal{S}$ *for some $c_2 \geq 1$ independent of R.*
(A5) *On any N_R, we have $\mathcal{S} \geq \frac{m_0}{c_2} H > 0$.*

Proof. Let $\gamma = \min_{|x|=1} w_0(\mathbf{x})$ and $L = \sup_{\mathbb{R}^n} |Dw_0|$. Then $\gamma > w_0(0) = 0$ because w_0 is strictly convex and $Dw_0(0) = 0$. Again by convexity, we have that $w_0(\mathbf{x}) \geq \gamma|\mathbf{x}|$ for all $\mathbf{x}$ with $|\mathbf{x}| \geq 1$. Therefore, w_0 satisfies

$$\gamma|\mathbf{x}| \leq w_0(\mathbf{x}) \leq L|\mathbf{x}|, \qquad \forall\, |\mathbf{x}| \geq 1. \tag{12}$$

If we set $c_0 = L + 1$, we find that the ball of center $(0, c_0R)$ and radius R lies above the graph of w_0, for any $R > 0$.

Now we want to bend upwards the graph of w_0 in the region $R \leq |x| \leq R + 1$ in such a way that the normal to the graph becomes horizontal for $|x| = R + 1$. We consider the function $\psi_R : [0, R + 1] \to \mathbb{R}$ defined as follows

$$\psi_R(r) = \begin{cases} 0 & 0 \leq r \leq R \\ 1 - (1 - (r - R)^4)^{1/4} & R \leq r \leq R + 1. \end{cases}$$

This function is chosen in such a way that the graph of $z = \psi_R(r)$ for $R \leq r \leq R + 1$ belongs to the curve $(z - 1)^4 + (r - R)^4 = 1$. Then we consider the radially symmetric function $\Psi_R(\mathbf{x}) = \psi_R(|\mathbf{x}|)$, for $|\mathbf{x}| \leq R + 1$. Then

$\Psi_R \in C^2(B_{R+1}(0))$, the gradient $D\Psi_R(\mathbf{x})$ points in radial direction and becomes unbounded as $|\mathbf{x}| \to R+1$. However, the curvatures of the graph of Ψ_R are defined also for $|\mathbf{x}| = R+1$ and are given by $\lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = (R+1)^{-1}$ at those points.

Now let us define $\tilde{w}_R(\mathbf{x}) = w_0(\mathbf{x}) + \Psi_R(\mathbf{x})$, for $|x| \le R+1$. Then $\tilde{w}_R$ is convex, satisfies $w_0(\mathbf{x}) \le \tilde{w}_R(\mathbf{x}) \le c_0|\mathbf{x}|$ and is of class C^2 except on the boundary of $B_{R+1}(0)$ where the derivatives blow up. Since the first derivatives of w_0 are bounded, it follows from (6) that the normal to the graph of $\tilde{w}_R$ is asymptotic to the normal to the graph of Ψ_R as $|\mathbf{x}| \uparrow R+1$ and therefore tends to $(\frac{\bar{\mathbf{x}}}{|\bar{\mathbf{x}}|}, 0)$ for any $\bar{\mathbf{x}} \in \partial B_{R+1}(0)$. Similarly, using (7), (8) and the explicit expression of Ψ_R, we find that the curvatures of the graph of $\tilde{w}_R$ satisfy $\lambda_1 \to 0$, $\lambda_i \to (R+1)^{-1}$ for $i = 2, \dots, n$ as $|\mathbf{x}| \to R+1$.

Let us now define $N_R = \partial\Omega_R$, where

$$\Omega_R = \{\, (\mathbf{x}, x_{n+1}) \;:\; |\mathbf{x}| \le R+1,\ \tilde{w}_R(\mathbf{x}) \le x_{n+1} \le 2c_0(R+1) - \tilde{w}_R(\mathbf{x}) \,\}.$$

Then N_R consists of the two symmetric graphs of $\tilde{w}_R$ and of $2c_0(R+1) - \tilde{w}_R$, and of a portion of the boundary of the cylinder $B_{R+1}(0) \times \mathbb{R}$. By the properties of $\tilde{w}_R$ observed above, the different parts meet in such a way that N_R is C^2. By construction, N_R satisfies (A1), (A2) and (A3).

To check the remaining conditions, we first observe that on the cylindrical part of N_R property (A4) is trivial, while (A5) follows from assumption (H3) on the speed function. On the graphical part, the properties can be deduced from (7), (8), our assumptions on w_0 and the special form of Ψ_R. Roughly speaking, on the points with $|x| \in [R+1-\varepsilon, R+1]$, for $\varepsilon > 0$ small enough, the curvatures are close to the ones of the cylinder and therefore (A4)–(A5) are satisfied with a constant c_2 independent of R. If $|x| \in [R, R+1-\varepsilon]$, then the first and second derivatives of $\tilde{w}_R$ are uniformly bounded, which implies (A4). If (A5) were violated, we would find points on N_R where the ratio $\mathcal{S}/H$ is arbitrarily small as we let $R \to \infty$ in our procedure. By assumption (H3) on $\mathcal{S}$, this implies that the pinching ratio λ_h/λ_n becomes arbitrarily small for some $h \ge 2$. Since we are considering regions with bounded gradient, (7) and (8) show that such a pinching ratio is comparable to the corresponding one for the eigenvalues μ_i of $D^2\tilde{w}_R = D^2 w_0 + D^2\Psi_R$. The eigenvalues of $D^2\Psi_R$ are positive and are all equal except one because ψ_R is radially symmetric. Using this, it is easy to see that the pinching ratio μ_h/μ_n for $h \ge 2$ cannot be arbitrarily smaller than the corresponding one for $D^2 w_0$. Since w_0 satisfies the desired property by assumption (ID), we conclude that N_R satisfies (A5). □

Theorem 7. *Let the speed $\mathcal{S}$ satisfy conditions* (H1)–(H4) *and let the function $w_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy* (ID) *for some $m_0 > 0$. Then there exists a solution $w : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ of equation* (5) *with initial value w_0. The supremum of $|Dw(\cdot, t)|$ and $|D^2 w(\cdot, t)|$ is bounded for every $t \ge 0$ by constants depending only on M_0. The curvatures of the graph of $w(\cdot, t)$ belong for every $t \ge 0$ to the cone $\Gamma(m_0')$ defined in formula* (3) *for a suitable* $0 < m_0' \le m_0$.

Proof. For any $R > 0$ suitably large, we consider the evolution under the flow (1) of the hypersurface N_R given by the previous lemma. More precisely, we consider the solution $\mathbf{F}_R : N_R \times [0, T) \to \mathbb{R}^{n+1}$ of the flow (1) with initial value the identity, and denote by $N_R(t) = \mathbf{F}_R(N_R, t)$ the solution at time t. By the results of [4], such a solution exists on a finite time interval, remains convex and contracts to a point at the singular time. By the avoidance principle and property (A2) of the previous lemma, its interval of existence is at least as long as the one of a sphere of radius R.

Let us choose T_R to be the time taken by a contracting sphere to go from radius R to radius $R/2$, that is $T_R = \frac{3}{8}R^2\mathcal{S}(1, 1, \dots, 1)^{-1}$. Again by the avoidance principle, the sphere of center $\mathbf{p}_0 := (0, c_0 R)$ and radius $R/2$ is enclosed by $N_R(t)$ for all $t \in [0, T_R]$. Then we can apply a well known technique, first introduced in [20], to estimate $\mathcal{S}$ on $N_R(t)$. We write $Y_R(\mathbf{p}, t) = \mathbf{F}_R(\mathbf{p}, t) - \mathbf{p}_0$ to denote the position vector of the flow with respect to $\mathbf{p}_0$. Then, by the convexity of $N_R(t)$, we have

$$2\,\langle Y_R(\mathbf{p}, t), \nu(\mathbf{p}, t)\rangle - R/2 \ge \frac{R}{2}$$

fot all $t \in [0, T_R]$. We then define

$$u(\mathbf{p}, t) := \frac{\mathcal{S}(\mathbf{p}, t)}{2\,\langle Y_R(\mathbf{p}, t), \nu(\mathbf{p}, t)\rangle - R/2}.$$

By maximum principle techniques we obtain (see the proof of Theorem 5 in [5])

$$u(\mathbf{p},t) \le \max\left\{\max_{N_R} u,\, \frac{8}{R^2}\right\}.$$

Let us set $M^\star = \sup_{M_0} \mathcal{S}$. By property (A4), we can estimate

$$\max_{N_R} u \le \frac{2}{R}\max_{N_R}\mathcal{S} \le \frac{c}{R}M^\star,$$

where we denote by c constants depending only on the initial data and on the values of c_0, c_1, c_2 in Lemma 6. It follows that, for R large, we have $u(\mathbf{p},t) \le cM^\star/R$, and we obtain

$$\begin{aligned} S(\mathbf{p},t) &\le 2\left\langle Y(\mathbf{p},t), \nu(\mathbf{p},t)\right\rangle \frac{c}{R}M^\star \le 2\,\mathrm{diam}(N_R(t))\frac{c}{R}M^\star \\ &\le 2\,\mathrm{diam}(N_R)\frac{c}{R}M^\star \le cM^\star, \end{aligned} \tag{13}$$

where we have used property (A3) and the fact that the diameter of $N_R(t)$ decreases with time.

We now study the behaviour of the ratio $H/\mathcal{S}$ on the approximating flows $N_R(t)$. We can compute, using Proposition 4,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{H}{\mathcal{S}} &= \mathcal{S}^{-1}\left(\dot{\mathcal{S}}g^\star\left(Hess_\nabla H\right) + \frac{\partial^2\mathcal{S}}{\partial h_l^k \partial h_r^s}\left\langle \nabla h_l^k, \nabla h_r^s\right\rangle + \dot{\mathcal{S}}\left(\mathcal{W}^2\right)H\right) - H\mathcal{S}^{-2}\left(\dot{\mathcal{S}}g^\star\left(Hess_\nabla\mathcal{S}\right) - \dot{\mathcal{S}}\left(\mathcal{W}^2\right)\mathcal{S}\right) \\ &= \dot{\mathcal{S}}g^\star\left(Hess_\nabla \frac{H}{\mathcal{S}}\right) + 2\mathcal{S}^{-1}\dot{\mathcal{S}}g^\star\left(\nabla\frac{H}{\mathcal{S}}, \nabla\mathcal{S}\right) + \mathcal{S}^{-1}\frac{\partial^2\mathcal{S}}{\partial h_l^k \partial h_r^s}\left\langle \nabla h_l^k, \nabla h_r^s\right\rangle. \end{aligned}$$

Since $\mathcal{S}$ is concave by assumption (H2), we conclude that the maximum of $H/\mathcal{S}$ is nonincreasing. By property (A5), we have

$$\mathcal{S} \ge \frac{c_0}{m_2}H > 0 \tag{14}$$

on $N_R(t)$ for all $t \in [0, T_R]$, that is, the curvatures of $N_R(t)$ belong to the cone $\Gamma(m_0/c_2)$. In addition, we see that estimate (13) on the speed implies a uniform bound on the curvature on $N_R(t)$:

$$|A|^2 \le H^2 \le \left(\frac{c_2}{m_0}\right)^{-2}\mathcal{S}^2 \le c(M^\star)^2. \tag{15}$$

We now show that suitable subsets of $N_R(t)$ are graphs with uniformly bounded gradient. To this purpose, let $\mathbf{p}' = (\mathbf{x}', x'_{n+1})$ and $\mathbf{p}'' = (\mathbf{x}'', x''_{n+1})$ be any two points in $N_R(t) \cap (B_{R/4}(0) \times [0, c_0R])$ for some $t \in [0, T_R]$. Since the points belong to a convex hypersurface enclosing the ball of center $(0, c_0R)$ and radius $R/2$, we easily obtain

$$0 \le x'_{n+1}, x''_{n+1} < c_0 R, \qquad \frac{|x''_{n+1} - x'_{n+1}|}{|\mathbf{x}'' - \mathbf{x}'|} \le 4c_0.$$

Thus, $N_R(t) \cap (B_{R/4}(0) \times [0, c_0R])$ is a graph of a convex function with Lipschitz constant not greater than $4c_0$; let us denote this function as $w_R(\mathbf{x}, t)$, defined for $(\mathbf{x}, t) \in B_{R/4}(0) \times [0, T_R]$.

We now want to find some local uniform lower bound on the curvature of the approximating surfaces. As a first step, we will show that on each $N_R(t)$ the curvature remains bounded away from zero at least at one point. We will prove this by finding a sphere which touches the surface from outside.

We first remark that, for any R, the function w_R is increasing in t; in particular, by (12), we have $w_R(\mathbf{x}, t) \ge \gamma|\mathbf{x}|$ for any $|\mathbf{x}| \ge 1$ and $t \in [0, T_R]$. From (5) and (13) we also have

$$w_R(0,t) \le \sqrt{1 + 16c_0^2}\, cM^\star t.$$

Let us set $\rho = 1/\gamma + \gamma/4$. Then we have $\rho \ge 1$ and $(\rho - \gamma/2)^2 = \rho^2 - 1$. For any fixed t_0 such that

$$t_0 \in \left(0, \frac{\gamma}{2cM^\star\sqrt{1 + 16c_0^2}}\right), \tag{16}$$

we consider the function

$$\phi_R(\mathbf{x}) = w_R(\mathbf{x}, t_0) + \sqrt{\rho^2 - |\mathbf{x}|^2},$$

for $\mathbf{x} \in \mathbb{R}^n$, $|\mathbf{x}| \le 1$. Then, for every $\mathbf{x}$ such that $|\mathbf{x}| = 1$, we have

$$\phi_R(\mathbf{x}) \ge \gamma + \sqrt{\rho^2 - 1} = \gamma + \left|\rho - \frac{\gamma}{2}\right| \ge \frac{\gamma}{2} + \rho > \phi_R(0).$$

It follows that the minimum of ϕ_R over the unit ball is attained at a point $\bar{\mathbf{x}}$ in the interior. By the definition of ϕ_R, this means that at $\bar{\mathbf{x}}$ the graph of $w_R(\cdot, t_0)$ is touched from below by a sphere of radius ρ. Then at such a point each principal curvature of the graph of $w_R(\cdot, t_0)$ is at least $1/\rho$ and therefore $\mathcal{S} \ge (\rho)^{-n}\mathcal{S}(1, \dots, 1) =: \sigma_0$, where σ_0 only depends on the data of our problem and not on R.

To simplify the notation, let us denote by $\mathcal{S}_R(\mathbf{x}, t)$ the function $\mathcal{S}$ evaluated at the point $(x, w_R(\mathbf{x}, t))$ of the graph of $w_R(\cdot, t)$. Let us fix arbitrary $\tilde{R} > 1$ and $\tilde{T}_2 > \tilde{T}_1 > 0$. We want to show that for R enough large $\mathcal{S}_R(\mathbf{x}, t)$ is bounded from below by a positive constant, independent on R, for every $(\mathbf{x}, t) \in B_{\tilde{R}}(0) \times [\tilde{T}_1, \tilde{T}_2]$.

To prove this, we shall use Andrews' Harnack-type inequality. It is not restrictive to assume that $\tilde{T}_1/2$ satisfies the same bound imposed on t_0 in (16). Then, by the previous step, there exists $\bar{\mathbf{x}} \in B_1(0)$ such that

$$\mathcal{S}_R(\bar{\mathbf{x}}, \tilde{T}_1/2) \ge \sigma_0 > 0. \tag{17}$$

Consider now any given $(\mathbf{x}, t) \in B_{\tilde{R}}(0) \times [\tilde{T}_1, \tilde{T}_2]$. Let $\bar{\mathbf{p}}, \mathbf{p} \in N_R$ be the points such that

$$\mathbf{F}_R(\bar{\mathbf{p}}, \tilde{T}_1/2) = (\bar{\mathbf{x}}, w_R(\bar{\mathbf{x}}, \tilde{T}_1/2)), \qquad \mathbf{F}_R(\mathbf{p}, t) = (\mathbf{x}, w_R(\mathbf{x}, t)).$$

We want to apply (11) with $t_1 = \tilde{T}_1/2$ and $t_2 = t$. To this purpose, we have to estimate the distance (with respect to the intrinsic metric of $N_R(\tilde{T}_1/2)$) between the two points $\mathbf{F}_R(\bar{\mathbf{p}}, \tilde{T}_1/2)$ and $\mathbf{F}_R(\mathbf{p}, \tilde{T}_1/2)$. We first observe that (1) and (13) imply

$$|\mathbf{F}_R(\mathbf{p}, \tilde{T}_1/2) - \mathbf{F}_R(\mathbf{p}, t)|_{\mathbb{R}^{n+1}} \le cM^\star(t - \tilde{T}_1/2) < cM^\star\tilde{T}_2.$$

Let $\mathbf{x}'$ be the first n components of $\mathbf{F}_R(\mathbf{p}, \tilde{T}_1/2)$. Then

$$|\mathbf{x}'| \le |\mathbf{x}| + |\mathbf{x}' - \mathbf{x}| \le \tilde{R} + |\mathbf{F}_R(\mathbf{p}, \tilde{T}_1/2) - \mathbf{F}_R(\mathbf{p}, t)|_{\mathbb{R}^{n+1}} < \tilde{R} + cM^\star\tilde{T}_2.$$

Therefore, if $R > 4(\tilde{R} + cM^\star\tilde{T}_2)$, we have that $\mathbf{F}_R(\mathbf{p}, \tilde{T}_1/2)$ belongs to the graph of $w_R(\cdot, \tilde{T}_1/2)$. By the Lipschitz continuity of w_R, we deduce that intrinsic distances on the graph can be estimated by distances in $\mathbb{R}^{n+1}$

$$\begin{aligned} d_{N_R(\tilde{T}_1/2)}(\bar{\mathbf{p}}, \mathbf{p}) &\le (1 + 4c_0)|\bar{\mathbf{x}} - \mathbf{x}'| \le (1 + 4c_0)(|\bar{\mathbf{x}}| + |\mathbf{x}'|) \\ &\le (1 + 4c_0)(1 + \tilde{R} + cM^\star\tilde{T}_2) =: \tilde{K}. \end{aligned}$$

Then we deduce from (11), (14) and (17),

$$\mathcal{S}_R(\mathbf{x}, t) \ge \sigma_0\sqrt{\frac{\tilde{T}_1}{2\tilde{T}_2}}\exp\left(-\frac{c_2\tilde{K}^2}{2m_0\tilde{T}_1}\right),$$

for every $R > 4(\tilde{R} + cM^\star\tilde{T}_2)$ and $(\mathbf{x}, t) \in B_{\tilde{R}}(0) \times [\tilde{T}_1, \tilde{T}_2]$, proving the desired bound. This shows that, on any compact subset of $\mathbb{R}^n \times (0, +\infty)$, the graphs of w_R are such that the speed $\mathcal{S}$, hence $|A|^2$ as well, is bounded from below by a positive constant independent on R. This ensures that on any such compact region the curvatures of the graph remain inside an open subset of Γ_+ where $\mathcal{S}$ is smooth and the flow is uniformly parabolic, by Proposition 3.

Since we have uniform bounds from above on Dw_R and D^2w_R, a subsequence of w_R converges as $R \to \infty$ locally uniformly on $\mathbb{R}^n \times [0, +\infty)$ to a convex function $\bar{w}$. Using standard parabolic regularity theory we obtain that the convergence of the subsequence is in C^∞ and that the limit graph w is a smooth solution of the flow with positive curvature everywhere and which inherits all the uniform curvature bounds we have proved for $N_R(t)$. □

4 Maximum principle

In this section we derive some estimates on the solutions of (1) using maximum principle techniques. These results apply to the solution constructed in the previous section, giving a more explicit bound on the slope of the graph and on the supremum of the speed during the flow. However, they hold under much weaker assumptions on the speed and so they can be regarded as apriori estimates for solutions of a larger class of flows than the one considered in the rest of the paper.

We first prove a version of maximum principle adapted to the study of graphical solutions of the flow (1). As usual in the noncompact case, we need an apriori restriction on the growth rate at infinity of the function we want to estimate. We need here a different argument from the case of the mean curvature flow in [9], where the corresponding result used the monotonicity formula.

Theorem 8. *Let $\mathcal{S}$ satisfy hypothesis* (H1), *and let* $\mathbf{F} : \mathcal{M}\times[0,\tau] \to \mathbb{R}^{n+1}$ *be a smooth solution of problem* (1)–(2) *such that $M_t = F(\mathcal{M}, t)$ is the graph of a function over $\mathbb{R}^n$ for any $t \in [0, \tau]$. Suppose in addition that the matrix $\dot{\mathcal{S}}$ is positive definite on M_t and that it is bounded uniformly for $t \in [0, \tau]$. Let f be a smooth function defined on $\mathcal{M} \times [0, \tau]$, which satisfies the inequality*

$$\frac{\partial f}{\partial t} \le \dot{\mathcal{S}}g^\star\left(Hess_\nabla f\right) + \dot{\mathcal{S}}g^\star\left(\mathbf{a}, \nabla f\right), \tag{18}$$

where $\mathbf{a}$ *is some smooth vector field. Suppose that*

$$\sup_{\mathcal{M}\times[0,\tau]} |\mathbf{a}| < +\infty, \qquad \sup_{\mathcal{M}\times[0,\tau]} f - e^{k_0|\mathbf{F}|} < +\infty \tag{19}$$

for a suitable $k_0 > 0$. Then we have that

$$\sup_{\mathcal{M}\times[0,\tau]} f = \sup_{\mathcal{M}\times\{0\}} f.$$

Proof. Let us set

$$c_1 = \sup_{\mathcal{M}\times[0,\tau]} |\mathbf{a}|, \qquad c_2 = n \sup_{\mathcal{M}\times[0,\tau]} |\dot{\mathcal{S}}|.$$

Setting $\mathbf{y} = \mathbf{F}(\mathbf{p}, t)$, we define

$$\phi(\mathbf{p}, t) = e^{\alpha t + k_1\sqrt{1+|\mathbf{y}|^2}}, \tag{20}$$

where $\alpha > 0$ and $k_1 > k_0$, with k_0 the constant appearing in (19). We claim that ϕ is a supersolution of (18) if α large enough. We first observe that

$$\nabla_j |\mathbf{y}|^2 = 2\langle \mathbf{y}, e_j\rangle, \qquad \nabla_i\nabla_j|\mathbf{y}|^2 = 2g_{ij} - 2h_{ij}\langle \mathbf{y}, \nu\rangle, \tag{21}$$

and also, by the homogeneity of $\mathcal{S}$,

$$\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^\star Hess_\nabla\right)|\mathbf{y}|^2 = -2\mathrm{tr}\dot{\mathcal{S}}. \tag{22}$$

Then we find

$$\begin{aligned}
&\frac{\partial\phi}{\partial t} - \dot{\mathcal{S}}g^\star\left(Hess_\nabla\phi\right) + \dot{\mathcal{S}}g^\star\left(\mathbf{a}, \nabla\phi\right)\\
&= \alpha\phi + \frac{k_1\phi}{2\sqrt{1+|\mathbf{y}|^2}}\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^\star Hess_\nabla\right)|\mathbf{y}|^2 - \frac{k_1^2\phi}{4(1+|\mathbf{y}|^2)}\dot{\mathcal{S}}g^\star\left(\nabla|\mathbf{y}|^2, \nabla|\mathbf{y}|^2\right) + \frac{k_1\phi}{2\sqrt{1+|\mathbf{y}|^2}}\dot{\mathcal{S}}g^\star\left(\mathbf{a}, \nabla|\mathbf{y}|^2\right)\\
&\ge \alpha\phi - c_2k_1\frac{\phi}{\sqrt{1+|\mathbf{y}|^2}} - c_2k_1^2\phi\frac{|\mathbf{y}|^2}{1+|\mathbf{y}|^2} - c_1c_2k_1\phi\frac{|\mathbf{y}|}{\sqrt{1+|\mathbf{y}|^2}}\\
&\ge \phi(\alpha - c_2k_1(1 + k_1 + c_1)),
\end{aligned}$$

which is positive for α large enough. For any $\eta > 0$, we now define

$$f_\eta = f - \eta\phi.$$

Our choice of k_1 implies that $f_\eta(\mathbf{p}, t) \to -\infty$ as $\mathbf{p} \to \infty$ uniformly for $t \in [0, \tau]$. Therefore, f_η assumes a strict maximum on $M \times [0, \tau]$. Since f_η is a strict subsolution of (18), the maximum can only be attained at the initial time. We deduce that

$$f(\mathbf{p}, t) \leq \sup_{\mathcal{M}} f(\mathbf{p}, 0) + \eta\phi(\mathbf{p}, t),$$

from which the assertion follows, by the arbitrariness of $\eta > 0$. □

We now consider the function on M_t

$$v(\mathbf{p}, t) := \langle \nu(\mathbf{p}, t), \omega \rangle^{-1},$$

where $\omega = -\mathbf{e}_{n+1}$. Our assumption that M_t can be written as a smooth graph implies that v is well defined and our choice of orientation implies that $v > 0$.

Lemma 9. *Along a graphical solution of* (1)*, with* $\mathcal{S}$ *satisfying* (H1)*,* v *satisfies the equation*

$$\frac{\partial v}{\partial t} = \dot{\mathcal{S}}g^\star (Hess_\nabla v) - 2v^{-1}\dot{\mathcal{S}}g^\star (\nabla v, \nabla v) - v\dot{\mathcal{S}}\left(\mathcal{W}^2\right). \tag{23}$$

In addition,

$$\frac{\partial}{\partial t}(\mathcal{S}v) = \dot{\mathcal{S}}g^\star (Hess_\nabla \mathcal{S}v) - 2v^{-1}\dot{\mathcal{S}}g^\star (\nabla(\mathcal{S}v), \nabla v). \tag{24}$$

Proof. We compute the time derivative of v using Proposition 4

$$\frac{\partial v}{\partial t} = -v^2\left\langle \frac{\partial}{\partial t}\nu, \omega\right\rangle = -v^2\langle\nabla\mathcal{S}, \omega\rangle,$$

and the gradient of v^{-1}

$$\nabla_j\langle\nu, \omega\rangle = h_j^k\left\langle\frac{\partial\mathbf{F}}{\partial x_k}, \omega\right\rangle.$$

Moreover we have

$$\begin{aligned}\nabla_i\nabla_j v &= \nabla_i\left(-v^2\left\langle\nabla_j\nu, \omega\right\rangle\right) = \nabla_i\left(-v^2\left\langle h_j^k\frac{\partial\mathbf{F}}{\partial x_k}, \omega\right\rangle\right)\\ &= -v^2\left\langle\nabla h_{ij}, \omega\right\rangle - vh_{ik}h_j^k + 2v^{-1}\nabla_i v\nabla_j v.\end{aligned}$$

Then the evolution equation for v in the statement is a consequence of the following identity

$$\begin{aligned}\dot{\mathcal{S}}g^\star(Hess_\nabla v) &= -v^2\left\langle\dot{\mathcal{S}}^{ij}\left(\nabla h_{ij}\right), \omega\right\rangle + v\dot{\mathcal{S}}\left(\mathcal{W}^2\right) + 2v^{-1}\dot{\mathcal{S}}g^\star(\nabla v, \nabla v)\\ &= -v^2\langle\nabla\mathcal{S}, \omega\rangle + 2v^{-1}\dot{\mathcal{S}}g^\star(\nabla v, \nabla v) + v\dot{\mathcal{S}}g^\star\left(\mathcal{W}^2\right).\end{aligned}$$

Combining the formula for the evolution of v with the one for $\mathcal{S}$ in (10) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{S}v) &= v\dot{\mathcal{S}}g^\star(Hess_\nabla\mathcal{S}) + \mathcal{S}\dot{\mathcal{S}}g^\star(Hess_\nabla v) - 2v^{-1}\mathcal{S}\dot{\mathcal{S}}g^\star(\nabla v, \nabla v)\\ &= \dot{\mathcal{S}}g^\star(Hess_\nabla\mathcal{S}v) - 2\dot{\mathcal{S}}g^\star(\nabla\mathcal{S}, \nabla v) - 2v^{-1}\mathcal{S}\dot{\mathcal{S}}g^\star(\nabla v, \nabla v)\\ &= \dot{\mathcal{S}}g^\star(Hess_\nabla\mathcal{S}v) - 2v^{-1}\dot{\mathcal{S}}g^\star(\nabla(\mathcal{S}v), \nabla v).\end{aligned}$$

□

Remark 10. The previous result is analogous to Lemma 3.1 and Lemma 4.1 in [9] in the case of the mean curvature flow. In that paper, these estimates are used to prove the preservation of the graph property and the long time existence of the solution. Here these properties do not play an equally crucial role, because the long time existence of a graphical solution follows already from the construction of the previous section. However, they allow us to give a more explicit bound on the Lipschitz constant of the graphs and on the speed, as shown in the next corollary. We also remark that the previous lemma holds under weaker hypotheses on the speed than the ones in the rest of the paper; therefore, it suggests that long time existence for Lipschitz graphical solutions could be derived for more general speeds if a local existence result were available.

Corollary 11. *Let w be the function constructed in Theorem 7, whose graph is a solution of the flow* (1) *under assuptions* (H1)–(H4) *and* (ID). *Then the Lipschitz constant of $w(\cdot, t)$ is a nonincreasing function of t. In addition, the speed is uniformly bounded by $\sqrt{1+L_0^2}\,(\sup_{t=0} \mathcal{S})$, where L_0 is the Lipschitz constant of $w(\cdot, 0)$.*

Proof. When computed on the graph of w, the quantities v and $\mathcal{S}v$ are uniformly bounded by Theorem 7. The vector field $v^{-1}\nabla v$, occurring in equation (24), is also uniformly bounded, by the definition of v and the property that the graph of $w(\cdot, t)$ has bounded curvature. Therefore, we can apply the maximum principle from Theorem 8 to the evolution equations for v and $v\mathcal{S}$ to obtain that these quantities are monotone. We further observe that $w(\cdot, t)$ has Lipschitz constant L if and only if $v(\cdot, t) \le \sqrt{1+L^2}$ everywhere; therefore, since $v \ge 1$ by definition, the maximum principle implies:

$$\mathcal{S}(\mathbf{p}, t) \le \mathcal{S}(\mathbf{p}, t)v(\mathbf{p}, t) \le \sup_{t=0} \mathcal{S}v \le \sqrt{1+L_0^2}\,(\sup_{t=0} \mathcal{S}).$$

The assertion follows. □

5 Asymptotic behaviour

In this section we study the long time behaviour of the solutions to the equation (1). As in the case of the mean curvature flow, see [9, 16, 19], it turns out that M_t moves out to infinity as time increases, and that it converges to a smooth limit after an appropriate rescaling.

As in [9], the convergence of the rescaled flow will be proved under an additional assumption on the initial value which controls the possible oscillations at infinity: there exist constants $0 < \delta < 1$ and $K_0 > 0$ such that

$$\langle \mathbf{F}_0, \nu\rangle^2 \le K_0 \left(1 + |\mathbf{F}_0|^2\right)^{1-\delta}. \tag{25}$$

Let us first show that this condition is preserved by the flow, provided we allow the constant K to depend on time.

Proposition 12. *If the initial data satisfy* (25) *for some constant $\delta > 0$, then there exists $K(t) > 0$ such that*

$$\langle \mathbf{F}, \nu\rangle^2 \le K(t) \left(1 + |\mathbf{F}|^2\right)^{1-\delta}.$$

Proof. If we set $f := \langle \mathbf{F}, \nu\rangle$, then we have

$$\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star} Hess_{\nabla}\right) f = -2\mathcal{S} + \dot{\mathcal{S}}\left(\mathcal{W}^2\right) f. \tag{26}$$

Let B be the constant such that (4) holds on the cone $\Gamma(m_0')$ where the curvatures of our solution are contained, by Theorem 7. Since our solution has bounded curvature, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star} Hess_{\nabla}\right) f^2 &\le -2\dot{\mathcal{S}}g^{\star}(\nabla f, \nabla f) - 4\mathcal{S}f + 2B|A|^2 \\ &\le -2\dot{\mathcal{S}}g^{\star}(\nabla f, \nabla f) + C(1+f^2),\end{aligned}$$

where C denotes any constant independent of t. Let us then consider the function $\eta_1 = 1 + |\mathbf{y}|^2 + 2nBt$. Then we find, using (4) and (22),

$$\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star} Hess_{\nabla}\right) \eta_1 \ge 0. \tag{27}$$

We now set $\rho = \eta_1^{\delta-1}$ where δ is the constant in assumption (25). By (27) we have

$$\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star} Hess_{\nabla}\right) \rho = (\delta-1)\,\eta_1^{\delta-2}\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star} Hess_{\nabla}\right) \eta_1 - (\delta-1)(\delta-2)\,\eta_1^{\delta-3}\dot{\mathcal{S}}g^{\star}(\nabla\eta_1, \nabla\eta_1) \le 0$$

and using the evolution equation of f^2

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star}Hess_{\nabla}\right) f^2\rho &= \rho\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star}Hess_{\nabla}\right) f^2 + f^2\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star}Hess_{\nabla}\right)\rho - 4f\dot{\mathcal{S}}g^{\star}(\nabla f, \nabla\rho) \\ &\leq C\left(1+f^2\right)\rho - \frac{2}{\rho}\dot{\mathcal{S}}g^{\star}\left(\nabla\left(f^2\rho\right), \nabla\rho\right) + 2f^2\rho^{-1}\dot{\mathcal{S}}g^{\star}(\nabla\rho, \nabla\rho).\end{aligned}$$

Estimate (4), together with $|\mathbf{y}|^2 \leq \eta_1$ and $\eta_1^{-1} < 1$, gives

$$\dot{\mathcal{S}}g^{\star}(\nabla\rho, \nabla\rho) = (\delta-1)^2\eta_1^{-2}\rho^2\dot{\mathcal{S}}g^{\star}\left(\nabla|\mathbf{y}|^2, \nabla|\mathbf{y}|^2\right) \leq 4B\eta_1^{-1}\rho^2 \leq 4B\rho^2$$

and we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \dot{\mathcal{S}}g^{\star}Hess_{\nabla}\right) f^2\rho &\leq -\frac{2}{\rho}\dot{\mathcal{S}}g^{\star}\left(\nabla\left(f^2\rho\right), \nabla\rho\right) + C\rho + \left[C + 2\rho^{-2}\dot{\mathcal{S}}g^{\star}(\nabla\rho, \nabla\rho)\right] f^2\rho \\ &\leq -\frac{2}{\rho}\dot{\mathcal{S}}g^{\star}\left(\nabla\left(f^2\rho\right), \nabla\rho\right) + C\left(1+f^2\rho\right).\end{aligned}$$

Since $f^2\rho$ grows at most polynomially in space and the vector field $\nabla\rho$ is uniformly bounded, we can apply the maximum principle (Theorem 8) to deduce that the function $e^{-Ct}(1+f^2\rho)$ is bounded by its supremum at initial time, which is not larger than $1+K_0$ by assumption (25). We conclude that

$$\begin{aligned}\langle\mathbf{F}, \nu\rangle^2 &\leq (1+K_0)e^{Ct}\left(1+|\mathbf{F}|^2+2nBt\right)^{1-\delta} \\ &\leq (1+K_0)e^{Ct}(1+2nBt)^{1-\delta}(1+|\mathbf{F}|^2)^{1-\delta}\end{aligned}$$

as required. □

As in [9], we now define

$$\tilde{\mathbf{F}} = \frac{1}{\sqrt{1+2t}}\mathbf{F}$$

with time parameter $s = \frac{1}{2}\log(1+2t)$ for $s \in [0, \infty)$. We denote with a tilde all the geometric quantities associated with the rescaled flow. It is easily checked that the rescaled flow satisfies

$$\frac{\partial}{\partial s}\tilde{\mathbf{F}} = -\tilde{\mathcal{S}}\tilde{\nu} - \tilde{\mathbf{F}}. \tag{28}$$

The next theorem describes the asymptotic behaviour of our solutions to problem (5) under assumption (25), showing that the solution $\tilde{M}_s$ of the normalized equation (28) converges to a limiting surface $\tilde{M}$ such that

$$\tilde{\mathbf{F}}^{\perp} = -\tilde{\mathcal{S}}\tilde{\nu}.$$

Such an equation characterizes the selfsimilar solutions of (1) which are homothetically expanding.

Theorem 13. *Let us consider problem* (1)–(2), *where the speed* $\mathcal{S}$ *satisfies assumptions* (H1)–(H4) *and the initial hypersurface* M_0 *satisfies* (ID) *and assumption* (25). *Then there exists a solution to the flow for all positive times and the corresponding solution of the normalized flow converges for* $s \to \infty$ *to a limiting selfsimilar expanding surface satisfying*

$$\tilde{\mathcal{S}} + \left\langle\tilde{\mathbf{F}}, \tilde{\nu}\right\rangle = 0.$$

Proof. Using equations (10) and (26) one can compute that on the rescaled solution we have

$$\left(\frac{\partial}{\partial s} - \widetilde{\dot{\mathcal{S}}}g^{\star}Hess_{\nabla}\right)\left(\tilde{\mathcal{S}} + \left\langle\tilde{\mathbf{F}}, \tilde{\nu}\right\rangle\right) = \left[\widetilde{\dot{\mathcal{S}}}\left(\mathcal{W}^2\right) - 1\right]\left(\tilde{\mathcal{S}} + \left\langle\tilde{\mathbf{F}}, \tilde{\nu}\right\rangle\right).$$

In addition, from equation (23) we deduce

$$\left(\frac{\partial}{\partial s} - \widetilde{\dot{\mathcal{S}}}g^{\star}Hess_{\nabla}\right)\tilde{v} = -2\tilde{v}^{-1}\widetilde{\dot{\mathcal{S}}}g^{\star}(\nabla\tilde{v}, \nabla\tilde{v}) - \widetilde{\dot{\mathcal{S}}}\left(\mathcal{W}^2\right)\tilde{v}.$$

We define the function $h := \left(\tilde{\mathcal{S}} + \langle \mathbf{F}, \tilde{v}\rangle\right)\tilde{v}$, which satisfies

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right) h &= \left[\tilde{\dot{\mathcal{S}}}\left(\mathcal{W}^2\right) - 1\right] h - 2\tilde{v}^{-2}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\tilde{v}, \nabla\tilde{v}) h - \tilde{\dot{\mathcal{S}}}\left(\mathcal{W}^2\right) h - 2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla\left(\tilde{\mathcal{S}} + \langle \mathbf{F}, \tilde{v}\rangle\right), \nabla\tilde{v}\right)\\ &= -h - 2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla h, \nabla\tilde{v}).\end{aligned}$$

We can also estimate

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right) h^2 &= 2h\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right) h - 2\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla h, \nabla h)\\ &\leq -2h^2 - 2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla h^2, \nabla\tilde{v}\right).\end{aligned}$$

Note that, by (22), the function $\eta_\sigma(\tilde{\mathbf{y}}) := 1 + \sigma|\tilde{\mathbf{y}}|^2$, with $\sigma \in (0,1)$, satisfies

$$\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right)\eta_\sigma(\tilde{\mathbf{y}}) \geq -2\sigma\left[|\tilde{\mathbf{y}}|^2 + nB\right]$$

and if $\rho := \eta_\sigma^{\varepsilon-1} e^{\gamma s}$ with $0 < \varepsilon < \delta \leq 1$ and $\gamma > 0$, then

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right)\rho &= (\varepsilon - 1)\eta_\sigma^{-1}\rho\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right)\eta_\sigma + \gamma\rho\\ &< \gamma\rho + 2(1-\varepsilon)[1 + \sigma nB]\rho,\end{aligned}$$

where we used $\sigma|\tilde{\mathbf{y}}|^2 < \eta_\sigma$ and $\eta_\sigma^{-1} \leq 1$. This allows us to compute the evolution equation for $\phi := h^2\rho$

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right)\phi &\leq -2\phi - 2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla h^2, \nabla\tilde{v}\right)\rho + \gamma\phi + 2(1-\varepsilon)\phi\\ &\quad +2(1-\varepsilon) nB\sigma\phi - 2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla h^2, \nabla\rho\right).\end{aligned}$$

To estimate this expression we use the equality

$$\begin{aligned}-2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla h^2, \nabla\tilde{v}\right)\rho - 2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla h^2, \nabla\rho\right) &= 2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\rho, \nabla\tilde{v}) h^2\\ +2\rho^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\rho, \nabla\rho) h^2 &- 2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla\phi, \tilde{v}^{-1}\nabla\tilde{v} + \rho^{-1}\nabla\rho\right).\end{aligned}$$

Then the estimates

$$\begin{aligned}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\rho, \nabla\rho) &= (1-\varepsilon)^2\sigma^2\eta_\sigma^{-2}\rho^2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla|\tilde{\mathbf{y}}|^2, \nabla|\tilde{\mathbf{y}}|^2\right)\\ &\leq 4(1-\varepsilon)^2\sigma^2\eta_\sigma^{-2}\rho^2 B|\tilde{\mathbf{y}}|^2 \leq 4B(1-\varepsilon)^2\sigma\eta_\sigma^{-1}\rho^2\end{aligned}$$

and $|\nabla\tilde{v}| = \tilde{v}^2|\nabla\langle\tilde{v}, \omega\rangle| \leq |\tilde{A}|\tilde{v}^2$ imply

$$\begin{aligned}2\tilde{v}^{-1}\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\rho, \nabla\tilde{v}) &\leq 2\tilde{v}^{-1}\left(\tilde{\dot{\mathcal{S}}} g^{\star}(\nabla\rho, \nabla\rho)\right)^{\frac{1}{2}}|\nabla\tilde{v}|\\ &\leq 4(1-\varepsilon)\sqrt{B}\sigma^{\frac{1}{2}}\eta_\sigma^{-\frac{1}{2}}|\tilde{A}|\tilde{v}\rho.\end{aligned}$$

Finally we can estimate

$$\begin{aligned}\left(\frac{\partial}{\partial s} - \tilde{\dot{\mathcal{S}}} g^{\star} Hess_{\nabla}\right)\phi &\leq -2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla\phi, \tilde{v}^{-1}\nabla\tilde{v} + \rho^{-1}\nabla\rho\right)\\ &\quad +4(1-\varepsilon)\sqrt{B}\eta_\sigma^{-\frac{1}{2}}\sigma^{\frac{1}{2}}|\tilde{A}|\tilde{v}\phi + 8(1-\varepsilon)^2 B\sigma\eta_\sigma^{-1}\phi\\ &\quad +\gamma\phi - 2\varepsilon\phi + 2(1-\varepsilon) nB\sigma\phi\\ &\leq -2\tilde{\dot{\mathcal{S}}} g^{\star}\left(\nabla\phi, \tilde{v}^{-1}\nabla\tilde{v} + \rho^{-1}\nabla\rho\right) + \left[C\sigma^{\frac{1}{2}} + \gamma - 2\varepsilon\right]\phi,\end{aligned}$$

for a suitable $C > 0$ independent of s. Choosing σ and γ in a suitable way with respect to ε, we can make the last term non positive. Now we check that the maximum principle Theorem 8 can be applied to the equation for ϕ. The function

$$\phi = \left(\tilde{\mathcal{S}} + \left\langle \tilde{\mathbf{F}}, \tilde{\nu} \right\rangle\right)^2 \tilde{\nu}^2 \left(1 + \sigma |\tilde{\mathbf{y}}|^2\right)^{\varepsilon-1} e^{\gamma s}$$

satisfies the required growth assumption on any interval $[0, \tau]$; in fact, it even decays at infinity by our requirement $0 < \varepsilon < \delta$ and by Proposition 12. The vector field $\tilde{\nu}^{-1}\nabla\tilde{\nu} + \rho^{-1}\nabla\rho$ is uniformly bounded by the previous computations. Then the maximum principle implies that ϕ is bounded by its initial values, and therefore we have

$$\sup_{\tilde{M}_s} \frac{(\tilde{\mathcal{S}} + \langle \tilde{\mathbf{F}}, \tilde{\nu} \rangle)^2 \tilde{\nu}^2}{(1 + \sigma |\tilde{\mathbf{F}}|^2)^{1-\varepsilon}} \leq \sup_{M_0} \frac{(\mathcal{S} + \langle \mathbf{F}, \nu \rangle)^2 \nu^2}{(1 + \sigma |\mathbf{F}|^2)^{1-\varepsilon}} e^{-\gamma s},$$

for suitable constants $\sigma > 0$ and $\gamma > 0$. This implies that $\tilde{\mathcal{S}} + \left\langle \tilde{\mathbf{F}}, \tilde{\nu} \right\rangle$ decays to zero exponentially fast on compact sets as $s \to \infty$, and the assertion follows. □

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