

More on microstate geometries of 4d black holes

M. Bianchi¹, J.F. Morales², L. Pieri^{1,3} and N. Zinnato¹

¹ *Università di Roma Tor Vergata and I.N.F.N, Dipartimento di Fisica,
Via della Ricerca Scientifica, I-00133 Rome, Italy*

² *I.N.F.N - sezione di Roma 2
and Università di Roma Tor Vergata, Dipartimento di Fisica
Via della Ricerca Scientifica, I-00133 Roma, Italy*

³ *center for Research in String Theory, School of Physics and Astronomy,
Queen Mary University of London,
Mile End Road, London, E1 4NS, United Kingdom*

E-mail: bianchi@roma2.infn.it, morales@roma2.infn.it,
lorenzo.pieri@roma2.infn.it, natale.zinnato@gmail.com

ABSTRACT: We construct explicit examples of microstate geometries of four-dimensional black holes that lift to smooth horizon-free geometries in five dimensions. Solutions consist of half-BPS D-brane atoms distributed in \mathbb{R}^3 . Charges and positions of the D-brane centers are constrained by the bubble equations and boundary conditions ensuring the regularity of the metric and the match with the black hole geometry. In the case of three centers, we find that the moduli spaces of solutions includes disjoint one-dimensional components of (generically) finite volume.

KEYWORDS: black holes, D-branes, micro-states

Contents

1	Introduction	2
2	Black holes from intersecting D3-branes	3
2.1	The asymptotic geometry	5
2.1.1	Orthogonal branes	7
2.1.2	Branes at angles	7
3	Microstate geometries	8
3.1	The eleven dimensional lift	8
3.2	The bubble equations	9
4	Fuzzballs of orthogonally intersecting branes	11
4.1	Three centers	11
4.1.1	Scaling solutions	12
4.1.2	Non-scaling solutions	13
5	Fuzzballs of branes at angles	14
5.1	Three centers	14
6	Conclusions	16
A	The ten dimensional solution and its 4d reduction	16
A.1	The ten dimensional solution	16
A.2	The four dimensional model	18
A.3	The basic solutions	19
A.3.1	L solutions	20
A.3.2	K solutions	20
A.3.3	M solutions	20
A.4	Sub-family of solutions	21
A.4.1	No scalars: IWP solution	21
A.4.2	One complex scalar: SWIP solutions	22
A.4.3	Two complex scalars	23

1 Introduction

Black holes are classical solutions of Einstein’s equations with curvature singularities hidden behind event horizons. According to the “no-hair theorem”, the solutions are unique in four dimensions, once the mass, charge and angular momentum are specified. At the classical level black holes are absolutely black and have zero statistical entropy $S = \log(1) = 0$. In a quantum theory however a black hole radiates as a black body with a finite temperature and an entropy given by one quarter of the area of its event horizon. To explain the microscopic origin of this entropy remains a primary task for any serious contender to a quantum theory of gravity.

In string theory, black holes can be realised in terms of D-branes intersecting in the internal space. The micro-states can be represented (and counted) in terms of excitations of the open strings connecting the building brane bits. Alternatively, one may think of the geometry generated by the excited brane state as the gravity representation of the micro-state. Since the micro-state geometry describes a pure state with zero entropy, it should have no horizon. This line of ideas motivates the “fuzzball” proposal that associates to every black hole micro-state a regular and horizon-free solution of classical gravity. The solutions, known as “fuzzballs” or “micro-state geometries”, share with the would-be black hole the mass, charges and angular momentum but differ from it in the interior [1–3]. The black hole horizon and its entropy arise from a coarse graining superposition of the micro-state geometries.

In the last years, a large class of four and five dimensional black hole micro-state geometries have been produced [4–13]. The micro-state geometries are typically coded in smooth, horizon-less geometries with no closed time-like curves (CTC’s) in five or six dimensions. This is the best one can achieve, since no-go theorems in four dimensions exclude the existence of non-singular asymptotically flat soliton solutions¹. This is not the case in five or higher dimensions where the existence of Chern-Simons interactions and spatial sections with non-trivial topologies circumvent the no-go result [16]. From a four-dimensional perspective, the finiteness of the higher dimensional Riemann tensor (and its derivatives) results into a finite effective action with curvature divergences compensated by the singular behaviour of the scalars and gauge fields.

A black hole with finite area in four dimensions can be realised in several different frames. Popular choices include bound states of D1-D5-KK-p, of D0-D2-D4-D6 branes [17–23] or of intersecting D3-branes, wrapping three cycles in $T^2 \times T^2 \times T^2$ [24–26]. After reduction down to four dimensions, the solution can be viewed as a supersymmetric vacuum of the $\mathcal{N} = 2$ truncation of $\mathcal{N} = 8$ supergravity involving the gravity multiplet and three vector multiplets characterizing the complex structures of the internal torus. In a microscopic world-sheet description of the D3-brane system [25, 26], the harmonic functions characterising the gravity solution are sourced by disk diagrams with a boundary ending on a single, two or four different branes [25, 27, 28]. Higher multipole modes are generated by extra insertions of untwisted open string fields on the disk boundaries. The micro-state multiplicities can be computed by counting open strings in the D1-D5-p-KK

¹Regular solutions with AdS asymptotics have been recently found in [14, 15]

system or vacua in the quantum mechanics associated to the D0-D2-D4-D6 realisation of the black hole [29, 30].

The aim of this paper is to construct explicit examples of micro-state geometries of four-dimensional BPS black holes. We follow the Bena-Warner ansatz [5, 16, 19] and look for regular five-dimensional geometries generated by distributions of half-BPS D-brane atoms in \mathbb{R}^3 . The regularity of the five dimensional geometry is coded in the so called bubble equations that we generalise to account for the case of branes at angles. Boundary conditions at infinity further restrict the choices leading generically to a moduli space that consists of disjoint components. There are two classes of solutions. Scaling solutions are configuration that can be rigidly scaled (see [18, 31–33] for some results). The other class includes solutions where the distances between the centers are bounded by the charges.

The micro-state geometries that we find carry in general non-zero angular momentum. This may look surprising, since single-center BPS black holes in four dimensions cannot carry angular momentum, because rotations of a black hole horizon are not compatible with supersymmetry. In our case, like for the multi center BPS black holes with non zero angular momentum considered in [34], the angular momentum is generated by the crossed electric and magnetic fields of charges separated on \mathbb{R}^3 . In the spirit of the fuzzball proposal, the black hole can be viewed as the superposition of an ensemble of geometries with all allowed angular momenta. The statistical average exposes zero angular momentum, even though each micro-state can carry some².

The plan of the paper is as follows. In Section 2 we present the BPS solutions describing systems of intersecting D3-branes on T^6 from the four dimensional perspective. We consider both cases of orthogonal and of intersecting D3-branes at angles. The Bena-Warner ansatz is introduced in Section 3. The ansatz is generalised to accomodate for non-orthogonally intersecting D3-branes. In section 4 and 5 solutions to the bubble equations and to the boundary conditions are found for the 3-center case with orthogonal or D3-branes intersecting at angles. A preliminary discussion of the counting of the number of micro-states with fixed charges is presented in the Conclusions. In Appendix A we derive the four dimensional solution from dimensional reduction of the intersecting D3-branes solution in ten dimensional type IIB supergravity.

2 Black holes from intersecting D3-branes

In this paper we consider a family of BPS solutions describing systems of intersecting D3-branes on T^6 from the four dimensional perspective. We refer the reader to the appendix for details on the ten dimensional solution and its reduction down to four dimensions. The four dimensional solution has been first explicitly derived in [19] in the D0-D2-D4-D6 type IIA frame and lifted to a system of M5 branes in M-theory. Here, for convenience, we consider the very symmetric formulation in terms of D3-branes, but the results can be easily translated into the type IIA and M-theory frame.

²Here we have in mind a canonical ensemble interpretation. We thank Iosif Bena for clarifications on this point.

The four-dimensional geometries can be viewed as solutions of an $\mathcal{N} = 2$ truncation of $\mathcal{N} = 8$ supergravity involving the gravity multiplet and three vector multiplets. The scalars U^I in the vector multiplets, usually referred as STU, parametrises the complex structures of the three internal T^2 's and span the moduli space $\mathcal{M}_{STU} = [SL(2, R)/U(1)]^3 \subset E_{7(+7)}/SU(8) = \mathcal{M}_{\mathcal{N}=8}$. Setting $16\pi G = 1$, the lagrangian can be written as

$$\mathcal{L} = \sqrt{g_4} \left(R_4 - \sum_{I=1}^3 \frac{\partial_\mu U_I \partial^\mu \bar{U}_I}{2(\text{Im } U_I)^2} - \frac{1}{4} F_a \mathcal{I}^{ab} F_b - \frac{1}{4} F_a \mathcal{R}^{ab} \tilde{F}_b \right) \quad (2.1)$$

where

$$U_I = (\sigma + is, \tau + it, \nu + iu) \quad F_a = dA_a \quad \tilde{F}_a = *_4 F_a \quad (2.2)$$

are the three complex scalars in the vector multiplets characterising the complex structures of the internal $T^2 \times T^2 \times T^2$. In these variables the kinetic functions read

$$\mathcal{I}^{ab} = stu \begin{pmatrix} 1 + \frac{\sigma^2}{s^2} + \frac{\tau^2}{t^2} + \frac{\nu^2}{u^2} & -\frac{\sigma}{s^2} & -\frac{\tau}{t^2} & -\frac{\nu}{u^2} \\ -\frac{\sigma}{s^2} & \frac{1}{s^2} & 0 & 0 \\ -\frac{\tau}{t^2} & 0 & \frac{1}{t^2} & 0 \\ -\frac{\nu}{u^2} & 0 & 0 & \frac{1}{u^2} \end{pmatrix} \quad \mathcal{R}^{ab} = \begin{pmatrix} 2\sigma\tau\nu & -\tau\nu & -\nu\sigma & -\sigma\tau \\ -\tau\nu & 0 & \nu & \tau \\ -\nu\sigma & \nu & 0 & \sigma \\ -\sigma\tau & \tau & \sigma & 0 \end{pmatrix} \quad (2.3)$$

The solutions will be written in terms of eight harmonic functions

$$\{V, L_I, K_I, M\} \quad (2.4)$$

on \mathbb{R}^3 . It is convenient to introduce the combinations

$$\begin{aligned} Z_I &= L_I + \frac{|\epsilon_{IJK}| K_J K_K}{2V}, \\ \mu &= \frac{M}{2} + \frac{L_I K_I}{2V} + \frac{|\epsilon_{IJK}| K_I K_J K_K}{6V^2}. \end{aligned} \quad (2.5)$$

Here ϵ_{IJK} characterise the triple intersections among the three T_I^2 2-cycles in T^6 .

The solutions can then be written as

$$\begin{aligned} ds^2 &= -e^{2U} (dt + w)^2 + e^{-2U} \sum_{i=1}^3 dx_i^2, \\ A_a &= (A_0, A_I) = w_a + a_a (dt + w) \\ U_I &= -b_I + i (V e^{2U} Z_I)^{-1} \end{aligned} \quad (2.6)$$

with

$$\begin{aligned} b_I &= \frac{K_I}{V} - \frac{\mu}{Z_I}, \quad a_0 = -\mu V^2 e^{4U}, \quad a_I = V e^{4U} \left(-\frac{Z_1 Z_2 Z_3}{Z_I} + K_I \mu \right) \\ *_3 dw_0 &= dV, \quad *_3 dw_I = -dK_I, \quad *_3 dw = \frac{1}{2} (V dM - M dV + K_I dL_I - L_I dK_I) \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} e^{-4U} &= \mathcal{I}_4(L_i, V, K_I, M) \equiv Z_1 Z_2 Z_3 V - \mu^2 V^2 \\ &= L_1 L_2 L_3 V - K_1 K_2 K_3 M + \frac{1}{2} \sum_{I>J}^3 K_I K_J L_I L_J - \frac{MV}{2} \sum_{I=1}^3 K_I L_I - \frac{1}{4} M^2 V^2 - \frac{1}{4} \sum_{I=1}^3 K_I^2 L_I^2 \end{aligned} \quad (2.8)$$

2.1 The asymptotic geometry

For general choices of the eight harmonic functions, the solution (2.6) is singular. Both naked and ‘horizon-dressed’ curvature singularities can be present. The generic solution is characterised by a mass \mathfrak{M} , associated to the Killing vector $\xi^{(t)M}\partial_M = \partial_t$, four electric charges Q_a and four magnetic charges P_a . Introducing the symplectic vector

$$\mathcal{F} = \begin{pmatrix} F_a \\ \frac{\delta\mathcal{L}}{\delta F_a} \end{pmatrix} = \begin{pmatrix} F_a \\ \star_4 I^{ab} F_b - R^{ab} F_b \end{pmatrix} \quad (2.9)$$

one finds for the charges

$$\begin{aligned} \mathfrak{M} &= -\frac{1}{8\pi G} \int_{S_\infty^2} \star_4 d\xi^{(t)} \\ \begin{pmatrix} P_a \\ Q^a \end{pmatrix} &= -\frac{1}{4\pi} \int_{S_\infty^2} \mathcal{F} \end{aligned} \quad (2.10)$$

with $\xi^{(t)} = \xi_M^{(t)} dx^M$ and S_∞^2 the two sphere at infinity. Solutions with extra symmetries arise for special choices of the harmonic functions. Axially symmetric solutions are characterised by the existence of an additional Killing vector $\xi^{(\phi)M}\partial_M = \partial_\phi$ associated to rotations around an axis in \mathbb{R}^3 , and carry an extra quantum number, the angular momentum J given by³

$$J = -\frac{1}{16\pi G} \int_{S_\infty^2} \star_4 d\xi^{(\phi)} \quad (2.11)$$

with $\xi^{(\phi)} = \xi_M^{(\phi)} dx^M$. Spherical symmetric solutions are invariant under rotations around the origin and are characterized by zero angular momentum.

In this paper we consider fuzzballs of spherically symmetric black holes. The harmonic functions specifying the general spherically symmetric solution can be written in the single-center form

$$V = v_0 + \frac{v}{r} \quad L_I = l_{0I} + \frac{l_I}{r} \quad K_I = k_{0I} + \frac{k_I}{r} \quad M = m_0 + \frac{m}{r} \quad (2.12)$$

and describe a general system of intersecting D3-branes wrapping three cycles on $T^2 \times T^2 \times T^2$ with one leg on each of the three T^2 . The absence of Dirac-Misner strings requires that w vanishes at infinity or, equivalently, that $\star_3 dw \sim r^{-3}$ at infinity leading to the constraint

$$v_0 m - m_0 v + k_{I0} l_I - l_{I0} k_I = 0 \quad (2.13)$$

For simplicity we take $m_0 = m = 0$. For this choice one finds

$$e^{-4U} = V L_1 L_2 L_3 - \frac{1}{4} \left(\sum_{I=1}^3 K_I L_I \right)^2 \quad (2.14)$$

³Being a surface integral, the expression for J holds true even when $\xi^{(\phi)}$ is only an asymptotic Killing vector.

Poles and zeros of this function are associated to horizons and curvature singularities respectively. If $e^{-4U} > 0$ for all $r > 0$ the solution describes a black hole with near horizon geometry $AdS_2 \times S^2$ and entropy proportional to

$$\lim_{r \rightarrow 0} r^4 e^{-4U} = \mathcal{I}_4 = v l_1 l_2 l_3 - \frac{1}{4} \left(\sum_{I=1}^3 k_I l_I \right)^2 > 0 \quad (2.15)$$

If e^{-4U} has zeros for some positive r , the solution exposes a naked singularity.

The charges of the solution (or its fuzzball) are computed by the integrals (2.17) evaluated in the asymptotic geometries (2.12). Writing the three-dimensional metric in spherical coordinates

$$ds^2 = -e^{2U} (dt + w)^2 + e^{-2U} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (2.16)$$

and setting $G = (16\pi)^{-1}$ one finds for the charges⁴

$$\begin{aligned} \mathfrak{M} &= 8\pi r^2 \partial_r e^{2U} \\ \begin{pmatrix} P_a \\ Q^a \end{pmatrix} &= \begin{pmatrix} (v, -k_I)^T \\ -r^2 \mathcal{I}^{ab} \partial_r a_b \end{pmatrix} \end{aligned} \quad (2.17)$$

where we used the fact that at infinity $w = 0$ and $F_a = dw_a + da_a dt$. On the other hand the angular momentum of the fuzzball is computed by the integral (2.11). We notice that the evaluation of this integral requires a more detailed knowledge of the asymptotic geometry since the angular momentum arises from the first dipole mode in the expansion of the harmonic function. Indeed, denoting

$$H = h_0 + \frac{h_1}{r} + \frac{\vec{h}_2 \cdot \vec{x}}{r^3} \quad (2.18)$$

one finds for the angular momentum

$$\vec{J} = 4\pi \left[m_0 \vec{v}_2 - v_0 \vec{m}_2 + l_{0I} \vec{k}_{2I} - k_{0I} \vec{l}_{2I} \right] \quad (2.19)$$

We anticipate here that apart from the scaling solutions, all fuzzball solutions we will find here carry a non-trivial angular momentum. We observe that for orthogonal branes angular momentum is carried by K-components (see Appendix A.3 for details) corresponding to open string condensates on disks with boundary on two different D3-brane stacks. Indeed, an explicit microscopic description of the general supergravity solution exists if the harmonic functions satisfy the boundary conditions [25]

$$m_2 + \sum k_{I2} = 0 \quad (2.20)$$

As we will see, only scaling solutions in the list of examples we find satisfy this restriction on the dipole modes.

⁴In our conventions $\star_4 dr \wedge dt = e^{-2U} r^2 \sin \theta d\theta \wedge d\phi$ and $\int \sin \theta d\theta \wedge d\phi = 4\pi$.

2.1.1 Orthogonal branes

We first consider the supergravity solution characterised by the harmonic functions

$$V = 1 + \frac{v}{r} \quad L_I = 1 + \frac{l_I}{r} \quad K_I = M = 0 \quad (2.21)$$

describing a system of four stacks of D3-branes intersecting orthogonally on T^6 . At large distances one finds

$$\begin{aligned} e^{-2U} &= \sqrt{VL_1L_2L_3} = 1 + \frac{(v + l_1 + l_2 + l_3)}{2r} + \dots \\ a_I &= -L_I^{-1} = -1 + \frac{l_I}{r} + \dots \quad a_0 = 0 \\ U_I &= i(Ve^{2U}L_I)^{-1} = i + \dots \end{aligned} \quad (2.22)$$

leading to

$$\mathfrak{M} = 4\pi(v + l_1 + l_2 + l_3) \quad (2.23)$$

$$\begin{pmatrix} P_a \\ Q^a \end{pmatrix} = \begin{pmatrix} (v, 0, 0, 0)^T \\ (0, l_1, l_2, l_3)^T \end{pmatrix} \quad (2.24)$$

The extremal Reissner Nordstrom solution corresponds to the choice $l_I = v = Q/2$, or equivalently

$$L_I = V = 1 + \frac{l}{r} \quad M = K_I = 0 \quad (2.25)$$

after the identification $F_{\text{RN}} = \frac{1}{2}(*F_0 + \sum_{I=1}^3 F_I)$.

2.1.2 Branes at angles

We next consider the supergravity solution characterised by the harmonic functions

$$V = 1 + \frac{v}{r} \quad L_I = 1 + \frac{l_I}{r} \quad K_1 = g + \frac{k_1}{r} \quad K_2 = g \quad K_3 = M = 0 \quad (2.26)$$

The absence of Dirac-Misner strings (2.13) requires $k_1 = g(l_1 + l_2)$. The resulting solution is equivalent after a duality transformation to the solution found in [24] describing a system of D3 branes intersecting at a non-trivial angle between the branes at infinity, parametrised by g .

The asymptotic solution at large r becomes

$$\begin{aligned} e^{2U} &= 1 - \frac{v + l_1 + l_2 + l_3}{2r} + \dots \quad U_I = \left(i, i, \frac{1}{g-i}\right) + \dots \\ w_a &= (v \cos \theta d\phi, -k_1 \cos \theta d\phi, 0, 0) + \dots \\ a_a &= \left(\frac{gl_3}{r}, \frac{(1+g^2)l_1}{r}, \frac{l_2 - g^2l_1}{r}, \frac{l_3}{r}\right) + \dots \\ \mathcal{I}^{ab} &= \begin{pmatrix} 1 & 0 & 0 & -g \\ 0 & \frac{1}{1+g^2} & 0 & 0 \\ 0 & 0 & \frac{1}{1+g^2} & 0 \\ -g & 0 & 0 & 1+g^2 \end{pmatrix} \quad \mathcal{R}^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{g}{1+g^2} & 0 \\ 0 & \frac{g}{1+g^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (2.27)$$

with dots denoting higher order terms in the expansion for large r . Plugging (2.27) into (2.17) one finds for the charges⁵

$$\mathfrak{M} = 4\pi (v + l_1 + l_2 + l_3) \quad (2.29)$$

$$\begin{pmatrix} P_a \\ Q^a \end{pmatrix} = \begin{pmatrix} (v, -g(l_1 + l_2), 0, 0)^T \\ (0, l_1, l_2, l_3)^T \end{pmatrix} \quad (2.30)$$

Due to different conventions and duality frames, it is not easy to compare the charges with the corresponding ones in [24].

3 Microstate geometries

In this section we review the Bena-Warner multi-Taub NUT ansatz for fuzzi-ball geometries of four- and five-dimensional black holes. We slightly generalise the ansatz to accommodate for non-orthogonal brane intersections and derive the corresponding bubble equations. In the next section we present explicit horizon-free solutions with three centers.

3.1 The eleven dimensional lift

The four-dimensional solution (2.6) lifts to an eleven dimensional solution representing a systems of intersecting M5-branes with four electric and four magnetic charges. The eleven dimensional metric is given by [5]:

$$ds^2 = ds_5^2 + ds_{T^6}^2 \quad (3.1)$$

where

$$\begin{aligned} ds_5^2 &= -(Z_1 Z_2 Z_3)^{-\frac{2}{3}} [dt + \mu(d\Psi + w_0) + w]^2 + (Z_1 Z_2 Z_3)^{\frac{1}{3}} [V^{-1}(d\Psi + w_0)^2 + V d\vec{x}^2] \\ ds_{T^6} &= \sum_{I=1}^3 \left(\frac{Z_1 Z_2 Z_3}{Z_I^3} \right)^{\frac{1}{3}} (dy_I^2 + d\tilde{y}_I^2) \end{aligned} \quad (3.2)$$

in which the coordinates associated to $\mathbb{R} \times S^1 \times T^6$ are respectively $\{t, \vec{x}, \Psi, y_I, \tilde{y}_I\}$ with $I = 1, 2, 3$. Micro-states of the four dimensional black holes can be generically defined as smooth geometries with no horizons or curvature singularities in eleven dimensions carrying the same mass and charges as the corresponding black hole. Regular solutions can be constructed in terms of multi-center harmonic functions (V, L_I, K_I, M) with the positions of the centers and the charges chosen such that Z_I are finite and $\mu = 0$ near the centers. Under these assumptions one finds that the eleven dimensional metric (3.1) near

⁵For the central charges $Z_I = e^{-U} (2\text{Im}U_I)^{-1} r^2 \partial_r U_I$, $Z_4 = i e^{-U} r^2 \partial_r U$ one finds at infinity

$$\begin{aligned} Z_1 &= 4\pi [2g l_1 + i(v + l_1 - l_2 - l_3)] & Z_2 &= 4\pi [-2g l_1 + i(v - l_1 + l_2 - l_3)] \\ Z_3 &= 4\pi i(v - l_1 - l_2 + l_3) & Z_4 &= 4\pi i(v + l_1 + l_2 + l_3) \end{aligned} \quad (2.28)$$

showing the saturation of the BPS bound $\mathfrak{M} = |Z_4| \geq |Z_i|$ for $i \neq 4$, when v and l_I all have the same sign and g is sufficiently small.

the centers is $\mathbb{R} \times T^6 \times \mathbb{R}^4 / \mathbb{Z}_{|q_i|}$. To avoid orbifold singularities we will henceforth take $|q_i| = 1$. Moreover, the absence of horizons and closed time-like curves requires that

$$Z_I V > 0 \quad \text{and} \quad e^{2U} > 0 \quad (3.3)$$

Let us remark that the condition $Z_I V > 0$ near the centers requires

$$Z_I V|_{r_i=0} = q_i \left(l_{0I} + \sum_{j \neq i} \frac{l_{I,j}}{r_{ij}} \right) + l_{I,i} \left(v_0 + \sum_{j \neq i} \frac{q_j}{r_{ij}} \right) + C_{IJK} k_i^J \left(k_0^K + \sum_{j \neq i} \frac{k_j^K}{r_{ij}} \right) > 0 \quad (3.4)$$

It turns out that these necessary conditions often are enough to ensure the positivity of both $Z_I V$ and e^{2U} on the whole \mathbb{R}^3 . In the next section we look for explicit solutions of these requirements satisfying the boundary conditions (2.12). We stress that the resulting solutions are regular everywhere in five dimensions and fall off at infinity to $\mathbb{R}^{1,3} \times S^1$. The four-dimensional fuzzball solution follows from reduction of this five-dimensional geometry down to four dimensions where the apparent singularity in the geometry is balanced by a blow up of the scalar fields.

3.2 The bubble equations

We consider N-center harmonic functions following the ansatz

$$\begin{aligned} V &= v_0 + \sum_{i=1}^N \frac{q_i}{r_i} \quad , \quad L_I = l_{0I} + \sum_{i=1}^N \frac{l_{I,i}}{r_i} \\ K^I &= k_0^I + \sum_{i=1}^N \frac{k_i^I}{r_i} \quad , \quad M = m_0 + \sum_{i=1}^N \frac{m_i}{r_i} \end{aligned} \quad (3.5)$$

with $r_i = |\mathbf{y}_i - \mathbf{x}|$ and \mathbf{y}_i the position of the i^{th} center. We notice that $(\ell_{I,i}, m_i)$ and (q_i, k_i^I) describe the electric and magnetic fluxes of the four dimensional gauge fields through the sphere encircling the i^{th} - centers, so Dirac quantisation requires that they be quantised. Here we adopt units such that they are all integers. Alternatively, one can think of the eight charges as parametrising the number of D3-branes wrapping one of the eight three-cycles with a leg on each of the three tori T^2_j in the factorisation $T^6 = T^2_1 \times T^2_2 \times T^2_3$. In other words, in our units each charge describes the number of D3-branes of a certain kind.

We look for regular five dimensional geometries behaving as $\mathbb{R} \times \text{Taub-NUT}$ near the centers. It is easy to see that w vanishes near the centers, so the Taub-NUT geometry factorises if Z_I are finite and μ vanishes near the centers, i.e.

$$\begin{aligned} Z_I|_{r_i \approx 0} &\approx \zeta_i^I \\ \mu|_{r_i \approx 0} &\approx 0 \end{aligned} \quad (3.6)$$

with ζ_i^I some finite constants. The conditions that Z_I is finite near the centers can be solved by taking

$$\begin{aligned} \ell_{I,i} &= -\frac{|\epsilon_{IJK}| k_i^J k_i^K}{2 q_i} \\ m_i &= \frac{k_i^1 k_i^2 k_i^3}{q_i^2} \end{aligned} \quad (3.7)$$

The vanishing of μ near the centers boils down to the so called *bubble equations*

$$\sum_{j=1}^N \Pi_{ij}^{(1)} \Pi_{ij}^{(2)} \Pi_{ij}^{(3)} \frac{q_i q_j}{r_{ij}} + v_0 \frac{k_i^1 k_i^2 k_i^3}{q_i^2} - \sum_{I=1}^3 l_{0I} k_i^I - |\epsilon_{IJK}| \frac{k_{0I} k_i^J k_i^K}{2 q_i} - m_0 q_i = 0 \quad (3.8)$$

with

$$\Pi_{ij}^{(I)} = \frac{k_i^I}{q_i} - \frac{k_j^I}{q_j} \quad r_{ij} = |\mathbf{y}_i - \mathbf{y}_j| \quad (3.9)$$

Indeed the conditions (3.7) ensure that both Z_I and μ are finite near the centers while the bubble equations follows from the requirement that μ vanishes near the center. The bubble equations ensure also the absence of Dirac-Misner strings. To see this, we notice that using the bubble equations, the w function defined by (2.7) can be written in the form

$$\begin{aligned} *_3 dw &= \sum_{i,j=1}^N (q_{[i} m_{j]} + k_{[i}^I l_{j],I}) \frac{1}{r_i} d\frac{1}{r_j} + \frac{1}{2} \sum_{i=1}^N \left(v_0 m_i - m_0 q_i - \sum_{I=1}^3 (l_{0I} k_i^I - k_{I0} l_{i,I}) \right) d\frac{1}{r_i} \\ &= \frac{1}{2} \sum_{i,j=1}^N \Pi_{ij}^{(1)} \Pi_{ij}^{(2)} \Pi_{ij}^{(3)} q_i q_j \left(\frac{1}{r_j} - \frac{1}{r_{ij}} \right) d\frac{1}{r_i} \end{aligned} \quad (3.10)$$

where in the second line we used equations (3.7), (3.8) and $A_{[BC]}$ means $\frac{1}{2}(A_{BC} - A_{CB})$. The solution can be written in the form

$$w = \frac{1}{4} \sum_{i,j=1}^N \Pi_{ij}^{(1)} \Pi_{ij}^{(2)} \Pi_{ij}^{(3)} q_i q_j \omega_{ij} \quad (3.11)$$

in terms of the one forms ω_{ij} defined via the relation

$$*_3 d\omega_{ij} = \left(\frac{1}{r_j} - \frac{1}{r_{ij}} \right) d\frac{1}{r_i} - \left(\frac{1}{r_i} - \frac{1}{r_{ij}} \right) d\frac{1}{r_j}, \quad (3.12)$$

that is

$$\omega_{ij} = \frac{(\mathbf{n}_i + \mathbf{n}_{ij}) \cdot (\mathbf{n}_j - \mathbf{n}_{ij})}{r_{ij}} d\phi_{ij} \quad (3.13)$$

with

$$\mathbf{n}_i = \frac{\mathbf{x} - \mathbf{y}_i}{r_i} \quad \mathbf{n}_{ij} = \frac{\mathbf{y}_i - \mathbf{y}_j}{r_{ij}} \quad d\phi_{ij} = \frac{\mathbf{n}_{ij} \times \mathbf{n}_i \cdot d\mathbf{x}}{r_i [1 - (\mathbf{n}_{ij} \cdot \mathbf{n}_i)^2]}.$$

It is easy to see that ω_{ij} is free of Dirac-Misner singularities. Indeed along the dangerous lines connecting any two centers the numerator of (3.13) always vanish so no string-like singularity arises. One can also see that near the centers w goes to a constant and exact form.

Finally we notice that if the coefficients k_i^I satisfy the relation

$$v_0 m_i - \sum_{I=1}^3 l_{0I} k_i^I + k_{0I} l_{Ii} - m_0 q_i = 0 \quad (3.14)$$

the system of equations is invariant under overall rescalings of the center positions $\mathbf{y}_i \rightarrow \lambda \mathbf{y}_i$. These solutions are known as “scaling solutions”. Multiplying equation (3.14) by the positions of the centers \vec{y}_i and summing one finds that the scaling solutions satisfy

$$m_0 \vec{v}_2 - v_0 \vec{m}_2 + l_{0I} \vec{k}_{2I} - k_{0I} \vec{l}_{2I} = 0 \quad (3.15)$$

and therefore according to (2.19) they carry zero angular momentum.

4 Fuzzballs of orthogonally intersecting branes

We look for regular geometries with the asymptotics (2.21), i.e.

$$l_{0I} = v_0 = 1 \quad m_0 = m = k_{0I} = k_I = 0 \quad (4.1)$$

For concreteness we take $q_i = 1$. The charges of the fuzzball solutions are then

$$\begin{aligned} P_0 &= N \\ Q_I &= - \sum_{i=1}^N \frac{|\epsilon_{IJK}| k_i^J k_i^K}{2} \end{aligned} \quad (4.2)$$

The solution is specified by the the positions \mathbf{y}_i of the centers and the fluxes k_i^I . The positions of the centers are constrained by the bubble equations

$$\sum_{j \neq i}^N \frac{k_{ij}^{(1)} k_{ij}^{(2)} k_{ij}^{(3)}}{r_{ij}} + k_i^1 k_i^2 k_i^3 - \sum_{I=1}^3 k_i^I = 0 \quad (4.3)$$

with $k_{ij}^{(I)} = k_i^I - k_j^I$ while the match of the asymptotic geometries requires

$$\sum_{i=1}^N k_i^I = \sum_{i=1}^N k_i^1 k_i^2 k_i^3 = 0 \quad (4.4)$$

In addition, the absence of horizons and of closed time-like curves requires

$$Z_I V > 0 \quad \text{and} \quad e^{2U} > 0 \quad (4.5)$$

We will consider solutions with three centers. Configurations with one or two centers fail to meet the requirement $Q_I > 0$.

4.1 Three centers

The bubble equations (4.3) for three centers can be solved in general by taking

$$r_{12} = \frac{\Pi_{12} r_{23}}{\Pi_{23} - r_{23} (\Gamma_2 - \Lambda_2)} \quad r_{13} = \frac{\Pi_{13} r_{23}}{-\Pi_{23} + r_{23} (\Gamma_1 + \Gamma_2 - \Lambda_1 - \Lambda_2)} \quad (4.6)$$

with

$$\Pi_{ij} = \prod_{I=1}^3 (k_i^I - k_j^I) \quad \Gamma_i = \sum_{I=1}^3 k_i^I \quad \Lambda_i = k_i^1 k_i^2 k_i^3 \quad (4.7)$$

A solution given by (4.6) makes sense if the distances r_{ij} between the three centers are positive and they satisfy the triangle inequalities. This restricts significantly the choices for the k_i^I . A quick scan over the integers shows that boundary conditions are solved only if at least one of the fluxes k_i^I vanishes. Without loss of generality the general solution can then be parametrised in the form (up to permutations of rows and columns)

$$k_i^I = \begin{pmatrix} -\kappa_1 \kappa_2 & -\kappa_1 \kappa_3 & \kappa_1 (\kappa_2 + \kappa_3) \\ \kappa_3 & \kappa_2 & -\kappa_2 - \kappa_3 \\ -\kappa_4 & \kappa_4 & 0 \end{pmatrix} \quad (4.8)$$

Consequently the harmonic functions takes the general form

$$\begin{aligned} V &= 1 + \sum_{i=1}^3 \frac{1}{r_i} & M &= \kappa_1 \kappa_2 \kappa_3 \kappa_4 \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \\ L_1 &= 1 + \kappa_4 \left(\frac{\kappa_3}{r_1} - \frac{\kappa_2}{r_2} \right) & L_2 &= 1 + \kappa_1 \kappa_4 \left(-\frac{\kappa_2}{r_1} + \frac{\kappa_3}{r_2} \right) \\ L_3 &= 1 + \kappa_1 \left(\frac{\kappa_2 \kappa_3}{r_1} + \frac{\kappa_2 \kappa_3}{r_2} + \frac{(\kappa_2 + \kappa_3)^2}{r_3} \right) & K_1 &= \kappa_1 \left(-\frac{\kappa_2}{r_1} - \frac{\kappa_3}{r_2} + \frac{\kappa_2 + \kappa_3}{r_3} \right) \\ K_2 &= \frac{\kappa_3}{r_1} + \frac{\kappa_2}{r_2} - \frac{\kappa_2 + \kappa_3}{r_3} & K_3 &= \kappa_4 \left(-\frac{1}{r_1} + \frac{1}{r_2} \right) \end{aligned}$$

The charges and distances between the centers reduce to

$$\begin{aligned} Q_1 &= \kappa_4 (\kappa_3 - \kappa_2) & Q_2 &= \kappa_1 \kappa_4 (\kappa_3 - \kappa_2) & Q_3 &= \kappa_1 (\kappa_2^2 + 4\kappa_2 \kappa_3 + \kappa_3^2) \\ r_{12} &= \frac{2\kappa_1 \kappa_4 (\kappa_2 - \kappa_3)^2 r_{23}}{\kappa_1 \kappa_4 (2\kappa_2^2 + 5\kappa_2 \kappa_3 + 2\kappa_3^2) + (\kappa_2 + \kappa_4 - \kappa_1 \kappa_3 + \kappa_1 \kappa_2 \kappa_3 \kappa_4) r_{23}} \\ r_{13} &= \frac{\kappa_1 \kappa_4 (2\kappa_2 + \kappa_3) (\kappa_2 + 2\kappa_3) r_{23}}{\kappa_1 \kappa_4 (2\kappa_2^2 + 5\kappa_2 \kappa_3 + 2\kappa_3^2) - (\kappa_1 - 1) (\kappa_2 + \kappa_3) r_{23}}. \end{aligned} \quad (4.9)$$

4.1.1 Scaling solutions

The scaling solution corresponds to the choice

$$\kappa_2 = 0 \quad \kappa_1 = 1 \quad \kappa_3 = \kappa_4 = \kappa \quad (4.10)$$

One finds

$$\begin{aligned} k_i^I &= \begin{pmatrix} 0 & -\kappa & \kappa \\ \kappa & 0 & -\kappa \\ -\kappa & \kappa & 0 \end{pmatrix} & r_{12} &= r_{23} = r_{13} = \ell \\ P_0 &= 3 & Q_1 &= Q_2 = Q_3 = \kappa^2 \end{aligned} \quad (4.11)$$

for any given ℓ . The regularity conditions become

$$\begin{aligned} e^{-4U} &= 1 + \frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{r_1 r_2 r_3} + \frac{\kappa^2 (r_1 r_2 + r_1 r_3 + r_2 r_3 + 3r_1 + 3r_2 + 3r_3)}{r_1 r_2 r_3} \\ &+ \frac{\kappa^4 (r_1 + r_2 + r_3 + 9)}{r_1 r_2 r_3} + \frac{\kappa^6 (2r_1 r_2 + 2r_1 r_3 + 2r_2 r_3 + r_1 r_2 r_3 - r_1^2 - r_2^2 - r_3^2)}{r_1^2 r_2^2 r_3^2} > 0 \\ Z_1 V &= 1 + \frac{r_1 r_2 + r_1 r_3 + r_2 r_3 + \kappa^2 (2r_2 + 2r_3 - r_1 + r_2 r_3)}{r_1 r_2 r_3} > 0 \end{aligned} \quad (4.12)$$

The conditions $Z_2V > 0$ and $Z_3V > 0$ follow from $Z_1V > 0$ and the permutation symmetry of the system. The two conditions can be shown to be satisfied using the inequality

$$r_1 + r_2 - r_3 \geq 0 \quad (4.13)$$

holding for any point $x \in \mathbb{R}^3$ if r_i denotes the distance from the point to the three vertices of an equilateral triangle. This inequality can be proved using triangle inequalities [35].

We conclude that the five-dimensional geometry defined by the multi-center solution is regular everywhere. We notice that the fluxes satisfy the scaling condition (3.14) and consistently a rigid rescaling of the positions of the centers generate a new solution. More precisely, the moduli space of solutions with this charge is spanned by a single continuous parameter ℓ and permutations of the rows or columns of the matrix (4.11). There are 12 inequivalent choices corresponding to the $3!$ permutations of the entries in the first line in (4.11) times the two choices for the position of the 0 in the second line. The remaining entries are determined by the conditions that the sum along rows and columns of the matrix k_i^I should vanish. The number 12 matches the number of apostles and the degeneracy of four-charge black hole micro-states with the minimal unit of charge $P_0 = Q_I = 1$ [29]! Moreover the solutions satisfy $\sum_{I=1}^3 \vec{k}_2^I = \vec{m}_2 = 0$ and therefore according to (2.19) and (2.20) they carry zero angular momentum and admit a microscopic description in terms of orthogonal intersecting D3-branes along the lines of [25].

This solution can be shown⁶ to be related by dualities to a known supertube solution originally found in [36].

4.1.2 Non-scaling solutions

The analysis above can be repeated for more general choices of the fluxes but regularity conditions in general can only be verified numerically. Here we list some illustrative examples of the type of solutions one finds.

- $\kappa_2 = 0, \kappa_1 = \kappa_3 = 1, \kappa_4 = \kappa$:

$$k_i^I = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -\kappa & \kappa & 0 \end{pmatrix} \quad r_{13} = r_{23} \quad r_{12} = \frac{2\kappa r_{23}}{2\kappa + (\kappa - 1)r_{23}}$$

$$P_0 = 3 \quad Q_1 = Q_2 = \kappa \quad Q_3 = 1 \quad (4.14)$$

Interestingly, triangle inequalities in this case do not constrain r_{23} that can take arbitrarily large value.

- $\kappa_2 = \kappa_4 = \kappa, \kappa_1 = 1, \kappa_3 = 2\kappa$

$$k_i^I = \begin{pmatrix} -\kappa & -2\kappa & 3\kappa \\ 2\kappa & \kappa & -3\kappa \\ -\kappa & \kappa & 0 \end{pmatrix} \quad r_{12} = \frac{r_{23}}{10 + r_{23}} \quad r_{13} = r_{23}.$$

$$P_0 = 3 \quad Q_1 = Q_2 = \kappa^2 \quad Q_3 = 13\kappa^2 \quad (4.15)$$

As before r_{23} can take arbitrarily large value.

⁶We thank Iosif Bena for pointing this out to us.

- $\kappa_2 = 0, \kappa_1 = 3\kappa, \kappa_3 = 2\kappa, \kappa_4 = \kappa$

$$\begin{aligned}
k^I_i &= \begin{pmatrix} 0 & -3\kappa & 3\kappa \\ \kappa & 0 & -\kappa \\ -2\kappa & 2\kappa & 0 \end{pmatrix} & r_{12} &= \frac{12\kappa^2 r_{23}}{12\kappa^2 - r_{23}} & r_{13} &= \frac{6\kappa^2 r_{23}}{6\kappa^2 - r_{23}} \\
P_0 &= 3 & Q_1 &= 2\kappa^2 & Q_2 &= 6\kappa^2 & Q_3 &= 3\kappa^2 \\
r_{23} &< 6(2 - \sqrt{2})\kappa^2 & & & & & &
\end{aligned} \tag{4.16}$$

We notice that triangle inequality in this case impose an upper bound on r_{23} leading to a moduli space of finite volume.

5 Fuzzballs of branes at angles

We look for regular fuzzball geometries with the asymptotics (2.26), i.e.

$$l_{0I} = v_0 = 1 \quad m_0 = m = k_{03} = k_3 = k_2 = 0 \quad k_{01} = k_{02} = g \quad k_1 = g(l_1 + l_2) \tag{5.1}$$

This describes a fuzzball of a non-rotating black hole with charges

$$\begin{aligned}
P_0 &= N \\
Q_I &= -\frac{|\epsilon_{IJK}|}{2} \sum_{i=1}^N \frac{k_i^J k_i^K}{q_i}
\end{aligned} \tag{5.2}$$

The fuzzball is specified by the parameters k_i^I describing the magnetic fluxes through the two-spheres encircling the centers and the positions \mathbf{y}_i . The positions of the centers are constrained by the bubble equations

$$\sum_{j \neq i}^N \frac{k_{ij}^{(1)} k_{ij}^{(2)} k_{ij}^{(3)}}{r_{ij}} + k_i^1 k_i^2 k_i^3 - \sum_{I=1}^3 k_i^I - g k_i^2 k_i^3 - g k_i^1 k_i^3 = 0 \tag{5.3}$$

while the match of the asymptotic geometries requires

$$\sum_{i=1}^N k_i^2 = \sum_{i=1}^N k_i^3 = \sum_{i=1}^N k_i^1 k_i^2 k_i^3 = 0 \quad \sum_{i=1}^N k_i^1 = g(Q_1 + Q_2) \tag{5.4}$$

In addition, the absence of horizons and of closed time-like curves requires

$$Z_I V > 0 \quad \text{and} \quad e^{2U} > 0 \tag{5.5}$$

5.1 Three centers

The bubble equations (5.3) for three centers can be solved in general by taking

$$r_{12} = \frac{\Pi_{12} r_{23}}{\Pi_{23} - r_{23} (\Gamma_2 - \Lambda_2 + \Omega_2)} \quad r_{13} = -\frac{\Pi_{13} r_{23}}{\Pi_{23} - r_{23} (\Gamma_1 + \Gamma_2 - \Lambda_1 - \Lambda_2 + \Omega_1 + \Omega_2)} \tag{5.6}$$

with

$$\Pi_{ij} = \prod_{I=1}^3 (k_i^I - k_j^I) \quad \Gamma_i = \sum_I k_i^I \quad \Lambda_i = k_i^1 k_i^2 k_i^3 \quad \Omega_i = g k_i^2 k_i^3 + g k_i^1 k_i^3 \quad (5.7)$$

We consider the case

$$k_i^I = \begin{pmatrix} 0 & -\kappa_1 \kappa_1 + g \kappa_3 (\kappa_1 + \kappa_2) \\ \kappa_2 & 0 & -\kappa_2 \\ -\kappa_3 & \kappa_3 & 0 \end{pmatrix} \quad (5.8)$$

with $\kappa_1, \kappa_2, \kappa_3$ three positive integers and g a rational number. One finds

$$\begin{aligned} Q_1 &= \kappa_2 \kappa_3 & Q_2 &= \kappa_1 \kappa_3 & Q_3 &= \kappa_1 \kappa_2 + g \kappa_2 \kappa_3 (\kappa_1 + \kappa_2) \\ r_{12} &= \frac{2 \kappa_1 \kappa_2 \kappa_3 r_{23}}{2 \kappa_1 \kappa_2 \kappa_3 + g \kappa_2 \kappa_3^2 (\kappa_1 + \kappa_2) - (\kappa_1 - \kappa_3 + g \kappa_1 \kappa_3) r_{23}} \\ r_{13} &= \frac{2 [\kappa_1 \kappa_2 \kappa_3 + g \kappa_2 \kappa_3^2 (\kappa_1 + \kappa_2)] r_{23}}{2 \kappa_1 \kappa_2 \kappa_3 + g \kappa_2 \kappa_3^2 (\kappa_1 + \kappa_2) - [\kappa_1 - \kappa_2 + g \kappa_3 (\kappa_1 + \kappa_2)] r_{23}} \end{aligned} \quad (5.9)$$

The harmonic functions become

$$V = 1 + \sum_{i=1}^3 \frac{1}{r_i} \quad L_I = 1 + \frac{Q_I}{r_I} \quad M = 0 \quad (5.10)$$

$$K_1 = g + \left(-\frac{\kappa_1}{r_2} + \frac{\kappa_1 + \kappa_1}{r_3} \right) \quad K_2 = g + \kappa_2 \left(\frac{1}{r_1} - \frac{1}{r_3} \right) \quad K_3 = \kappa_3 \left(-\frac{1}{r_1} + \frac{1}{r_2} \right)$$

Some examples are

- $\kappa_1 = \kappa_2 = \kappa_3 = \kappa = (2g)^{-1}$

$$\begin{aligned} k_i^I &= \begin{pmatrix} 0 & -\kappa & 2\kappa \\ \kappa & 0 & -\kappa \\ -\kappa & \kappa & 0 \end{pmatrix} & r_{12} &= \frac{4 \kappa^2 r_{23}}{6 \kappa^2 - r_{23}} & r_{13} &= \frac{4 \kappa^2 r_{23}}{3 \kappa^2 - r_{23}} \\ Q_0 &= 3 & Q_1 &= Q_2 = \kappa^2 & Q_3 &= 2 \kappa^2 \\ r_{23} &< \frac{9 - \sqrt{57}}{2} \kappa^2. \end{aligned} \quad (5.11)$$

- $\kappa_1 = \kappa_2 = \kappa, \kappa_3 = 2\kappa, g = (4\kappa)^{-1}$

$$\begin{aligned} k_i^I &= \begin{pmatrix} 0 & -\kappa & 2\kappa \\ \kappa & 0 & -\kappa \\ -2\kappa & 2\kappa & 0 \end{pmatrix} & r_{12} &= \frac{8 \kappa^2 r_{23}}{12 \kappa^2 + r_{23}} & r_{13} &= \frac{8 \kappa^2 r_{23}}{6 \kappa^2 - r_{23}} \\ Q_0 &= 3 & Q_1 &= Q_2 = Q_3 = 2 \kappa^2 \\ r_{23} &< \kappa^2 (-11 + \sqrt{145}). \end{aligned} \quad (5.12)$$

We notice that in the two examples considered here distances are bounded by triangle inequalities leading to moduli spaces of finite volume.

6 Conclusions

We have constructed explicit examples of micro-state geometries of four-dimensional black holes. Following Bena, Gibbons and Warner, we have considered solutions consisting of half-BPS D-brane atoms with centers in \mathbb{R}^3 . Charges and positions of the centers are constrained by the bubble equations that ensure that the metric uplifts to a horizon-less and CTC free metric in five dimensions and boundary conditions that grant the match of the fuzzball and by the black hole geometries at infinity. As a result, divergences coming from curvature singularities in the four dimensional metric are compensated by the singular behaviours of the scalars and gauge fields, leading to a finite (higher-derivative) string effective action.

We have considered the case of three centers in some details and found that there are two broad classes of solutions. Scaling solutions and non-scaling ones. The moduli space of scaling solutions is described by a disjoint union of 12 one-dimensional components spanned by a single parameter (up to rigid rotations and translations of the systems) describing the distances between the centers. These solutions carry zero angular momentum and admit a microscopic description in terms of intersecting D3-branes along the lines of [25]. Non-scaling solution are described by disjoint unions of one-dimensional components with (generically) a finite volume bounded by the charges of the system.

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A The ten dimensional solution and its 4d reduction

In this appendix we collect some details on the dimensional reduction down to four dimensions of the eight harmonic family of BPS solutions describing a general system of intersecting D3-branes on T^6 .

A.1 The ten dimensional solution

The eight harmonic family of BPS solutions associated to D3-branes intersecting on T^6 is characterised by a metric g_{MN} and a four form Ramond field C_4 of the form [25]

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu + \sum_{I=1}^3 h_{mn}^I dy_I^m dy_I^n \\ C_4 &= C_{\mu,mnp} dx^\mu \wedge dy_1^m \wedge dy_2^n \wedge dy_3^p, \end{aligned} \tag{A.1}$$

with $\mu = 0, \dots, 3$, $m = 1, 2$. x^μ are the coordinates along the four-dimensional space time and $y_I^m = (y_I, \tilde{y}_I)$ span a $T^2 \times T^2 \times T^2$ torus with $I = 1, 2, 3$ labelling the three two-torus. More precisely we write ⁷

$$ds^2 = -e^{2U} (dt + w)^2 + e^{-2U} \sum_{i=1}^3 dx_i^2 + \sum_{I=1}^3 \frac{1}{\text{Im} U_I} |dy_I + U_I d\tilde{y}_I|^2$$

$$C_4 = A_\Lambda \gamma^\Lambda = A_a \gamma^a + A_{\dot{a}} \gamma^{\dot{a}} \quad (\text{A.3})$$

with α^Λ one forms in four dimensions and γ_Λ three forms in the internal torus. $\Lambda = (mnp) = (a, \dot{a})$ is a collective index labelling the 8 different three cycles $[mnp]$ on T^6 entering in the solution

$$\begin{aligned} \gamma^a &= (dy_1 \wedge dy_2 \wedge dy_3, d\tilde{y}_1 \wedge dy_2 \wedge dy_3, dy_1 \wedge d\tilde{y}_2 \wedge dy_3, dy_1 \wedge dy_2 \wedge d\tilde{y}_3) \\ \gamma^{\dot{a}} &= (d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3, dy_1 \wedge d\tilde{y}_2 \wedge d\tilde{y}_3, d\tilde{y}_1 \wedge dy_2 \wedge d\tilde{y}_3, d\tilde{y}_1 \wedge d\tilde{y}_2 \wedge dy_3) \end{aligned} \quad (\text{A.4})$$

All functions entering in the metric and four form can be written in terms of eight harmonic functions

$$\{V, L_I, K_I, M\} \quad (\text{A.5})$$

on \mathbb{R}^3 or equivalently in terms of the following combination

$$\begin{aligned} Z_I &= L_I + \frac{|\varepsilon_{IJK}| K_J K_K}{2V}, \\ \mu &= \frac{M}{2} + \frac{L_I K_I}{2V} + \frac{|\varepsilon_{IJK}| K_I K_J K_K}{6V^2}. \end{aligned} \quad (\text{A.6})$$

with ε_{IJK} characterising the triple intersections between two cycles on T^6 . One finds

$$\begin{aligned} e^{-4U} &= Z_1 Z_2 Z_3 V - \mu^2 V^2, \\ U_I &= \text{Re} U_I + i \text{Im} U_I = -b_I + i (V e^{2U} Z_I)^{-1} \quad b_I = \frac{K_I}{V} - \frac{\mu}{Z_I}, \\ A_a &= (\alpha, \alpha^I - b^I \alpha) \\ A_{\dot{a}} &= \left(\beta - b^1 b^2 b^3 \alpha - \beta^I b_I + \frac{1}{2} |\varepsilon_{IJK}| \alpha^I b^J b^K, \beta^I + |\varepsilon_{IJK}| (\alpha b^J b^K - 2b^J \alpha^K) \right) \end{aligned} \quad (\text{A.7})$$

Finally the one-forms $\alpha, \alpha^I, \beta, \beta^I$ are defined in terms of the eight harmonic function via

$$\begin{aligned} \alpha &= w_0 - \mu V^2 e^{4U} (dt + w), \\ \alpha_I &= -\frac{dt + w}{Z_I} + b_I w_0 + w_I, \\ \beta &= -v_0 + \frac{e^{-4U}}{V^2 Z_1 Z_2 Z_3} (dt + w) - b_I v_I + b_1 b_2 b_3 w_0 + \frac{|\varepsilon_{IJK}|}{2} b_I b_J w_K, \\ \beta_I &= -v_I + \frac{|\varepsilon_{IJK}|}{2} \left\{ \frac{\mu (dt + w)}{Z_J Z_K} + b_J b_K w_0 + 2b_J w_K \right\} \end{aligned} \quad (\text{A.8})$$

⁷ In matrix form

$$h_{mn}^I = \frac{1}{\text{Im} U_I} \begin{pmatrix} 1 & \text{Re} U_I \\ \text{Re} U_I & |U_I|^2 \end{pmatrix}, \quad h_I^{mn} = \frac{1}{\text{Im} U_I} \begin{pmatrix} |U_I|^2 & -\text{Re} U_I \\ -\text{Re} U_I & 1 \end{pmatrix}. \quad (\text{A.2})$$

and

$$\begin{aligned} *_3 dw_0 &= dV, \quad *_3 dw_I = -dK_I, \quad *_3 dv_0 = dM, \quad *_3 dv_I = dL_I \\ *_3 dw &= \frac{1}{2} (VdM - MdV + K_I dL_I - L_I dK_I) \end{aligned} \quad (\text{A.9})$$

A.2 The four dimensional model

After reduction to four dimensions, the ten dimensional solution can be viewed as a solution of a $\mathcal{N} = 2$ truncation of maximal supersymmetric supergravity involving the gravity multiplet and three vector multiplets. The four dimensional model arises as a dimensional reduction of the ten dimensional lagrangian

$$\mathcal{L} = \sqrt{g_{10}} \left(R_{10} - \frac{1}{4 \cdot 5!} F_{M_1 \dots M_5} F^{M_1 \dots M_5} \right) \quad (\text{A.10})$$

Plugging the ansatz (A.1) into (A.10) and taking all fields varying only along the four-dimensional spacetime one finds

$$\mathcal{L} = \sqrt{g_4} \left(R_4 - \sum_{I=1}^3 \frac{\partial_\mu U_I \partial^\mu \bar{U}_I}{2 (\text{Im } U_I)^2} - \frac{1}{4 \cdot 2!} F_{\mu\nu, \Lambda} F^{\mu\nu, \Lambda} \right) \quad (\text{A.11})$$

It is convenient to introduce a metric $\mathcal{H}^{\Lambda\Sigma}$ and its inverse to raise and lower the Λ indices. One writes

$$\mathcal{H}^{abc, def} = h_1^{ad} h_2^{be} h_3^{cf} \quad (\text{A.12})$$

or in matrix form

$$\mathcal{H}^{\Lambda\Sigma} = \begin{pmatrix} \mathcal{H}_1 & \mathcal{H}_2 \\ \mathcal{H}_2^T & \mathcal{H}_3 \end{pmatrix}, \quad (\text{A.13})$$

with \mathcal{H}_1^{ab} , $\mathcal{H}_2^{a\dot{a}}$, $\mathcal{H}_3^{\dot{a}\dot{b}}$ 4×4 matrices. The self-duality condition of the five form field in ten dimensions

$$F_{\mu\nu abc} = \frac{\sqrt{g_{10}}}{2} \varepsilon_{\mu\nu\rho\sigma abcdef} F^{\rho\sigma def}, \quad (\text{A.14})$$

reduces to

$$F_{abc} = \varepsilon_{abcdef} \tilde{F}^{def} \quad \Leftrightarrow \quad F_\Lambda = \varepsilon_{\Lambda\Sigma} \tilde{F}^\Sigma \quad (\text{A.15})$$

with $\tilde{F} = *_4 F$ and $\varepsilon_{\Lambda\Sigma}$ an block off-diagonal antisymmetric matrix with the only non-trivial components

$$\varepsilon_{0\dot{0}} = -\varepsilon_{I\dot{I}} = -\varepsilon_{\dot{0}0} = \varepsilon_{\dot{I}I} = 1 \quad (\text{A.16})$$

In components

$$F_a = \varepsilon_{a\dot{a}} \tilde{F}^{\dot{a}} \quad F_{\dot{a}} = \varepsilon_{\dot{a}a} \tilde{F}^a \quad (\text{A.17})$$

These self-duality relations can be used to express the components $F_{\dot{a}}$ in terms of the Poincare' duals of F_a . Indeed, inverting the first equation in (A.17) one finds⁸

$$F_{\dot{a}} = -(\mathcal{H}_3^{-1})_{\dot{a}\dot{b}} \left(\varepsilon^{\dot{b}c} \tilde{F}_c + \mathcal{H}_2^{\dot{b}c} F_c \right) \quad (\text{A.18})$$

⁸Here we use $*_4^2 = -1$.

with $\varepsilon^{\dot{a}a} = \text{diag}(1, -1, -1, -1)$ the inverse of $\varepsilon_{a\dot{a}}$. Using these relations one can write

$$F_\Lambda F^\Lambda = \mathcal{L}_{stu} + \mathcal{L}_{top} \quad (\text{A.19})$$

with $\mathcal{L}_{top} = -2\varepsilon^{\dot{a}a}\tilde{F}_{\dot{a}}F_a$ a total derivative,

$$\mathcal{L}_{stu} = 2(F_a\mathcal{I}^{ab}F_b + F_a\mathcal{R}^{ab}\tilde{F}_b) \quad (\text{A.20})$$

and ⁹

$$\mathcal{I}^{ab} = stu \begin{pmatrix} 1 + \frac{\sigma^2}{s^2} + \frac{\tau^2}{t^2} + \frac{\nu^2}{u^2} & -\frac{\sigma}{s^2} & -\frac{\tau}{t^2} & -\frac{\nu}{u^2} \\ -\frac{\sigma}{s^2} & \frac{1}{s^2} & 0 & 0 \\ -\frac{\tau}{t^2} & 0 & \frac{1}{t^2} & 0 \\ -\frac{\nu}{u^2} & 0 & 0 & \frac{1}{u^2} \end{pmatrix} \quad \mathcal{R}^{ab} = \begin{pmatrix} 2\sigma\tau\nu & -\tau\nu & -\sigma\nu & -\sigma\tau \\ -\tau\nu & 0 & \nu & \tau \\ -\sigma\nu & \nu & 0 & \sigma \\ -\sigma\tau & \tau & \sigma & 0 \end{pmatrix} \quad (\text{A.21})$$

where

$$U_I = (\sigma + is, \tau + it, \nu + iu) \quad (\text{A.22})$$

Discarding the total derivative term, the four-dimensional Lagrangian can then be written as

$$\mathcal{L} = \sqrt{g_4} \left(R_4 - \sum_{I=1}^3 \frac{\partial_\mu U_I \partial^\mu \bar{U}_I}{2(\text{Im } U_I)^2} - \frac{1}{4} F_a \mathcal{I}^{ab} F_b - \frac{1}{4} F_a \mathcal{R}^{ab} \tilde{F}_b \right) \quad (\text{A.23})$$

The equations of motion read

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} &= \frac{1}{2(\text{Im } U_I)^2} \left(\partial_\mu \bar{U}_I \partial_\nu U_I - \frac{1}{2}g_{\mu\nu} (\partial U_I)^2 \right) + \frac{1}{2}\mathcal{I}^{ab} \left(F_{a\mu\sigma} F_{b\nu}^\sigma - \frac{1}{4}g_{\mu\nu} F_a F_b \right) \\ &\quad + \frac{1}{2}\mathcal{R}^{ab} \left(F_{a\mu\sigma} \tilde{F}_{b\nu}^\sigma - \frac{1}{4}g_{\mu\nu} F_a \tilde{F}_b \right) \\ \nabla_\mu \left\{ \mathcal{I}^{ab} F_b^{\mu\nu} + \mathcal{R}^{ab} \tilde{F}_b^{\mu\nu} \right\} &= 0 \\ -\nabla_\mu \frac{\nabla^\mu U_I}{(\text{Im } U_I)^2} &= i \frac{\partial_\mu U_I \partial^\mu \bar{U}_I}{(\text{Im } U_I)^3} + \frac{1}{2} F_a \frac{\partial \mathcal{I}^{ab}}{\partial \bar{U}_I} F_b + \frac{1}{2} F_a \frac{\partial \mathcal{R}^{ab}}{\partial \bar{U}_I} \tilde{F}_b \end{aligned} \quad (\text{A.24})$$

A.3 The basic solutions

A family of supersymmetric solutions to equations (A.24) is given in [25]. These solutions can be viewed as made of three different types of solutions, referred as K, L or M. In the following we display a representative of solution in each type.

⁹Equivalently

$$\begin{aligned} \mathcal{I}^{ab} &\equiv \mathcal{H}_1^{ab} + \varepsilon^{ab} (\mathcal{H}_3^{-1})_{\dot{b}\dot{c}} \varepsilon^{\dot{c}b} - \mathcal{H}_2^{ab} (\mathcal{H}_3^{-1})_{\dot{c}\dot{d}} \mathcal{H}_2^{\dot{d}b} \\ \mathcal{R}^{ab} &\equiv \varepsilon^{ab} (\mathcal{H}_3^{-1})_{\dot{b}\dot{c}} \mathcal{H}_2^{\dot{c}b} + \mathcal{H}_2^{ab} (\mathcal{H}_3^{-1})_{\dot{b}\dot{c}} \varepsilon^{\dot{c}b} \end{aligned}$$

A.3.1 L solutions

The L class of solutions can be represented by the choice

$$\begin{aligned} V \equiv L(x), \quad M = K_I = 0, \quad L_I = 1 \quad \Rightarrow \quad Z_I = 1, \quad \mu = 0 \\ \mathcal{I} = \text{diag} \left(L^{-3/2}, L^{-1/2}, L^{-1/2}, L^{-1/2} \right) \quad \mathcal{R} = 0 \end{aligned} \quad (\text{A.25})$$

The solution can be written as

$$\begin{aligned} ds^2 &= -L^{-\frac{1}{2}} dt^2 + L^{\frac{1}{2}} \sum_{i=1}^3 dx_i^2 \\ A_0 &= w_0 \quad *_3 dw_0 = dL \\ U_1 &= U_2 = U_3 = i L^{-\frac{1}{2}} \end{aligned} \quad (\text{A.26})$$

A.3.2 K solutions

The K solutions correspond to the choice

$$\begin{aligned} K_3 = -M \equiv K(x), \quad L_I = V = 1, \quad K_1 = K_2 = 0 \quad \Rightarrow \quad Z_I = 1, \quad \mu = 0 \\ \mathcal{I} = \begin{pmatrix} 1 + K^2 & 0 & 0 & K \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ K & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -K & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (\text{A.27})$$

The solution is given by

$$\begin{aligned} ds^2 &= -(dt + w)^2 + \sum_{i=1}^3 dx_i^2, \\ U_1 &= U_2 = i \quad U_3 = -K + i \\ A_0 &= A_3 = 0 \quad A_1 = A_2 = -w \quad *_3 dw = -dK \end{aligned} \quad (\text{A.28})$$

A.3.3 M solutions

The M solutions correspond to the choice

$$\begin{aligned} K_2 = M \equiv M(x), \quad L_I = V = 1, \quad K_1 = K_3 = 0 \quad \Rightarrow \quad \mu = M \quad Z_I = 1, \\ \mathcal{I} = a^{3/2} \begin{pmatrix} 1 + 2\frac{M^2}{a} & -\frac{M}{a} & 0 & -\frac{M}{a} \\ -\frac{M}{a} & a^{-1} & 0 & 0 \\ 0 & 0 & a^{-1} & 0 \\ -\frac{M}{a} & 0 & 0 & a^{-1} \end{pmatrix} \quad \mathcal{R} = \begin{pmatrix} 0 & 0 & -M^2 & 0 \\ 0 & 0 & M & 0 \\ -M^2 & M & 0 & M \\ 0 & 0 & M & 0 \end{pmatrix} \end{aligned} \quad (\text{A.29})$$

with $a = 1 - M^2$. The solution is given by

$$\begin{aligned} ds^2 &= -\frac{dt^2}{\sqrt{1 - M^2}} + \sqrt{1 - M^2} \sum_{i=1}^3 dx_i^2, \\ U_1 &= U_3 = M + i\sqrt{1 - M^2} \quad U_2 = i\sqrt{1 - M^2} \\ A_0 &= -\frac{M dt}{1 - M^2} \quad A_1 = A_3 = -\frac{dt}{1 - M^2} \quad A_2 = w_2 \quad *_3 dw_2 = -dM \end{aligned} \quad (\text{A.30})$$

A.4 Sub-family of solutions

For completeness, we list some interesting sub-families of solutions included in the eight-harmonic class.

A.4.1 No scalars: IWP solution

Einstein-Maxwell theory can be embedded in four dimensional supergravity by restricting to geometries with a trivial internal square metric

$$V e^{2U} Z = 1 \quad b_I = 0 \quad (\text{A.31})$$

These equations can be solved in terms of two harmonic functions $\text{Re } H$ and $\text{Im } H$ via the identifications

$$\begin{aligned} V &= L_1 = L_2 = L_3 = \text{Im } H \\ -M &= K_1 = K_2 = K_3 = \text{Re } H \end{aligned} \quad (\text{A.32})$$

The general solution reduces to¹⁰

$$\begin{aligned} ds^2 &= -|H|^{-2} (dt + w)^2 + |H|^2 d\vec{x}^2 \\ A_0 &= w_0 - \frac{\text{Re } H}{|H|^2} (dt + w) \\ A_1 &= A_2 = A_3 = w_1 - \frac{\text{Im } H}{|H|^2} (dt + w) \end{aligned} \quad (\text{A.34})$$

with H a complex harmonic function

$$H = \text{Re } H + i \text{Im } H \quad \nabla^2 H = 0 \quad (\text{A.35})$$

and w and w_0, w_1 one forms defined as

$$\begin{aligned} dw &= i *_3 [H dH^* - H^* dH] \\ dw_0 &= *_3 d\text{Im } H \quad dw_1 = *_3 d\text{Re } H \end{aligned} \quad (\text{A.36})$$

We notice that the contribution to the stress energy tensor of gauge fields A_I exactly match that of A_0 , so we can replace the four gauge fields by a single one given by

$$A = 2w_0 - \frac{2\text{Re } H}{|H|^2} (dt + w)$$

The resulting solution is known in the General Relativity literature as IWP (after Israel, Wilson and Perjes [37, 38]) and includes very well known examples of solutions of Maxwell-Einstein gravity:

¹⁰ In our conventions the Einstein-Maxwell lagrangian reads

$$\mathcal{L} = \sqrt{g} [R - \frac{1}{4} F^2] \quad (\text{A.33})$$

- $AdS_2 \times S^2$. The harmonic function H reads¹¹

$$H = \frac{1}{\sqrt{x_1^2 + x_2^2 + (x_3 - iL)^2}} \quad (\text{A.38})$$

The geometry is regular everywhere. An infinite class of regular IWP geometries, obtained as bubbling of $AdS_2 \times S^2$ and parametrised by a string profile function has been recently constructed in [14].

- Kerr-Newman solution with $M = Q = q$, $P = 0$, $J = qL$. The harmonic function H reads

$$H = 1 + \frac{q}{\sqrt{x_1^2 + x_2^2 + (x_3 - iL)^2}} \quad (\text{A.39})$$

The geometry has a naked curvature singularity at the zero of H .

- Reissner-Nordstrom with $M = Q = q$, $P = J = 0$. The harmonic function H reads

$$H = 1 + \frac{q}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \quad (\text{A.40})$$

The geometry has a curvature singularities at the zero of H .

- Charged Taub-NUT with $M = Q = b_1$, $P = -b_2$, $J = 0$. The harmonic function H reads

$$H = 1 + \frac{b_1 + i b_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \quad (\text{A.41})$$

The geometry has no curvature singularities but it has a Dirac-Misner string-like singularity.

A.4.2 One complex scalar: SWIP solutions

Next, we consider a solutions with single active scalar field, let us say U_1 , with $U_2 = U_3 = i$. These conditions can be solved in terms of two complex harmonic functions H_1 , H_2 after the identification

$$L_1 = V = \text{Re } H_2, \quad L_2 = L_3 \equiv \text{Im } H_1 \quad K_1 = -M = \text{Re } H_1 \quad K_2 = K_3 \equiv -\text{Im } H_2 \quad (\text{A.42})$$

¹¹ Global coordinates are defined by

$$(x_1, x_2, x_3) = \left(\sqrt{(\rho^2 + L^2)(1 - \chi^2)} \cos \phi, \sqrt{(\rho^2 + L^2)(1 - \chi^2)} \sin \phi, \rho \chi \right) \quad (\text{A.37})$$

with $\rho \in (-\infty, \infty)$, $\chi \in [-1, 1]$, $\phi \in [0, 2\pi]$. These coordinates cover twice the flat space with the points (ρ, χ) and $(-\rho, -\chi)$ identified.

For this choice the general solution reduces to

$$\begin{aligned}
ds^2 &= - [\text{Im} (H_1 \bar{H}_2)]^{-1} (dt + w)^2 + \text{Im} (H_1 \bar{H}_2) \sum_{i=1}^3 dx_i^2, \\
U_1 &= \frac{H_1}{H_2}, \quad U_2 = U_3 = i, \\
A_0 &= w_0 + \frac{\text{Im} H_2}{\text{Im} (H_1 \bar{H}_2)} (dt + w), \quad A_1 = w_1 - \frac{\text{Im} H_1}{\text{Im} (H_1 \bar{H}_2)} (dt + w) \\
A_2 &= A_3 = w_2 - \frac{\text{Re} H_2}{\text{Im} (H_1 \bar{H}_2)} (dt + w) \\
*_3 dw &= -\text{Re} (H_1 d\bar{H}_2 - \bar{H}_2 dH_1) \\
*_3 dw_0 &= \text{Re} dH_2, \quad *_3 dw_1 = -\text{Re} dH_1, \quad *_3 dw_2 = \text{Im} dH_2.
\end{aligned} \tag{A.43}$$

The IWP class corresponds to the choice $H_1 = iH_2 = H$. See for instance [39] for more information on the SWIP solution.

A.4.3 Two complex scalars

This solution corresponds to the choice

$$L_3 = L_2 \quad K_3 = K_2 \tag{A.44}$$

leading to

$$\begin{aligned}
Z_1 &= L_1 + \frac{K_2^2}{V} & Z_2 = Z_3 &= L_2 + \frac{K_1 K_2}{V} \\
\mu &= \frac{M}{2} + \frac{K_1 K_2^2}{V^2} + \frac{K_1 L_1}{2V} + \frac{K_2 L_2}{V}
\end{aligned} \tag{A.45}$$

The solution reads

$$\begin{aligned}
ds^2 &= -e^{-2U} (dt + w)^2 + e^{2U} \sum_{i=1}^3 dx_i^2, \quad e^{-4U} = Z_1 Z_2^2 V - \mu^2 V^2 \\
U_1 &= -b_1 + i(e^{2U} V Z_1)^{-1} & U_2 = U_3 &= -b_2 + i(e^{2U} V Z_2)^{-1} \\
A_0 &= w_0 - \mu V^2 e^{4U} (dt + w) & A_1 &= w_1 + V e^{4U} (dt + w) (Z_2^2 - K_1 \mu) \\
A_2 &= A_3 = w_2 + V e^{4U} (dt + w) (Z_1 Z_2 - K_1 \mu)
\end{aligned} \tag{A.46}$$

with

$$\begin{aligned}
*_3 dw &= \frac{1}{2} (V dM - M dV + K_1 dL_1 - L_1 dK_1 + 2K_2 dL_2 - 2L_2 dK_2) \\
*_3 dw_0 &= dV, \quad *_3 dw_1 = -dK_1 \quad *_3 dw_2 = -dK_2
\end{aligned} \tag{A.47}$$

and

$$b_1 = \frac{K_1 L_1 - 2K_2 L_2 - MV}{2(K_2^2 + V L_1)} \quad b_2 = b_3 = -\frac{K_1 L_1 + MV}{2(K_1 K_2 + V L_2)} \tag{A.48}$$

The SWIP solution is recovered for $L_1 = V$ and $K_1 = M$ after the identifications (A.42).

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