$\mathcal{N} = 2$ Moduli of AdS$_4$ vacua: A fine-print study

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ABSTRACT

We analyze the moduli spaces near $\mathcal{N} = 4$ supersymmetric AdS$_4$ vacua of string theory by combining and comparing various approaches: (a) the known exact solutions of Type IIB string theory with localized 5-brane sourcees; (b) the holographically dual 3d quiver gauge theories; (c) gauged supergravity; and (d) the representations of the superconformal algebra $\mathfrak{osp}(4|4)$. Short multiplets containing the marginal $\mathcal{N} = 2$ deformations transform in the $(2; 0)$, $(0; 2)$ or $(1; 1)$ representations of the R-symmetry group $SU(2)_H \times SU(2)_C$. The first two are classified by the chiral rings of the Higgs and Coulomb branches, while the latter contain mixed-branch operators. We identify the origin of these moduli in string theory, matching in particular the operators of the chiral rings with open strings on the magnetized 5-brane sources. Our results provide new evidence for the underlying holographic duality. The existence of a large number of bound-state moduli highlights the limitations of effective supergravity.

1 Introduction

Along with supersymmetry, moduli are ubiquitous in string theory. Both must be lifted in the real world, but may manifest themselves at low or intermediate energies $\ll M_{\mathrm{Planck}}$ and thereby allow contact with observations. Understanding how this happens is one of the main themes of string phenomenology.

Most earlier studies employed either effective supergravities, or gauged supergravities which are consistent truncations in flux-compactification backgrounds (for a review of flux compactifications see e.g. [1, 2, 3]). One of our motivations for the present work is to explore the limitations of these approaches. We will consider a class of AdS$_4$ vacua with $\mathcal{N} = 4$ supersymmetry for which improved tools are now at our disposal:

(a) Ten-dimensional solutions of Type IIB string theory with fully localized NS5- and D5-brane sources [4, 5]; (b) The dual three-dimensional gauge theories based on linear or circular quivers [6]; (c) Corresponding $\mathcal{N} = 4$ gauged supergravities [7, 8, 9, 10]; and last but not least (d) A complete list of unitary representations of the superconformal algebra $\mathfrak{osp}(4|4)$ [11, 12].

We will provide a detailed dictionary between representations, string excitations, fields of the quiver theory and fields of gauged supergravities, identifying in particular the ones that contain $\mathcal{N} = 2$ supersymmetric moduli. As will be clear, gauged supergravity retains some, but not all pertinent information about the solutions. It misses in particular most of the massless moduli.

To be sure, these AdS$_4$ solutions are far from realistic vacua of string theory. In addition to a plethora of moduli, they also contain scalar fields with $(\mathrm{mass})^2 < 0$, though above the Breitenlohner-Freedman stability bound. One would have to worry about all these modes in any attempt to uplift these vacua to more realistic de Sitter backgrounds.

One of the most satisfying results of our analysis is the match between string excitations of the solutions and chiral fields in the Higgs or Coulomb branches of the dual gauge theories. This parallels, but is different, from the analysis in [13]. We will exhibit in particular a selection rule for allowed representations of the $SU(2)_H \times SU(2)_C$ R-symmetry, whose origin is strikingly different on the two sides. This is a non-trivial test of holographic duality.

The outline of the paper is as follows: In section 2 and in appendix A we briefly review the relevant features of the Type IIB solutions of [4, 5], as well as the analysis of supersymmetric vacua in gauged $\mathcal{N} = 4$ supergravity by Louis and Triendl [10]. We then examine the excitations around these backgrounds, comparing the two approaches. In section 3 we first review the multiplets of $\mathcal{N} = 4$ superconformal symmetry classified in [11, 12], and identify the ones that contain putative $\mathcal{N} = 2$ moduli (there are no $\mathcal{N} = 4$ moduli, as shown in [14]). We match these multiplets to the excitations of section 2, summarizing our conclusions in a table. In section 4 we introduce the 'good' quiver gauge theories of Gaiotto and Witten [15], conjectured to flow to interacting CFTs that are dual to the above AdS$_4$ backgrounds. We show how the spectrum of chiral operators on the Higgs branches of the electric and magnetic quivers matches nicely with expectations from string theory, comment on mixed-branch operators and count moduli in some examples.

Note added

In one higher dimension, the similar problem of identifying $\mathcal{N} = 1$ moduli of $\mathcal{N} = 2$ $D = 4$ SCFT's with AdS$_5$ holographic duals has been recently addressed in [16] from the perspective of generalized exceptional geometry.

2   AdS$_4$ vacua of string theory

In this section we review those features of the $\mathcal{N} = 4$ AdS$_4$ solutions that will be useful later, compare them with the solutions of gauged supergravity, and derive some properties of the small-fluctuation spectrum. There is a large literature on AdS$_4$ compactifications with fluxes and branes, mostly in Type IIA string theory (a non-exhaustive list is [17]-[23]). What is special about the solutions of [4, 5] is that they have fully localized (as opposed to smeared) brane sources. A general method of searching for such solutions with the help of pure-spinor equations has been proposed recently in [24].

2.1   Review of the $\mathcal{N} = 4$ Type IIB solutions

We begin with the $\mathcal{N} = 4$ AdS$_4$ compactifications of Type IIB string theory found in [4, 5]. The geometry of these solutions has the warped form (AdS$_4\times$S$^2\times$S$^2{}'$)$\times_w\Sigma$, where the base Σ is an open Riemann surface, which is either the disk or the annulus. Superconformal symmetry $\mathfrak{osp}(4|4)$ is realized as isometries of the AdS$_4\times$S$^2\times$S$^2{}'$ fiber. R-symmetry, in particular, is realized as isometries of the two 2-spheres.

In addition to the metric, the solutions feature non-trivial dilaton, 5-form and complex 3-form backgrounds. As shown by D'Hoker, Estes and Gutperle [25, 26] all these backgrounds can be expressed in terms of two harmonic functions $h_i : \Sigma \to \mathbb{C}$ $(i = 1, 2)$, which are positive in the interior of Σ and vanish at alternating parts of the boundary. This latter property ensures that generic points of $\partial\Sigma$ are regular interior points of the 10-dimensional geometry. The explicit expressions of all background fields are given in appendix A .

A key feature of the solutions is that they have localized D5-brane and NS5-brane sources (at finite distance) on the internal space $\mathcal{M}_6 := ($S$^2\times$S$^2{}') \times_w \Sigma$. The D5-branes wrap the S^2 fiber and are localized in the transverse space S$^2{}' \times_w \Sigma$, whereas the NS5-branes wrap the S$^2{}'$ fiber and are localized in S$^2 \times_w \Sigma$. These cycles are homologically trivial, so there is no need for tadpole cancellation by orientifolds or anti-branes.

We focus on the solutions with Σ the infinite strip $0 \leq \mathrm{Im} z \leq \pi/2$ (the case of the annulus is a simple extension). The singularities labelled by $a \in \{1, \cdots, p\}$ have D5-brane charge N_a and are located at $\mathrm{Re} z = \delta_a$ on the upper strip boundary, while the singularities labelled by $\hat{a} \in \{1, \cdots, \hat{p}\}$ have NS5-brane charge $\hat{N}_{\hat{a}}$ and position $\mathrm{Re} z = \hat{\delta}_{\hat{a}}$ on the lower boundary. Both the 5-brane charges and their positions are continuous parameters of the supergravity solutions but in string theory they are all quantized. This is obvious for the 5-brane charges, but more subtle for their positions $\{\delta_a, \hat{\delta}_{\hat{a}}\}$. It turns out that these latter can be related to

the D3-brane charges of the 5-brane stacks [4]

$$\ell_a = -\sum_{\hat{a}=1}^{\hat{p}} \hat{N}_{\hat{a}} \frac{2}{\pi} \arctan(e^{-\delta_a + \hat{\delta}_{\hat{a}}}) \ , \qquad \hat{\ell}_{\hat{a}} = \sum_{a=1}^{p} N_a \frac{2}{\pi} \arctan(e^{-\delta_a + \hat{\delta}_{\hat{a}}}) \ , \qquad (2.1)$$

where ℓ_a is the D3 charge of a D5-brane in the ath stack and $\hat{\ell}_{\hat{a}}$ the D3 charge of a NS5-brane in the $\hat{a}^{\text{th}}$ stack. The above equations can be used to solve for all source positions in terms of the charges $\{N_a, \ell_a, \hat{N}_{\hat{a}}, \hat{\ell}_{\hat{a}}\}$ which are quantized.

Thanks to this dimensional transmutation (charges transmuting to geometric positions) the solutions of [4, 5] have only discrete but no continuous moduli. As we will discuss in section 3, the absence of continuous $\mathcal{N} = 4$ superconformal moduli follows in all generality from the study of unitary $\mathfrak{osp}(4|4)$ representations [14]. Charge quantization removes therefore a potential contradiction with this general result.

The data $\{N_a, \ell_a\}$ and $\{\hat{N}_{\hat{a}}, \hat{\ell}_{\hat{a}}\}$ can be repackaged conveniently in two Young diagrams ρ and $\hat{\rho}$. The diagram ρ has N_a rows of $|\ell_a|$ boxes (with the $|\ell_a|$ arranged in descending order). Likewise $\hat{\rho}$ has $\hat{N}_{\hat{a}}$ rows of $\hat{\ell}_{\hat{a}}$ boxes. In the Type IIB solutions ρ and $\hat{\rho}$ are not independent: they have the same total number of boxes by conservation of D3-brane charge (one can indeed check that $\sum_{\hat{a}} \hat{N}_{\hat{a}} \hat{\ell}_{\hat{a}} = -\sum_a N_a \ell_a := N$), and they furthermore automatically satisfy the partial ordering condition

$$\rho^T > \hat{\rho} \ . \qquad (2.2)$$

Here ρ^T is the transposed Young diagram (with columns and rows interchanged) and the above condition means that the first k rows of the left diagram contain more boxes than the first k rows of the right diagram, for all $k = 1, \cdots, \hat{p} - 1$.[1] In the dual linear-quiver gauge theory, where ℓ_a, $\hat{\ell}_{\hat{a}}$ become the linking numbers of the 5-branes, this condition ensures that the gauge symmetry can have a non trivial Higgs branch, see section 4. Such theories were called 'good' by Gaiotto and Witten [15] and conjectured to flow to strongly-interacting fixed points in the infrared. That the same conditions also arise in string theory is a nice check of holographic duality [4].

The gauge symmetries on the worldvolumes of the 5-branes correspond to global flavor symmetries of the dual field theory. The D5-brane symmetry $U(N_1) \times \cdots \times U(N_p)$ is manifest when the CFT is realized as an 'electric quiver' gauge theory, while the NS5-brane symmetry $U(\hat{N}_1) \times \cdots \times U(\hat{N}_{\hat{p}})$ is manifest in the Lagrangian of the mirror 'magnetic quiver'. At the origin of the Higgs branch, the electric- and magnetic-quiver gauge theories are expected to flow to the same CFT where these flavor symmetries coexist.

It is important here to note that global symmetries do not determine the SCFT uniquely. There is extra information contained in the D3-brane charges or linking numbers $\{\ell_a, \hat{\ell}_{\hat{a}}\}$ of the 5-branes. We will refer to this extra information as the 'fine print'. Figure 1 shows an

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[1]This condition has appeared earlier in related contexts [27][28]. It is also natural in the Brieskorn-Slodowy theory of transverse slices [29, 30] reviewed in section 4. It would be interesting to see if this condition can be obtained from some appropriate version of K theory along the lines of [31][32].
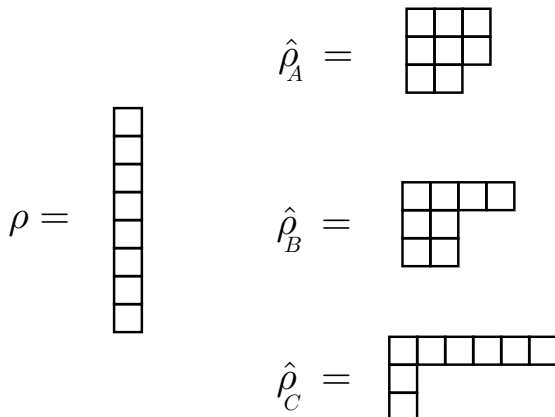
Figure 1: Three theories (A, B, C) with the same ρ and different $\hat{\rho}$. These theories admit the same continuous global symmetry $SU(8) \times \left(U(2) \times U(1)\right)/U(1)$ but different fine print. The gravitational dual solutions all consist of one stack of $N = 8$ D5-branes and two stacks of $\hat{N}_1 = 2$ and $\hat{N}_2 = 1$ NS5-branes, but they differ in the distribution of the 8 units of D3-brane charge.

example of three theories with the same 5-brane charges $\{N_a, \hat{N}_{\hat{a}}\}$, and hence the same global flavor symmetry, [2]

2.2 Gauged supergravity

A different approach to supersymmetric AdS$_4$ vacua is the approach of gauged supergravity [33, 7, 8, 9, 10].[3] For the case at hand one starts with $\mathcal{N} = 4$ supergravity coupled to m vector multiplets in 4d Minkowski spacetime, and then deforms this theory by gauging a subgroup of the global symmetry group SL(2)$\times$SO(6, m). Gauging is elegantly achieved by the introduction of an embedding tensor that obeys a set of quadratic constraints [33]. The result is not an effective low-energy theory, but its solutions are guaranteed to be solutions of any higher-dimensional theory of which the gauged supergravity is a consistent truncation.

The existence of maximally-supersymmetric AdS$_4$ vacua has been studied systematically within this formalism by Louis and Triendl [10]. Before gauging, the scalar fields of the $\mathcal{N} = 4$ supergravity take values in the coset space $\mathcal{M} = [\text{SL}(2)/\text{SO}(2)] \times [\text{SO}(6, m)/\text{SO}(6) \times SO(m)]$. Gauging generates a potential that lifts some of these flat directions. Louis and Triendl showed that whenever maximally supersymmetric AdS$_4$ vacua exist the following is true:

- The gauge group is of the form $\text{G}_+ \times \text{G}_- \times \text{G}_0^v$, where $\text{G}_\pm \subset \text{SO}(3, m_\pm)$ and $\text{G}_0^v \subset \text{SO}(q)$ with $m_+ + m_- + q = m$. Furthermore, the gauging of G_+ employs the electric components

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[2]In principle some theories could be distinguished by extra discrete symmetries. This does not seem to be the case for the examples of the figure, even though we cannot rule out the possibility that some higher-form discrete symmetry emerges in the infrared. Note in passing that a diagonal electric and a diagonal magnetic $U(1)$ act trivially and can be modded out of the global-symmetry groups. but with different fine print. These theories have different operator spectra and AdS$_4$ duals as will become clear later on.

[3]There is a vast literature on gauged supergravity and stabilization of moduli by fluxes, the above references are just closer to the contents of this paper. A recent review is [34].

of three graviphotons while the gauging of G$_-$ employs the magnetic components of the other three graviphotons.

- The vacua break spontaneously G$_\pm$ to their maximal compact subgroups SO(3)$_\pm \times H_\pm$ where $H_\pm \subset$ SO($m_\pm$). The only continuous moduli are the Goldstone bosons of these broken gauge symmetries that are eaten by the corresponding massive vector bosons and are hence unphysical. As a result, there are no $\mathcal{N} = 4$ preserving continuous moduli.

Let us compare these conclusions with the findings of the previous subsection. Clearly SO(3)$_+\times$SO(3)$_-$ is the $\mathcal{N} = 4$ R-symmetry and it is natural to identify H_+ and H_- with the flavor symmetries realized, respectively, on D5-branes and NS5-branes. The extra factor G$_0^v$ can be a priori attributed to either type of 5-branes, or may come from extra bulk vector bosons. The absence of continuous moduli in this description is remarkable. For instance SL(2, R) transformations of Type IIB supergravity which activate the RR axion field take us outside the class of solutions given in appendix A. Such transformations can generate new ('orbifold equivalent') solutions [5], but they are discretized by the fact that all 5-brane charges must remain integer. The fact that gauged supergravity has no continuous SL(2, R) moduli shows that the embedding-tensor formalism knows about integrality of charges.

Of course gauged supergravity cannot know everything about the solutions, since it is a truncation of the full string theory. It is unclear, in particular, whether it can store the 'fine print' data of the previous subsection, i.e. distinguish vacua with the same unbroken gauge symmetry. In gauged supergravity this fine print can only come from inequivalent embeddings $H_+ \times H_- \times$G$_0^v \subset$ SO(6, m), or from multiple solutions of the quadratic constraints. In many previous studies the number of vector multiplets was small (usually $m = 6$) so the possibilities were restricted. The question deserves further scrutiny.

What is certain is that gauged supergravity misses many scalar excitations including some with (mass)$^2 \leq 0$ which correspond to marginal and relevant deformations of the dual CFT. Apart from the multiplets of the graviton and the vector bosons of unbroken gauge symmetries, the only other surviving fields are the multiplets of massive gauge bosons corresponding to the broken non-compact gauge symmetries. These are in representations $(1; 0)$ or $(0; 1)$ of the R-symmetry group, i.e. they carry a vector index under either SO(3)$_+$ or SO(3)$_-$.[4] We will soon see that they belong (in the notation of [12], modified only by halving the spins) to the superconformal multiplets $B_1[0]_2^{(2;0)}$ or $B_1[0]_2^{(0;2)}$ which provide some, but not all, of the $\mathcal{N} = 2$ superconformal moduli. Thus gauged supergravity is not a reliable tool for addressing the problem of moduli stabilization.

2.3 Properties of vacuum excitations

The complete spectrum of small excitations can be in principle derived by expanding around the backgrounds of [4, 5]. In practice this is a formidable task. We will limit ourselves to

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[4]For illustration, consider gauging G$_+$=SO(3, 2) $\subset$SO(3, m_+) where $m_+ = 7$. Note that 3+7 = 10 is precisely the dimension of the gauge group. The maximal compact subgroup of this latter is SO(3)$\times$U(1). The six non-compact generators transform in the $(3, \pm)$ representations of this unbroken gauge group.

some generic features that are easy to extract.

We will use certain facts about representations of the superconformal algebra $\mathfrak{osp}(4|4)$, postponing a more systematic discussion to the following section. Unitary representations are decsribed by the spacetime spin of the highest-weight state, and by its $SO(3)_+ \times SO(3)_-$ R-symmetry spins which are denoted $(R; R')$. When not otherwise qualified, 'spin' means spacetime spin. Apart from the long representation $L[0]_1^{(0;0)}$ which corresponds to a massive supergraviton, there are three series of representations with maximum spin ≤ 2: [12]

- the $B_1[0]_R^{(R;0)}$ and $B_1[0]_{R'}^{(0;R')}$ series with max. spin ≤ 1,

- the $B_1[0]_{R+R'}^{(R;R')}$ series $(RR' \neq 0)$ with max. spin $\leq 3/2$,

- the $A_2[0]_{R+R'+1}^{(R;R')}$ series with max. spin ≤ 2.

Barring excited-string modes, single-particle states of the theory are either the 10d graviton multiplet, or the lowest-lying modes of open strings living on the 5-branes. Both have spins not exceeding 2, and are therefore organized in the above representations.

Consider first the open strings which contain spins ≤ 1 and belong to the $B_1[0]_R^{(R;0)}$ series. We concentrate on the D5-branes, the discussion of NS5-branes is mirror symmetric. Strings on the ath D5-brane stack transform in the adjoint representation of $U(N_a)$, while those stretching from the ath to the bth stack transform in the bifundamental $(N_a, \bar{N}_b)$ representation of $U(N_a) \times U(N_b)$. There are no open strings stretching between D5-branes and NS5-branes, so CFT operators charged under both electric and magnetic flavor groups can only correspond to multi-string states.

We now focus on the spin-1 component of the multiplet which transforms in the $(R-1;0)$ representation of $SO(3)_+$, see eq. (3.6) of the following section. Clearly $R-1$ is the angular momentum of the string on S^2, the 2-sphere wrapped by all the D5-branes. We can constrain the range of R for D5-brane strings by the following argument: A D5-brane of the ath stack carries ℓ_a units of D3-brane charge, which must be induced by ℓ_a units of internal monopole flux in the 2-sphere direction [35]. Therefore open strings from a to b feel a monopole field of strength $\ell_a - \ell_b$ in appropriate units. Their spin-1 components are classified by the scalar monopole harmonics on S^2, which have $R - 1 \geq \frac{1}{2}|\ell_a - \ell_b|$, see e.g. ref. [36]. This leads then to the following selection rule:

$$R = \frac{1}{2}|\ell_a - \ell_b| + n \qquad (n = 1, 2, \cdots) \qquad \text{for } (ab) \text{ open strings .} \qquad (2.3)$$

We will see that this prediction agrees beautifully with the analysis of the CFT side. It is important also to note that the S^2 angular momentum can be half-integer, a well-known fact of physics in the background of magnetic poles [37, 38].

We come next to the closed-string modes. Barring again string excitations, these include the $\mathcal{N} = 4$ supergraviton in four dimensions, its Kaluza Klein excitations, and massive spin-3/2 or spin-1 multiplets corresponding to the supersymmetries and R-symmetry generators broken by the compactification. All closed strings are of course flavor singlets. In addition,

since 10d closed-string states are either tensors or spinors on all three (pseudo)spheres of the $AdS_4 \times S^2 \times S^{2}{}'$ fiber simultaneously, the quantum numbers R and R' of the highest-weight state (which is always scalar) must be integer.[5] These simple facts will also emerge automatically on the CFT side.

Gauged supergravity retains three types of fields: (i) the massless 4d graviton coupling to the stress tensor multiplet, $A_2[0]_1^{(0;0)}$ in the notation of ref.[12]; (ii) the massless vector bosons of compact symmetries coupling to conserved-current multiplets $B_1[0]_1^{(1;0)}$ or $B_1[0]_1^{(0;1)}$, and (iii) the massive vector bosons of broken non-compact symmetries, which couple to the $B_1[0]_2^{(2;0)}$ or $B_1[0]_2^{(0;2)}$ multiplets (whose spin-1 components are vectors of $SO(3)_+$ or $SO(3)_-$). We will see that these last multiplets contain marginal $\mathcal{N} = 2$ supersymmetric operators, but they are not the only ones. Marginal $\mathcal{N} = 2$ operators are also contained in the spin-3/2 multiplet $B_1[0]_2^{(1;1)}$ which is truncated out in gauged $\mathcal{N} = 4$ supergravity. We expect such multiplets to exist in half-supersymmetric vacua of $\mathcal{N} = 8$ supergravity, where four of the gravitini become massive, but have not checked this explicitly.

An example of a deformation that excites the truncated modes is the TsT deformation of Lunin and Maldacena [39]. For the backgrounds of interest, the two commuting Killing isometries are the azimuthal rotations of S^2 and $S^{2}{}'$. If τ is the complexified Kähler modulus of the 2-torus generated by these Killing isometries, the deformed solutions have

$$\tau \to \tau' = \frac{\tau}{1 + \gamma\tau} \ . \tag{2.4}$$

where γ is the real deformation parameter. Besides the metric and B-field components that are encoded in τ, several other supergravity fields are also deformed – explicit general formulae have been worked out by Imeroni [40] but are not needed here. One crucial remark is that in the original backgrounds B has no 2-torus component, so $\tau = B + i\sqrt{G}$ is purely imaginary. This guarantees that the deformed solutions do not develop singularities at the loci where the torus degenerates [39].

The TsT deformation breaks the $SO(4)$ R-symmetry to $U(1) \times U(1)$ which is compatible with at most $\mathcal{N} = 2$ supersymmetry. Generally-speaking we expect all four supersymmetries to be broken by this deformation[6]. The point we want to make however here is different: this is a deformation of the classical background that is truncated away in the gauged $\mathcal{N} = 4$ supergravity description. Indeed, this deformation is generated by a neutral scalar operator of scaling dimension $\Delta = 3$ in the dual CFT, and there is no such operator in the $\mathcal{N} = 4$ stress-tensor multiplet $A_2[0]_1^{(0;0)}$.

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[5]There can be an exception to this rule if a 2-sphere is transpersed by magnetic flux of a gauge field under which the closed string is electrically charged. This situation does not arise in the solutions of interest.

[6]In the dual field theory the TsT deformation inserts charge-dependent phases in front of field products, $fg \to \exp(i\gamma(Q_1(f)Q_2(g) - Q_1(g)Q_2(f))$ where Q_1, Q_2 are the two $U(1)$ charges being used. When one of these charges is the R-charge, different components in the product of two superfields acquire different phases. For general charges supersymmetry is thus broken.

3 Superconformal Multiplets

The $\mathcal{N} = 4$ superconformal algebra is $\mathfrak{osp}(4|4)$. Its maximal compact subalgebra includes spatial rotations and the R-symmetry algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)' \simeq \mathfrak{so}(4)_R$. Unitary representations of this algebra have been classified, see Dolan [11], and Cordova, Dumitrescu and Intriligator who classified representations of superconformal algebras in all dimensions [12].

We adopt the conventions of this latter work and denote the highest-weight states in a given representation by $[j]_\Delta^{(R;R')}$, where Δ is the scaling dimension and (j, R, R') are the spins with respect to space rotations and the two R-symmetry factors. Contrary to [12], our spins will be however canonically normalized. For instance $[1]_\Delta^{(0;1)}$ is a vector field in the (singlet, vector) representation of R symmetry. When this is a superconformal primary, the $\mathcal{N} = 4$ representation built on it is denoted as in reference [12], e.g. $A_1[1]_3^{(0;1)}$ for a short representation in the A_1 series, or $L[1]_{\Delta \geq 3}^{(0;1)}$ for a long representation.

We start with a short review of superconformal multiplets, their spin content and matching string-theory modes, then decompose them under $\mathcal{N} = 2$ supersymmetry and identify the ones that contain $\mathcal{N} = 2$ moduli.

3.1 $\mathcal{N} = 4$ multiplets

The 8 independent Poincaré supercharges can be labelled by the projections of the three spins, $Q_\pm^{(\pm;\pm)}$. The primary conformal operators are obtained by acting with these supercharges on the primary superconformal operators. There exist 128 bosonic and 128 fermionic raising operators transforming as follows under $\mathfrak{su}(2)_{\text{space}} \oplus [\mathfrak{su}(2) \oplus \mathfrak{su}(2)']_{\text{R-symmetry}}$:

$$
\begin{aligned}
1\,(1) : \quad & (0, 0, 0) \\
Q\,(8) : \quad & \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \\
Q^2\,(28) : \quad & \left[(1, 1, 0) \oplus \text{perms} \right] \oplus (0, 0, 0) \\
Q^3\,(56) : \quad & \left[\left(\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \oplus \text{perms} \right] \oplus \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \\
Q^4\,(70) : \quad & \left[(2, 0, 0) \oplus \text{perms} \right] \oplus (1, 1, 1) \oplus \left[(1, 1, 0) \oplus \text{perms} \right] \oplus (0, 0, 0) \\
Q^5\,(56) : \quad & \left[\left(\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \oplus \text{perms} \right] \oplus \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \\
Q^6\,(28) : \quad & \left[(1, 1, 0) \oplus \text{perms} \right] \oplus (0, 0, 0) \\
Q^7\,(8) : \quad & \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right) \\
Q^8\,(1) : \quad & (0, 0, 0)
\end{aligned}
$$

Here 'perms' stands for all inequivalent permutations of the three spins, and the number of operators at each level is given in parenthesis. The scaling dimension of Q^n is $n/2$. Tensoring these representations with those of a superconformal primary gives the spin content at each level of a long superconformal representation. This rombus is truncated in short multiplets because of the appearance of null states.

Short $\mathcal{N} = 4$ multiplets come in three different series called A_1, A_2 and B_1

$$A_1[j]_{1+j+R+R'}^{(R;R')} \quad (j > 0) \; , \qquad A_2[0]_{1+R+R'}^{(R;R')} \; , \quad \text{and} \quad B_1[0]_{R+R'}^{(R;R')} \; . \tag{3.1}$$

The subscript $1, 2$ indicates the level of the first null states. The $A_{1,2}$ multiplets appear in the decomposition of long multiplets at unitarity threshold, so their scaling dimension can change continuously by recombination. The B_1 multiplets, on the other hand, are separated from the unitarity threshold by a gap. Those with $R < 1$ or $R' < 1$ don't appear in the decomposition of any long representation. Their dimension cannot therefore change without breaking $\mathcal{N} = 4$ superconformal symmetry, and is in this sense absolutely protected [12].

The most basic multiplets are the $(16+16)$-component multiplet of the $\mathcal{N} = 4$ graviton in four dimensions, and the $(8+8)$-component of the unbroken electic and magnetic gauge bosons. The former includes the graviton, six vectors and two scalars, while the latter include a vector and six scalar fields. These fields couple to the stress-tensor and flavor-current multiplets of the dual SCFT,

$$\textbf{stress}: \quad A_2[0]_1^{(0;0)} = [0]_1^{(0;0)} \oplus [0]_2^{(0;0)} \oplus [1]_2^{(1;0)} \oplus [1]_2^{(0;1)} \oplus [2]_3^{(0;0)} \oplus \text{fermions} \tag{3.2}$$

$$\textbf{e} - \textbf{flavor}: \quad B_1[0]_1^{(1;0)} = [0]_1^{(1;0)} \oplus [1]_2^{(0;0)} \oplus [0]_2^{(0;1)} \oplus \text{fermions} \tag{3.3}$$

$$\textbf{m} - \textbf{flavor}: \quad B_1[0]_1^{(0;1)} = [0]_1^{(0;1)} \oplus [1]_2^{(0;0)} \oplus [0]_2^{(1;0)} \oplus \text{fermions} \tag{3.4}$$

Notice the absence of a $\Delta = 3$ scalar in the graviton multiplet, in agreement with our earlier claim that the TsT mode is not part of the 4d supergravity spectrum.

The vector multiplets (3.3) and (3.4) of the conserved flavor-symmetry currents belong to absolutely-protected B_1 representations. This implies that flavor symmetries cannot break continuously in $\mathcal{N} = 4$ superconformal theories. Equivalently, gauge symmetries in the AdS$_4$ bulk cannot be broken à la Higgs-Brout-Englert without also breaking $\mathcal{N} = 4$ supersymmetry, a conclusion confirmed by the analysis of Louis and Triendl [10].

The stress-tensor multiplet, on the other hand, belongs to the A_2 series of representations and does not enjoy the same kinematic protection. At unitarity threshold one finds

$$L[0]_1^{(0;0)} = A_2[0]_1^{(0;0)} \oplus B_1[0]_2^{(1;1)} \; . \tag{3.5}$$

One may expect this recombination to occur when two decoupled theories, with separately-conserved energy-momentum tensors, are made to interact very weakly so that only the total energy-momentum is exactly conserved. This still cannot happen continuously because $\mathcal{N} = 4$ superconformal theories do not have *any* continuous moduli. It can however happen after a 'small' renormalization-group flow as will be discussed elsewhere. [7]

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[7]C. Bachas and I. Lavdas, work in progress. A slightly massive graviton also occurs in the Karch-Randall model of locally-localized gravity [41]. The limit of vanishing graviton mass is, however, a decompactification limit [42].

The next interesting set of representations are $B_1[0]_R^{(R;0)}$ and $B_1[0]_{R'}^{(0;R')}$ for generic R, R'. Their superconformal primaries are always annihilated by 4 out of the 8 Poincaré supercharges, so they are also 1/2 BPS. The superconformal primary with spin labels $(0, R, 0)$ for example is annihilated by the supercharges $Q_\pm^{(+;\pm)}$. These multiplets contain $16R - 8$ bosonic and as many fermionic fields, all with spins ≤ 1,

$$B_1[0]_R^{(R;0)} = [0]_R^{(R;0)} \oplus [1]_{R+1}^{(R-1;0)} \oplus [0]_{R+1}^{(R-1;1)} \oplus [0]_{R+2}^{(R-2;0)} \oplus \text{fermions} . \tag{3.6}$$

These are the representations in which open BPS strings on D5-brane stacks transform. Note that the spin-1 component of the multiplet has S^2 angular momentum $R - 1$, as anticipated in the previous section. When $R = 2$ it is a vector of $\text{SO}(3)_+$, precisely like the vector bosons corresponding to broken non-compact electric symmetries in the gauged-supergravity description, $\text{SO}(3, m_+) \to \text{SO}(3)_+ \times G_+$ (see section 2.2). Similar statements hold of course for the mirror NS5-brane excitations which have Kaluza-Klein momenta $(0; R')$.

At the level of group theory it is possible to generate the representation (3.6) by taking tensor products of the simplest ultrashort representation

$$B_1[0]_{1/2}^{(1/2;0)} = [0]_{1/2}^{(1/2;0)} \oplus [\tfrac{1}{2}]_1^{(0;1/2)} . \tag{3.7}$$

This is a free hypermultiplet corresponding to a free superfield $H^a = q^a + \theta_\alpha^{a\dot{a}} \zeta_{\dot{a}}^\alpha$ with q^a a scalar doublet of $\mathfrak{su}(2)$, and $\zeta_{\dot{a}}^\alpha$ a spinor doublet of $\mathfrak{su}(2)'$. The product of two hypermultiplets gives a conserved vector current multiplet

$$H^a H^b = q^a q^b + \theta_\alpha^{\{a\dot{a}} q^{b\}} \zeta_{\dot{a}}^\alpha - \theta_\alpha^{a\dot{a}} \theta_\beta^{b\dot{b}} \zeta_{\dot{a}}^\alpha \zeta_{\dot{b}}^\beta \quad \to \quad B_1[0]_1^{(1;0)} , \tag{3.8}$$

while, more generally, the product of R identical hypermultiplets gives

$$H^{a_1} \cdots H^{a_R} \quad \to \quad B_1[0]_R^{(R;0)} .$$

The absence of spin 3/2 components in these representations follows from the identity[8]

$$\text{Sym}_{abc} \, \text{Sym}_{\alpha\beta\gamma} \left[\theta_\alpha^{a\dot{a}} \theta_\beta^{b\dot{b}} \theta_\gamma^{c\dot{c}} \right] = 0 .$$

All this can be repeated for twisted hypermultiplets $\tilde{H}^a = \tilde{q}^{\dot{a}} + \theta_\alpha^{a\dot{a}} \tilde{\zeta}_a^\alpha$ whose products give the mirror representations $B_1[0]_{R'}^{(0;R')}$.

The B_1 multiplets with $RR' \neq 0$ contain spin-3/2 components and are 1/4 BPS (they continue up to level Q^6). They must be dual to Kaluza-Klein modes of the 10d gravitini, those that do not arrange themselves inside spin-2 towers. The lowest-lying such multiplet, $B_1[0]_1^{(1/2;1/2)}$, has as its top component a conserved supercurrent. It is present in backgrounds with enhanced supersymmetry, but not in the solutions studied here which break half of the $\mathcal{N} = 8$ supersymmetries.

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[8]The reader may be amused to note that in the context of the quark model this same identity shows why the mysterious Δ^{++} resonance requires the existence of three colors.

$\mathcal{N}=4$ **Multiplet**	**String mode**	**Gauged SUGRA**
$A_2[0]_1^{(0;0)}$	Graviton	yes
$B_1[0]_1^{(1;0)}$	D5 gauge bosons	yes
$B_1[0]_1^{(0;1)}$	NS5 gauge bosons	yes
$B_1[0]_R^{(R>1;0)}$	Open F-strings, $R \in \frac{1}{2}|\ell_a - \ell_b| + \mathbb{N}$ Closed strings, $R \in \mathbb{N}$	only $R = 2$
$B_1[0]_{R'}^{(0;R'>1)}$	Open D-strings, $R' \in \frac{1}{2}|\hat{\ell}_{\hat{a}} - \hat{\ell}_{\hat{b}}| + \mathbb{N}$ Closed strings, $R' \in \mathbb{N}$	only $R' = 2$
$B_1[0]_{R+R'}^{(R\geq 1;R'\geq 1)}$	Kaluza Klein gravitini $(R,R'\in\mathbb{N})$	no
$A_2[0]_{1+R+R'}^{(R>0;R'>0)}$	Kaluza Klein gravitons $(R,R'\in\mathbb{N})$	no
$A_1[j>0]_{1+j+R+R'}^{(R;R')}$	Stringy excitations	no

Table 1: The short $\mathcal{N}=4$ superconformal multiplets in the notation of ref. [12], possible dual single-string excitations in the Type IIB solutions of [4][5], and the fate of these excitations after the gauged-supergravity truncation. In the middle column $\mathbb{N} = \{1, 2, \cdots\}$ is the set of non-zero natural numbers. A yes/no entry in the third column indicates that the excitation survives/does not survive in gauged $\mathcal{N}=4$ supergravity. Among the $R=2$ or $R'=2$ modes some may (but need not) survive.

The next entry in the list (3.1) are the representations $A_2[0]_{\Delta_0}^{(R;R')}$ with $R + R' > 0$ and $\Delta_0 = 1 + R + R'$. These all contain spins up to and including 2 and are hence Kaluza-Klein excitations of the graviton. Like the stress-tensor multiplet, these multiplets can also recombine into long representations at the unitarity threshold.

The last set of short super-conformal multiplets is $A_1[j]_{\Delta_0}^{(R;R')}$ with $\Delta_0 = 1 + j + R + R'$ and $j > 0$. These contain spins higher than 2 and can only be dual to excited string states, or to multiparticle states. They end at level Q^4 if $R = R' = 0$, at level Q^6 if $RR' = 0$, and at level Q^7 otherwise. With some abuse of terminology, these special semi-short multiplets may be termed accordingly 1/2, 1/4 or 1/8 BPS. An example of such a representation is $A_1[1]_2^{(0;0)}$. It contains a conserved vector current at the lowest level and a conserved spin-3 current at level Q^4, and can become long by eating $A_1[\frac{1}{2}]_{5/2}^{(1/2;1/2)}$. States in the A_1 representations are interesting for the study of supersymmetric black holes, but they are outside our scope here.

For later reference we have collected in table 1 the short $\mathcal{N}=4$ superconformal multiplets and corresponding single-particle fields in the AdS$_4$ backgrounds.

3.2 Marginal Deformations

Marginal deformations are generated by scalar operators of dimension $\Delta = 3$. To preserve maximal supersymmetry they must be top components of $\mathcal{N} = 4$ multiplets. Inspection of all short multiplets shows that such operators do not exist [14], so $\mathcal{N} = 4$ SCFT$_3$ have no fully superconformal moduli. We have seen that the same conclusion has been reached from gauged supergravity, and also from the explicit Type IIB solutions. The proof based on representations of $\mathfrak{osp}(4|4)$ settles definitely the issue.

Actually, the inspection of multiplets is tricky because some of the 3d superfields have top (or rather 'dead end') components at intermediate levels [14]. An example is the scalar $[0]_2^{(0;0)}$ in the stress-tensor multiplet (3.2) which is annihilated by all supercharges, and can be used to trigger a universal $\mathcal{N} = 4$ mass deformation[9]. No such dead-end components arise however at $\Delta = 3$. The only other relevant $\mathcal{N} = 4$ deformations reside in the electric and magnetic flavor-current multiplets $B_1[0]_1^{(1;0)}$ and $B_1[0]_1^{(0;1)}$ and correspond to triplets of flavor masses and Fayet-Iliopoulos terms [47][48].

As shown in ref. [14] $\mathcal{N} = 3$ also does not allow fully superconformal moduli[10]. The maximal supersymmetry that allows them is $\mathcal{N} = 2$. This is enough supersymmetry to protect some marginal operators againts quantum corrections, making the problem technically tractable. We focus henceforth on $\mathcal{N} = 2$.

The $\mathcal{N} = 2$ Poincaré supercharges are spinors Q and $\bar{Q}$ with R-charge, respectively, $r = -1$ and $r = 1$. The unitary multiplets are two-sided: they are obtained by imposing independent unitarity bounds and shortening conditions for Q and $\bar{Q}$. All possible representations are listed in pages 67-69 of ref. [12]. The most relevant multiplets are the conserved stress tensor, vector current, and 'superpotential' multiplets

$$\textbf{stress tensor}: \quad A_1\bar{A}_1[1]_2^{(0)} = [1]_2^{(0)} \oplus [\tfrac{3}{2}]_{5/2}^{(\pm 1)} \oplus [2]_3^{(0)} , \tag{3.9}$$

$$\textbf{vector current}: \quad A_2\bar{A}_2[0]_1^{(0)} = [0]_1^{(0)} \oplus [\tfrac{1}{2}]_{3/2}^{(\pm 1)} \oplus [0]_2^{(0)} \oplus [1]_2^{(0)} , \tag{3.10}$$

$$\textbf{superpotential}: \quad L\bar{B}_1[0]_r^{(r>0)} = [0]_r^{(r)} \oplus [\tfrac{1}{2}]_{r+\frac{1}{2}}^{(r-1)} \oplus [0]_{r+1}^{(r-2)} . \tag{3.11}$$

These couple respectively to the graviton multiplet, the vector boson multiplets, and the hypermultiplets of 4d $\mathcal{N} = 2$ supergravity. The first two representations are self-conjugate, while the third is paired with the antichiral multiplet $B_1\bar{L}[0]_{-r}^{(r<0)}$.

The local structure of the superconformal manifold of 3d $\mathcal{N} = 2$ CFTs is essentially the same as for 4d $\mathcal{N} = 1$ theories, and is well understood [49, 50]. Firstly, candidate moduli only

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[9]This deformation exists for any $\mathcal{N} \geq 4$. It has been discussed by many authors, especially in the context of the ABJM theory. An incomplete list of references is [43][44][45][46].

[10]In principle on can obtain $\mathcal{N} = 3$ theories from $\mathcal{N} = 4$ by turning on a 'quantised' super potential, that in $\mathcal{N} = 2$ notation reads $W = kTr(\Phi^2)$, and integrating out the massive modes. This produces Chern-Simons and supersymmetry related terms. In the brane setup it corresponds to combining the original 5-branes into (1,k) 5-branes.

exist in the marginal superpotential multiplet $L\bar{B}_1[0]_2^{(2)}$ and its conjugate [14]. They may, however, fail to be exactly marginal because these multiplets are not absolutely protected and can recombine in long multiplets at the unitarity threshold. The power of the superconformal algebra is that it points to a unique culprit: the only possible recombination is with a vector current

$$L\bar{B}_1[0]_2^{(2)} \oplus B_1\bar{L}[0]_{-2}^{(2)} \oplus A_2\bar{A}_2[0]_1^{(0)} \;\; \to \;\; L\bar{L}[0]_1^{(0)} \;. \tag{3.12}$$

This is the only mechanism by which a marginal operator $\mathcal{L}$ can become marginally irrelevant in 3d $\mathcal{N}=2$ providing the 'longitudinal' component of a previously conserved current $\mathcal{J}_\mu$ [11]

$$\partial_\mu \mathcal{J}^\mu = 0 \quad \to \quad \partial_\mu \mathcal{J}^\mu = \mathcal{L} \;. \tag{3.13}$$

In the end, as shown using only superconformal perturbation theory in [51], the superconformal manifold $\mathcal{M}_c$ is the Kähler quotient of the space $\{\lambda_i\}$ of complex marginal couplings by the complexified global (flavor) symmetry group G,

$$\mathcal{M}_c = \{\lambda_i | D^a = 0\}/G = \{\lambda_i\}/G^{\mathbb{C}} \;. \tag{3.14}$$

Here $D^a = 0$ is the moment-map condition

$$D^a = \lambda^i T_{i\bar{j}}^a \bar{\lambda}^{\bar{j}} + O(\lambda^3) \;=\; 0 \tag{3.15}$$

with T^a the generators of the global group in the representation of the marginal couplings. This condition follows directly from conformal perturbation theory, while the extra quotient by G is just the identification of theories obtained by G-transformation of the deforming operator [49, 50].

The story is more familiar in the context $\mathcal{N}=1$ SCFTs in four dimensions, following the pioneering work of Leigh and Strassler [52] and further explored in [49, 50, 53, 54]. From the perspective of the dual supergravity the recombination (3.12) is the familiar 'Higgsing' of a gauge symmetry which is allowed by $\mathcal{N}=2$, but not by $\mathcal{N}=4$ supersymmetry. [12]

When the CFT has no flavor symmetries the supergravity has no vector multiplets. Marginal deformations are dual to hypermultiplets and since $\mathcal{N}=2$ forbids a superpotential they are, in this case, exactly marginal. This agrees with the fact that the quotient (3.14) is in this case the trivial quotient. When the supergravity couples to vector multiplets the analysis of the vacua is more involved, see for example [64].

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[11] As shown in [55]-[59], this generalises to generic spin s currents in (super)conformal theories,

$$\partial \mathcal{J}_{(s,\Delta=s+D-2)} = 0 \quad \to \quad \partial \mathcal{J}_{(s,\Delta=s+D-2+\gamma)} = \mathcal{L}_{(s-1,\Delta=s+D-1+\gamma)} \;,$$

provided that the required 'Goldstone' or Stückelberg modes $\mathcal{L}_{(s-1,\Delta=s+D-1+\gamma)}$ exist in the spectrum.

[12] $\mathcal{N}=4$ supersymmetry is however compatible with a pantagruelic Higgs mechanism, termed *La Grande Bouffe* in [55]-[59], whereby higher spin currents are violated by the interactions and acquire anomalous dimensions γ in the boundary (S)CFT, holographic dual to mass-shifts in the bulk AdS (super)gravity. The prototypical example is the long Konishi super-multiplet, originally studied in e.g. [60]-[63].

3.3 $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$

We are interested in marginal deformations of $\mathcal{N} = 4$ theories preserving $\mathcal{N} = 2$. Such deformations only exist in the $\mathcal{N} = 2$ superpotential multiplet

$$L\bar{B}_1[0]_2^{(2)} = [0]_2^{(2)} \oplus [\tfrac{1}{2}]_{5/2}^{(1)} \oplus [0]_3^{(0)} \tag{3.16}$$

and its conjugate $B_1\bar{L}$. Since our starting point is $\mathcal{N} = 4$ supersymmetric, we must search for $\mathcal{N} = 4$ multiplets that contain such marginal $\mathcal{N} = 2$ operators. The $\mathcal{N} = 2$ subalgebra has a unique embedding in $\mathcal{N} = 4$, and commutes with a $u(1)_F$ that acts as an "accidental flavor symmetry",

$$\mathfrak{osp}(2|4) \oplus \mathfrak{u}(1)_F \subset \mathfrak{osp}(4|4) \tag{3.17}$$

In an appropriate basis the $\mathcal{N} = 2$ R-symmetry is generated by $\mathfrak{J}_3 + \mathfrak{J}_3'$, where $\mathfrak{J}_3$ and $\mathfrak{J}_3'$ are the canonically-nomalized Cartan generators of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)'$, while the "accidental" commuting $\mathfrak{u}(1)_F$ is generated by $\mathfrak{J}_3 - \mathfrak{J}_3'$. We are interested in the decomposition of $\mathcal{N} = 4$ multiplets under this embedding.

The relevant $\mathcal{N} = 4$ representations must contain a $\Delta = 2$ scalar component, which is the lowest component of (3.16). This gives the following candidate list

$$B_1[0]_{R+R'}^{(R;R')} \quad \text{with} \ \ R + R' = 1, 2 \ , \qquad \text{or} \qquad A_2[0]_{R+R'+1}^{(R;R')} \quad \text{with} \ \ R + R' = 0, 1 \ . \tag{3.18}$$

The lowest entries of the list are the familar stress-tensor and vector-current multiplets, eqs. (3.2) - (3.4), whose $\mathcal{N} = 2$ decomposition reads:

$$A_2[0]_1^{(0;0)} = \underbrace{A_2\bar{A}_2[0]_1^{(0)(0)}}_{\text{vector current}} \oplus \underbrace{A_1\bar{A}_1[\tfrac{1}{2}]_{3/2}^{(0)(1)} \oplus A_1\bar{A}_1[\tfrac{1}{2}]_{3/2}^{(0)(-1)}}_{\text{supercurrents}} \oplus \underbrace{A_1\bar{A}_1[1]_2^{(0)(0)}}_{\text{stress tensor}} , \tag{3.19}$$

$$B_1[0]^{(1;0)} = \underbrace{A_2\bar{A}_2[0]_1^{(0)(0)}}_{\text{vector current}} \oplus \underbrace{L\bar{B}_1[0]_1^{(1)(1)} \oplus B_1\bar{L}[0]_1^{(-1)(-1)}}_{\text{chiral+antichiral}} , \tag{3.20}$$

$$B_1[0]^{(0;1)} = A_2\bar{A}_2[0]_1^{(0)(0)} \oplus L\bar{B}_1[0]_1^{(1)(-1)} \oplus B_1\bar{L}[0]_1^{(-1)(1)}. \tag{3.21}$$

These are the familiar decompositions of the $\mathcal{N} = 4$ graviton and vector multiplets in terms of $\mathcal{N} = 2$ multiplets. The conserved current in the graviton multiplet is that of the 'accidental' $\mathfrak{u}(1)_F$ symmetry. We indicated the $\mathfrak{u}(1)_F$ charge of representations in blue fonts next to the $\mathcal{N} = 2$ R-symmetry charge.

None of these multiplets contains the sought-for marginal superpotential (3.16). One can also rule out the representations $A_2[0]_2^{(R;R')}$ with $R + R' = 1$ from the list, because the $\Delta = 2$ scalars in this representation have $\mathcal{N} = 2$ R-charge $r = 0, \pm 1$, but not $r = 2$ as required. This leaves then the representations $B_1[0]_2^{(R;R')}$ with $R + R' = 2$ as the only ones containing marginal $\mathcal{N} = 2$ superconformal deformations.

The decomposition of these representations in terms of $\mathcal{N} = 2$ multiplets is as follows[13] [we only exhibit one member of each mirror pair; the c.c. of $X\bar{Y}[j]_\Delta^{(r)(f)}$ is $Y\bar{X}[j]_\Delta^{(-r)(-f)}$]:

$$B_1[0]_2^{(2;0)} = L\bar{L}[0]_2^{(0)(0)} \oplus \left[L\bar{A}_2[0]_2^{(1)(1)} \oplus \boxed{L\bar{B}_1[0]_2^{(2)(2)}} \oplus c.c. \right] , \qquad (3.22)$$

$$B_1[0]_2^{(1;1)} = L\bar{L}[0]_2^{(0)(0)} \oplus \left[L\bar{L}[0]_2^{(0)(2)} \oplus L\bar{A}_2[0]_2^{(1)(1)} \oplus L\bar{A}_2[0]_2^{(1)(-1)} \oplus \boxed{L\bar{B}_1[0]_2^{(2)(0)}} \oplus c.c. \right]$$

$$\oplus \left[L\bar{L}[\tfrac{1}{2}]_{5/2}^{(0)(1)} \oplus L\bar{A}_1[\tfrac{1}{2}]_{5/2}^{(1)(0)} \oplus c.c. \right] \oplus L\bar{L}[0]_3^{(0)(0)} , \qquad (3.23)$$

$$B_1[0]_2^{(3/2;1/2)} = \left[L\bar{L}[0]_2^{(0)(1)} \oplus L\bar{A}_2[0]_2^{(1)(2)} \oplus L\bar{A}_2[0]_2^{(1)(0)} \oplus \boxed{L\bar{B}_1[0]_2^{(2)(1)}} \oplus c.c. \right]$$

$$\oplus L\bar{L}[\tfrac{1}{2}]_{5/2}^{(0)(0)} \oplus \left[L\bar{A}_2[0]_3^{(2)(1)} \oplus c.c. \right] . \qquad (3.24)$$

All of them contain the (boxed) marginal superpotential $L\bar{B}_1$, so we should look for these multiplets in the original $\mathcal{N} = 4$ theory.

The breaking $\mathcal{N} = 4 \to \mathcal{N} = 2$ requires that two spin-1/2 Goldstini be eaten by the two spin-3/2 gravitini which acquire a mass. The corresponding recombination reads

$$A_1\bar{A}_1[\tfrac{1}{2}]_{3/2}^{(0)} \oplus L\bar{A}_2[0]_2^{(1)} \oplus c.c. \quad \to \quad L\bar{L}[\tfrac{1}{2}]_{3/2}^{(0)} , \qquad (3.25)$$

where $L\bar{A}_2[0]_2^{(1)}$ is the $\mathcal{N} = 2$ Goldstino multiplet. All of the above muliplets contain, in addition to the deforming superpotential, a candidate Goldstino as required for consistency. Note also that the self-conjugate representation (3.23) respects the accidental $\mathfrak{u}(1)_F$ symmetry, while among all flat directions in other multiplets a linear combination will be lifted by the $\mathfrak{u}(1)_F$ moment-map condition.

To summarize, we have identified the three $\mathcal{N} = 4$ superconformal multiplets that contain candidate $\mathcal{N} = 2$ moduli. From table 1 we see that the first can arise from either open or closed strings, and may survive the gauged-supergravity truncation, the second can come from Kaluza-Klein gravitini, while the third violates our angular-momentum selection rule of section 3.1 and can only be an exotic multi-particle state. We will now go to the CFT side of the correspondence, where the first kind of deformations will be identified with standard superpotential deformations on the Higgs branch of the electric-quiver gauge theory (or its mirror dual), while the others involve mixed-branch operators.

4 $\mathcal{N} = 4$ quiver gauge theories

The evidence that the AdS$_4$ solutions of [4][5] are holographic duals to the 'good' $\mathcal{N} = 4$ quiver gauge theories of [15] is that their symmetries match, and that the condition (2.2) is

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[13]In comparing with the tables in ref. [12] the reader should be warned that these are only valid for generic values of the spins. For small R, R' some of the states are actually missing.
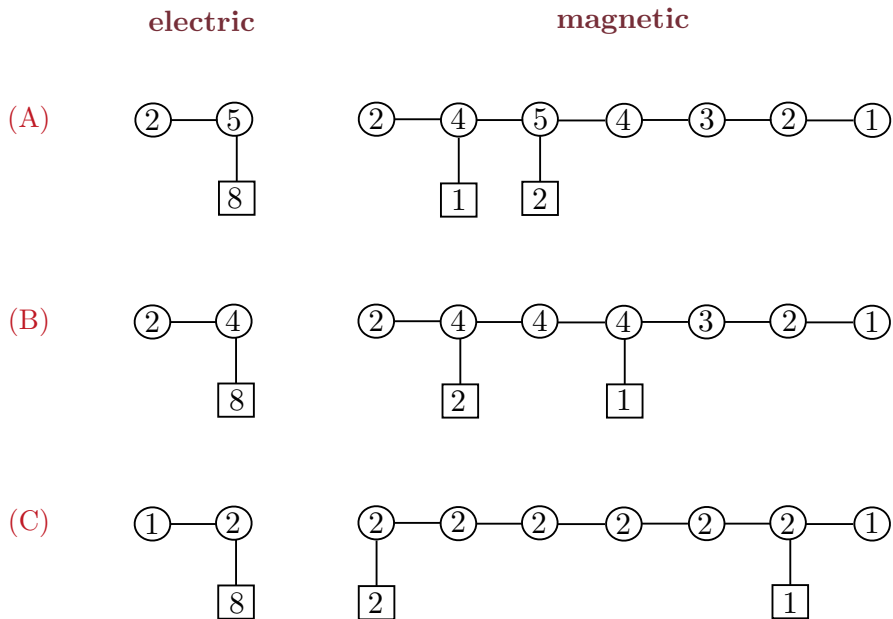
Figure 2: The electric and magnetic quivers corresponding to the theories (A, B, C) of section 2. The infrared limits of these theories have the same flavor symmetry but different operator content.

automatically obeyed on the string-theory side. In this section we will find more evidence for this correspondence. To make the paper self-contained, we start by recalling some simple facts about $\mathcal{N} = 4$ quiver theories.

4.1 Generalities

The fields of $\mathcal{N} = 4$ gauge theories are vector multiplets and hypermultiplets, and the group of R symmetries is usually denoted $\mathrm{SU}(2)_H \times \mathrm{SU}(2)_C$ (for Higgs and Coulomb).[14] A vector multiplet contains a vector, A_μ, and three scalars $\phi^{1,2,3}$ transforming as a vector of $\mathrm{SU}(2)_C$, while a hypermultiplet contains two complex scalars, $H = (q^+, q^-)$, which transform as a doublet of $\mathrm{SU}(2)_H$.

The vector fields are in the adjoint representation of the gauge group which is a product of unitary factors, $\prod_i U(n_i)$, while the hypermultiplets transform either in fundamental or in bifundamental representations. The precise representation content is described by a linear quiver like those depicted in figure 2. A circular node (n_i) stands for a gauge-group factor $U(n_i)$, a square node $\boxed{N_a}$ for N_a hypermultiplets in the fundamental of the associated gauge-group factor, and a horizontal link for a hypermultiplet in the bifundamental of the adjacent circular nodes. The $\mathcal{N} = 4$ Lagrangian contains the Yang-Mills term but no Chern-Simons terms,[15] and it is completely fixed by the quiver data modulo the gauge couplings which are free parameters with dimension $[\mathrm{mass}]^{1/2}$.

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[14] This is the same as the group $\mathrm{SO}(3)_+ \times \mathrm{SO}(3)_-$ of earlier sections.

[15] Chern-Simons terms are necessary when there exist three different types of 5-brane, in which case the maximal supersymmetry is $\mathcal{N} = 3$ [65]. The $\mathcal{N} = 4$ theories of interest here have equivalent Chern-Simons realizations [66], which we will not need in this paper.

The relation of the quiver to the data $\{N_a, \ell_a\}$ and $\{\hat{N}_{\hat{a}}, \hat{\ell}_{\hat{a}}\}$ of the supergravity solutions follows from the engineering of the gauge theory on flat-space branes [6]. The electric quiver is realized by D3-branes suspended between NS5-branes and intersecting D5-branes, while for the magnetic quiver the roles of NS5 and D5 are exchanged. It follows from these constructions that the electric quiver has N_a fundamental hypermultiplets at the $i = \ell_a$ gauge node (counting from right to left), while the magnetic quiver has $\hat{N}_{\hat{a}}$ fundamental hypermultiplets at the $\hat{i} = \hat{\ell}_{\hat{a}}$ gauge node (counting from left to right). The number of gauge nodes in the electric quiver is the total number $\hat{k} = \sum_{\hat{a}} \hat{N}_{\hat{a}}$ of NS5-branes minus one. Reading the complete NS5-brane data $\{\hat{N}_{\hat{a}}, \hat{\ell}_{\hat{a}}\}$ from the electric quiver, is possible but slightly more involved (see e.g. ref. [4]). By moving around the 5-branes one obtains the dual magnetic quiver with a total number of gauge nodes $\sum_a N_a - 1 := k - 1$.

The condition (2.2) for a 'good theory' ensures that the number of hypermultiplets suffices to completely Higgs the gauge symmetry. The vacuum manifold thus contains a pure Higgs branch, along which only the hypermultiplet v.e.vs are non-zero, and a pure Coulomb branch which is isomorphic to the pure Higgs branch of the mirror quiver. These branches can be viewed as complex varieties described by chiral rings of holomorphic functions. In the language of $\mathcal{N} = 2$ supersymmetry, vectors decompose into pairs (V, Φ) of vectors and chiral multiplets in the adjoint representation of the gauge group, while hypermultiplets contain two chiral fields $(q, \tilde{q})$ in complex-conjugate representations. [16] The Yang-Mills Lagrangian includes a superpotential term for each $U(n_i)$ factor, $W = \sum_\alpha q_\alpha^T \Phi \tilde{q}_\alpha$ where Φ is the adjoint chiral field and the sum runs over all chiral multiplets in the (anti)fundamental representation of $U(n_i)$. Since on the Higgs branch $\langle \Phi \rangle = 0$, the non-trivial F-flatness conditions are the n_i^2 equations $\sum_\alpha q_\alpha \tilde{q}_\alpha^T = 0$ for each gauge node. The chiral ring consists of all gauge-invariant combinations of qs and $\tilde{q}$s modulo these F-flatness conditions.

4.2 Chiral-ring operators as open strings

The chiral-ring operators on the Higgs branch are singlets of $SU(2)_C$. Their dimension is equal to their $\mathcal{N} = 2$ R-symmetry charge which is half the number of chiral fields in a product. It is easy to see that all such operators are highest-weights in short representations $B_1[0]_R^{(R;0)}$ of the $\mathcal{N} = 4$ algebra, where $\Delta = R$ is the $SU(2)_H$ spin. These representations survive the RG flow to the infrared SCFT.

One can picture these operators as oriented strings with string-bits being the links of the quiver diagram, as in figure 3. The usefulness of this perspective was demonstrated by Assel [13] who mapped chiral and monopole operators to strings in the flat-brane setup. Here we will map them to strings in the near-horizon geometry. The string orientation indicates whether one picks the q or $\tilde{q}$ chiral field in a hypermultiplet: the q for up- or left-pointing string bits, and the $\tilde{q}$ for down- or right-pointing string bits. Gauge-group indices are automatically summed while flavor indices are free, so strings can be either closed, or have endpoints on the

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[16]Compared to our previous notation, $q := q^+$ is the upper component of the $SU(2)_H$ doublet, and $\tilde{q} := (q^-)^\star$ is the complex conjugate of the lower component.
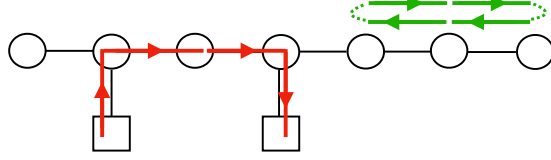
Figure 3: Two chiral operators on the Higgs branch of the magnetic quiver of theory B. The open string operator (in red) is in the bifundamental of the flavor group $U(2) \times U(1)$, while the closed string operator (in green) is a flavor singlet. Both contain marginal superpotential deformations since they have length=4, and hence belong to $B_1[0]_2^{(2;0)}$ multiplets.

square nodes which we identify with the stacks of D5-branes. The same holds for the dual magnetic quiver, where the square nodes are NS5-branes and the relevant $\mathcal{N} = 4$ multiplets are $B_1[0]_{R'}^{(0;R')}$. Clearly the dimension Δ is half the total length of the string.

To make the notation lighter it is convenient to label the D5-brane stacks by their linking number, indicating the circular node to which they attach. The quiver data is then specified by the set of $\hat{k} - 1$ non-negative integers $S = \{N_\ell\}$, some of which can be zero, and by the set of corresponding gauge-group ranks $s = \{n_\ell\}$. For example the magnetic quiver of theory A in figure 2 has $S = \{0, 1, 2, 0, 0, 0, 0\}$ and $s = \{2, 4, 5, 4, 3, 2, 1\}$. We also consider the q and $\tilde{q}$ as matrices, so that the superpotential at the ℓth circular node reads

$$W = \mathrm{tr}(\Phi_\ell \, \tilde{q}_{\ell, \square} \, q_{\square, \ell} + \Phi_\ell \, \tilde{q}_{\ell, \ell+1} \, q_{\ell+1, \ell} + \Phi_\ell \, q_{\ell, \ell-1} \, \tilde{q}_{\ell-1, \ell}) \tag{4.1}$$

where the subscripts here indicate the link, with gauge and flavor indices suppressed (the label '$\square, \ell$' denotes the vertical link). The corresponding F-term condition can be drawn as a linear relation between strings cut-open at the circular node, as shown in figure 4. Notice that this relation identifies flavor-singlet combinations of open strings with closed strings.

Having set the notation, let us enumerate some chiral operators by increasing length:

- **$\Delta = 1$**. These operators belong to conserved-current multiplets of $\mathcal{N} = 4$. There exist length-2 open strings in the adjoint representation of each $U(N_\ell)$ flavor-group factor, and one length-2 closed string for each horizontal node of the quiver, i.e. in total $\hat{k} - 2$ horizonatl nodes. These strings are subject to $\hat{k} - 1$ F-term conditions, one for each $tr\Phi_\ell$. The number of independent operators is therefore precisely the dimension of the flavor group $\left(\prod_\ell U(n_\ell)\right)/U(1)$ as expected. Note that the overall $U(1)$ which acts trivially on the fields consistently decouples.

- **$\Delta = 3/2$**. There are no closed strings of length 3, in accordance with the rule on the gravity side, see section 2.3, that spin-0 closed strings must have integer S^2 angular momenta. Open strings of length 3 exist for every neighbouring pair of square nodes,

and they are linearly independent. They transform in the bifundamental representation of $U(N_\ell) \times U(N_{\ell+1})$. Of the six quivers of figure 2 only the magnetic quiver of theory A has such chiral operators.

- $\boldsymbol{\Delta = 2}$. There are several possibilities for strings of length 4. The corresponding chiral operators belong to the short multiplets $B_1[0]_2^{(2;0)}$ (or $B_1[0]_2^{(0;2)}$ for the Coulomb branch or, alternatively, the Higgs branch of the dual quiver) which contain marginal $\mathcal{N} = 2$ superpotential deformations.

  The open strings transform either in the second symmetric product of the flavor group, or in bifundamental representations of next-to-nearest-neighbor group factors $U(N_\ell) \times U(N_{\ell+2})$, when such exist. Of the six quivers of figure 2 only the magnetic quiver of theory B has such chiral operators. The operators in the second symmetric product of the adjoint representation are bound states of two open strings. We will argue shortly that this completes the list of independent $\Delta = 2$ chiral operators.

One can now see the emerging general pattern. At level $\Delta = R$ there exist single-string chiral operators that are either closed strings, or open strings in the bifundamental of $U(N_\ell) \times U(N_{\ell'})$. They are subject to the following selection rules:

$$R = 1 + n \quad \text{for closed strings}, \qquad R = \frac{1}{2}|\ell - \ell'| + n \quad \text{for open strings} \qquad (4.2)$$

where $n = 1, 2 \cdots$. These are precisely the rules derived from Type IIB string theory in section 2.3, in perfect agreement with the conjectured holographic duality. They have been also obtained as scaling dimensions of monopole operators on the Coulomb branch of the magnetic quiver in agreement with mirror symmetry [67].

For linear quivers, it is actually possible to choose a basis in which all chiral operators are multiparticle bound states of open strings. This follows from the F-term condition in figure 4

$$-\tilde{q}_{\ell,\ell+1}\, q_{\ell+1,\ell} \;=\; q_{\ell,\ell-1}\, \tilde{q}_{\ell-1,\ell} + \tilde{q}_{\ell,\square}\, q_{\square,\ell} \;, \qquad (4.3)$$
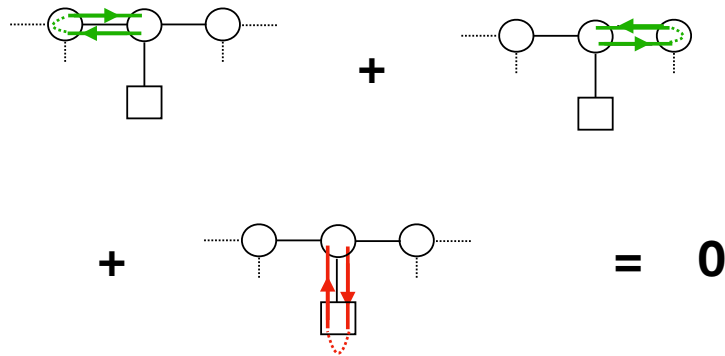

Figure 4: Graphical representation of the F-flatness conditions on the Higgs branch, as linear relations among cut-open string segments. The dotted red semicircle in the third term stands for a summation over the free flavor indices of the open strings.

which can be used to "fold and slide" any closed string along the horizontal part of the quiver. If the second term on the right-hand-side were absent, the closed string would eventually hit the quiver boundary and annihilate. Because of the second term, the process is accompanied by the "emission" of open strings, *qed.*

This argument does not work for circular quivers, which have no boundary and which can support irreducible closed winding strings.

4.3   Counting techniques

Although the logic is clear, the actual counting of $\Delta > 2$ operators on the pure Higgs or pure Coulomb branch can become quickly cumbersome. In this section we mention some elegant techniques developped by both physicists and mathematicians.

The moduli spaces of the family of $\mathcal{N} = 4$ quivers considered here have a description in terms of nilpotent orbits and Slodowy slices, see [68] and references therein. Let us review the basic points. For a given partition ρ of N we denote by $\overline{\mathcal{O}}_\rho$ the closure of the nilpotent orbit associated to this partition. The orbit consists of all $N \times N$ nilpotent matrices whose Jordan normal form has Jordan blocks of sizes given by the partition, and the closure includes the orbits of all smaller partitions as well. We also denote by $\mathcal{S}_\rho$ the Slodowy slice associated to this partition, namely the transverse slice to the orbit $\mathcal{O}_\rho$ in the space which is freely generated by adjoint-valued variables.

Let the Higgs branch be $\mathcal{H}$ and the Coulomb branch be $\mathcal{C}$, and denote the electric theory by a subscript e and the magnetic theory by a subscript m. Then the Higgs branch $\mathcal{H}_e$ of the electric theory and the Coulomb branch $\mathcal{C}_m$ of the magnetic theory are given by the intersection [29, 30]

$$\mathcal{H}_e = \mathcal{C}_m = \mathcal{S}_\rho \cap \overline{\mathcal{O}}_{\hat{\rho}^T} ~, \tag{4.4}$$

while the Higgs branch $\mathcal{H}_m$ of the magnetic theory and the Coulomb branch $\mathcal{C}_e$ of the electric theory are given by the mirror intersection

$$\mathcal{H}_m = \mathcal{C}_e = \mathcal{S}_{\hat{\rho}} \cap \overline{\mathcal{O}}_{\rho^T}. \tag{4.5}$$

As an example, we can compute three of the moduli spaces for the partitions of Figure 1 by noticing that for the partition ρ in this figure, the Slodowy slice is the whole freely-generated adjoint-valued variety, and therefore the Higgs branch of the electric theory becomes the closure of the corresponding nilpotent orbit.

$$\mathcal{H}_e = \mathcal{C}_m = \overline{\mathcal{O}}_{\hat{\rho}^T}. \tag{4.6}$$

The latter have simple descriptions as algebraic varieties with relations given by conditions on matrices. Explicitly, for models A, B, C of Figure 2 we have

$$\begin{aligned}
\mathcal{H}_e^A &= \mathcal{C}_m^A = \overline{\mathcal{O}}_{\hat{\rho}_A^T} = \left\{ M_{8 \times 8} : \text{tr}\,(M) = \text{tr}\,(M^2) = 0,\, M^3 = 0,\, rk(M) \leq 5 \right\}, &\text{(4.7a)} \\
\mathcal{H}_e^B &= \mathcal{C}_m^B = \overline{\mathcal{O}}_{\hat{\rho}_B^T} = \left\{ M_{8 \times 8} : \text{tr}\,(M) = \text{tr}\,(M^2) = 0,\, M^3 = 0,\, rk(M) \leq 4 \right\}, &\text{(4.7b)} \\
\mathcal{H}_e^C &= \mathcal{C}_m^C = \overline{\mathcal{O}}_{\hat{\rho}_C^T} = \left\{ M_{8 \times 8} : \text{tr}\,(M) = \text{tr}\,(M^2) = 0,\, M^3 = 0,\, rk(M) \leq 2 \right\}, &\text{(4.7c)}
\end{aligned}$$

where $rk(M)$ is the rank of the matrix M.

In the language used in the previous subsection all electric quivers in Figure 2 have two hypermultiplets. Call for short $(q, \tilde{q})$ the fields in the fundamental of $SU(8)$, and $(u, \tilde{u})$ those corresponding to the horizontal link. The F-flatness conditions are $\tilde{u}u = u\tilde{u} + \tilde{q}q = 0$. It follows that the meson matrix $M = q\tilde{q}$ obeys the conditions $\mathrm{tr}\,(M) = \mathrm{tr}\,(M^2) = M^3 = 0$. Furthermore its rank cannot exceed the rank of the gauge group under which the quarks q and $\tilde{q}$ are charged. This explains the equations (4.7).

The global symmetry on these branches is $SU(8)$. This is seen in the electric quivers as flavor symmetry, and it manifests itself in the magnetic quivers from the balanced A_7 Dynkin diagram.[17] Denoting the highest-weight fugacities of $SU(8)$ by μ_i, $i = 1, ..., 7$ and the highest-weight fugacity of $SU(2)_{H(C)}$ by $t_{1(2)}$, we find for all three moduli spaces that the operators up to dimension $\Delta = R = 2$ transform in the following representations

$$1 + \mu_1\mu_7 t_1^2 + (\mu_1^2\mu_7^2 + \mu_2\mu_6 + \mu_1\mu_7)t_1^4. \tag{4.8}$$

At $\Delta = 1$ we find the adjoint of $SU(8)$, and at $\Delta = 2$ there are $1232 + 720 + 63$ distinct operators, the same number as in the symmetric traceless product of two adjoints.[18] This agrees with our statement that there is a basis consisting of only open strings.

The other branches have the global symmetry $\bigl(U(1) \times U(2)\bigr)/U(1) \simeq SU(2) \times U(1)$ with, again, a universal sector in the adjoint representation at dimension 1, but different content at higher dimensions. This is clear from our earlier enumeration. Letting μ and α be the two fugacities of $SU(2) \times U(1)$, we find the following content up to dimension 2

$$A \;\; : \;\; 1 + (\mu^2 + 1)t_2^2 + \mu(\alpha + \alpha^{-1})t_2^3 + (\mu^4 + \mu^2 + 1)t_2^4 \;, \tag{4.9a}$$

$$B \;\; : \;\; 1 + (\mu^2 + 1)t_2^2 + (\mu^4 + \mu^2 + 1)t_2^4 + \mu(\alpha + \alpha^{-1})t_2^4 \;, \tag{4.9b}$$

$$C \;\; : \;\; 1 + (\mu^2 + 1)t_2^2 + (\mu^4 + \mu^2 + 1)t_2^4. \tag{4.9c}$$

These calculations can be checked from the Coulomb branch of the electric quivers. In addition to the universal adjoint, and (adjoint)^2 sectors, we have new operators in the bi-fundamental of $SU(2) \times U(1)$, with $\Delta = 3/2, 2$ and $7/2$ for theories A,B and C, as explained.

For the conformal manifold one must consider all $\Delta = 2$ operators, including the product of D5-brane strings in the adjoint of the electric flavor group with NS5-brane strings in the adjoint of the magnetic flavour group. Notwithstanding additional mixed-branch operators, the number of candidate moduli even in these simple models is formidable! Furthermore, the majority of marginal fields are 2-string bound states which would go undetected by a semiclassical analysis on the gravity side.

The actual conformal manifold is the quotient (3.14) – (3.15) of the space of all candidate moduli by the complexified flavor group. This removes $63+3+1 + 1 = 68$ directions, i.e.

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[17] A balanced node is one that has the number of flavors equal to twice the number of colors. For linear or circular quivers this means $2n_a = n_{a+1} + n_{a-1} + N_a$. Amusingly this is nothing but the condition for vanishing β functions of the $D = 4$ SYM theory with the same quiver.

[18] We already checked that the trace of the product representation, $\mathrm{tr}\,M^2$, is indeed zero.

one direction per each generator of the flavor symmetry and one for the accidental $U(1)_F$ at the enhanced $\mathcal{N} = 4$ supersymmetry point. Denote the space of all $U(1)$ symmetries which are preserved by the conformal deformation by $\mathcal{P}$. The $U(1)_R$ in the $\mathcal{N} = 2$ superconformal algebra is a particular linear combination of all $U(1)$s in $\mathcal{P}$. The remaining $U(1)$ charges in $\mathcal{P}$ can be arranged in a convenient orthogonal basis $\mathcal{O}$. The conformal manifold is given by the Kähler quotient with respect to $U(1)$ symmetries in $\mathcal{O}$. Notice that at least two charges of opposite sign are necessary to solve $U(1)$ D-term conditions. The particular choice of $U(1)_R$ as the diagonal subgroup of $U(1)_H \times U(1)_C$ implies that there must exist at least one marginal operator from the Higgs branch and one from the Coulomb branch, or else the conformal manifold is trivial.

To calculate the manifold one proceeds with the methods presented in [50], with the help of plethystic techniques as in [69]. To be more specific, one computes the set of holomorphic functions from the Hilbert series of this moduli space with a Molien Weyl integral over the group of global symmetries. In the examples at hand, both the dimension of the group and the number of marginal operators is large, and the integrals cannot be done explicitly. Here we will only sketch the calculation, which we illustrate in appendix B for a simpler case, $U(1)$ theory with one flavor, for which explicit formulae are available.

The formula uses the characters for the representations of the moduli, which can be found in standard references such as Lie Online. The representations are rather large, hence we will adopt a short-hand notation to denote the expression for the integral. The Haar measure for a group G is denoted $d\mu(G)$, and the character of a representation of $SU(8) \times SU(2)$ by the Dynkin labels $[n_1, \ldots, n_7; n]$. With these notational conventions, the integral for (the most interesting) theory B takes the form

$$\oint \frac{dz}{z} \oint \frac{dw}{w} \int d\mu_{SU(8)} d\mu_{SU(2)} \, \mathcal{Z}$$

where

$$\begin{aligned}
\mathcal{Z} &= \mathrm{PE}\Big[([2,0,0,0,0,0,2;0] + [0,1,0,0,0,1,0;0] + [1,0,0,0,0,0,1;0]) z^2 q^2 \\
&\quad + \ ([\vec{0};4] + [\vec{0};2] + [\vec{0};1](w + w^{-1}) + [\vec{0};0]) z^{-2} q^2 + [1,0,0,0,0,0,1;2]q^2 \Big] \ . \quad (4.10)
\end{aligned}$$

Here PE is the plethystic exponential [69], w is the fugacity of the $U(1)$ magnetic-flavor group, and z is the fugacity of the accidental $U(1)_F$ symmetry which is always present at the points of moduli space where supersymmetry is enhanced to $\mathcal{N} = 4$. We have also introduced a fugacity q to keep track of the number of couplings in the problem, which can be identified with the residual R symmetry.

4.4 Mixed-branch operators

So far we have considered chiral operators on the Higgs branch where the moduli space is classical, or on the Coulomb branch which is the Higgs branch of the mirror quiver. These operators organize themselves in $B_1[0]_2^{(2;0)}$ and $B_1[0]_2^{(0;2)}$ multiplets of $\mathfrak{osp}(4|4)$. There exist, however, also mixed-branch operators. The simplest are products of a current multiplet of the
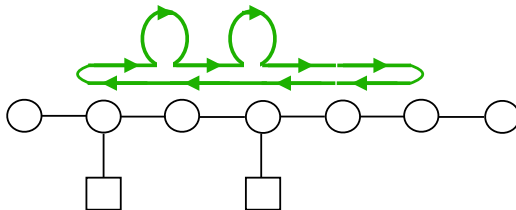
Figure 5: Mixed-branch operators are gauge-invariant products of chiral fields from both hyper-multiplets and vector multiplets. The former are linear string bits while the latter, in the adjoint representation of the gauge group, are denoted by bubbles. The closed string in the figure has 8 linear segments and 2 bubbles, and transforms in the representation $B_1[0]_6^{(4;2)}$.

electric flavor group with a current multiplet of the magnetic flavor group. These marginal $B_1[0]_2^{(1;1)}$ operators are bound states of a string on the D5-branes with a string on the NS5-branes. They are part of the universal spectrum of the CFT.

A natural question to ask is whether there exist single-string (non-factorizable) mixed-branch operators. Let us consider gauge-invariant products of chiral fields which, in addition to the $q, \tilde{q}$ coming from hypermultiplets, also involve the fields Φ_ℓ sitting in the $\mathcal{N} = 4$ vector multiplets. In the ultraviolet theory, which is free, the Φ_ℓ belong to the representation $B_1[0]_1^{(0;1)}$ which contains a triplet of scalars and the conserved topological current $\epsilon^{\mu\nu\rho}F_{\nu\rho}$. Each Φ_ℓ adds therefore one unit of $SU(2)_C$ spin, and raises the scaling dimension by 1. The counting is valid in the free theory, but because the $\mathcal{N} = 4$ multiplet in which such operators sit, is absolutely protected, we may conjecture that it continues to hold in the strongly-coupled low-energy theory as well.

Such operators can be again depicted as strings which, in addition to line segments, contain 'bubbles' at circular nodes, see figure 5. Each line segment stands for a hypermultiplet and contributes $\Delta = R = 1/2$ to the operator dimension and the $SU(2)_H$ spin, while each bubble stands for a Φ_ℓ insertion and contributes $\Delta = R' = 1$ to the operator dimension and the $SU(2)_C$ spin. The resulting operator is in the $B_1[0]_{R+R'}^{(R;R')}$ multiplet, with R' the number of bubbles and $2R$ the number of line segments of the string.

How many of these operators are independent? In addition to the conditions (4.3) we must now also impose the F-term conditions of $q, \tilde{q}$ which read

$$q_{\Box,\ell} \, \Phi_\ell = \Phi_\ell \, \tilde{q}_{\ell,\Box} = 0 \ , \qquad q_{\ell+1,\ell} \, \Phi_\ell \sim \Phi_{\ell+1} \, q_{\ell+1,\ell} \ , \qquad \tilde{q}_{\ell,\ell+1} \, \Phi_{\ell+1} \sim \Phi_\ell \, \tilde{q}_{\ell,\ell+1} \ . \qquad (4.11)$$

The '$\sim$' sign in the above relations allows for different gauge couplings of the factor groups. The second and third relations show that bubbles can move freely around the string, while the first equation shows that if they encounter a vertical segment they vanish. To avoid this from happening, such mixed-branch single-string operators must thus be closed. This fits nicely with the analysis in section 3.1 where we argued that $B_1[0]_{R+R'}^{(R;R')}$ multiplets with $RR' \neq 0$ are either multiparticle states or Kaluza-Klein descendants of the gravitini.

Such single-closed-string mixed-branch operators exist in theories based on circular quivers (whose dual supergravity backgrounds were found in [5]). For linear quivers they vanish by a

similar argument as the one used in section 4.3. As explained there, one can 'fold and slide' the closed string until it reaches the boundary of the linear quiver, where it is annihilated. It can be seen that the bubble insertions go along for the ride, their role being only to annihilate any open strings emitted in the process.

A   The $\mathcal{N} = 4$ supergravity solutions

The local form of the most general Type IIB solutions with $\mathfrak{osp}(4|4)$ symmetry was derived in [25, 26]. It depends on two real non-negative functions h_1 and h_2 that are harmonic on the surface Σ. The coordinates on this latter are $(z, \bar{z})$. Changing slightly the notation of the above references we define the auxiliary functions

$$W = \partial_z \partial_{\bar{z}}(h_1 h_2) \qquad \text{and} \qquad \mathcal{U}_i = 2h_1 h_2 |\partial_z h_i|^2 - h_i^2 W \ . \tag{A.1}$$

The 10-dimensional manifold is a fibration of $\text{AdS}_4 \times \text{S}^2_{(1)} \times \text{S}^2_{(2)}$ over Σ with metric

$$\frac{4}{\alpha'} \, ds^2 = \rho_4^2 \, ds^2_{\text{AdS}} + \rho_1^2 \, ds^2_{(1)} + \rho_2^2 \, ds^2_{(2)} + 4\rho^2 \, dz d\bar{z} \ , \tag{A.2}$$

where α' is the Regge slope, $ds^2_{(i)} = d\vartheta_i^2 + \sin \vartheta_i^2 d\varphi_i^2$ are the metrics of the unit-radius 2-spheres (also called S^2 and $\text{S}^{2\prime}$ in the main text) and the scale factors read

$$\rho_1^8 = 16 h_1^8 \frac{\mathcal{U}_2 W^2}{\mathcal{U}_1^3} \ , \qquad \rho_2^8 = 16 h_2^8 \frac{\mathcal{U}_1 W^2}{\mathcal{U}_2^3} \ , \tag{A.3}$$

$$\rho^8 = \frac{\mathcal{U}_1 \mathcal{U}_2 W^2}{h_1^4 h_2^4} \ , \qquad \rho_4^8 = 16 \frac{\mathcal{U}_1 \mathcal{U}_2}{W^2} \ . \tag{A.4}$$

The solutions have a non-trivial dilaton field

$$e^{2\phi} = \sqrt{\frac{\mathcal{U}_2}{\mathcal{U}_1}} \ , \tag{A.5}$$

as well as antisymmetric Neveu-Schwarz and Ramond-Ramond 3-form fields

$$\frac{4}{\alpha'} \, \text{Re}(f_3) = \omega^{(1)} \wedge d\beta_1 \qquad \text{and} \qquad \frac{4}{\alpha'} \, \text{Im}(f_3) = \omega^{(2)} \wedge d\beta_2 \tag{A.6}$$

with $\omega^{(i)} = d\cos\vartheta_i \wedge d\varphi_i$ the 2-sphere volume forms and

$$\beta_1 = 2i\frac{h_1}{\mathcal{U}_1}h_1h_2(\partial_z h_1\partial_{\bar{z}}h_2 - \partial_z h_2\partial_{\bar{z}}h_1) + 2\widetilde{h}_2 \ , \tag{A.7}$$

$$\beta_2 = 2i\frac{h_2}{\mathcal{U}_2}h_1h_2(\partial_z h_1\partial_{\bar{z}}h_2 - \partial_z h_2\partial_{\bar{z}}h_1) - 2\widetilde{h}_1 \ . \tag{A.8}$$

Here $\tilde{h}_i$ are the functions dual to h_i, i.e. in terms of two meromorphic functions $\mathcal{A}_i$:

$$h_1 = -i(A_1 - \bar{A}_1) \ , \quad h_2 = A_2 + \bar{A}_2 \ ,$$

$$\widetilde{h}_1 = A_1 + \bar{A}_1 \ , \quad \widetilde{h}_2 = i(A_2 - \bar{A}_2) \ . \tag{A.9}$$

Finally, there is a non-trivial self-dual Ramond-Ramond 5-form

$$(\frac{4}{\alpha'})^2 f_5 = -4\rho_4^4\, \omega^{\text{AdS}} \wedge \mathcal{F} + 4\rho_1^2\rho_2^2\, \omega^{(1)} \wedge \omega^{(2)} \wedge (*_2\mathcal{F}) \ , \tag{A.10}$$

where ω^{AdS} is the volume form of the unit-radius AdS$_4$, and $\rho_4^4\mathcal{F}$ is a closed one-form given by the following formidable-looking expressions:

$$\mathcal{F} = \rho_4^{-4}\, d\mathcal{J} \ , \quad \text{with} \quad \mathcal{J} = 3(\mathcal{C} + \bar{\mathcal{C}}) - 3\mathcal{D} + i\frac{h_1 h_2}{W}(\partial_z h_1\partial_{\bar{z}}h_2 - \partial_z h_2\partial_{\bar{z}}h_1) \ ,$$

$$\partial\mathcal{C} = A_1\partial_z A_2 - A_2\partial_z A_1 \ , \quad \text{and} \quad \mathcal{D} = A_1\bar{A}_2 + A_2\bar{A}_1 \ . \tag{A.11}$$

The 1-form $*_2\mathcal{F}$ in (A.10) is the dual of $\mathcal{F}$ on the Riemann surface Σ.

The explicit solutions found in [4] correspond to the following choices for the meromorphic functions A_i (here rescaled by a factor $4/\alpha'$)

$$iA_1 = \sum_{a=1}^{p} N_a \log\left(\frac{1 + ie^{z-\delta_a}}{1 - ie^{z-\delta_a}}\right) \ , \qquad A_2 = \sum_{\hat{a}=1}^{\hat{p}} \hat{N}_{\hat{a}} \log\left(\frac{1 + e^{-z+\hat{\delta}_{\hat{a}}}}{1 - e^{-z+\hat{\delta}_{\hat{a}}}}\right) \ , \tag{A.12}$$

where z parametrizes the infinite strip $0 \leq \text{Im}z \leq \pi/2$. These solutions are holographically dual to linear-quiver gauge theories at their infrared fixed points. To find the supergravity backgrounds for circular quivers one simply sums over an infinite array, $A_i \to \sum_n A_i(z + nt)$, before identifying z periodically ($z \equiv z + t$) [5].

The function A_1 in (A.12) has logarithmic singularities at $z = \delta_a + i\pi/2$ on the upper strip boundary, while A_2 has logarithmic singularities at $z = \hat{\delta}_{\hat{a}}$ on the lower boundary. These are the locations, respectively, of (stacks of N_a) D5-branes and (stacks of $\hat{N}_{\hat{a}}$) NS5-branes. Both the coefficients $\{N_a, \hat{N}_{\hat{a}}\}$ and the positions $\{\delta_a, \hat{\delta}_{\hat{a}}\}$ of these singularities on the boundary are continuous parameters of the supergravity solution, but in string theory they are quantized. This is because the former are five-brane charges and the latter are related to D3-brane charges of the five-brane stacks by the following equations [4]

$$\ell_a = -\sum_{\hat{a}=1}^{\hat{p}} \hat{N}_{\hat{a}}\frac{2}{\pi}\arctan(e^{-\delta_a+\hat{\delta}_{\hat{a}}}) = -\frac{i}{\pi}A_2(z = \delta_a + \frac{i\pi}{2}) \ , \tag{A.13}$$

$$\hat{\ell}_{\hat{a}} = \sum_{a=1}^{p} N_a \frac{2}{\pi} \arctan(e^{-\delta_a + \hat{\delta}_{\hat{a}}}) = \frac{1}{\pi} A_1(z = \hat{\delta}_{\hat{a}}) \;, \tag{A.14}$$

where ℓ_a is the D3-brane charge (or linking number) of a D5-brane in the ath stack and $\hat{\ell}_{\hat{a}}$ is the D3-brane charge of a NS5-brane in the $\hat{a}$th stack. Note that one equation is trivial, because the total D3-brane charge is zero, reflecting the arbitrariness in choosing the origin of the Rez axis. The remaining $(p + \hat{p} - 1)$ equations determine the $(p + \hat{p} - 1)$ 5-brane positions. So the solutions (A.12) have no continuous moduli.

Near a NS5-brane singularity, $z - \hat{\delta}_{\hat{a}} = re^{i\theta}$ ($r \ll 1$), the meromorphic functions read

$$A_1 = \pi \hat{\ell}_{\hat{a}} + c_1 \, re^{i\theta} + O(r^2) \;, \qquad A_2 = -\hat{N}_{\hat{a}} \log(re^{i\theta}) + c_2 + O(r) \;, \tag{A.15}$$

where the expansion coefficients c_1, c_2 are both real,

$$c_1 = \sum_{a=1}^{p} \frac{N_a}{\cosh(\delta_a - \hat{\delta}_{\hat{a}})} \;, \qquad c_2 = \hat{N}_{\hat{a}} \log 2 + \sum_{\hat{b} \neq \hat{a}} \hat{N}_{\hat{b}} \log \left(\frac{1 + e^{\hat{\delta}_{\hat{b}} - \hat{\delta}_{\hat{a}}}}{1 - e^{\hat{\delta}_{\hat{b}} - \hat{\delta}_{\hat{a}}}} \right) \;. \tag{A.16}$$

Inserting (A.15) in (A.1)-(A.11) gives the following singular metric and dilaton:

$$ds^2 \simeq \alpha' c_1^2 \hat{N}_{\hat{a}}^2 \, (-\frac{\log r}{r^2})^{-1/4} \left(\frac{dr^2}{r^2} + ds_{\mathrm{S}^3}^2 + (-\log r)[ds_{\mathrm{AdS}}^2 + ds_{(2)}^2] \right) \tag{A.17}$$

$$e^{2\phi} \simeq \frac{\hat{N}_{\hat{a}}^2}{c_1^2} \, (-\frac{\log r}{r^2})^{1/2} \tag{A.18}$$

Note in particular that the singularity is at finite distance since the integral $\int \frac{dr}{r} (-\frac{\log r}{r^2})^{-1/8}$ converges at $r \simeq 0$. Similar expressions hold for the D5-brane singularities.

B Conformal manifold of $U(1)$ with 1 flavor

The simplest set of examples are those conformal manifolds which have points on their moduli spaces with free theories, or more generally, with BPS spectra that are generated by free operators. Although these theories correspond to singular supergravity backgrounds, the study of their moduli demonstrates the method outlined in section 4.3.

Consider the case of SQED which is well known to have a one (quaternionic) dimensional Coulomb branch and no Higgs branch.[19] Details of this theory are conveniently summarized in [70] and are presented below. The theory starts as an interacting theory in the UV and flows to a free theory in the IR. Even though the theory is free in the IR, the free operators are in fact non perturbative objects which carry a magnetic charge. The Coulomb branch is a copy of $\mathbb{H} = \mathbb{C}^2$, and the ring of holomorphic BPS operators is the set

$$O_{m,k} = V_m \phi^k, \qquad m \in \mathbb{Z}, \quad k = 0, 1, 2, \ldots \qquad \Delta(m,k) = \frac{1}{2}|m| + k \tag{B.1}$$

---

[19]This means that there can be no conformal manifold corresponding to the maximal embedding 3.17, but there is one if we pick the $\mathcal{N} = 2$ R-symmetry to lie entirely inside $SU(2)_C$.

with V_m a bare monopole operator of magnetic charge m and conformal dimension $\Delta(m) = \frac{1}{2}|m|$ and ϕ is a complex scalar in the vector multiplet with conformal dimension 1. $\Delta(m, k)$ is the conformal dimension of $O_{m,k}$. The ring is freely generated, as the moduli space is $\mathbb{H}$ and this implies non trivial relations that these operators satisfy. They can be written as

$$O_{m,k} O_{m',k'} = O_{m+m',k+k'} \, \phi^{\Delta(m)+\Delta(m')-\Delta(m+m')} \tag{B.2}$$

or perhaps a slightly simpler expression for bare monopole operators,

$$V_m V_{m'} = V_{m+m'} \, \phi^{\Delta(m)+\Delta(m')-\Delta(m+m')} \tag{B.3}$$

from which one can derive relations like

$$V_m = (V_1)^m, \quad V_{-m} = (V_{-1})^m, \qquad m > 0 \tag{B.4}$$

and perhaps the most crucial relation

$$V_1 V_{-1} = \phi \tag{B.5}$$

where V_1 and V_{-1} are the generators of the ring. The global symmetry of this theory is $Sp(1) = SU(2)$ under which the monopole operators V_1 and V_{-1} transform in the fundamental (doublet) representation. The set of holomorphic BPS operators can be recast into a collection of representations of $Sp(1)$ with fixed conformal dimension. We have a spin j representation of $Sp(1)$ for every operator of conformal dimension j. This can be summarized by the highest weight generating function (HWG) [71]

$$\frac{1}{1 - \mu t} \tag{B.6}$$

where t is the fugacity for the conformal dimension (or alternatively the fugacity for $SU(2)_C$) and μ is the fugacity for the highest weight $n = 2j$ of the representation with spin j. For example we list the operators up to spin 2 of $Sp(1)$.

$$
\begin{aligned}
V_0 &\quad 0 \\
V_1, V_{-1} &\quad \tfrac{1}{2} \\
V_2, \phi, V_{-2} &\quad 1 \\
V_3, V_1\phi, V_{-1}\phi, V_{-3} &\quad \tfrac{3}{2} \\
V_4, V_2\phi, \phi^2, V_{-2}\phi, V_{-4} &\quad 2
\end{aligned}
$$

At conformal dimension 2 we find 5 different operators which can be added with arbitrary coefficients $\lambda_{ij}, i, j = 1, 2, 3$ symmetric such that $\lambda_{ii} = 0$ as marginal deformations, breaking supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$. Recalling that the moduli space is $\mathbb{C}^2$ with a $U(2) = U(1)_r \times SU(2)$ isometry that preserves $\mathcal{N} = 2$ supersymmetry, where $U(1)_r$ is the $\mathcal{N} = 2$ R symmetry, we find that the $SU(2)$ symmetry rotates the 5 parameters and leaves a 2 complex dimensional conformal manifold. To compute this manifold we proceed with the methods that

are presented in [50], with the help of plethystic techniques as in [69]. The moduli space of conformal deformation is given by the Kähler quotient of the gauge theory of $SU(2)$ coupled to 1 chiral multiplet in the spin 2 representation. To compute the set of holomorphic functions we compute the Hilbert series of this moduli space with the Molien Weyl integral

$$\oint_{|z|=1} dz \frac{1-z^2}{2\pi i z} \text{PE}\left[(z^4 + z^2 + 1 + z^{-2} + z^{-4})q\right] = \frac{1}{(1-q^2)(1-q^3)} \qquad \text{(B.7)}$$

where the fugacity q measures the number of λ's, to be consistent with equation (B.6). This computation has the interpretation that the conformal manifold is a copy of $\mathbb{C}^2$ and is freely generated by two invariants which are quadratic and cubic in λ, say $c_2 = \lambda_{ij}\lambda_{ij}$ and $c_3 = \lambda_{ij}\lambda_{jk}\lambda_{ki}$. The functions c are globally defined on the conformal manifold as they are invariant under the $SU(2)$ rotations. These invariants can be used to construct any globally defined function on the conformal manifold, and in particular any physical quantity on the conformal manifold must be a function of these two invariants.

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