AN INFINITE NATURAL SUM

PAOLO LIPPARINI

ABSTRACT. As far as algebraic properties are concerned, the usual addition on the class of ordinal numbers is not really well behaved; for example, it is not commutative, nor left cancellative etc. In a few cases, the *natural Hessenberg sum* is a better alternative, since it shares most of the usual properties of the addition on the naturals.

A countably infinite iteration of the natural sum has been used in a recent paper by Väänänen and Wang, with applications to infinitary logics. We present a detailed study of this infinitary operation, showing that there are many similarities with the ordinary infinitary sum, and providing connections with certain kinds of infinite mixed sums.

1. INTRODUCTION

There are different ways to extend the addition operation from the set ω of natural numbers to the class of ordinals. The standard way is to take $\alpha + \beta$ as the ordinal which represents the order type of α with a copy of β added at the top. This operation can be introduced by the customary inductive definition and satisfies only few of the familiar properties shared by the addition on the naturals.

On the other hand, again on the class of the ordinals, one can define the *(Hessenberg) natural sum* $\alpha \# \beta$ of α and β by expressing α and β in Cantor normal form and "summing linearly". See below for further details. The resulting operation $\#$ is commutative, associative and cancellative. It can be given an inductive definition as follows.

$$0 \# 0 = 0$$

$$\alpha \# \beta = \sup_{\substack{\alpha' < \alpha \\ \beta' < \beta}} \{S(\alpha \# \beta'), S(\alpha' \# \beta)\} \tag{1}$$

where S denotes *successor*.

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It is relevant that the natural sum, too, admits an order theoretical definition. If α, β and γ are ordinals, γ is said to be a *mixed sum* of α and β if there are disjoint subsets A and B of γ such that $\gamma = A \cup B$ and A, B have order type, respectively, α and β, under the order induced by γ. P. W. Carruth [C] showed that $\alpha \# \beta$ is the largest mixed sum of α and β. He also found many applications.

In Väänänen and Wang [VW] the authors define a countably infinite extension of $\#$ by taking supremum at the limit stage. They provide applications to infinitary logics. Subsequently, we have found applications to compactness of topological spaces in the spirit of [Li], in particular, with respect to Frolík sums.

Carruth Theorem, as it stands, cannot be generalized to such an infinite natural sum. Indeed, every countably infinite ordinal is an "infinite mixed sum" of countably many 1's, hence in the infinite case the maximum is not necessarily attained. See Definition 4.1 and the comment after Theorem 4.2.

However, we show that Carruth Theorem can indeed be generalized, provided we restrict ourselves to certain well behaved infinite mixed sums. In order to provide this generalization, we need a finer description of the countably infinite natural sum. We show that any infinite natural sum can be computed in two steps: in the first step one takes the natural sum of some sufficiently large *finite* set of summands. In the second step one adds the infinite ordinary sum of the remaining summands. In other words, the infinite natural sum and the more usual infinite sum differ only for a finite "head" and they agree on the remaining "tail". This is used in order to show that the infinite natural sum of a sequence is the maximum of all possible infinite mixed sums made of elements from the sequence, provided one restricts only to mixed sums satisfying an appropriate finiteness condition.

In the end, we show that the infinite natural sum can be actually computed as some finite natural sum, and can be expressed in terms of the Cantor normal forms of the summands. We show that a sequence has only a finite number of mixed sums satisfying an additional convexity property. This extends a classical theorem by Sierpinski [S1], asserting that one gets only a finite number of values for the sum of some fixed countable sequence of ordinals, by changing their order.

2. Natural sums

We now give more details about the definitions hinted above and list some simple facts about the natural sums. Here and below sums, products and exponentiations will be always intended in the ordinal sense. See, e. g., the books Bachmann [B] and Sierpinski [S2] for a detailed introduction to ordinal operations. Recall that every ordinal $\alpha > 0$ can be expressed in a unique way in *Cantor normal form* as follows

$$\alpha = \omega^{\xi_k} r_k + \omega^{\xi_{k-1}} r_{k-1} + \cdots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0 \tag{2}$$

for integers $k \geqslant 0$, $r_k, \dots, r_0 > 0$ and ordinals $\xi_k > \xi_{k-1} > \cdots > \xi_1 > \xi_0$.

Definition 2.1. The natural sum $\alpha \mathbin{\#} \beta$ of two ordinals α and β is the only operation satisfying

$$\alpha \mathbin{\#} \beta = \omega^{\xi_k}(r_k + s_k) + \cdots + \omega^{\xi_1}(r_1 + s_1) + \omega^{\xi_0}(r_0 + s_0)$$

whenever

$$\alpha = \omega^{\xi_k} r_k + \cdots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0$$
$$\beta = \omega^{\xi_k} s_k + \cdots + \omega^{\xi_1} s_1 + \omega^{\xi_0} s_0$$

and $k, r_k, \dots, r_0, s_k, \dots, s_0 < \omega$, $\xi_k > \cdots > \xi_1 > \xi_0$.

The definition is justified by the fact that we can represent every nonzero α and β in Cantor normal form and then insert some more null coefficients for convenience just in order to make the indices match. The null coefficients do not affect the ordinals, hence the definition is well-posed. See, e. g., [B, S2] for further details.

An elegant way to introduce the natural sum is obtained by expressing equation (2) in a conventional way as $\alpha = \sum_{\xi \in F} \omega^\xi$, where F is the finite *multiset* which contains each ξ_ℓ exactly r_ℓ times. This is justified by the fact that, say, $\omega^\xi + \omega^\xi = \omega^\xi 2$. In this way, $\alpha = 0$ is expressed by summing over the empty multiset. Of course, when expanding the above summation, one should be careful to consider the terms with larger exponents first, that is, write them on the left. If $\alpha = \sum_{\xi \in F} \omega^\xi$ and $\beta = \sum_{\xi \in T} \omega^\xi$, then $\alpha \mathbin{\#} \beta$ is defined as $\sum_{\xi \in F \cup T} \omega^\xi$, where in the union $F \cup T$ we take into account multiplicities. In this note, however, we shall follow the more conventional notations.

It can be shown by induction on $(\max\{\alpha, \beta\}, \min\{\alpha, \beta\})$, ordered lexicographically, that Definition 2.1 is equivalent to the definition given by means of equations (1). This shall not be needed in what follows.

Notice that the assumption $\xi_k > \cdots > \xi_1 > \xi_0$ in Definition 2.1 is necessary, since, for example, $(1 + \omega) \mathbin{\#} (1 + \omega 0)$ is $\omega \mathbin{\#} 1 = \omega + 1$, while summing "linearly" we would obtain $2 + \omega = \omega$. However, the assumption that $\xi_k > \cdots > \xi_1 > \xi_0$ can be relaxed to $\xi_k \geqslant \cdots \geqslant \xi_1 \geqslant \xi_0$.

Proposition 2.2. *Let α, β and η be ordinals.*

(1) The operation $\#$ is commutative, associative, both left and right cancellative and strictly monotone in both arguments.
(2) $\sup\{\alpha, \beta\} \leqslant \alpha + \beta \leqslant \alpha \# \beta$.
(3) If $\alpha, \beta < \omega^\eta$, *then* $\alpha \# \beta < \omega^\eta$.
(4) If $\beta < \omega^\eta$, *then* $\alpha \# \beta < \alpha + \omega^\eta$.
(5) If $\beta < \omega^\eta$, *then* $(\alpha \# \beta) + \omega^\eta = \alpha + \omega^\eta$.

Proof. Everything is almost immediate from Definition 2.1.

For example, to prove (4), let $\alpha = \omega^{\xi_k} r_k + \dots + \omega^\eta r + \dots$ with, as usual, the exponents of ω in decreasing order, and where we can allow r to be 0. Then $\alpha + \omega^\eta = \omega^{\xi_k} r_k + \dots + \omega^\eta (r+1)$, while, if $\beta < \omega^\eta$, then $\alpha \# \beta = \omega^{\xi_k} r_k + \dots + \omega^\eta r + \dots$, since β does not contribute to summands where the exponent of ω is $\geqslant \eta$. Thus surely $\alpha + \omega^\eta > \alpha \# \beta$, with no need to compute explicitly those summands which are $< \omega^\eta$. □

Parentheses are usually necessary in expressions involving *both* $+$ and $\#$; for example, $(1 \# 0) + \omega = \omega \neq \omega + 1 = 1 \# (0 + \omega)$, or $(1 + 0) \# \omega = \omega + 1 \neq \omega = 1 + (0 \# \omega)$.

Definition 2.3. Suppose that $(\alpha_i)_{i<\omega}$ is a countable sequence of ordinals, and set $S_n = \alpha_0 \# \dots \# \alpha_{n-1}$, for every $n < \omega$. The *natural sum* of $(\alpha_i)_{i<\omega}$ is

$$\mathop{\#}_{i<\omega} \alpha_i = \sup_{n<\omega} S_n$$

The above natural sum is denoted by $\sum^{\#}_{i<\omega} \alpha_i$ in [VW].

In the above notation $\alpha_0 \# \dots \# \alpha_{n-1}$ we conventionally allow $n = 0$, and assume that 0 is the outcome of such an "empty" sum. Notice that the notation is not ambiguous, in view of Proposition 2.2(1).

Proposition 2.4. *Let* α_i, β_i *be ordinals and* $n, m < \omega$.
(1) $\sum_{i<\omega} \alpha_i \leqslant \#_{i<\omega} \alpha_i$
(2) If $\beta_i \leqslant \alpha_i$, *for every* $i < \omega$, *then* $\#_{i<\omega} \beta_i \leqslant \#_{i<\omega} \alpha_i$
(3) If $n < m$, *then* $S_n \leqslant S_m$; *equality holds if and only if* $\alpha_n = \dots = \alpha_{m-1} = 0$.
(4) $S_n \leqslant \#_{i<\omega} \alpha_i$; *equality holds if and only if* $\alpha_i = 0$, *for every* $i \geqslant n$.
(5) If π *is a permutation of* ω, *then* $\#_{i<\omega} \alpha_i = \#_{i<\omega} \alpha_{\pi(i)}$
(6) More generally, suppose that $(F_h)_{h<\omega}$ *is a partition of* ω *into finite subsets, say,* $F_h = \{j_1, \dots, j_{r(h)}\}$, *for every* $h \in \omega$. *Then*

$$\mathop{\#}_{i<\omega} \alpha_i = \mathop{\#}_{h<\omega} \mathop{\#}_{j \in F_h} \alpha_j = \mathop{\#}_{h<\omega} (\alpha_{j_1} \# \alpha_{j_1} \# \dots \alpha_{j_{r(h)}})$$

Proof. (1)-(4) are immediate from the definitions and Proposition 2.2 (1)-(2).

Clause (5) is a remark in the proof of [VW, Proposition 4.4]. Anyway, (5) is the particular case of (6) when all the F_h's are singletons.

To prove (6), define, for $h < \omega$

$$T_h = \left(\mathop{\#}_{j \in F_0} \alpha_j \right) \# \ldots \# \left(\mathop{\#}_{j \in F_{h-1}} \alpha_j \right)$$

Thus the right-hand of the equation in (6) is $\sup_{h<\omega} T_h$. For $h < \omega$, let $m = \max_{0 \leqslant \ell < h} F_\ell$. The maximum exists since each F_ℓ is finite, and we are considering only a finite number of F_ℓ's at a time. Then each summand in the expansion of T_h appears in S_{m+1} (taking into account multiplicities), hence, by (4) and monotonicity of the natural sum, $\#_{i<\omega} \alpha_i \geqslant S_{m+1} \geqslant T_h$. Hence $\#_{i<\omega} \alpha_i \geqslant \sup_{h<\omega} T_h$. The reverse inequality is similar and easier. □

The assumption that each F_h is finite in condition (6) above is necessary. For example, take $\alpha_i = 1$, for every $i < \omega$, thus $\#_{i<\omega} \alpha_i = \omega$. Suppose that there is some infinite $F_{\bar{h}}$. Then $\#_{j \in F_{\bar{h}}} \alpha_j = \omega$. If $\omega \setminus F_{\bar{h}} \neq \emptyset$, then $\#_{h<\omega} \left(\#_{j \in F_h} \alpha_j \right) \geqslant \omega \# 1 > \omega$.

Not everything from Proposition 2.2 generalizes to infinite sums. For example, the operation $\#_{i<\omega} \alpha_i$, though monotone, as stated in (2) above, is *not* strictly monotone. E. g., $\#_{i<\omega} 2 = \#_{i<\omega} 1 = \omega$. Actually, $\#_{i<\omega} \alpha_i = \omega$, for every choice of the α_i's such that $\alpha_i < \omega$, for every $i < \omega$, and such that there are infinitely many nonzero α_i's.

Condition (5) above can be interpreted as a version of commutativity, and (6) as a version of the generalized commutative-associative law. However, not all forms of associativity hold. We have seen that we cannot associate infinitely many summands inside some natural sum. Similarly, we are not allowed to "associate inside out". Indeed, $\omega + 1 = 1 \# \#_{i<\omega} 1 \neq \#_{i<\omega} 1 = \omega$. This is a general and well-known fact. For infinitary operations, some very weak form of generalized associativity implies some form of absorption.

Example 2.5. Suppose that $\oplus$ is a binary operation on some set X, and $a \in X$ is such that $a \oplus x \neq x$, for every $x \in X$. There is no infinitary operation $\bigoplus$ on X such that

$$x_0 \oplus \bigoplus_{i \in \omega} x_{i+1} = \bigoplus_{i \in \omega} x_i$$

for every sequence $(x_i)_{i \in \omega}$ of elements of X. Indeed, taking $x_i = a$, for every $i \in \omega$, and letting $x = \bigoplus_{i \in \omega} x_i$, we get $a \oplus x = x$, a contradiction.

3. Computing the infinite natural sum

Theorem 3.1. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, then there is $m<\omega$ such that the following hold, for every $n \geqslant m$.*

$$\mathop{\#}_{n\leqslant i<\omega} \alpha_i = \sum_{n\leqslant i<\omega} \alpha_i \tag{3}$$

$$\begin{aligned} \mathop{\#}_{i<\omega} \alpha_i &= (\alpha_0 \# \dots \# \alpha_{n-1}) + \mathop{\#}_{n\leqslant i<\omega} \alpha_i \\ &= (\alpha_0 \# \dots \# \alpha_{n-1}) + \sum_{n\leqslant i<\omega} \alpha_i \end{aligned} \tag{4}$$

Proof. Let ξ be the smallest ordinal such that the set $\{i \in \omega \mid \alpha_i \geqslant \omega^\xi\}$ is finite. Let m be the smallest index such that $\alpha_i < \omega^\xi$, for every $i \geqslant m$. The definition of ξ assures the existence of such an m. If $\xi = 0$, then all but finitely many α_i's are 0 and the proposition is trivial.

Suppose that ξ is a successor ordinal, say $\xi = \varepsilon + 1$. By the minimality of ξ, the set $\{i \in \omega \mid \alpha_i \geqslant \omega^\varepsilon\}$ is infinite, hence unbounded in ω. Then $\#_{n\leqslant i<\omega} \alpha_i \geqslant \sum_{n\leqslant i<\omega} \alpha_i \geqslant \omega^\varepsilon \omega = \omega^{\varepsilon+1} = \omega^\xi$. Suppose that ξ is limit. By the definition of ξ, we have that, for every $\varepsilon < \xi$, there are infinitely many $i < \omega$ such that $\alpha_i \geqslant \omega^\varepsilon$. In particular, we can choose such an i with $i \geqslant n$. Then $\sum_{n\leqslant i<\omega} \alpha_i \geqslant \alpha_i \geqslant \omega^\varepsilon$. Since this holds for every $\varepsilon < \xi$, we get $\#_{n\leqslant i<\omega} \alpha_i \geqslant \sum_{n\leqslant i<\omega} \alpha_i \geqslant \sup_{\varepsilon<\xi} \omega^\varepsilon = \omega^\xi$. The inequality $\#_{n\leqslant i<\omega} \alpha_i \geqslant \sum_{n\leqslant i<\omega} \alpha_i \geqslant \omega^\xi$ is proved, no matter whether ξ is successor or limit.

On the other hand, because of the definition of m, if $i \geqslant n \geqslant m$, then $\alpha_i < \omega^\xi$. By Proposition 2.2(3), $\alpha_n \# \dots \# \alpha_{\ell-1} < \omega^\xi$, for every $\ell \geqslant n$. Hence $\sum_{n\leqslant i<\omega} \alpha_i \leqslant \#_{n\leqslant i<\omega} \alpha_i = \sup_{\ell<\omega} (\alpha_n \# \dots \# \alpha_{\ell-1}) \leqslant \omega^\xi$. In conclusion,

$$\mathop{\#}_{n\leqslant i<\omega} \alpha_i = \sum_{n\leqslant i<\omega} \alpha_i = \omega^\xi \tag{5}$$

thus we have proved (3).

Let us now prove (4). The inequality $\#_{i<\omega} \alpha_i \geqslant (\alpha_0 \# \dots \# \alpha_{n-1}) + \#_{n\leqslant i<\omega} \alpha_i$ is trivial, since every "partial sum" on the right is bounded by the partial sum on the left having the same length, by Proposition 2.2(2). For the other direction, and recalling that S_ℓ denotes $\alpha_0 \# \dots \# \alpha_{\ell-1}$, observe that, by associativity, for every $\ell \geqslant n$, we have $S_\ell = S_n \# \alpha_n \# \dots \# \alpha_{\ell-1} < S_n + \omega^\xi = S_n + \#_{n\leqslant i<\omega} \alpha_i$, where the strict inequality follows from repeated applications of Proposition 2.2(4), since $\alpha_n, \dots, \alpha_{\ell-1} < \omega^\xi$. The last identity is from equation (5). Since $\#_{i<\omega} \alpha_i = \sup_{\ell<\omega} S_\ell$ and since S_ℓ is increasing, we get $\#_{i<\omega} \alpha_i \leqslant S_n + \#_{n\leqslant i<\omega} \alpha_i$.

The identity $\#_{i<\omega} \alpha_i = (\alpha_0 \# \ldots \# \alpha_{n-1}) + \sum_{n \leqslant i<\omega} \alpha_i$ is now immediate from (3). It can be also proved in a way similar to above. □

Notice that the sum $+$ in equation (4) cannot be replaced by a natural sum $\#$, that is, we do not have, in general, $\#_{i<\omega} \alpha_i = S_n \# \#_{n \leqslant i<\omega} \alpha_i$, nor we have $\#_{i<\omega} \alpha_i = S_n \# \sum_{n \leqslant i<\omega} \alpha_i$. This is similar to the argument in Example 2.5: just take $\alpha_i = 1$, for every $i \in I$; then $\#_{i<\omega} \alpha_i = \omega$ but $S_n \# \#_{n \leqslant i<\omega} \alpha_i = S_n \# \sum_{n \leqslant i<\omega} \alpha_i = n \# \omega = \omega + n$. However, in Corollary 5.1 we shall show that the computation of a countable natural sum can be actually reduced to the computation of some finite natural sum.

Remark 3.2. Notice that equation (3) in Theorem 3.1, together with Proposition 2.4(5), imply that if $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, m is given by Theorem 3.1, and $n \geqslant m$, then $\sum_{n \leqslant i<\omega} \alpha_i = \#_{n \leqslant i<\omega} \alpha_i = \sum_{n \leqslant i<\omega} \alpha_{\pi(i)}$, for every permutation π of $[n, \omega)$. Actually, equation (5) in the proof of Theorem 3.1 shows that it is enough to assume that π is a bijection from $[n, \omega)$ to $[n', \omega)$, for some $n' \geqslant m$ (equation (5) does not hold if $\xi = 0$, but this case is trivial).

The result in the present remark can be obtained also as a consequence of a theorem by Sierpinski [S1], asserting that a countable sum of nondecreasing ordinals is invariant under permutations. Just notice that every sequence of ordinals is nondecreasing from some point on. On the other hand, Sierpinski's result is immediate from equation (5). Thus parts of the present note can be seen as an extension of results from [S1] to natural sums.

4. Some kinds of mixed sums

The definition of a mixed sum of two ordinals can be obviously extended to deal with infinitely many ordinals.

Definition 4.1. Let $(\alpha_i)_{i \in I}$ be any sequence of ordinals (with no restriction on the cardinality of I). An ordinal γ is a *mixed sum* of $(\alpha_i)_{i \in I}$ if there are pairwise disjoint subsets $(A_i)_{i \in I}$ of γ such that $\bigcup_{i \in I} A_i = \gamma$ and, for every $i \in I$, A_i has order type α_i, with respect to the order induced on A_i by γ.

In the above situation, we say that γ is a mixed sum of $(\alpha_i)_{i \in I}$ *realized by* $(A_i)_{i \in I}$, or simply that $(A_i)_{i \in I}$ is a *realization of* γ. Notice that α_i can be recovered by A_i, as embedded in γ.

Notice that we could have given the above definition just under the assumption that γ and the α_i's are linearly ordered sets, not necessarily well ordered. In this respect, notice that any finite mixed sum of well ordered sets is itself necessarily well ordered; however, in case I is infinite, the α_i's could "mix themselves" to a non well ordered set. For example, starting with countably many 1's, we could obtain *every* countably infinite linear order as a mixed sum. Throughout this note, however, and no matter how interesting the general case of linear orders is, we shall always assume that γ is an ordinal, that is, well ordered.

Theorem 4.2. (Carruth [C], Neumer [N]) *For every $n < \omega$ and ordinal numbers $\alpha_0, \dots, \alpha_n$, the largest mixed sum of $(\alpha_i)_{i \leqslant n}$ exists and is $\alpha_0 \mathbin{\#} \alpha_1 \mathbin{\#} \dots \mathbin{\#} \alpha_n$.*

As we hinted in the introduction, and contrary to the finite case, the set of all the mixed sums of an infinite sequence of ordinals need not have a maximum. If we take $\alpha_i = 1$ for every $i < \omega$, then every infinite countable ordinal is a mixed sum of $(\alpha_i)_{i \in \omega}$, thus the supremum of all the mixed sums of $(\alpha_i)_{i \in \omega}$ is ω_1, which is *not* a mixed sum of $(\alpha_i)_{i \in \omega}$. Hence there is some interest in restricting ourselves to well-behaved mixed sums

Definition 4.3. We say that γ is a *left-finite* mixed sum of $(\alpha_i)_{i \in I}$ if γ can be realized as a mixed sum by $(A_i)_{i \in I}$ in such a way that, for every $\delta < \gamma$, the set $\{i \in I \mid A_i \cap \delta \neq \emptyset\}$ is finite; in words, for every $\delta < \gamma$, the predecessors of δ are all taken from finitely many A_i's.

Given a realization $(A_i)_{i \in I}$ of γ and $i \in I$, we say that A_i is *convex (in γ)* if $[a, a']_\gamma = \{\delta \in \gamma \mid a \leqslant \delta \leqslant a'\} \subseteq A_i$, whenever $a < a' \in A_i$.

We say that γ is a *piecewise convex* (resp., an *almost piecewise convex*) mixed sum of $(\alpha_i)_{i \in I}$ if γ can be realized in such a way that all the A_i's (resp., all but a finite number of the A_i's) are convex in γ. For brevity, we shall write *pw-convex* in place of piecewise convex.

If γ is a pw-convex mixed sum of $(\alpha_i)_{i \in I}$, as realized by $(A_i)_{i \in I}$, then, for every $i \neq j \in I$ and $\delta, \varepsilon \in A_i$, $\delta', \varepsilon' \in A_j$, we have that $\delta < \delta'$ if and only if $\varepsilon < \varepsilon'$. In this way, if each A_i is nonempty, the order on γ induces an order (in fact, a well order) on I. Hence we can reindex $(A_i)_{i \in I}$ as $(A_{\pi(\iota)})_{\iota < \theta}$ for some ordinal θ and some bijection $\pi : \theta \to I$ in such a way that $\delta < \delta'$, whenever $\delta \in A_{\pi(\iota)}$, $\delta' \in A_{\pi(\iota')}$ and $\iota < \iota'$. Then an easy induction shows that $\gamma = \sum_{\iota < \theta} \alpha_{\pi(\iota)}$. If in addition γ is left finite, then necessarily $\theta \leqslant \omega$.

Conversely, if $\gamma = \sum_{\iota < \theta} \alpha_{\pi(\iota)}$, for some reindexing of the α_i's, then trivially γ is a pw-convex mixed sum of $(\alpha_i)_{i \in I}$, and if $\theta \leqslant \omega$, then γ is also left finite. We have proved the next proposition.

Proposition 4.4. *Suppose that $(\alpha_i)_{i \in I}$ is a sequence of ordinals, and $\alpha_i > 0$, for every $i \in I$. Then γ is a pw-convex (pw-convex and left-finite) mixed sum of $(\alpha_i)_{i \in I}$ if and only if there are some ordinal θ (with $\theta \leqslant \omega$) and a bijection $\pi : \theta \to I$ such that $\gamma = \sum_{\iota < \theta} \alpha_{\pi(\iota)}$.*

Remark 4.5. There might be infinitely many left-finite mixed sums of the same sequence. Indeed, take $\alpha_i = \omega$, for every $i < \omega$. Since ω is the union of countably many disjoint countably infinite sets, we see that ω is a (necessarily left-finite) mixed sum of $(\alpha_i)_{i<\omega}$. By moving just one copy of ω "to the bottom" we get that also $\omega + \omega$ is a left-finite mixed sum of $(\alpha_i)_{i<\omega}$. Iterating, for every $n < \omega$ we get ωn as a left-finite mixed sum of $(\alpha_i)_{i<\omega}$. Also ω^2 is a left-finite mixed sum of $(\alpha_i)_{i<\omega}$; by Proposition 4.4 it is the only one which is left-finite and pw-convex; actually, it is the only one which is left-finite and almost pw-convex.

Theorem 4.6. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, then $\#_{i<\omega} \alpha_i$ is a mixed sum of $(\alpha_i)_{i<\omega}$. In fact, $\#_{i<\omega} \alpha_i$ is the largest left-finite mixed sum of $(\alpha_i)_{i<\omega}$, and also the largest left-finite and almost pw-convex mixed sum of $(\alpha_i)_{i<\omega}$.*

Proof. By equation (4) in Theorem 3.1, we have $\#_{i<\omega} \alpha_i = (\alpha_0 \# \dots \# \alpha_{n-1}) + \sum_{n\leqslant i<\omega} \alpha_i$, for some $n < \omega$. By Theorem 4.2, $\gamma_1 = \alpha_0 \# \dots \# \alpha_{n-1}$ is a mixed sum of $\alpha_0, \dots, \alpha_{n-1}$. By the easy part of Proposition 4.4, $\gamma_2 = \sum_{n\leqslant i<\omega} \alpha_i$ is a left-finite pw-convex mixed sum of $(\alpha_i)_{n\leqslant i<\omega}$. Putting the members of the realization of γ_1 at the bottom, and the members of the realization of γ_2 at the top, we realize $\#_{i<\omega} \alpha_i = \gamma_1 + \gamma_2$ as a left-finite and almost pw-convex mixed sum of $(\alpha_i)_{i<\omega}$.

To finish the proof of the theorem it is enough to show that if γ is any left-finite mixed sum of $(\alpha_i)_{i<\omega}$, then $\gamma \leqslant \#_{i<\omega} \alpha_i$. Let γ be a left-finite mixed sum of $(\alpha_i)_{i<\omega}$ as realized by $(A_i)_{i<\omega}$. If all but a finite number of the α_i's are 0, then the result is immediate from Theorem 4.2. Otherwise, left finiteness implies that γ is a limit ordinal. If $\gamma' < \gamma$, then $(\gamma' \cap A_i)_{i<\omega}$ witnesses that γ' is a mixed sum of $(\beta_i)_{i\in I}$, where, for every $i < \omega$, β_i is the order type of $\gamma \cap A_i$; thus $\beta_i \leqslant \alpha_i$. Left finiteness implies that only a finite number of the β_i's are nonzero, thus, again by Theorem 4.2, $\gamma' \leqslant \beta_{i_1} \# \dots \# \beta_{i_\ell}$, for certain distinct indices $i_1, \dots, i_\ell$. Taking $n = \sup\{i_1, \dots, i_\ell\}$, we get $\gamma' \leqslant \beta_{i_1} \# \dots \# \beta_{i_\ell} \leqslant \alpha_{i_1} \# \dots \# \alpha_{i_\ell} \leqslant \alpha_0 \# \dots \# \alpha_n < \#_{i<\omega} \alpha_i$. Since γ is limit and $\gamma' \leqslant \#_{i<\omega} \alpha_i$, for every $\gamma' < \gamma$, we get $\gamma \leqslant \#_{i<\omega} \alpha_i$. □

5. Expressing sums in terms of the normal form

The proof of Theorem 3.1 gives slightly more. Let α and ξ be ordinals, and express α in Cantor normal form as $\omega^{\eta_k} r_k + \dots + \omega^{\eta_0} r_0$. The ordinal $\alpha^{\restriction \xi}$, in words, *$\alpha$ truncated at the ξ^{th} exponent of ω*, is $\omega^{\eta_k} r_k + \dots + \omega^{\eta_\ell} r_\ell$, where ℓ is the smallest index such that $\ell \geqslant \xi$. The above definition should be intended in the sense that $\alpha^{\restriction \xi} = 0$ in case that $\alpha < \omega^\xi$.

Corollary 5.1. *Suppose that $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, and let ξ be the smallest ordinal such that $\{i < \omega \mid \alpha_i \geqslant \omega^\xi\}$ is finite. Enumerate those α_i's such that $\alpha_i \geqslant \omega^\xi$ as $\alpha_{i_0}, \dots, \alpha_{i_h}$, with $i_0 < \dots < i_h$ (the sequence might be empty). If $\xi > 0$, then*

$$\#_{i<\omega} \alpha_i = (\alpha_{i_0} \# \dots \# \alpha_{i_h}) + \omega^\xi = \alpha_{i_0}^{\restriction \xi} \# \dots \# \alpha_{i_h}^{\restriction \xi} \# \omega^\xi \quad \textit{and} \tag{6}$$

$$\sum_{i<\omega} \alpha_i = \alpha_{i_0} + \dots + \alpha_{i_h} + \omega^\xi = \alpha_{i_0}^{\restriction \xi} + \dots + \alpha_{i_h}^{\restriction \xi} + \omega^\xi\,; \tag{7}$$

moreover, for every $\varepsilon < \xi$, we have

$$\#_{i<\omega} \alpha_i = \#_{i<\omega} \alpha_i^{\restriction \varepsilon} \qquad \textit{and} \qquad \sum_{i<\omega} \alpha_i = \sum_{i<\omega} \alpha_i^{\restriction \varepsilon}$$

Proof. The ξ defined in the statement of the present corollary is the same as the ξ defined in the proof of Theorem 3.1; and the α_{i_h} defined here is the same as a_{m-1} in that proof (if the sequence of the α_{i_ℓ}'s is not empty). Equation (5) in the proof of Theorem 3.1 gives $\#_{m\leqslant i<\omega} \alpha_i = \omega^\xi$. By commutativity and associativity of $\#$, and using Proposition 2.2(5), equation (4) in Theorem 3.1 becomes exactly the first identity in equation (6). The second identity is easy ordinal arithmetic, noticing that $\alpha + \omega^\xi = \alpha^{\restriction \xi} + \omega^\xi$ and $(\alpha \# \beta)^{\restriction \xi} = \alpha^{\restriction \xi} \# \beta^{\restriction \xi}$, for every α and β.

The proof of (7) is similar, using the fact that $\sum_{i<\omega} \alpha_i = \alpha_0 + \dots + \alpha_{m-1} + \sum_{m\leqslant i<\omega} \alpha_i$. Then one should use the identity $\beta + \gamma + \omega^\xi = \gamma + \omega^\xi$, holding whenever $\beta < \omega^\xi$. Indeed, if $\gamma < \omega^\xi$, then all sides are equal to ω^ξ; otherwise, if $\gamma \geqslant \omega^\xi$, then β is absorbed by γ, since it is already absorbed by the leading term in the Cantor normal expression of γ. See [S1].

To prove the last two identities, notice that if $\varepsilon < \xi$, then ξ is also the least ordinal such that $\{i < \omega \mid \alpha_i^{\restriction \varepsilon} \geqslant \omega^\xi\}$ is finite. Hence we can apply (6) twice to get $\#_{i<\omega} \alpha_i^{\restriction \varepsilon} = (\alpha_{i_0}^{\restriction \varepsilon})^{\restriction \xi} \# \dots \# (\alpha_{i_h}^{\restriction \varepsilon})^{\restriction \xi} \# \omega^\xi = \alpha_{i_0}^{\restriction \xi} \# \dots \# \alpha_{i_h}^{\restriction \xi} \# \omega^\xi = \#_{i<\omega} \alpha_i$. The last identity is proved in the same way, using equation (7). □

Notice that Corollary 5.1 furnishes a method to compute $\#_{i<\omega} \alpha_i$ and $\sum_{i<\omega} \alpha_i$ in terms of the Cantor normal forms of the α_i's, in fact, of just finitely many α_i's, once ξ has been determined.

One cannot expect that, for every sequence $(\alpha_i)_{i\in\omega}$ of ordinals, there is some permutation of ω such that $\#_{i<\omega} \alpha_i = \sum_{i<\omega} \alpha_{\pi(i)}$. The counterexample has little to do with infinity: just take two ordinals α_0 and α_1 such that $\alpha_0 \# \alpha_1 \neq \alpha_0 + \alpha_1$ and $\alpha_0 \# \alpha_1 \neq \alpha_1 + \alpha_0$, for example, $\alpha_0 = \alpha_1 = \omega + 1$. Then, setting $\alpha_i = 0$, for $i > 1$, we have $\alpha_0 \# \alpha_1 = \#_{i<\omega} \alpha_i \neq \sum_{i<\omega} \alpha_{\pi(i)}$, for every permutation π. Of course, we can arrange things in order to have some really infinite sum, e. g., take $\alpha_0 = \alpha_1 = \omega^2 + \omega$ and $\alpha_i = 1$, for $i \geqslant 2$.

Lemma 5.2. *Suppose that γ is a mixed sum of α_1 and α_2 with $\alpha_1 < \alpha_2 = \omega^\xi$, for some $\xi > 0$, and the mixed sum is realized by A_1, A_2 in such a way that A_2 is cofinal in γ. Then $\gamma = \omega^\xi$.*

Proof. $\gamma \geqslant \alpha_2 = \omega^\xi$ is trivial. For the other direction, let $\delta < \gamma$. Thus δ is a mixed sum of β_1, β_2, where β_1, β_2 are the order types of, respectively, $A_1 \cap \delta$, $A_2 \cap \delta$. Since $\delta < \gamma$, A_2 is cofinal in γ and A_2 has order type ω^ξ, then $A_2 \cap \delta$ has order type $< \omega^\xi$. Moreover $A_1 \cap \delta$ has order type $\leqslant \alpha_1 < \omega^\xi$. Since δ is a mixed sum of β_1 and β_2, then, by Theorem 4.2, $\delta \leqslant \beta_1 \# \beta_2$. We have showed that $\beta_1, \beta_2 < \omega^\xi$, hence $\delta \leqslant \beta_1 \# \beta_2 < \omega^\xi$, by Proposition 2.2(3). Since $\delta < \omega^\xi$, for every $\delta < \gamma$, then $\gamma \leqslant \omega^\xi$. □

Theorem 5.3. *If $(\alpha_i)_{i \in I}$ is a sequence of ordinals, then there are at most a finite number of left-finite and almost pw-convex mixed sums of $(\alpha_i)_{i \in I}$.*

Proof. If there is some left-finite mixed sum of $(\alpha_i)_{i \in I}$, then necessarily all but countably many α_i's are 0. Hence it is no loss of generality to assume that $|I| \leqslant \omega$. If all but a finite number of the α_i's are 0, then the corollary follows from a theorem by Läuchli [Lä], asserting that a finite set of ordinals has only a finite number of mixed sums. In conclusion, we can assume that $I = \omega$ and $\alpha_i \neq 0$, for every $i < \omega$.

Suppose that γ is a left-finite and almost pw-convex mixed sum of $(\alpha_i)_{i<\omega}$, as realized by $(A_i)_{i<\omega}$. Assume the notation in Corollary 5.1. We shall show that γ is a mixed sum of $\alpha_{i_0}, \dots, \alpha_{i_h}, \omega^\xi$. This will give the result by the mentioned theorem from Läuchli [Lä].

Since γ is realized as an almost pw-convex mixed sum, there is n such that A_i is convex, for every $i \geqslant n$. Without loss of generality, choose $n > i_h$, or, which is the same, $n \geqslant m$, where m is given by Theorem 3.1. Let $A = \bigcup_{i \geqslant n} A_i$, thus γ is realized as a finite mixed sum by $A_0, \dots, A_{n-1}, A$. Moreover, by the left finiteness of the realization $(A_i)_{i \in I}$, and since we have assumed that the A_i's are nonempty, we get that A is cofinal in γ. The order type of A is $\sum_{n \leqslant i < \omega} \alpha_{\pi(i)}$, for some permutation of π of $[n, \omega)$, by Proposition 4.4, since $A = \bigcup_{i \geqslant n} A_i$, the A_i's are convex (in γ, hence, a fortiori, in A), for $i \geqslant n$, and the realization $(A_i)_{i<\omega}$ is left-finite. By Remark 3.2 and equation (5) in the proof of Theorem 3.1, we get that A has order type ω^ξ. Now let $F = \{0, \dots, n-1\} \setminus \{i_0, \dots, i_h\}$, and set $A' = A \cup \bigcup_{j \in F} A_j$. By repeated applications of Lemma 5.2, we get that A' has order type ω^ξ. Since

$\gamma = A' \cup A_{i_0} \cup \cdots \cup A_{i_h}$, then γ is a mixed sum of $\omega^\xi, \alpha_{i_0}, \dots, \alpha_{i_h}$, what we wanted to prove. □

In view of Proposition 4.4, Theorem 5.3 extends a classical result by Sierpinski [S1], asserting that $\sum_{i<\omega} \alpha_{\pi(i)}$ assumes only finitely many values, π varying among all permutations of ω.

By Remark 4.5, the assumption of almost pw-convexity is necessary in Theorem 5.3.

Remark 5.4. A *shifted sum* is defined in the same way as a mixed sum, except that we do not require the A_i's to be pairwise disjoint. See [Li] for applications of finite shifted sums. Since every mixed sum is a shifted sum and, on the other hand, given a shifted sum of $(\alpha_i)_{i\in I}$, there is always some larger mixed sum of $(\alpha_i)_{i\in I}$, we get that Theorem 4.2 holds for shifted sums in place of mixed sums. Then the proof of Theorem 4.6, too, carries over to get the result for shifted sums in place of mixed sums. On the other hand, the analogue of Theorem 5.3 does not hold for shifted sums. Indeed, let $\alpha_i = \omega$, for $i < \omega$. Then ω can be realized as a left-finite pw-convex shifted sum of $(\alpha_i)_{i<\omega}$ by $A_i = [i, \omega)$. Then we can get infinitely many left-finite pw-convex shifted sums of $(\alpha_i)_{i<\omega}$ arguing as in Remark 4.5.

Remark 5.5. It is clear that, when restricted to the class of ordinals, the surreal number addition is the natural sum. See Alling [A], Ehrlich [E] and Gonshor [G] for information about the surreal numbers. Corollary 5.1 suggests the possibility of extending the infinitary natural sum to the class of those surreal numbers which have positive coefficients in their Conway normal representation. Recall that every surreal number x can be uniquely expressed in *Conway normal form* as $x = \sum_{s\in S} \omega^s r_s$, where S is a reverse-well-ordered set of surreal numbers and the r_s are nonzero real numbers. In case x is an ordinal the Conway and the Cantor normal forms coincide.

Let $(x_i)_{i<\omega}$ be a countable sequence of surreal numbers with normal forms $x_i = \sum_{s\in S_i} \omega^s r_{s,i}$ and such that all the $r_{s,i}$'s are positive (we shall show later that this request can be somewhat weakened). Let $S = \bigcup_{i<\omega} S_i$ and define a subset S^* of S by declaring $s \in S^*$ if and only if $\{t \in S \mid t \geqslant s\}$ is reverse-well-ordered. It might well happen that $S^* = \emptyset$; however, in any case, S^* is reverse-well-ordered. For $s \in S$, let $c_s = \sum_i r_{s,i}$, where the sum is taken among all $i < \omega$ such that $s \in S_i$. This might be either a finite sum or a countably infinite sum of positive real numbers; in the latter case we consider it as an infinite sum in the sense of classical analysis. We allow the possibility $c_s = \infty$, that is, $c_s = \omega$ in the surreal sense.

Suppose first that $S = S^*$. In this case we define

$$\#_{i<\omega} x_i = \sum_{s \in S} \omega^s c_s \tag{8}$$

This is a well-defined surreal number, since $S = S^*$, hence S is reverse-well-ordered. Strictly speaking, equation (8) does not necessarily give a normal form representation, due to the possibility that some c_s is ω. Formally, $\#_{i<\omega} x_i = \sum_{s \in T} \omega^s c_s + \sum_{s \in U} \omega^{s+1}$, where $U = \{s \in S \mid c_s = \omega\}$ and $T = S \setminus U$.

Notice also that, in order to give the definition in equation (8) we only need that $S = S^*$ and that the sums $\sum_i r_{s,i}$ are well-defined in the sense of classical analysis, no matter whether all the $r_{s,i}$ are positive.

Now suppose that $S \neq S^*$ and let $\bar{s}$ be the surreal number $\{S \setminus S^* \mid \varnothing\}$. In this case we define

$$\#_{i<\omega} x_i = \omega^{\bar{s}} + \sum_{s \in S^*} \omega^s c_s \tag{9}$$

The definition in equation (9) makes sense just under the assumption that $c_s = \sum_i r_{s,i}$ is well-defined, for every $s \in S^*$. However, it is natural to ask that there is $s \in S \setminus S^*$ such that $\sum_i r_{t,i}$ is well-defined *and strictly positive*, for every $t \in S \setminus S^*$ with $t \geqslant s$. In case that there is $s \in S \setminus S^*$ such that $\sum_i r_{t,i}$ is well-defined and strictly negative, for every $t \in S \setminus S^*$ with $t \geqslant s$, a more natural definition would be $\#_{i<\omega} x_i = -\omega^{\bar{s}} + \sum_{s \in S^*} \omega^s c_s$.

In the special case when each x_i is an ordinal, the definitions given by (8) and (9) coincide with Definition 2.3, by Corollary 5.1.

The definition given in the present remark is quite tentative, and is not the only possible one. Notice that the definition of $\#_{i<\omega} x_i$ given here for surreal numbers does not satisfy the analogue of Proposition 2.4(2). Indeed, let $x_i = \omega^i$ and $y_i = \omega^{\omega-2}$, for $i < \omega$. Then $x_i < y_i$, for every $i < \omega$; however, $\#_{i<\omega} x_i = \omega^\omega > \omega^{\omega-1} = \omega^{\omega-2}\omega = \#_{i<\omega} y_i$.

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Dipartimento Diaccio di Matematica, Viale della Ricerca Scientifica, II Università di Roma (Tor Vergata), I 00133 Rome, Italy

URL: http://www.mat.uniroma2.it/~lipparin

E-mail address: lipparin@axp.mat.uniroma2.it