THE OPERATOR ALGEBRA CONTENT OF THE RAMANUJAN-PETERSSON PROBLEM

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Dedicated to Professor Henri Moscovici on the occasion of his 70th anniversary

ABSTRACT. Let G be a discrete countable group, and let Γ be an almost normal subgroup. In this paper we investigate the classification of (projective, with 2-cocycle $\varepsilon \in H^2(G, \mathbb{T})$) unitary representations π of G into the unitary group of the Hilbert space $l^2(\Gamma, \varepsilon)$ that extend the (projective, with 2-cocycle ε) unitary left regular representation of Γ. Representations with this property are obtained by restricting to G (projective) unitary square integrable representations of a larger semisimple Lie group $\overline{G}$, containing G as dense subgroup and such that Γ is a lattice in $\overline{G}$. This type of unitary representations of of G appear in the study of automorphic forms.

We obtain a classification of such (projective) unitary representations and hence we obtain that the Ramanujan-Petersson problem regarding the action of the Hecke algebra on the Hilbert space of Γ-invariant vectors for the unitary representation $\pi \otimes \overline{\pi}$ is an intrinsic problem on the outer automorphism group of the skewed, crossed product von Neumann algebra $\mathcal{L}(G \rtimes_\varepsilon L^\infty(\mathcal{G}, \mu))$, where $\mathcal{G}$ is the Schlichting completion of G and μ is the canonical Haar measure on $\mathcal{G}$.

1. INTRODUCTION, DEFINITIONS, AND MAIN RESULTS

Let G be a discrete group and let Γ be an almost normal subgroup. In this paper we investigate the classification, up to unitary equivalence, of (projective, with 2-cocycle $\varepsilon \in H^2(G, \mathbb{T})$) unitary representations π of G on Hilbert spaces H_π, with the property that π restricted to Γ is unitarily equivalent to the (projective, with 2-cocycle ε) unitary left regular representation of Γ.

* Member of the Institute of Mathematics "S. Stoilow" of the Romanian Academy

* Florin Rădulescu is supported in part by PRIN-MIUR and by a Grant of the Romanian National Authority for Scientific Research, project number PN-II-ID-PCE-2012-4-0201.

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Throughout this paper we assume that the cocycle ε corresponds to a finite central extension of the group G, compatible with Γ, and to a central character on the center of the central extension as in the papers [2], [28] (the comments after the proof of Lemma 6.1) and [26], Remark 37. There exists a canonical (projective) left regular representation Γ associated to this data, that is compatible with the action of G. This construction is reviewed in Definition 1.

When the cocycle ε is trivial, such representations appear naturally in the study of automorphic forms or Maass forms, where one studies the associated vector spaces of $\pi(\Gamma)$-invariant vectors ([20], [34], [11]). The space of $\pi(\Gamma)$-invariant vectors, briefly referred to as the space of Γ-*invariant vectors* in this paper, is naturally a Hilbert space. In general this is not a subspace of the original Hilbert space; it is related to the original Hilbert space if one considers a rigged Hilbert space structure ([12]) on H_π (see e. g. the construction in [30]).

Given such a unitary representation π, one constructs a new unitary representation $\widetilde{\pi}$ of G, admitting proper Γ-invariant vectors, which extends to a profinite completion of G. The block matrix coefficients of this new representation, corresponding to the reduction to the subspace of Γ invariant vectors, are the usual Hecke operators ([4], [15], [30]).

In this paper we analyze the representation $\pi \otimes \overline{\pi}$ and its extension $\mathrm{Ad}\pi$ acting, by conjugation, on the space of bounded linear operators $B(H_\pi)$. Here $\overline{\pi}$ is the complex conjugated representation associated with the representation π. The representation $\pi \otimes \overline{\pi}$ is a (proper) unitary representation since $|\varepsilon|^2 = 1$. In this paper we will make the assumption that the groups G and Γ have i.c.c. (infinite conjugacy classes), so that the associated von Neumann algebras are factors (i.e. have trivial centers, see [36], [33]).

Consequently, in the case considered above, the Hilbert space of Γ-invariant vectors is canonically identified with the L^2-space, associated to the unique trace on the type II_1 factor commutant $\pi(\Gamma)'$ (see e.g [16]). This construction corresponds to the analysis of Hecke operators on Maass forms ([26],[28]). We note that different, but related approaches, are also considered in the papers [9], [8], [6].

The unitary representation $\widetilde{\pi \otimes \overline{\pi}}$, associated to the unitary representation $\pi \otimes \overline{\pi}$ considered above, is induced by a representation into the automorphism group of a larger type II_1 factor. The additional structure on the underlying unitary representation is used to determine canonical tensor product decompositions of such representations (see e.g. Corollary 9).

The data from a representation π as above, which is used to construct Hecke operators, is transformed into data coming from a canonical unitary representation of the Schlichting completion $\mathcal{G}$ of G (see [35]). Splittings of canonical representations associated to G are proven to be in one-to-one correspondence with unitary representations π, up to the obvious equivalence relation on the space of such representations.

The main source of such unitary (projective) representations arises from the square summable (projective) unitary representations $\widetilde{\Pi}$ of a semisimple Lie group $\overline{G}$, containing G as a dense subgroup and so that Γ is a lattice in $\overline{G}$ (see [13], Section 3.3 and also [2]), into the unitary group $\mathcal{U}(H_{\widetilde{\Pi}})$ of a Hilbert space $H_{\widetilde{\Pi}}$. The restriction $\pi = \widetilde{\Pi}|_G$ has the property that $\pi|_\Gamma$ is a multiple of the (projective) left regular representation, with multiplicity

$$D_\pi = \dim_{\{\pi(\Gamma)\}''}(H_{\widetilde{\Pi}}) \in (0, \infty)$$

given by the Murray-von Neumann dimension of the Hilbert space $H_{\widetilde{\Pi}}$ as left module over the von Neumann algebra $\{\pi(\Gamma)\}''$ generated by the image of the group representation. The dimension is proportional by $\mathrm{Vol}(\overline{G}/\Gamma)^{-1}$ to the Plancherel coefficient of the square integrable representation $\widetilde{\Pi}$ (see [1] and [13], Section 3.3). In general the dimension D_π is not always an integer (see e.g [13], [2], [27] for more details on the non-integer case).

The classification of the (projective) unitary representations π of the group G with $D_\pi = 1$ is analyzed in detail in the first sections. This corresponds to the case where $\pi|_\Gamma$ is unitarily equivalent to the (projective) left regular representation.

In the last section we also consider the situation $D_\pi \neq 1$. Although this is similar to the previous case, it requires additional formalism. In Theorem 35 (i) and formula (91), we introduce an alternative method to construct the unitary representation of G that plays the essential role in the analysis of $D_\pi = 1$ (cf. Theorem 6 (ii)) in the equivalent description of a unitary representation π as above.

We prove that the classification problem of (projective) unitary representations π of G as above, up to the equivalence relation corresponding to conjugation by unitary operators, is determined by the first cohomology group

$$H^1_\alpha(G, \mathcal{U}(P)) \tag{1}$$

of the group G, with values in the unitary group $\mathcal{U}(P)$ of a type II_1 factor P, associated with a homomorphism $\alpha : G \to \mathrm{Aut}(P)$.

Both objects P and α are constructed canonically, starting from a fixed unitary (projective) representation π of G as above, through a process similar to an infinite, simultaneous, Jones's basic construction ([16]), which we describe below (see formulae (5) and (14)). Since α is a lifting of a canonical group homomorphism (see Definition 3) it follows that up to isomorphism, the group $H^1_\alpha(G,\mathcal{U}(P))$ is a canonical object associated with the given inclusion $\Gamma \subseteq G$.

For a discrete group N we denote by $\mathcal{L}(N)$ the associated von Neumann algebra ([33], [32]). We describe below the necessary modifications for the case when we are also given a group 2-cocycle ε defining an element in $H^2(N,\mathbb{T})$, under additional assumptions (see [2], [28], ([26], Remark 37)).

Definition 1 (Projective, left regular representation)**.** Let N, ε be as above. In addition assume that there is a central extension, that is a surjective group homeomorphism, $\Psi : N_0 \to N$ by a finite (abelian) group $Z \subseteq N_0$ (the kernel of Ψ), and a unitary, complex valued character χ of Z with the following property:
(i). The class of the 2- cocycle ε in $H^2(N,\mathbb{T})$ is determined by the character χ.

Let $\ell^2_\chi(N)$ be the subspace of $\ell^2(N_0)$ defined by the formula:

$$\ell^2_\chi(N) = \{\zeta \in \ell^2(N_0) | \zeta(s^{-1}n_0) = \chi(s)\zeta(n_0), \forall z \in Z, n_0 \in N_0\}. \tag{2}$$

Let λ_{N_0} be the left regular representation associated with group N_0, acting on $\ell^2(N_0)$. The projection P_χ from $\ell^2(N_0)$ onto $\ell^2_\chi(N)$ is given by the formula

$$P_\chi = \frac{1}{|Z|}\sum_{s\in Z}\overline{\chi(z)}\lambda_{N_0}(z). \tag{3}$$

Then P_χ is a central projection in the type II von Neumann algebra $\mathcal{L}(N_0)$.

We let $\mathcal{L}_\chi(N)$ be the type II_1 factor $P_\chi\mathcal{L}(N_0)$ acting on $\ell^2_\chi(N)$. Then $\mathcal{L}_\chi(N)$ is isomorphic to the II_1 factor $\mathcal{L}(N,\varepsilon)$ associated to the group N and the 2-cocycle ε (we refer to [38], [22], [3] for the definition of $\mathcal{L}(N,\varepsilon)$).

When no confusion is possible, we will continue to denote the factor $\mathcal{L}_\chi(N)$ by $\mathcal{L}(N)$ and the space $\ell^2_\chi(N)$ by $\ell^2(N)$. We will use the same notational pattern for skewed crossed algebras by the group N, if the action extends to N_0 and the action of the center Z is trivial.

Definition 2. We will assume that the group G admits a central extension $\Psi : G_0 \to G$ by an abelian group Z_0 as in Definition 1. We assume that the groups $\Psi^{-1}(\Gamma_1)$ have a fixed center $Z \subseteq \Gamma \cap Z_0$ for all $\Gamma_1 \in \mathcal{S}$.

We assume that the (projective, corresponding to $\varepsilon \in H^2(G,\mathbb{T})$) unitary irreducible representations π of G that we are considering in this paper all extend to G_0 and induce a unitary character χ_0 on Z_0. Let $\chi = \chi_0|_Z$. We assume that the 2-cocycle ε in $H^2(G, \mathbb{T})$ corresponds to the character χ_0. We assume that the representation π has the property that $\pi|_\Gamma$ is a multiple of the projective, left regular representation λ_χ of the group Γ. Consequently, the von Neumann algebra $\{\pi(\Gamma)\}''$ is isomorphic to $\mathcal{L}(\Gamma, \epsilon) = \mathcal{L}_\chi(\Gamma)$. When no confusion is possible we will omit the reference to the 2-cocycle ε an we will denote the later algebra simply by $\mathcal{L}(\Gamma)$.

The example that we have in mind here is the case $G_0 = \mathrm{SL}(2, \mathbb{Q}(\sqrt{p}))$, $G = \mathrm{PGL}_2(\mathbb{Z}[1/p])$, $\Gamma_0 = \mathrm{SL}(2, \mathbb{Z})$ and $\Gamma = \mathrm{PSL}_2\mathbb{Z})$. Hence the center group Z is isomorphic to $\mathbb{Z}_2$ and the character χ is the unique non-trivial character of $\mathbb{Z}_2$. The (projective) unitary representations of $G = PGL_2(\mathbb{Z}[1/p])$ that correspond to this character are the representations π_n, introduced below, in the discrete series of $SL(2, \mathbb{R})$ where n is an odd natural number $n \geq 3$.

For σ in G, let

$$\Gamma_\sigma = \sigma\Gamma\sigma^{-1} \cap \Gamma$$

be the finite index subgroup of Γ associated to σ. Let $\mathcal{S}$ be the downward directed lattice of finite index subgroups of Γ, generated by the subgroups Γ_σ, $\sigma \in G$. We assume that $\mathcal{S}$ separates the elements of Γ. For simplicity, for reasons concerning the construction of the tower of commutant algebras below, we will assume that all the groups in $\mathcal{S}$ have the i.c.c. property.

We will assume in this paper that the following equality

$$[\Gamma : \Gamma_\sigma] = [\Gamma : \Gamma_{\sigma^{-1}}], \tag{4}$$

holds true for all $\sigma \in G$.

By Jones's index theory, the existence of a unitary representation π as above automatically implies the equality of the indices, since $\pi(\sigma)$ conjugates the subfactors $\{\pi(\Gamma_{\sigma^{-1}})\}''$ and $\{\pi(\Gamma_\sigma)\}''$ in $\{\pi(\Gamma)\}''$. Let K be the profinite completion of the subgroup Γ with respect to the family of finite index subgroups $\mathcal{S}$, and let μ be the corresponding Haar measure on K ([35], [37]). The equality of the indices proves that the (partial) action by left and right multiplication by elements in G preserves the Haar measure on K.

Then, the type II_1 factor

$$P = \mathcal{L}(\Gamma \rtimes_\varepsilon L^\infty(K, \mu)), \tag{5}$$

entering in the definition of the cohomology group describing the classification of the (projective) unitary representations π as above, is the reduced skewed (by the 2-cocycle ε) crossed product von Neumann algebra ([33], [21], [3], [22], [38]) associated to the probability measure preserving action of Γ, by left translations on (K, μ). In the setting of Definition 2, the action of the center group Z is trivial. We let P_0 be the reduced crossed product von Neumann algebra

$$P_0 = \mathcal{L}(\Gamma_0 \rtimes L^\infty(K, \mu)). \tag{6}$$

Then, since p_χ is a central projection we have, using the constructions in the previous definitions,

$$P \cong p_\chi P_0 \subseteq B(\ell^2_\chi(\Gamma)). \tag{7}$$

When no confusion is possible, we will use in the rest of the paper the crossed product notation $\mathcal{L}(\Gamma \rtimes L^\infty(K, \mu))$ for the factor in formula (5) also in the case when the cocycle $\varepsilon \in H^2(\Gamma, \mathbb{T})$ is non-trivial.

To construct the homomorphism $\alpha : G \to \mathrm{Aut}(P)$, we identify the type II_1 factor P with the output at infinity in the simultaneous Jones's basic construction associated to all the inclusions

$$\mathcal{L}(\Gamma_0) \subseteq \mathcal{L}(\Gamma), \quad \Gamma_0 \in \mathcal{S}. \tag{8}$$

The homomorphism α is obtained from a homomorphism of G into the automorphism group of a larger von Neumann algebra associated to an infinite, simultaneous Jones's basic construction for pairs of isomorphic subfactors (see Proposition 13).

We represent the terms of the Jones's basic construction in formula (8) using commutant algebras. Since we are working in the case $D_\pi = 1$, we can use the canonical (anti-)isomorphism of the algebra $\mathcal{L}(\Gamma)$ (see e.g [7],[32]) with the commutant algebra $\pi(\Gamma)'$.

Since the commutant von Neumann algebras $\pi(\Gamma_\sigma)'$ are type II_1 factors, it follows that, for $\Gamma_0 \subseteq \Gamma_1$, $\Gamma_0, \Gamma_1 \in \mathcal{S}$, the embeddings

$$\pi(\Gamma_1)' \subseteq \pi(\Gamma_0)'$$

are trace preserving.

Consider the type II_1 factor obtained, by completion, from the trace preserving, inductive limit of the type II_1 factors $\pi(\Gamma_0)'$, $\Gamma_0 \in \mathcal{S}$:

$$\widetilde{\mathcal{A}}_\infty = \overline{\bigcup_{\Gamma_0 \in \mathcal{S}} \pi(\Gamma_0)'}. \tag{9}$$

In the above formula the closure is taken with respect to the weak topology, in the GNS construction associated to the unique trace on the tower of algebras.

Then $\mathcal{A}_\infty$ is a II_1 factor, anti-isomorphic to the simultaneous Jones basic construction for all the inclusions

$$\pi(\Gamma_\sigma)'' \subseteq \pi(\Gamma)'', \quad \sigma \in G.$$

For $\sigma \in G$, let K_σ be the closed subgroup of K defined by the formula

$$K_\sigma = K \cap \sigma K \sigma^{-1}. \tag{10}$$

Obviously K_σ is the closure, in the profinite completion, of the group Γ_σ.

We use the letter $\mathcal{R}$ to denote right convolutors. Then

$$\widetilde{\mathcal{A}}_\infty \cong \mathcal{R}(\Gamma \rtimes L^\infty(K, \mu)).$$

In the above isomorphism, the Jones's projection e_{Γ_σ}, associated to an inclusion as in formula (8), corresponds to the characteristic function

$$\chi_{K_\sigma} = \chi_{\overline{\Gamma_\sigma}} \in L^\infty(K, \mu).$$

By using the canonical anti-isomorphism between the algebras of left and respectively right convolutors ([7]), we obtain that the algebra $\widetilde{A}_\infty$, the inductive limit of II_1 factors in formula (9), is anti-isomorphic to the type II_1 factor $\mathcal{L}(\Gamma \rtimes L^\infty(K, \mu))$. Thus, in the sequel we let

$$\mathcal{A}_\infty = \mathcal{L}(\Gamma \rtimes L^\infty(K, \mu)) = P.$$

Using the above anti-isomorphism, the type II_1 factor $\pi(\Gamma_\sigma)'$ is identified with the type II_1 factor

$$\mathcal{A}_{\sigma\Gamma} = \mathcal{L}(\Gamma \rtimes l^\infty(\Gamma/\Gamma_\sigma)) \cong \{\mathcal{L}(\Gamma), \chi_{K_\sigma}\}'' \subseteq \mathcal{A}_\infty. \tag{11}$$

Note that, using the convention introduced in Definition 1, the above algebra (as well as the algebra introduced in the next formula) is a skewed crossed product algebra if the 2-cocycle $\varepsilon \in H^2(G, \mathbb{T})$ is non-trivial.

Clearly, using any choice of left representatives of cosets for $\Gamma_\sigma \subseteq \Gamma$ we obtain that

$$P \cong \mathcal{L}(\Gamma_\sigma \rtimes L^\infty(K_\sigma, \mu)) \otimes B(\ell^2(\Gamma/\Gamma_\sigma)). \tag{12}$$

Denote the Hilbert space of the (projective) unitary representation π by H_π. We consider the homomorphism $\mathrm{Ad}\,\pi$ of G into the inner automorphism group of the algebra $B(H_\pi)$. Note that, although π is a projective representation with non trivial cocycle ε, $\mathrm{Ad}\,\pi$ is a proper representation since $\varepsilon^2 = 1$. We observe that the automorphism $\mathrm{Ad}\,\pi(\sigma)$ leaves invariant the upward directed union of commutants $\bigcup\limits_{\Gamma_\sigma \in \mathcal{S}} \pi(\Gamma_\sigma)'$. Indeed, for $\sigma \in G$ we have that

$$\mathrm{Ad}\,\pi(\sigma)[\pi(\Gamma_{\sigma^{-1}})''] = \pi(\Gamma_\sigma)''.$$

Hence

$$\operatorname{Ad}\pi(\sigma)[\pi(\Gamma_{\sigma^{-1}})'] = \pi(\Gamma_\sigma)', \tag{13}$$

and therefore

$$\operatorname{Ad}\pi(\sigma)[\pi(\Gamma_{\sigma^{-1}} \cap \Gamma_0)'] = \pi(\Gamma_\sigma \cap \sigma\Gamma_0\sigma^{-1})', \quad \Gamma_0 \in \mathcal{S}, \sigma \in G.$$

Consequently, the (proper, as $\varepsilon^2 = 1$) unitary action $\operatorname{Ad}\pi$ induces a homomorphism $\widetilde{\alpha} : G \to \operatorname{Aut}(\widetilde{\mathcal{A}}_\infty)$, obtained by requiring that

$$\widetilde{\alpha}|_{\pi(\Gamma_{\sigma^{-1}} \cap \Gamma_0)'} = \operatorname{Ad}\pi(\sigma)|_{\pi(\Gamma_{\sigma^{-1}} \cap \Gamma_0)'}, \quad \Gamma_0 \in \mathcal{S}, \sigma \in G. \tag{14}$$

Using the above mentioned anti-isomorphism between the type II_1 factors $\mathcal{R}(\Gamma \rtimes L^\infty(K,\mu))$ and $P = \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))$, we construct from formula (14) a canonical homomorphism $\alpha : G \to \operatorname{Aut}(P)$ which defines the classifying cohomology group $H^1_\alpha(G,\mathcal{U}(P))$.

We construct below a canonical subgroup of $\operatorname{Out}(P)$. The unitary representations π as above are, up to unitary equivalence, in one-to-one correspondence with the liftings of the above mentioned subgroup to $\operatorname{Aut}(P)$ (see e.g. [7] for the relevant definitions concerning groups of automorphisms groups of von Neumann algebras).

Definition 3. For $\sigma \in G$, let $\theta_\sigma : \Gamma_{\sigma^{-1}} \to \Gamma_\sigma$ be the group isomorphism induced by σ. Using this isomorphism and the formula (12), when $\varepsilon \in H^2(G,\mathbb{T})$ is trivial it follows that any bijection between $\Gamma/\Gamma_{\sigma^{-1}}$ and Γ/Γ_σ induces an isomorphism

$$\widetilde{\theta_\sigma} \in \operatorname{Aut}(P), \tag{15}$$

with P as in formula (5). Then the isomorphisms $\widetilde{\theta_\sigma}$ define a group homomorphism

$$\Phi : G \to \operatorname{Out}(P) = \operatorname{Aut(P)}/\operatorname{Int}(P). \tag{16}$$

If $\varepsilon \in H^2(G,\mathbb{T})$ is non-trivial, we use the assumptions in Definitions 1 and 2. We define for $\widetilde{\sigma} \in G_0$ the analogous automorphism $\theta^0_{\widetilde{\sigma}}$ mapping the von Neumann algebra of the group $\Psi^{-1}(\Gamma_{\sigma^{-1}})$ into the von Neumann algebra of the group $\Psi^{-1}(\Gamma_\sigma)$, where $\sigma = \Psi(\widetilde{\sigma})$. Using the above bijection, we obtain an automorphism $\widetilde{\theta^0_{\widetilde{\sigma}}} \in \operatorname{Aut}(\widetilde{P})$.

In this case, we let $\widetilde{\theta_\sigma} \in \operatorname{Aut}(P)$ be defined by the formula

$$\widetilde{\theta_\sigma} = p_\chi \widetilde{\theta^0_{\widetilde{\sigma}}} p_\chi, \quad \widetilde{\sigma} \in G_0. \tag{17}$$

It is obvious that the right hand side of the above equation depends only on $\sigma = \Psi(\widetilde{\sigma}) \in G$. This automorphism is an abstract model for the automorphism introduced in formula (13).

The group homomorphism introduced in formula (16) is (in both cases) canonical, and it is independent on the choice of the coset representatives made in formula (12).

The classification of the (projective) representations π, up to unitary equivalence, is equivalent to the classification of the liftings of the map Φ to $\mathrm{Aut}(P)$. In particular, if there exists a (projective) unitary representation π as above, then Φ is liftable to a group homomorphism. In Section 6 we will prove the following:

Theorem 4. *Suppose $D_\pi = 1$ and let Φ be as in* (16)*. Then:*
(i) Any (projective) unitary representation π of G as above defines a lifting

$$\alpha_\pi = (\alpha_\pi(g))_{g \in G}$$

of Φ to a group homomorphism $\alpha_\pi : G \to \mathrm{Aut}(P)$, such that:

(a) *For every $\sigma \in G$, $\alpha_\pi(\sigma)$ maps $\mathcal{A}_{\sigma^{-1}\Gamma}$ onto $\mathcal{A}_{\sigma\Gamma}$ (see formula (11)).*
(b) $\alpha_\pi|_\Gamma = \mathrm{Id}_P$.

(ii) For any two (projective) unitary representations π, π' of G as above, the corresponding liftings α_π, $\alpha_{\pi'}$ are cocycle conjugated.
(iii) Two (projective) unitary representations π and π' of G, corresponding to the same 2-cocycle ε, are unitarily equivalent if and only if the corresponding liftings α_π, $\alpha_{\pi'}$ are conjugated by an automorphism in $\mathrm{Int}(P)$.

The above statement proves that if such a representation π exists, then the canonical obstruction ([7], [17], [10]) in $H^2(G, \mathcal{U}(P))$, associated to the homomorphism Φ into Out(P), vanishes.

Let $\mathcal{G}$ be the locally compact, totally disconnected group obtained as the Schlichting completion ([35]) of G with respect to the subgroups in $\mathcal{S}$. Consider the (skewed, by the 2-cocycle ε, determined by π) crossed product von Neumann algebra

$$\mathcal{M} = \mathcal{L}(G \rtimes_\varepsilon L^\infty(\mathcal{G}, \mu)). \tag{18}$$

Using the setting in Definition 2, let

$$\mathcal{M}_0 = \mathcal{L}(G_0 \rtimes_\varepsilon L^\infty(\mathcal{G}, \mu)). \tag{19}$$

Since the center Z_0 of G_0 acts trivially on $L^\infty(\mathcal{G}, \mu)$ it follows that

$$\mathcal{M} = p_\chi \mathcal{M}_0 \subseteq B(\ell^2_\chi(G)) \otimes B(L^2(\mathcal{G}, \mu)). \tag{20}$$

When no confusion is possible we will simply write the skewed crossed product as

$$\mathcal{M} = \mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu)).$$

Let $G^{\rm op}$ be the group G with opposite multiplication. We let $G \times G^{\rm op}$ act on K by partial isomorphism, by left and right multiplication. Thus the domain of $(\sigma, \sigma^{-1}) \in G \times G^{\rm op}$ is $K_{\sigma^{-1}}$. We also consider the 2-cocycle $\varepsilon \times \overline{\varepsilon}$ on $G \times G^{\rm op}$.

Definition 5. Consider the skewed, groupoid crossed product von Neumann algebra

$$\mathcal{B}_\infty = \mathcal{L}((G \times G^{\rm op}) \rtimes_{\varepsilon \times \overline{\varepsilon}} L^\infty(K, \mu)). \tag{21}$$

We also consider the crossed product von Neumann algebra

$$\mathcal{B}^0_\infty = \mathcal{L}((G_0 \times G_0^{\rm op}) \rtimes_{\varepsilon \times \overline{\varepsilon}} L^\infty(K, \mu)). \tag{22}$$

As above, the center $Z \times Z$ acts trivially on $L^\infty(K, \mu))$, and hence, identifying the projection corresponding to the character $\chi \times \overline{\chi}$ with

$$p_\chi \otimes p_{\overline{\chi}} \in C^*_{\rm red}(G_0) \otimes C^*_{\rm red}(G_0^{\rm op}),$$

we obtain that

$$\mathcal{B}_\infty = (p_\chi \otimes p_{\overline{\chi}}) \mathcal{B}^0_\infty. \tag{23}$$

In the rest of the paper, when no confusion is possible, the skewed von Neumann algebra crossed product defining $\mathcal{B}_\infty$ will simply be denoted by $\mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(K, \mu))$.

Let $\beta : G \to \mathrm{Aut}(\mathcal{M})$ be the canonical homomorphism of $G^{\rm op}$ into the automorphism group of $\mathcal{M}$, which acts by leaving $\mathcal{L}(G)$ invariant and acts by composition with right translation on $L^\infty(\mathcal{G}, \mu)$.

To prove Theorem 4 we show that unitary representations π as in the introduction are in one-to-one correspondence with tensor product splittings of the canonical right action $\beta = (\beta_g)_{g \in G}$. The case $D_\pi \neq 1$ is outlined in Section 7.

This proves that the homomorphisms α as above are tensor product factors of the homomorphism β. This is contained in the following theorem and its corollary, which give equivalent descriptions for representations π as above. Then $G^{\rm op}$ acts canonically on $\mathcal{M}$, leaving $\mathcal{L}(G)$ invariant and acting by left multiplication on the infinite measure space $\mathcal{G}$.

To state the next result, which will be proved in Sections 2 and 5, we introduce the (algebraic) operator system

$$\mathcal{SO} = \mathcal{SO}(\Gamma, G) = \mathbb{C}(G/\Gamma) \underset{\mathbb{C}(\Gamma\backslash G/\Gamma)}{\otimes} \mathbb{C}(\Gamma \setminus G). \tag{24}$$

We are not considering any norm on $\mathcal{SO}$, but rather view it as a vector space with internal operations.

Theorem 6. *The following statements are equivalent:*
(i) There exists a (projective) unitary representation $\pi : G \to \mathcal{U}(H_\pi)$ *such that* $\pi|_\Gamma$ *is unitarily equivalent to the left regular representation of* Γ.
(ii) There exists a representation $t : \mathcal{SO} \to \mathcal{L}(G)$ *of the (algebraic) operator system* $\mathcal{SO}$*, verifying the identities in the formulae (31), (32), (33).*
(iii) There exists a (projective, with 2-cocycle ε*) unitary representation* u *of the group* G^{op} *into* $\mathcal{B}_\infty = \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(K, \mu))$*, of the form*

$$u(\sigma) = \chi_K(X^{\Gamma\sigma\Gamma} \otimes \sigma^{-1})\chi_K, \quad \sigma \in G, \tag{25}$$

where, for each double coset $\Gamma\sigma\Gamma$*,* $\sigma \in G$*, the element* $X^{\Gamma\sigma\Gamma}$ *is a selfadjoint element in* $\mathcal{L}(G) \cap l^2(\Gamma\sigma\Gamma)$.
(iv) There exists a matrix unit $(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G} \subseteq \mathcal{M}$ *such that*

(a) $v_{\Gamma\sigma,\Gamma\sigma} = \chi_{\overline{\Gamma\sigma}} \in L^\infty(\mathcal{G}, \mu), \quad \sigma \in G,$
(b) $\beta_g(v_{\Gamma\sigma_1,\Gamma\sigma_2}) = v_{\Gamma\sigma_1 g,\Gamma\sigma_2 g}, \quad g \in G^{\mathrm{op}}, \Gamma\sigma_1, \Gamma\sigma_2 \in \Gamma\backslash G.$

Part (iii) in the above theorem is an abstract C^*-algebraic point of view for the representation π. A unitary representation as in part (iii) may be constructed directly also in the case $D_\pi \neq 1$ (see Theorem 35, Section 7). There is a one-to-one correspondence between the constructions introduced in points (i)-(iv) in the above theorem, which is made explicit in the proof.

The homomorphism $\alpha = \alpha_\pi$ acts on the corner $\chi_K\mathcal{M}\chi_K$ of the algebra $\mathcal{M}$ introduced in formula (18).

We denote by $\rho_{\Gamma\backslash G}$ the right quasi- regular representation of G into the unitary group associated with the Hilbert space $\ell^2(\Gamma\backslash G)$. Then $\mathrm{Ad}\,\rho_{\Gamma\backslash G}$ is a homomorphism from G into the inner automorphism group of $B(\ell^2(\Gamma\backslash G))$.

The following corollary will be proved in Section 2.

Corollary 7. *We assume that the equivalent conditions (i)-(iv) in the statement of Theorem 6 hold true. Recall that* $\chi_K\mathcal{M}\chi_K$ *is isomorphic to* P*. The* G^{op}*-equivariant matrix unit constructed in part (iv) of the preceding statement, yields a homomorphism*

$$G^{\mathrm{op}} \ni g \to \alpha_g \in \mathrm{Aut}(\chi_K\mathcal{M}\chi_K).$$

Then:

(i) The algebra $\mathcal{M}$ splits as

$$\mathcal{M} \cong \chi_K \mathcal{M} \chi_K \otimes B(l^2(\Gamma \setminus G)). \tag{26}$$

(ii) The homomorphism $(\beta_g)_{g \in G^{\mathrm{op}}}$ into the automorphism group of $\mathcal{M}$, splits in the tensor product form

$$\alpha_g \otimes \mathrm{Ad}\, \rho_{\Gamma \setminus G}(g), \quad g \in G^{\mathrm{op}}. \tag{27}$$

It may be easily observed that both homomorphisms α and β extend to a representation of $\mathcal{G}$. Indeed the C^*-algebra associated to $\mathcal{G}$ is generated by the elements $g \in G$ and by the characteristic functions of the form $\chi_{K_\sigma}, \sigma \in G$. To extend α, to such a characteristic function we make correspond the (renormalized) conditional expectation onto $\mathcal{A}_{\sigma\Gamma} = \{\pi(\Gamma_\sigma)\}'$.

The following result, which will be proved in Section 3, summarizes the relation of the above construction with the Hecke operators acting on the Hilbert space of Γ-invariant vectors for the representation $\pi \otimes \overline{\pi} \cong \mathrm{Ad}\ \pi$. Consider the canonical von Neumann conditional expectation (see e.g. [32])

$$E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))}(\,\cdot\,),$$

defined on $P = \mathcal{L}(\Gamma \rtimes L^\infty(K, \mu))$, with values onto $\mathcal{L}(\Gamma)$.

Theorem 8. *Assume that the four equivalent properties in the above theorem hold true. Consider the $*$-algebra representation for the Hecke algebra $\mathcal{H}_0$ constructed in* [26]*, acting on the Hilbert space of Γ-invariant vectors for the unitary representation $\pi \otimes \overline{\pi} \cong \mathrm{Ad}(\pi)$.*

Then the above $*$-algebra representation of the algebra $\mathcal{H}_0$ is unitarily equivalent to the representation of the Hecke algebra obtained by using the completely positive maps $\Psi_{\Gamma\sigma\Gamma}$ on $\mathcal{L}(\Gamma)$, defined by the formula

$$\Psi_{\Gamma\sigma\Gamma}(x) = E_{\mathcal{L}(\Gamma)}^{\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))}(\alpha_g(x)), \quad g \in \Gamma\sigma\Gamma, x \in \mathcal{L}(\Gamma). \tag{28}$$

In particular, the statement of the Ramanujan-Petersson conjectures for the Hecke algebra representation into the Hilbert space of Γ-invariant vectors for the representation $\pi \otimes \overline{\pi} \cong \mathrm{Ad}\ \pi$, is equivalent to the weak containment of the unitary representation of G induced by α on $L^2(P) \ominus \mathbb{C}1$, into the restriction to G of the left regular representation of the Schlichting completion $\mathcal{G}$ of G (see e. g. [19]).

Assume that π belongs to the discrete series of $\mathrm{PSL}_2(\mathbb{R})$, We consider the restriction $\pi|_{\mathrm{PGL}_2(\mathbb{Z}[1/p])}$, where p a prime number. Then, by [26], the Hecke operators associated to this representation, defined as above, are unitarily equivalent to the Hecke operators acting on Maass forms.

It is also possible to obtain a "coordinate free" version of the above statement. In this formulation we substitute the homomorphism α by a canonical homomorphism. The drawback is that the type II_1 factor on which it acts is defined only up to an isomorphism.

Recall that $\mathcal{M}$ is the reduced (skewed) crossed product von Neumann algebra $\mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu))$, with semifinite trace T induced by the G-invariant measure μ on $\mathcal{G}$. Recall that the homomorphism $\beta : G^{\mathrm{op}} \to \mathrm{Aut}(\mathcal{M})$, where each automorphism β_g acts by right translation by g on $L^\infty(\mathcal{G}, \mu)$, and acts identically on $\mathcal{L}(G)$, for $g \in G$.

We also consider the algebra $D \subseteq L^\infty(\mathcal{G}, \mu) \subseteq \mathcal{M}$ consisting of left K-invariant functions in $L^\infty(\mathcal{G}, \mu)$. Obviously $D \cong \ell^\infty(\Gamma \backslash G)$. We have (see Section 3 for the proof):

Corollary 9. *Assume that the equivalent conditions introduced above hold true. Then:*
(i) There exist a unitary representation $\theta : G \to \mathcal{U}(\mathcal{M})$ with the following properties:

(a) The representation θ is unitarily equivalent to the right quasi-regular representation $\rho_{\Gamma\backslash G}$ of G.

(b) For all $g, h \in G$ we have $\mathrm{Ad}(\theta_g)|_D = \beta_g|_D$ and $\beta_g(\theta_h) = \theta(ghg^{-1})$.
(ii) Let $\Theta \subseteq \mathcal{M}$ be the type I_∞ factor generated by the image of the representation θ. Let $\mathcal{A} = \Theta' \subseteq D'$. Obviously $\mathcal{L}(\Gamma) \subseteq \mathcal{A}$. Then $\mathcal{A}$ is a type II_1 factor isomorphic to $\mathcal{L}(\Gamma \rtimes K)$ with unique trace τ induced by μ.
(iii) The homomorphism $(\beta_g)_{g\in G}$ invariates Θ and hence it invariates $\mathcal{A}$. The homomorphism $(\beta_g|_{\mathcal{A}})_{g\in G}$ of G^{op} into $\mathrm{Aut}(\mathcal{A})$ is unitarily equivalent to the homomorphism α introduced in Corollary 7. It extends obviously to $\mathcal{G}$.

In particular, the Hecke algebra representation introduced in the previous theorem is unitarily equivalent to the representation of the Hecke algebra obtained by using the completely positive maps defined by the formula

$$\Psi^0_{[\Gamma\sigma\Gamma]}(x) = E^{\mathcal{A}}_{\mathcal{L}(\Gamma)}(\beta_\sigma(x)), \quad x \in \mathcal{A}, \sigma \in G.$$

(iv) The classification of the unitary representations θ as in (i) is determined by the cohomology group $H^1_{\beta|_{\mathcal{A}}}(G, \mathcal{U}(\mathcal{A}))$.

Thus, the unitary representation of G (extending to $\mathcal{G}$) determining the Hecke operators acting on the Γ-invariant vectors for the representation $\pi \otimes \overline{\pi}$, is an operator algebra object. The procedure to construct this operator algebra object is as follows. Consider the homomorphism $\beta : G^{\mathrm{op}} \to \mathrm{Aut}(\mathcal{M})$ described above, acting by right translation on $\mathcal{G}$ and trivially on $\mathcal{L}(G)$. There exists a type I_∞ factor $\Theta \subseteq \mathcal{M}$ such that $\beta_g|_\Theta$ is unitarily equivalent to the homomorphism $\mathrm{Ad}\rho_{\Gamma\backslash G}$ of G into $B(\ell^2(\Gamma\backslash G))$, induced by the right quasi-regular representation and such that both β_g and $\mathrm{Ad}\rho_{\Gamma\backslash G}$ act identically on the diagonal algebra $D = \ell^\infty(\Gamma\backslash G)$, and $\mathrm{Ad}\rho_{\Gamma\backslash G}(g^{-1}) \circ \beta_g$ is a homomorphism for every $g \in G$.

Then every β_g invariates $\mathcal{A} = \Theta'$ and the homomorphism $(\beta_g|_{\mathcal{A}})_{g\in G}$ into $\mathrm{Aut}(\mathcal{A})$ is conjugated to the homomorphism α into $\mathrm{Aut}(\mathcal{L}(\Gamma \rtimes K))$.

The goal for Ramanujan-Petersson problem is to prove that the unitary representation of G, induced by $\beta|_{\mathcal{A}}$ on the orthogonal complement of the scalars in $L^2(\mathcal{A}, \tau)$, is weakly contained in the restriction to G of the left regular representation of $\mathcal{G}$. This property obviously holds true for the unrestricted unitary representation induced by the homomorphism β on $L^2(\mathcal{M}, T)$. The reason why this property doesn't straightforwardly pass to a subalgebra is the fact that the two traces defining the Hilbert spaces $L^2(\mathcal{A}, \tau)$ and $L^2(\mathcal{M}, T)$ are different: one is finite and the other one is semifinite.

2. Construction of the G^{op}-equivariant splitting $\mathcal{M} \cong \mathcal{P} \otimes B(l^2(\Gamma\backslash G))$. Proof of Theorem 6.

This section is mainly concerned with proving part of Theorem 6 and Corollary 7. We use the matrix coefficients of the (projective) unitary representation π of G, which has the property that $\pi|_\Gamma$ is unitary equivalent to the left regular representation of Γ.

Let $\rho_{\,\Gamma\backslash G}$ be the right quasi-regular representation of G^{op} on $l^2(\Gamma\backslash G)$. We construct directly a G^{op}-equivariant embedding of $B(l^2(\Gamma\backslash G))$ into $\mathcal{M}$. Here $B(l^2(\Gamma\backslash G))$ is acted by $\mathrm{Ad}\rho_{\,\Gamma\backslash G}$. We prove that this equivariant representation of $B(l^2(\Gamma\backslash G))$ is splitting the algebra $\mathcal{M}$ G^{op}-equivariantly, in the sense of tensor products.

We establish a correspondence between unitary representations π as above and G^{op}-equivariant splittings of the form

$$\mathcal{M} \cong P \otimes B(l^2(\Gamma\backslash G)), \tag{29}$$

of the crossed product von Neumann algebra $\mathcal{M}$. In this identification the type II_1 factor P, introduced in formula (5), is the corner algebra $\chi_K\mathcal{M}\chi_K$ of $\mathcal{M}$ with unit χ_K, the characteristic function of K.

The homomorphism α is then identified, in this setting, with the action on the tensor product factor component $\chi_K\mathcal{M}\chi_K$, in the representation from formula (29), of the canonical embedding of the group G^{op} into the automorphism group of the algebra $\mathcal{M} \cong P \otimes B(l^2(\Gamma\backslash G))$.

We first recall the construction of the C^*-representation t of the Hecke algebra of double cosets of Γ in G into $\mathcal{L}(G)$, introduced in [26] (see also [28]). This representation was subsequently extended to arbitrary Murray-von Neumann dimensions in [27]. The representation will be used in the construction of the G^{op}-equivariant matrix unit corresponding to the G^{op}-equivariant embedding of $B(l^2(\Gamma\backslash G))$ into $\mathcal{M}$.

Let $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$ be the Hecke algebra of double cosets (see e.g. [5]). We let $\mathcal{H}_0$ act canonically on left and respectively right cosets, by left and respectively right multiplication. Let $\mathcal{H} \subseteq B(l^2(\Gamma\backslash G))$ be the uniform norm closure of $\mathcal{H}_0$. In the terminology introduced in [5], the C*-algebra $\mathcal{H}$ is the reduced Hecke von Neumann algebra associated to the inclusion $\Gamma \subseteq G$.

In [26] (see also [28] for another exposition of the construction, and see Section 7 for another proof of the extension of this construction to arbitrary D_π) we constructed, using the matrix coefficients of the representation π, a representation $t : \mathcal{H} \to \mathcal{L}(G)$ such that $t^{\Gamma\sigma\Gamma} = t([\Gamma\sigma\Gamma]) \in l^2(\Gamma\sigma\Gamma) \cap \mathcal{L}(G)$.

When the 2-cocycle ε is trivial, the precise formula for the representation t is as follows: let 1 be a trace vector for Γ in the Hilbert space $H_\pi \cong \ell^2(\Gamma)$ of the representation π. For a subset A of G, define

$$t^A = t(A) = \sum_{\theta \in A} \overline{\langle \pi(\theta)1, 1\rangle}\theta. \tag{30}$$

Then

$$[\Gamma\sigma\Gamma] \to t([\Gamma\sigma\Gamma]), \quad \sigma \in G,$$

extends to a representation of $\mathcal{H}$ into $\mathcal{L}(G)$.

We will prove this statement in the case when ε is trivial. When the representation π is projective ([26], Remark 37 and [28], Section 10), one has to replace the Hilbert space H_π by the Hilbert space ℓ^2_χ introduced in Definition 1, and to replace the sum in formula (30) by a sum over the preimage of the coset $\Gamma\sigma\Gamma$ in G_0 averaged by the character χ (we are using here the notation from the above mentioned definition).

We proved in [26] that formula (30) implies that the representation t of $\mathcal{H}_0$ extends to a representation of the larger operator system $\mathcal{SO}$ defined in formula (24). This system is canonically identified to the vector space

$$\mathbb{C}\{\sigma_1\Gamma\sigma_2 \mid \sigma_1, \sigma_2 \in G\}.$$

In [26] (see also [28]) we proved that the representation t of $\mathcal{H}_0$ extends to a representation $t : \mathcal{SO} \to \mathcal{L}(G)$. This representation is constructed using the matrix coefficients of the representation π as above in formula (30).

Let e be the neutral element of G, viewed as the identity element of the algebra $\mathcal{L}(G)$. We say that t is a representation of the operator system $\mathcal{SO}$ if the following sets of identities hold true:

$$t(\Gamma) = e. \tag{31}$$

$$t(\sigma\Gamma)^* = t(\Gamma\sigma^{-1}), \quad \sigma \in G. \tag{32}$$

$$t(\sigma_1\Gamma)t(\Gamma\sigma_2) = t(\sigma_1\Gamma\sigma_2), \quad \sigma_1, \sigma_2 \in G. \tag{33}$$

We proved in [26] (see also [27], [29]) that there exists a one-to-one correspondence between unitary representations π and representations t of the operator system $\mathcal{SO}$ with the above properties.

We identify the cosets of K with the cosets of Γ, by taking the closure in the profinite completion. Let $l^\infty(\Gamma\backslash G)$, $l^\infty(G/\Gamma)$ be the algebras of bounded left, and respectively right Γ-invariant functions, that is the algebras generated by the characteristic functions of left (respectively right) cosets of G by Γ. These algebras are identified with the subalgebras of $L^\infty(K\backslash\mathcal{G}, \mu)$ and $L^\infty(\mathcal{G}/K, \mu)$ of left, and respectively right, K-invariant bounded functions on $\mathcal{G}$. We denote by $l^2(\Gamma\backslash G)$, $l^2(G/\Gamma)$ the corresponding Hilbert spaces, and by $\rho_{\Gamma/G}$, $\lambda_{G/\Gamma}$, the corresponding quasi-regular unitary representations of G.

For notational simplicity, when no confusion is possible, we denote in the sequel the characteristic function $\chi_{\sigma K}$ of the coset $\sigma K = \overline{\sigma\Gamma} \subseteq \mathcal{G}$ by $\chi_{\sigma\Gamma}$, for $\sigma \in G$.

In Theorem 6 we prove that an alternative method to obtain representations t as above, is to the use of following G^{op}-equivariant matrix unit embedded in the crossed product algebra $\mathcal{M}$:

$$(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G} = (\chi_{\Gamma\sigma_1} t^{\Gamma\sigma_1\sigma_2^{-1}\Gamma} \chi_{\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}.$$

The G^{op}-equivariance of the matrix unit is assumed to hold true with respect to the adjoint of the right unitary representation $\rho_{\Gamma\backslash G}$ of G^{op} into $l^2(\Gamma \setminus G)$.

The existence of such a matrix unit implies that the von Neumann algebra $\mathcal{M}$ is G^{op}-equivariantly isomorphic to $\chi_K\mathcal{M}\chi_K \otimes B(l^2(\Gamma\backslash G))$. It also implies the isomorphism

$$\chi_K\mathcal{M}\chi_K = \chi_K(\mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu)))\chi_K \cong \mathcal{L}(\Gamma \rtimes L^\infty(K, \mu)) = P. \tag{34}$$

This is a consequence of the fact that the projections in the family χ_{gK}, where $g\Gamma$ runs over a family of coset representatives of Γ in G, are a partition of unity. The unit of the algebra $\chi_K\mathcal{M}\chi_K$ is identified with $\chi_K = \chi_{\overline{\Gamma}}$.

The $G^{\rm op}$-equivariant isomorphism

$$\mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu)) \cong \chi_K \mathcal{M} \chi_K \otimes B(l^2(\Gamma \backslash G))$$

holds true with respect to a tensor product representation of $G^{\rm op}$ into the automorphism group of $\chi_K \mathcal{M} \chi_K \otimes B(l^2(\Gamma \backslash G))$, of the form

$$\alpha_g \otimes {\rm Ad}\rho_{\Gamma \backslash G}(g), \quad g \in G^{\rm op}.$$

Recall that $\Gamma \subseteq G$ is a pair consisting of a discrete group G and an almost normal subgroup Γ, both assumed to be i.c.c. Let $\mathcal{S}$ be the downward directed class of subgroups of Γ generated by $\Gamma_\sigma = \sigma \Gamma \sigma^{-1} \cap \Gamma$, $\sigma \in G$. Assume that $\mathcal{S}$ separates the points of Γ. Let (K, μ) be the corresponding profinite completion of Γ. Let $(\mathcal{G}, \mu)$ be the Schlichting extension of G, as introduced in Section 2. Then μ is the Haar measure on $\mathcal{G}$, normalized by the condition that $\mu(K) = 1$.

Recall that we are assuming that for all $\sigma \in G$ the subgroups Γ_σ, $\Gamma_{\sigma^{-1}}$ have equal indices. In particular, the Haar measure on $\mathcal{G}$ is bivariant. Also, we assume that G acts ergodicaly on $\mathcal{G}$, and that all groups in $\mathcal{S}$ are i.c.c. Consequently, the reduced von Neumann algebra (skewed) crossed product factors

$$\mathcal{M} = \mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu)) \quad \text{and} \quad P = \mathcal{L}(\Gamma \rtimes L^\infty(K, \mu))$$

are type II_∞ (respectively II_1) factors. Recall that if the unitary representation π is projective with cocycle ε, then in the definitions of $\mathcal{M}$ and P we take the ε-skewed crossed product von Neumann algebras.

We consider the following outer action of $G^{\rm op}$ on $\mathcal{M}$ introduced in Section 1.

Definition 10. Let $\beta : G \to {\rm Aut}(\mathcal{M})$ be the canonical homomorphism of $G^{\rm op}$ into the automorphism group of $\mathcal{M}$, acting by leaving $\mathcal{L}(G)$ invariant and by composition with right translation on $L^\infty(\mathcal{G}, \mu)$.

Remark 11. Consider the canonical action by left and right multiplication of $G \times G^{\rm op}$ on $L^\infty(\mathcal{G}, \mu)$. To this action corresponds the reduced (skewed) crossed product von Neumann algebra

$$\mathcal{L}((G \times G^{\rm op}) \rtimes_{\varepsilon \times \overline{\varepsilon}} L^\infty(\mathcal{G}, \mu)).$$

As mentioned in the introduction, we use the notation

$$\mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(\mathcal{G}, \mu))$$

for the above (skewed) crossed product when no confusion is possible.

The above action reduces to a canonical groupoid action of $G \times G^{\rm op}$ on K. This is obtained by letting, for $g_1, g_2 \in G$, the domain of the transformation induced by $(g_1, g_2) \in G \times G^{\rm op}$, be

$$\{k \in K | g_1 k g_2^{-1} \in K\}.$$

Obviously, $G \times G^{\rm op}$ acts by measure preserving transformations on K. Let

$$\mathcal{B}_\infty = \mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(K, \mu))$$

be the corresponding (skewed) reduced von Neumann groupoid crossed product algebra. It is then obvious that

$$\chi_K \big[\mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(\mathcal{G}, \mu))\big] \chi_K \cong \big[\mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(K, \mu))\big]. \tag{35}$$

Theorem 6 shows that the data from the representation π is encoded in a (projective) unitary representation of a special form of G, into the unitary group of the algebra $\mathcal{B}_\infty$. Equivalently, this corresponds to a special $G^{\rm op}$-equivariant splitting of the algebra $\mathcal{M}$.

Proof of Theorem 6. The matrix unit defined in point (iv) in the statement of the theorem generates a copy of the type I_∞ algebra $B(l^2(\Gamma \setminus G))$ which is a subalgebra of $\mathcal{M}$. The diagonal algebra $l^\infty(\Gamma \setminus G) \subseteq B(l^2(\Gamma \setminus G))$ is a fixed algebra, independent of the choice of the matrix unit.

We consider the algebra $L^\infty(K \setminus \mathcal{G}, \mu)$. This algebra is the weak closure of the linear span of cosets of the form $\chi_{\overline{\Gamma\sigma}}$ with $\sigma \in G$. Then we have the isomorphism:

$$l^\infty(\Gamma \setminus G) \cong L^\infty(K \setminus \mathcal{G}, \mu) \subseteq \mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu)). \tag{36}$$

The left coset $K\sigma$ is $\overline{\Gamma\sigma}$, where the closure is taken in $\mathcal{G}$ for $\sigma \in G$. To simplify notation, recall that when no confusion is possible, we are denoting the characteristic function $\chi_{\sigma K}$ of the coset $\sigma K = \overline{\sigma\Gamma}$ by $\chi_{\sigma\Gamma}$ for $\sigma \in G$.

The requirement in part (iv) is therefore to find a $G^{\rm op}$-equivariant copy of $B(l^2(\Gamma \setminus G))$ inside $\mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu))$, with the prescribed diagonal algebra introduced in formula (36).

The implication (i) $\Rightarrow$ (ii) was proved in [26] (see also [27], [28]).

The converse implication (ii) $\Rightarrow$ (i) is the content of Proposition 58 in [26]. For convenience of the reader, we recall this proof here in the case where the representation is unitary (for the projective case see [26],[28]).

For a subset A of G we use the notation

$$t^A = \sum_{\theta \in A} t(\theta)\theta.$$

When the 2-cocycle ε is non-trivial, the above formula is replaced by a finite average over the group Z as in the first formula of Section 10 in [28] (see also Remark 37 in [26]). Let $\sigma \in G$ and let (s_i) be set of representatives for $\Gamma_{\sigma^{-1}}$ in Γ. We define

$$\pi(\sigma)s_i = [t^{\Gamma\sigma s_i}(\sigma s_i)^{-1}]^*, \quad i = 1, 2, \ldots, [\Gamma : \Gamma_{\sigma^{-1}}].$$

The element on the right hand side of the above equation belongs to $\mathcal{L}(\Gamma)$ ([26], [28], [29]).

The fact that $\pi(\sigma)$ is a representation follows form the identity

$$t(\theta_1\theta_2) = \sum_{\gamma\in\Gamma} t(\theta_1\gamma)t(\gamma^{-1}\theta_2), \quad \theta_1, \theta_2 \in G.$$

The above identity is in turn a consequence of the identity

$$t^{\sigma_1\Gamma}t^{\Gamma\sigma_2} = t^{\sigma_1\Gamma\sigma_2}.$$

To prove (iv) $\Rightarrow$ (ii) we proceed as follows. For σ in G, let

$$X^{\Gamma\sigma\Gamma} = \sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma} v_{\Gamma\sigma_1,\Gamma\sigma_2}, \tag{37}$$

with sum over all cosets $\Gamma\sigma_1, \Gamma\sigma_2 \in \Gamma \setminus G$ such that $\Gamma\sigma_1\sigma_2^{-1}\Gamma = [\Gamma\sigma\Gamma]$.

We recall that the G^{op}-equivariance of the matrix unit

$$(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2}$$

means that

$$\beta_g(v_{\Gamma\sigma_1,\Gamma\sigma_2}) = v_{\Gamma\sigma_1 g,\Gamma\sigma_2 g}, \quad g \in G.$$

Hence

$$\beta_g(X^{\Gamma\sigma\Gamma}) = X^{\Gamma\sigma\Gamma}, \quad g \in G^{\mathrm{op}}, \sigma \in G.$$

It follows from formula (37) that $X^{\Gamma\sigma\Gamma}$ belongs to the algebra $\mathcal{M}^G$ of fixed points for the action of G on $\mathcal{M}$. Since we assumed that G^{op} acts ergodicaly on $\mathcal{G}$, it follows that

$$\mathcal{M}^G = \mathcal{L}(G),$$

and hence

$$X^{\Gamma\sigma\Gamma} \in \mathcal{L}(G), \quad \sigma \in G.$$

Obviously, since $v_{\Gamma\sigma,\Gamma\sigma}$ is equal to $\chi_{\Gamma\sigma}$, it follows that the partial isometry $v_{\Gamma\sigma_1,\Gamma\sigma_2}$ will map the space of the projection $\chi_{\Gamma\sigma_2}$ onto $\chi_{\Gamma\sigma_1}$. Hence, using formula (37) defining $X^{\Gamma\sigma\Gamma}$, it follows that if $\Gamma\sigma_1, \Gamma\sigma_2$ are so that

$$[\Gamma\sigma_1\sigma_2^{-1}\Gamma] = [\Gamma\sigma\Gamma],$$

then

$$\chi_{\Gamma\sigma_1} X^{\Gamma\sigma\Gamma} \chi_{\Gamma\sigma_2} = v_{\Gamma\sigma_1,\Gamma\sigma_2}, \quad \sigma, \sigma_1, \sigma_2 \in G.$$

From formula (37) we also obtain that:

$$\chi_{\Gamma\alpha} X^{\Gamma\sigma\Gamma} \chi_{\Gamma\beta} = \delta_{[\Gamma\alpha\beta^{-1}\Gamma],[\Gamma\sigma\Gamma]} v_{\Gamma\alpha,\Gamma\beta}, \quad \alpha, \beta, \sigma \in G.$$

Here, we use the symbol δ to denote the Kronecker symbol.

Let θ be any element in G. Then the property that

$$\chi_{\Gamma\sigma_1} \theta \chi_{\Gamma\sigma_2} \neq 0,$$

is equivalent to the existence of $\gamma_1, \gamma_2 \in \Gamma$ such that

$$\theta\gamma_2\sigma_2 = \gamma_1\sigma_1.$$

This holds true if and only if $\theta \in \Gamma\sigma_1\sigma_2^{-1}\Gamma$. Consequently

$$X^{\Gamma\sigma\Gamma} \in \mathcal{L}(G) \cap l^2(\Gamma\sigma\Gamma).$$

We analyze the product of two elements $X^{\Gamma\sigma_1\Gamma}$ and $X^{\Gamma\sigma_2\Gamma}$ as in formula (37), corresponding to two double cosets. The product corresponds to a pairing of cosets in the product of the double cosets. Because $(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2}$ is a matrix unit, the same pairing of cosets shows up in the product formula for the corresponding double cosets $[\Gamma\sigma_1\Gamma]$ and $[\Gamma\sigma_2\Gamma]$ in the Hecke algebra $\mathcal{H}_0$.

This proves that the operators $X^{\Gamma\sigma\Gamma}$, $\sigma \in G$, which are a priori affiliated to $\mathcal{L}(G)$, have also the property that the correspondence

$$[\Gamma\sigma\Gamma] \to X^{\Gamma\sigma\Gamma}, \quad \sigma \in G \tag{38}$$

extends to a $*$-algebra representation of $\mathcal{H}_0 = \mathbb{C}(\Gamma \setminus G/\Gamma)$. Obviously, this is trace-preserving with respect to the traces τ on $\mathcal{L}(G)$ and the canonical trace $\langle \cdot [\Gamma], [\Gamma] \rangle$ on $\mathcal{H}_0$. Consequently, the $*$-representation in formula (38) extends to a C^*-representation of the reduced C^*-Hecke algebra

$$\mathcal{H} = C^*_{\text{red}}(\Gamma \setminus G/\Gamma) = \overline{\mathcal{H}_0}^{\|\cdot\|} \subseteq B(\ell^2(\Gamma \backslash G)).$$

Hence the elements

$$t^{\Gamma\sigma\Gamma} = X^{\Gamma\sigma\Gamma}, \quad \sigma \in G$$

are bounded.

For all $\sigma \in G$, we denote the coefficient of $\theta \in \Gamma\sigma\Gamma$ in $t^{\Gamma\sigma\Gamma} \in \mathcal{L}(G)$ by $t(\theta)$. Thus

$$t^{\Gamma\sigma\Gamma} = \sum_{\theta \in \Gamma\sigma\Gamma} t(\theta)\theta \in \mathcal{L}(G), \quad \sigma \in G.$$

The property that

$$\chi_{\Gamma\alpha} t^{\Gamma\alpha\beta^{-1}\Gamma} \chi_{\Gamma\beta} t^{\Gamma\beta\gamma^{-1}\Gamma} \chi_{\Gamma\gamma} = \chi_{\Gamma\alpha} t^{\Gamma\alpha\gamma^{-1}\Gamma} \chi_{\Gamma\alpha}$$

implies, when moving in the left side member the characteristic function $\chi_{\Gamma\beta}$ to the right (using the multiplication formula (80, Section 7)), a family of identities of the form

$$\sum t^{A_i} t^{B_i} = \sum t^{C_j}.$$

These family of identities, when summing over unions of cosets of subgroups in $\mathcal{S}$ whose unions are Γ-cosets, produces exactly the family of identities

$$t^{\sigma_1\Gamma} t^{\Gamma\sigma_2} = t^{\sigma_1\Gamma\sigma_2}, \quad \sigma_1, \sigma_2 \in G.$$

These identities are exactly the sufficient conditions that imply, as recalled in the introductory part of this section (see [26]), that the map $[\sigma\Gamma] \to t^{\sigma\Gamma}$, $\sigma \in G$, extends to a representation of the operator system $\mathcal{SO}$, as in property (ii) in the statement.

To prove (iii) $\Rightarrow$ (iv), we note that

$$u(\sigma) = (v_{\Gamma,\Gamma\sigma} \otimes 1) \cdot (1 \otimes \sigma^{-1}) \in \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)), \quad \sigma \in G. \tag{39}$$

Hence, using formula (25) it follows that

$$v_{\Gamma,\Gamma\sigma} = \chi_\Gamma(t^{\Gamma\sigma\Gamma})\chi_{\Gamma\sigma^{-1}} \tag{40}$$

is an isometry in the von Neumann algebra $\mathcal{M}$, with initial space χ_Γ and range the space of the projection $\chi_{\Gamma\sigma^{-1}}$.

We define

$$v_{\Gamma\sigma_1,\Gamma\sigma_2} = \beta_{\sigma_1}(v_{\Gamma,\Gamma\sigma_2\sigma_1^{-1}}), \quad \Gamma\sigma_2 \in \Gamma \setminus G. \tag{41}$$

By G^{op}-equivariance and because of formula (40), the expression in (41) is equal to

$$\chi_{\Gamma\sigma_1} t^{\Gamma\sigma_1\sigma_2^{-1}\Gamma} \chi_{\Gamma\sigma_2}.$$

Again, because of the G^{op}-equivariance, this is a partial isometry from $\chi_{\Gamma\sigma_2}$ onto $\chi_{\Gamma\sigma_1}$.

The property that the family of unitaries $(u(\sigma))_{\sigma\in G}$ is a representation of G translates into the fact that the family

$$(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G},$$

defined in formula (41), is a matrix unit. Since G^{op} acts on $\mathcal{M}$ by leaving $\mathcal{L}(G)$ invariant, and since it acts through by the Koopmann unitary representation by right translations on $L^\infty(\mathcal{G}, \mu)$, it follows that the above above matrix unit is G^{op}-equivariant.

The proof of the implication (i) $\Rightarrow$ (iii) is postponed to Section 5. A direct, alternative proof, valid also for $D_\pi \neq 1$, will be given in Theorem 35 in Section 7. $\square$

Proof of Corollary 7. Recall that

$$P=\chi_K\mathcal{M}\chi_K=\chi_{\overline{\Gamma}}(\mathcal{L}(G\rtimes L^\infty(\mathcal{G},\mu)))\chi_{\overline{\Gamma}}=\mathcal{L}(\Gamma\rtimes L^\infty(K,\mu)).$$

Here the unit of the above algebra is identified with the characteristic function $\chi_{\overline{\Gamma}}$. The construction of the G^{op}-equivariant splitting (29) is straightforward, once a G^{op}-equivariant matrix unit is given.

For $p\in P$, we define

$$\alpha_g(p)=v_{\Gamma,\Gamma g}\beta_g(p)v_{\Gamma g,\Gamma}. \tag{42}$$

This is consistent with the fact that $\beta_g(p)$ belongs to $\chi_{\Gamma g}\mathcal{M}\chi_{\Gamma g}$, for all $p\in P$ and $g\in G$. □

Remark 12. The relation between the various constructions in the preceding statement is summarized as follows. As above, let π be a (projective) unitary representation of G. Consider the representation t of the operator system $\mathcal{SO}$ introduced in formula (24). Then
(i). The formula for the (projective) unitary representation u of G into $\mathcal{B}_\infty$ is

$$u(\sigma)=\chi_K(t^{\Gamma\sigma\Gamma}\otimes\sigma^{-1})\chi_K,\quad \sigma\in G.$$

(ii). The formula that gives back the representation π from t (or u) is

$$\pi(\sigma)(x)=E^{\mathcal{L}(G)}_{\mathcal{L}(\Gamma)}(t^{\Gamma\sigma\Gamma}x\sigma^{-1}),\quad x\in l^2(\Gamma),\sigma\in G.$$

(iii). In the identification $\chi_K\mathcal{M}\chi_K\cong\mathcal{L}(\Gamma\rtimes L^\infty(K,\mu))$, the homomorphism α coincides with the homomorphisms α constructed in the introduction, using the simultaneous infinite Jones basic construction.
(iv) The restriction $\alpha|_\Gamma$ acts as the identity operator on the subalgebra

$$\mathcal{L}(\Gamma)\subseteq\mathcal{L}(\Gamma\rtimes L^\infty(K,\mu)).$$

Proof. The only non-obvious property is contained in (iv). We use the notation from the proof of the previous statement. Since G^{op} acts trivially on G, it follows that β_g acts trivially on $\mathcal{L}(\Gamma)$ for $g\in G$. It follows that α_γ acts trivially on $\mathcal{L}(\Gamma)$ for γ in Γ. Indeed, in this case, for every

$$x\in\mathcal{L}(\Gamma)\subseteq\chi_{\overline{\Gamma}}\mathcal{L}(\Gamma\rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}},$$

and for every $\gamma\in\Gamma^{\mathrm{op}}$, we have that

$$\alpha_\gamma(x)=v_{\Gamma,\Gamma}\beta_\gamma(x)v_{\Gamma,\Gamma}=x,$$

showing that $\alpha|_\Gamma$ acts identically on $\mathcal{L}(\Gamma)$. □

In the proof of Theorem 6 we showed that there exists a correspondence between representations π of G with the properties from (i) of the preceding theorem and unitary representations u of G, having an expression as in formula (39), with values in the subgroup of elements of $\mathcal{U}(\mathcal{B}_\infty)$ that are normalize $\mathcal{A}_\infty$. We state this separately in the following proposition.

Proposition 13. *We use the notation and the equivalent hypothesis from Theorem 6. Recall that*

$$\mathcal{A}_\infty = \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu)), \quad \mathcal{B}_\infty = \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)).$$

We embed G into the first component of $G \times G^{\mathrm{op}}$, and canonically extend this embedding to an embedding of $\mathcal{A}_\infty$ into $\mathcal{B}_\infty$. Let u be the (projective) unitary representation of G into $\mathcal{U}(\mathcal{B}_\infty)$, constructed in formula (39). Then
(i) For every $\sigma \in G$ the unitary $u(\sigma)$ normalizes $\mathcal{A}_\infty$.
(ii) The homomorphism α associated to π, constructed in Corollary 7 and mapping G into $\mathrm{Aut}(P = \mathcal{A}_\infty)$, is computed by the formula:

$$\alpha_\sigma = \mathrm{Ad}\, u(\sigma)|_{\mathcal{A}_\infty}, \quad \sigma \in G. \tag{43}$$

(iii) The homomorphism α extends to a homomorphism of $\mathcal{G}$ into $\mathrm{Aut}(\mathcal{A}_\infty)$.

Proof. We use the notation from the previous theorem and its proof. We recall that the formula (see formulae (65), (39)) for $u(\sigma)$ is

$$\begin{aligned} u(\sigma) &= \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma} \otimes 1)(1 \otimes \sigma^{-1})\chi_{\overline{\Gamma}} \\ &= \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma} \otimes 1)\chi_{\overline{\Gamma\sigma}}(1 \otimes \sigma^{-1}) = (v_{\Gamma,\Gamma\sigma} \otimes 1)(1 \otimes \sigma^{-1}), \quad \sigma \in G. \end{aligned}$$

Thus, for $x = \chi_{\overline{\Gamma}} x \chi_{\overline{\Gamma}} \in \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))$, which is identified to

$$\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu)) \otimes 1,$$

we have that $u(g)xu(g)^*$ is equal, with the above identification, to

$$u(g)(x \otimes 1)u(g)^* = (v_{\Gamma,\Gamma\sigma} \otimes 1)\chi_{\Gamma\sigma}(1 \otimes \sigma^{-1})[(\chi_{\overline{\Gamma}} x \chi_{\overline{\Gamma}}) \otimes 1](1 \otimes \sigma)(v_{\Gamma\sigma,\Gamma} \otimes 1).$$

This is thus equal to

$$(v_{\Gamma,\Gamma\sigma}\beta_\sigma(x)v_{\Gamma\sigma,\Gamma}) \otimes 1,$$

and so the unitary $u(\sigma) \in \chi_{\overline{\Gamma}}(\mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G})))\chi_{\overline{\Gamma}}$ normalizes P (which is identified to $P \otimes 1$) and

$$\alpha_\sigma(x) = \mathrm{Ad}u(\sigma)(x), \quad x \in P, \sigma \in G.$$

This completes the proof of properties (i) and (ii) in the statement.

To prove (iii) we proceed as follows. For $\sigma \in G$, let χ_{K_σ} be the characteristic function of the subgroup K_σ of K introduced in formula (10). Then the convolutor in $C^*(\mathcal{G})$ by the function $\mu(K_\sigma)^{-1}\chi_{K_\sigma}$ is a projection, denoted by p_σ. Let $\mathcal{A}_{\Gamma\sigma}$ be the subfactor introduced in formula (11). To extend α to a homomorphism of the locally compact, totally disconnected group $\mathcal{G}$, one represents the projection p_σ in $C^*(\mathcal{G})$ by the conditional expectation $E^{\mathcal{A}_\infty}_{\mathcal{A}_{\Gamma\sigma}}$. $\square$

3. The operator algebra representation of the Hecke algebra associated to the representation $\pi \otimes \overline{\pi}$. Proof of Theorem 8

In this section we describe the relation between the construction in the previous theorem and the construction of Hecke operators in [26].

First we prove that the Hecke operators associated to the unitary representation of G,

$$\pi \otimes \overline{\pi} \cong \mathrm{Ad}\,\pi,$$

constructed in [26], where π is as in part (i) of Theorem 6, may be constructed directly using a splitting of the ergodic action of the group $G \times G^{\mathrm{op}}$ on $\mathcal{G}$ as in part (iv) in Theorem 6, formulae (26) and (27). Let α be the homomorphism of G into the automorphism group of $P = \mathcal{L}(\Gamma \rtimes K)$, constructed in Corollary 7.

Proof of Theorem 8. Note that the definition of $\Psi_{\Gamma\sigma\Gamma}$ is independent of the choice of σ in $\Gamma\sigma\Gamma$, since by property (iv) in Remark 12 the homomorphism $\alpha|_\Gamma$ acts as the identity on $\mathcal{L}(\Gamma)$.

We also note that because $\mathcal{L}(\Gamma)$ is the space of Γ-invariant vectors in $P = \mathcal{L}(\Gamma \rtimes \mathcal{L}^\infty(K,\mu))$ for the homomorphism α, it follows that formula (28) defines indeed a representation of the Hecke algebra.

Recall that by formula (42) in the proof of Corollary 7 we have

$$\alpha_g(p) = v_{\Gamma,\Gamma g}\beta_g(p)v_{\Gamma g,\Gamma}, \quad p \in P.$$

Because of formula (43), we obtain that for all $\sigma \in G$ and

$$x \in \mathcal{L}(\Gamma) \subseteq \chi_{\overline{\Gamma}}\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}},$$

we have

$$\alpha_\sigma(x) = \chi_{\overline{\Gamma}}t^{\Gamma\sigma\Gamma}\chi_{\Gamma\sigma}x\chi_{\Gamma\sigma}t^{\Gamma\sigma\Gamma}\chi_{\overline{\Gamma}}.$$

Note that the last expression depends only on the coset $\Gamma\sigma \in \Gamma \setminus G$.

In particular, for every

$$x \in \mathcal{L}(\Gamma) = \chi_{\overline{\Gamma}}\mathcal{L}(\Gamma)\chi_{\overline{\Gamma}} \subseteq \chi_{\overline{\Gamma}}\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))\chi_{\overline{\Gamma}} = \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu)),$$

we obtain that

$$E^{\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))}_{\mathcal{L}(\Gamma)}(\alpha_\sigma(x)) = \sum_i \chi_{\overline{\Gamma}}t^{\Gamma\sigma\Gamma}\chi_{\Gamma\sigma s_i}x\chi_{\Gamma\sigma s_i}t^{\Gamma\sigma\Gamma}\chi_{\overline{\Gamma}}. \tag{44}$$

Here, the families (s_i), (t_i) are the Pimsner-Popa bases used in the proof of (i) $\Rightarrow$ (iii) in Theorem 6 from Section 5.

Then, the right hand term in the equation (44) is

$$\chi_{\overline{\Gamma}} t^{\Gamma\sigma\Gamma} x t^{\Gamma\sigma\Gamma} \chi_{\overline{\Gamma}}.$$

This expression is further equal to the value at x of the completely positive map constructed in [26], in correspondence with a representation π as in statement (i) of the previous theorem. $\square$

Consequently, the problem of determining the continuity of the representation Ψ of the Hecke algebra with respect to the norm on the C^*-reduced Hecke algebra ([5]), which is the essence of the Ramanujan-Petersson problem ([26]), is reduced to the analysis of the unitary representation induced by α on the Hilbert space $L^2(P,\tau) \ominus \mathbb{C}1$.

We recall that in Proposition 13 we introduced the following notation:

$$\mathcal{A}_\infty = \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu)), \quad \mathcal{B}_\infty = \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)).$$

We use the trivial embedding $G \times \{e\} \subseteq G \times G^{\mathrm{op}}$. We consider the reduced crossed product von Neumann algebras

$$\mathcal{C}_\infty = \mathcal{L}(G \rtimes L^\infty(\mathcal{G},\mu)) \subseteq \mathcal{D}_\infty = \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G},\mu)).$$

Note that

$$\mathcal{A}_\infty = \chi_K \mathcal{C}_\infty \chi_K \;; \quad B_\infty = \chi_K \mathcal{D}_\infty \chi_K. \tag{45}$$

We also consider the algebra $D \subseteq L^\infty(\mathcal{G},\mu) \subseteq \mathcal{C}_\infty$ consisting of left K-invariant functions in $L^\infty(\mathcal{G},\mu)$. Obviously $D \cong \ell^\infty(\Gamma\backslash G)$.

By $\rho_{\Gamma\backslash G}$ we denote the right regular representation of G into the unitary group associated with the Hilbert space $\ell^2(\Gamma\backslash G)$. With $\rho = \rho_G$ we denote the right regular representation of G. For $g \in G$, we identify the unitary element ρ_g with the unitary, left convolution operator in $\mathcal{L}(G^{\mathrm{op}}) \subseteq \mathcal{D}_\infty$, that corresponds to g. Then

$$\beta_g = \mathrm{Ad}(\rho_g), \quad g \in G, \tag{46}$$

defines an homomorphism from G^{op} into $\mathrm{Aut}(\mathcal{C}_\infty)$.

It is obvious that every β_g acts on $\mathcal{C}_\infty$ by right translation by g on $L^\infty(\mathcal{G},\mu)$. Each automorphism β_g acts identically on $\mathcal{L}(G)$.

We introduce below a different equivalent formulation to the four equivalent statements in Theorem 6. This is used to describe the homomorphism α introduced in Corollary 7, in a form that is independent of the choices made in the tensor product splitting defining α.

Lemma 14. *The equivalent statements in Theorem 6 are further equivalent to the following statement:*
(v) There exists a unitary representation $\theta : G \to \mathcal{U}(\mathcal{C}_\infty)$ with the following properties:

(v.1) The unitary representation θ of G is unitarily equivalent to the representation $\rho_{\Gamma\backslash G}$.

(v.2) For all g in G, the unitary element $\theta(g^{-1})\rho(g)$ belongs to the relative commutant $D' \cap \mathcal{C}_\infty$. Equivalently,

$$\mathrm{Ad}\theta_g|_D = \mathrm{Ad}\rho_g|_D, \quad g \in G.$$

(v.3) The unitaries

$$W_g = \theta(g^{-1})\rho(g), \quad g \in G,$$

define a unitary representation of G into the type II_1 von Neumann algebra

$$P_0 = \{\theta(G)\}' \cong \mathcal{A}_\infty.$$

Proof. We use the notation from the statement and proof of Theorem 6. We first prove that the equivalent conditions (i)-(iv) imply condition (v). Consider the selfadjoint elements constructed in formula (37). Because these elements generate a G^{op}-equivariant matrix unit, it follows that the formula

$$\theta_g = \sum_{\Gamma\sigma\in\Gamma\backslash G} \chi_{K\sigma g} X^{\Gamma\sigma\Gamma} \chi_{K\sigma}, \quad g \in G, \tag{47}$$

defines a unitary representation of G into $\mathcal{C}_\infty$ that is unitary equivalent to the left regular quasi-representation $\rho_{\Gamma\backslash G}$.

For $g \in G$, both $\mathrm{Ad}\rho_g$ and $\mathrm{Ad}\ \theta_g$ normalize D and induce the same action on the algebra D.

It remains to prove property (v.3). Formula (47) implies that

$$\rho_h(\theta_g)\rho_{h^{-1}} = \theta(hgh^{-1}), \quad h, g \in G. \tag{48}$$

The above formula obviously implies that

$$\theta(h^{-1})\rho_h \in \theta(G)', \quad h \in G,$$

and also that that $h \to \theta(h^{-1})\rho_h$ is a unitary representation of G. This completes the proof of the direct implication.

To prove the converse, we note that the unitary representation in part (iii) of Theorem 6 may be defined by the formula

$$u(g) = \chi_K W_g \chi_K, \quad g \in G.$$

□

We summarize the results from the previous lemma in the context of Corollary 7.

Lemma 15. *With the above introduced notation we define*

$$\widetilde{\beta}_g = \mathrm{Ad}(W_g)|_{\mathcal{C}_\infty}, \quad g \in G.$$

Then, we have the following G-invariant isomorphism:

$$\mathcal{C}_\infty \cong \mathcal{A}_\infty \otimes B(\ell^2(\Gamma \backslash G)).$$

Using the above identification and the homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\infty)$ introduced in Proposition 7, we obtain:

$$\mathrm{Ad}(\theta_g)\beta_g = \widetilde{\beta}_g = \alpha_g \otimes \mathrm{Id}_{B(\ell^2(\Gamma\backslash G))}.$$

We provide now a "coordinate free" description of the homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\infty)$. This is important because, as also mentioned above, the Ramanujan-Petersson problem for the action of the Hecke operators on Γ-invariant vectors for the unitary representation $\pi \otimes \overline{\pi}$ is equivalent to the weak containment of the unitary representation of G induced by α on $L^2(\mathcal{A}_\infty, \tau) \ominus \mathbb{C}1$, in the restriction to G of the unitary left regular representation of the Schlichting completion $\mathcal{G}$ of G.

Recall that the action of β_g on $\mathcal{C}_\infty$ is defined as follows. The automorphism β_g acts by right translation by g on $L^\infty(\mathcal{G}, \mu)$, and it acts identically on $\mathcal{L}(G)$.

Proof of Corollary 9. Parts (i), (ii) follow immediately from Lemma 14 and its proof (see formula (48)). Property (iii) follows from the representation in Lemma 15. This is because the homomorphisms $(\beta_g|_{\mathcal{A}})_{g\in G}$ and α are the same. The difference between the two cases is that in the case considered in this statement we no longer perform a splitting procedure to find explicitly the factor $\mathcal{A}$. On the other hand, in the present context the factor

$$\mathcal{L}(\Gamma) \cong \mathcal{L}(\Gamma \times \{e\}) \subseteq \mathcal{C}_\infty$$

is canonically embedded in the commutant algebra Θ', which by definition is equal to $\mathcal{A}$. The formula in part (iii) is then a direct consequence of the corresponding formula in Theorem 8.

To prove the uniqueness in part (iv), note that if θ^1 is another representation as in (i), then necessarily there exists a unitary $w \in D'$ such that

$$\theta^1_g = w\theta_g w^*, \quad g \in G.$$

We write the second condition in (b) for the representation θ^1. It follows that

$$\beta_h(w)\beta_h(\theta_g)\beta_h(w^*) = w\theta_{hgh^{-1}}w^*, \quad g,h \in G.$$

Hence

$$c(g) := w^*\beta_h(w) \in \Theta' = \mathcal{A}, \quad g \in G.$$

Thus c is defining a cocycle in the cohomology group $H^1_\beta(G,\mathcal{U}(\mathcal{A}))$. Its triviality would imply that $w\mathcal{A}w^* = \mathcal{A}$. □

Remark 16. Let π be a representation as in part (i) of Theorem 6. Since $\pi|_\Gamma$ is unitarily equivalent to the left regular representation of Γ, we identify the Hilbert space of the representation π with $\ell^2(\Gamma)$. We may also assume that the representation t constructed in formula (30) has the property that $t(\Gamma\sigma\Gamma)$ belongs to the full C^*-algebra $C^*(G)$ for all σ in G. This is possible by choosing a suitable cyclic trace vector in the formula defining t (see [26]).

Let $C^*((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G},\mu))$ be the full crossed product C^*- algebra and let $C^*(G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)$ be the full groupoid crossed product C^*-algebra. Let Π be the obvious representation of $C^*(G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)$ into $B(\ell^2(\Gamma))$. Then:

(i) The formula for the representation u in part (iii) of Theorem 6 defines a representation $\boldsymbol{U}$ of the group G, with values in the unitary group of the $C^*(G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)$. The explicit formula for $\boldsymbol{U}$ is

$$\boldsymbol{U}(\sigma) = \chi_K(t^{\Gamma\sigma\Gamma} \otimes \sigma^{-1})\chi_K. \tag{49}$$

(ii) The correspondence $\boldsymbol{\Psi}$ defined by

$$\Gamma\sigma\Gamma \to \chi_K(t^{\Gamma\sigma\Gamma} \otimes t^{\Gamma\sigma^{-1}\Gamma})\chi_K, \quad \sigma \in G, \tag{50}$$

extends by linearity to a unital $*$-representation of the Hecke algebra $\mathcal{H}_0$ into $C^*(G \times G^{\mathrm{op}}) \rtimes L^\infty(K,\mu)$.

(iii) Composing the representations $\boldsymbol{U}, \boldsymbol{\Psi}$ with representation Π, one obtains the representation π, and respectively the representation Ψ, of the Hecke algebra into $B(\ell^2(\Gamma))$ constructed in Theorem 8

Proof. The statements (i), (ii) follow from the results in Section 7. The last statement is a direct consequence of the formula (28) and of Remark 12. □

4. THE ALGEBRAIZATION OF THE SPACE OF INTERTWINERS AND JONES'S BASIC CONSTRUCTION FOR A PAIR OF ISOMORPHIC FACTORS

We first introduce an abstract setting that serves the purpose of describing the spaces of intertwiners between subalgebras of the form

$$\pi(\Gamma_0)', \quad \Gamma_0 \in \mathcal{S}.$$

Let $\sigma \in G$, and let (if the 2-cocycle ε is trivial)

$$\theta_\sigma : \Gamma_{\sigma^{-1}} \to \Gamma_\sigma,$$

be the group homomorphism implemented on $\Gamma_{\sigma^{-1}}$ by the adjoint map $\sigma \cdot \sigma^{-1}$ of the group element σ. We obviously have the intertwining property

$$\pi(\sigma)\pi(\gamma_0) = \pi(\theta_\sigma(\gamma_0))\pi(\sigma), \quad \gamma_0 \in \Gamma_{\sigma^{-1}}.$$

If $\varepsilon \in H^2(2, \mathbb{T})$ is non-trivial we use the setting introduced in Definition 3.

To describe the space of all intertwiners with properties as above, we introduce a construction similar to Jones's basic construction. Instead of working with a single subfactor, we start with a pair of subfactors of equal index, with a fixed isomorphism θ, mapping one subfactors onto the other one.

As in the case of Jones's simultaneous construction for the infinite family of subgroups $\Gamma_0 \in \mathcal{S}$, where the result was the inductive limit factor $\mathcal{A}_\infty$ in formula (9), we also perform an infinite simultaneous construction for the pairs of subgroups.

The inductive limit for the space of von Neumann algebra intertwiners between subgroups in $\mathcal{S}$ is the type II_1 factor

$$\mathcal{B}_\infty = \chi_K(\mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_K. \tag{51}$$

Then $\mathcal{B}_\infty$ contains $\mathcal{A}_\infty$. By construction, the II_1 factor $\mathcal{B}_\infty$ encodes the algebra structure of the spaces of intertwiners, corresponding to all subgroups in $\mathcal{S}$.

In the above correspondence between the space of intertwiners and the algebra $\mathcal{B}_\infty$, the (projective) unitary representation π then corresponds to a (projective) unitary representation u of G into the unitary group of the normalizer of $\mathcal{A}_\infty$ in $\mathcal{B}_\infty$. We have already observed (see Proposition 13, formula (43)) that

$$\mathrm{Ad}\, u(\sigma)[\mathcal{A}_\infty] = \mathcal{A}_\infty, \quad \sigma \in G.$$

The homomorphism α is obtained as the restriction to $\mathcal{A}_\infty$ of the adjoint representation $\mathrm{Ad}\, u$ of G:

$$\alpha(\sigma) = \mathrm{Ad}\, u(\sigma)|_{\mathcal{A}_\infty}, \quad \sigma \in G.$$

In the following lemma we present an abstract formalism which establishes a correspondence between intertwiners, such as $\pi(\sigma)$, $\sigma \in G$, of the algebras $\pi(\Gamma_{\sigma^{-1}})'$ and $\pi(\Gamma_\sigma)'$, with elements in the algebra associated to the crossed product introduced in formula (51).

We make essential use of the abstract formalism of spaces of intertwiners to see that the composition operation of two intertwiners corresponds to the product operation in the crossed product algebra $\mathcal{B}_\infty$, introduced in formula (51).

The above correspondence between the two product operations is used in the proof of (i) $\Rightarrow$ (iii) in Theorem 6. It shows that the expression in formula (64) defines a unitary representation of G. For an alternative verification, one could also use Theorem 35 (i) in Section 7.

The following two statements are probably known to specialists in subfactor theory. As we did not find a reference, we state and prove them directly.

We perform a construction which is analogous with Jones's construction of the first step in the basic construction ([16]). Recall that in that case of a single factor $N \subseteq M$, the first term of the basic construction is the algebra

$$\langle M, e_N \rangle = M e_N M^{\mathrm{op}} \cong N' \subseteq B(L^2(M, \tau)).$$

Here we reproduce Jones's basic construction for a pair of isomorphic subfactors of equal index. For a pair of isomorphic subfactors N_0, N_1, we replace the commutant algebra $N' \cong M e_N M$ with the space of intertwiners. This space also carries a natural Hilbert structure, which we describe below.

The following definition and lemma are also the subject of Appendix 1 in [26]. We reproduce the proofs from that paper in the actual context.

Definition 17. Let M be a II_1 factor with trace τ. Let N_0, N_1 be a pair of finite index subfactors of M, having the same index $[M : N_0] = [M : N_1]$. Assume we are given a fixed isomorphism $\theta : N_0 \to N_1$. Let e_{N_0}, e_{N_1} be the corresponding Jones projections onto the subfactors N_0, N_1. We introduce the following constructions:
(i) The space of θ-intertwiners is defined by the formula:

$$\mathrm{Int}_\theta(N_0, N_1) = \{X \in B(L^2(M, \tau)) \mid X n_0 = \theta(n_0) X,\ n_0 \in N_0\}.$$

(ii) Define $W_\theta : L^2(N_0, \tau) \to L^2(N_1, \tau)$ by

$$W_\theta(n_0) = \theta(n_0), \quad n_0 \in N_0.$$

Viewed as an element of $B(L^2(M, \tau))$, W_θ is a partial isometry mapping the projection e_{N_0} onto the projection e_{N_1}.

In analogy with the Hilbertian M-bimodules $M e_{N_0} M$ and $M e_{N_1} M$, which are the Jones basic construction terms from the inclusions $N_i \subseteq M$, $i = 0, 1$, we construct a Hilbertian M-bimodule $M W_\theta M$, corresponding to the pair of isomorphic subfactors.

Definition 18. The Hilbertian M-bimodule $MW_\theta M$ is obtained from the free M-bimodule $M \times M$, by taking the quotient corresponding to the relations:
(i)

$$W_\theta n_0 = \theta(n_0) W_\theta, \quad n_0 \in N_0, \tag{52}$$

(ii)

$$W_\theta = W_\theta e_{N_0} = e_{N_1} W_\theta. \tag{53}$$

(iii) The elements $mW_\theta m' \in MW_\theta M$ depend only on me_{N_1} and $e_{N_0}m'$.
(iv) The following formula defines a scalar product on $MW_\theta M$.

$$\langle mW_\theta m', aW_\theta a'\rangle = \tau(a^* m\theta(E_{N_0}(m'(a')^*), \quad m, m', a, a' \in M. \tag{54}$$

In the previous definition, we have to justify the compatibility of the four conditions that describe the subspace defining the quotient of the free module. This is done in the following lemma:

Lemma 19. *We use the notation from the previous statement. The assumption (iv) means implicitly that we are factorizing the free bimodule $M \times M$ by the vector space corresponding to vectors of zero norm with respect to the scalar product introduced in formula (54). The definition of the corresponding scalar product is compatible with the conditions (i)-(iii).*

Proof. Let $m, m', a, a' \in M$, $n_1 \in N_1$. Using the scalar product introduced in formula (54), we have to prove that

$$\langle mn_1 W_\theta m' - mW_\theta \theta^{-1}(n_1) m', aW_\theta a'\rangle = 0.$$

We have

$$\begin{aligned} \langle mn_1 W_\theta m', aW_\theta a'\rangle &= \tau(a^* mn_1 \theta(E_{N_0}(m'(a')^*) \\ &= \tau(a^* m\theta(\theta^{-1}(n_1))\theta(E_{N_0}(m'(a')^*))) \\ &= \tau(a^* m\theta(\theta^{-1}(n_1))(E_{N_0}(m'(a')^*))) \\ &= \tau(a^* m\theta(E_{N_0}(\theta^{-1}(n_1)m'(a')^*))). \end{aligned}$$

Here we use the fact that E_{N_0} is a conditional expectation and that $\theta^{-1}(n_1)$ belongs to N_0. □

Note that the scalar product corresponds exactly to the Stinespring dilation of the completely positive map $m \to \theta(E_{N_0}(m))$, viewed as a map from M with values into $N_1 \subseteq M$.

Definition 20. We refer to the above definitions. Let V be any unitary operator in $\mathrm{Int}_\theta(N_0, N_1)$. Then

$$\mathrm{Int}_\theta(N_0, N_1) = V(N_0)' = (N_1)'V.$$

Hence $\mathrm{Int}_\theta(N_0, N_1)$ carries a canonical scalar product, induced by the trace on the commutant algebra N_0', or equivalently by the trace on the commutant algebra N_1'. Let $L^2(\mathrm{Int}_\theta(N_0, N_1))$ be the Hilbert space completion. This is a Hilbertian M-bimodule.

Using the above definition, we prove in this more general setting that the correspondence in the Jones basic construction for a subfactor inclusion $N \subseteq M$ between N' and $Me_N M$, holds true also in the case of a pair of isomorphic subfactors.

Lemma 21. *Let $L^2(MW_\theta M)$ be the Hilbert space completion of $MW_\theta M$ with respect to the scalar product introduced in formula (54). Then:*
(i) The Hilbertian M-bimodules $L^2(\mathrm{Int}_\theta(N_0, N_1))$ and $L^2(MW_\theta M)$ are isomorphic by the following the M-bimodule anti-linear map:

$$\Phi_\theta : \mathrm{Int}_\theta(N_0, N_1) \to L^2(MW_\theta M),$$

defined by the relation

$$\langle mW_\theta m', \Phi_\theta(X)\rangle = \tau(X(m')m), \quad m, m' \in M. \tag{55}$$

(ii) Φ_θ extends to an isometry on the Hilbert space completion of $\mathrm{Int}_\theta(N_0, N_1)$, with respect to the given scalar products.

Proof. To prove (i), by bijectivity we may instead verify the converse. We have to check that, with the definition in formula (55),

$$X(n_0 m) = \theta(n_0)X(m), n_0 \in N_0, \quad m \in M.$$

By taking a trace of a product with an element $m' \in M$, we have to check that

$$\tau(X(n_0 m)m') = \tau(X(m)m'\theta(n_0)).$$

By using the above definition of $\Phi_\theta(X)$ this amounts to

$$\langle m'\sigma n_0 m, \theta(X)\rangle = \langle m'\theta(n_0)\sigma m, \theta(X)\rangle, \quad n_0 \in N, m, m' \in M.$$

This is true from the definition of the bimodule property of $MW_\theta M$. □

If $N^1 \subseteq N \subseteq M$ is a chain of subfactors, then the inclusion

$$N' \subseteq (N^1)',$$

may be interpreted in terms of the Jones bimodules structure of the bimodules associated to the basic construction as an inclusion

$$Me_{N^1}M \subseteq Me_N M.$$

We extend this to the case of pairs of subfactors. For simplicity, we assume that all indices are integer valued.

Definition 22. We use the definitions and notation assumed above. We restrict the isomorphism θ to an isomorphism of a smaller pair of subfactors $N_0^1 \subseteq N_0, N_1^1 \subseteq N_1$ such that $\theta(N_0^1) = N_1^1$. Assume that the inclusions $N_i^1 \subseteq N_i$ have equal, integer index.

It is obvious that

$$\mathrm{Int}_\theta(N_0, N_1) \subseteq \mathrm{Int}_\theta(N_0^1, N_1^1). \tag{56}$$

Since the index of the inclusion $N_0^1 \subseteq N_0$ is an integer, we may find a Pimsner-Popa basis (r_j) for $N_0^1 \subseteq N_0$, consisting of unitaries. Let e_{N_0}, $e_{N_0^1}$ be the corresponding Jones projection. Then e_{N_0} is given by

$$e_{N_0} = \sum_j r_j e_{N_0^1} r_j^*.$$

Hence, since

$$W_{\theta|N_0^1} = W_\theta e_{N_0^1},$$

we have a formal inclusion

$$MW_\theta M = MW_\theta e_{N_0} M \subseteq MW_\theta e_{N_0^1} M = MW_{\theta|N_0^1} M. \tag{57}$$

Lemma 23. *The maps Φ_θ and $\Phi_{\theta|N_0^1}$ are compatible with the inclusions in formulae (56) and (57)*

Proof. To prove part (ii), we have to check the compatibility with inclusion of the maps Φ. In the bimodule construction, this corresponds to replacing in the above bimodule the partial isometry W_θ by

$$W_\theta e_{N_0^1} = e_{N_1^1} W_\theta = W_{\theta|_{N_0^1}}.$$

When using the maps Φ_θ, $\Phi_{\theta|N_0^1}$ the above formal inclusion expresses exactly the compatibility of Φ_θ, $\Phi_{\theta|N_0^1}$ with the inclusion maps. $\square$

In the following lemma, we obtain an explicit formula for the map Φ_θ constructed above. As in the previous statements we assume that all subfactors have integer index. This condition is probably not necessary, but we consider it to have a simpler form of the statements.

Lemma 24. *In the context introduced above, let $k = [M : N_0]$ and assume that $(s_j)_{j=1}^k$ is a left Pimsner-Popa orthonormal basis for N_0 in M. Then Φ_θ has the following formula:*

$$\Phi_\theta(X) = \sum_j (X(s_j))^* W_\theta s_j, \quad X \in \mathrm{Int}_\theta(N_0, N_1). \tag{58}$$

Proof. We note that the decomposition

$$MW_\theta M^{\mathrm{op}} = \bigcup_j [MW_\theta s_j]$$

is orthogonal. Fix $X \in \mathrm{Int}_\theta(N_0, N_1)$. By the orthogonality property, we may assume that

$$\Phi_\theta(X) = \sum_j x_j W_\theta s_j.$$

By hypothesis, as left N_0-bimodule, M is the (N_0-orthogonal) sum of $N_0 s_j$. Then

$$X(n_0 s_j) = \theta(n_0) X(s_j), \quad n_0 \in N_0.$$

Denote $t_j = X(s_j)$. Then (t_j) is a Pimsner-Popa orthonormal basis for N_1 in M.

The relation between $\Phi_\theta(X)$ and X is:

$$\langle m_0 W_\theta m_1, \Phi_\theta(X) \rangle = \tau(X(m_1), m_0).$$

Hence

$$\langle X(m_1), m_0 \rangle = \langle m_0^* W_\theta m_1, \Phi_\theta(X) \rangle.$$

Hence, for a fixed j, taking $m_1 = s_j$ we obtain:

$$\langle t_j, m_0 \rangle = \langle X(s_j), m_0 \rangle = \langle m_0^* W_\theta s_j, \Phi_\theta(X) \rangle = \langle m_0^* W_\theta s_j, x_i W_\theta s_j \rangle.$$

Hence, it follows that for all $m_0 \in M$ we have

$$\langle t_j, m_0 \rangle = \langle m_0^*, x_j \rangle,$$

or that $\tau(t_j m_0^*) = \tau(x_j^* m_0^*)$ and hence $t_j = x_j^*$, leading to

$$\Phi_\theta(X) = \sum_j t_j^* W_\theta s_j.$$

□

We use the previous construction to transform the unitary representation π of G considered in part (i) of Theorem 6, into a unitary representation of G with values in the algebra $\mathcal{B}_\infty$. The next lemma translates for the product formula for intertwiners in the context of spaces of bimodules.

For simplicity, we assume from now on in this section that the factor M is a group von Neumann algebra (eventually skewed by a cocycle), and that all subfactors that are considered correspond to subgroups of finite index. We also assume that all isomorphisms between subfactors are induced by subgroup isomorphisms (this condition may certainly be relaxed, but in our main application this condition is verified anyway).

Lemma 25. *Consider two pairs of equal index subfactors N_0, N_1 and N_1, N_2 as above, and isomorphisms $\theta_0 : N_0 \to N_1$ and $\theta_1 : N_1 \to N_2$. Assume that there exists a pair (N_0^1, N_2^1) of equal finite index subfactors of M, $N_0^1 \subseteq N_0$, $N_2^1 \subseteq N_2$. Assume also that there exists an isomorphism θ_0^1 mapping N_0^1 onto N_2^1, such that the composition $\theta_1 \circ \theta_0$ is defined on N_0^1 and it is equal to θ_0^1. Then:*
(i) The composition operation

$$\mathrm{Int}_{\theta_0}(N_0, N_1) \times \mathrm{Int}_{\theta_1}(N_1, N_2) \to \mathrm{Int}_{\theta_0^1}(N_0^1, N_2^1) \tag{59}$$

is well-defined.

(ii) The product operation in formula (59) induces a product map

$$MW_{\theta_0}M \times MW_{\theta_1}M \to MW_{\theta_0^1}M, \tag{60}$$

compatible with the the maps $\Phi_{\theta_0}, \Phi_{\theta_1}, \Phi_{\theta_0^1}$ constructed in Lemma 21, corresponding to the composition of θ_1 with θ_0.

The image of $MW_{\theta_0}M \times MW_{\theta_1}M$ in (60) *is a sub-bimodule of $MW_{\theta_0^1}M$.*

Proof. The only verification that needs checking is the fact the product formula on products of bimodules corresponds to the composition of intertwiners. To verify the product formula, by linearity it is sufficient to consider simple intertwiners that map a set of coset representatives into a set of coset representatives. In this case the product formula is just a consequence of the enumeration of cosets for subgroups and of subgroups of subgroups □

Remark 26. More generally, one may consider a family, indexed by a set S, of pairs of equal index subfactors, along with isomorphisms mapping the first subfactors into the second:

$$N_0^s \xrightarrow{\theta_s} N_1^s, \quad s \in S.$$

By the technical assumption, the Jones projections for the subfactors N_0^s, N_1^s mutually commute for all $s \in S$.

We assume that for any two elements $\theta_{s_1}, \theta_{s_2}$ in the family $(\theta_s)_{s\in S}$, there exists a third element θ_{s_0} in the family such that the composition $\theta_{s_1} \circ \theta_{s_2}$ restricts to θ_{s_0}.

Let $\mathcal{B}_\infty$ be the union of all bimodules in the family. Then $\mathcal{B}_\infty$ has the associative $*$-algebra structure introduced in formula (60) . We define a trace on $\mathcal{B}_\infty$ by composing the trace on M with the projection onto the M-th component.

By performing the above construction, we obtain a von Neumann algebra $\mathcal{B}_\infty$, the simultaneous Jones's basic construction for the family of pairs of subfactors $(N_0^s \xrightarrow{\theta_s} N_1^s)_{s\in S}$. Then $\mathcal{B}_\infty$ is isomorphic to the algebra of all intertwiners in the family $(\theta_s)_{s\in S}$, containing all intertwiners $W_{\theta_s}, s \in S$, all the corresponding Jones projections, and the algebra M.

5. PROOF OF THE IMPLICATION (I) $\Rightarrow$ (III) IN THEOREM 6

In this section we complete the proof of Proposition 6 by using the construction from Section 4. A different proof follows from part (i) in Theorem 35. The latter proof has the advantage that it also works in the case $D_\pi \neq 1$. On the other hand, the realization of spaces of intertwiners in terms of bimodules, as in this section, will be used in Section 6, Proposition 29 (i), formula (73), to find a compact formula for cocycles for unitary perturbations of unitary representations as in part (iii) of Theorem 6.

Proof of the implication (i)⇒ (iii) in Theorem 6. To construct the unitary representation u of G into

$$\chi_{\overline{\Gamma}}(\mathcal{L}((G \times G^{\rm op}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_{\overline{\Gamma}},$$

we use Lemma 17 and Lemma 25 .

For every $\sigma \in G$ we consider the subgroup $\Gamma_\sigma \subseteq \Gamma$. If the 2-cocycle ε is trivial, we let the isomorphism

$$\theta_\sigma : \mathcal{L}(\Gamma_{\sigma^{-1}}) \to \mathcal{L}(\Gamma_\sigma)$$

be the conjugation by σ. In the general case (when ε is non-trivial we use the construction for θ_σ introduced in Definition 3). By taking inductive limit as in Remark 26, we obtain a type II$_1$ factor $\mathcal{B}_\infty$. We denote by W_{θ_σ} the isometry $W_\sigma = \theta_\sigma$ restricted to $\ell^2(\Gamma_{\sigma^{-1}})$ (for non-trivial ε we use the space $\ell^2_\chi(\Gamma_{\sigma^{-1}})$).

The bimodules constructed in Lemma 17 are of the form:

$$\mathcal{L}(\Gamma)(W_\sigma)e_{\Gamma_{\sigma^{-1}}}\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma)e_{\Gamma_\sigma}(W_\sigma)\mathcal{L}(\Gamma), \quad \sigma \in G.$$

We identify the partial isometry W_σ with the partial isometry

$$(\sigma \otimes \sigma^{-1}) e_{\Gamma_{\sigma^{-1}}} = e_{\Gamma_\sigma}(\sigma \otimes \sigma^{-1}), \quad \sigma \in G.$$

In this identification, the algebra $\mathcal{B}_\infty$ is generated by $\mathcal{L}(\Gamma) \otimes 1$, the partial isometries $(W_\sigma)_{\sigma \in G}$ and the Jones projections $(e_{\Gamma_\sigma})_{\sigma \in G}$.

Then $\mathcal{B}_\infty$ is isomorphic to a corner of the reduced von Neumann algebra crossed product $\mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu))$, as follows:

$$\mathcal{B}_\infty = \chi_{\overline{\Gamma}}(\mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_{\overline{\Gamma}}. \tag{61}$$

We recall that by Remark 11 it follows that using the groupoid action of $G \times G^{\mathrm{op}}$ on K, we obtain the isomorphism

$$\mathcal{B}_\infty \cong \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(K, \mu)), \tag{62}$$

with unit identified to $\chi_{\overline{\Gamma}}$.

For $\sigma \in G$, using the definition of the isomorphism θ_σ, we have that

$$\pi(\sigma) \in \mathrm{Int}_{\theta_\sigma}(\pi(\Gamma_{\sigma^{-1}}), \pi(\Gamma_\sigma)).$$

Let

$$s_i \in \mathcal{L}(\Gamma) \cong \pi(\Gamma)'', \quad i = 1, \dots, [\Gamma : \Gamma_\sigma],$$

be a system of Γ_σ-left coset representatives. More generally, we could only assume that the family $(s_i)$is a Pimsner-Popa basis for the subfactor inclusion $\mathcal{L}(\Gamma_\sigma) \subseteq \mathcal{L}(\Gamma)$. We let

$$t_i = \pi(\sigma) s_i, \quad i = 1, \dots, [\Gamma : \Gamma_\sigma].$$

With the identification for the space of intertwiners used in formula (58), each intertwining operator $\pi(\sigma)$, corresponds, because of Lemma 24, to

$$\sum_i t_i^* W_{\theta_\sigma} s_i \in \mathcal{B}_\infty. \tag{63}$$

Consequently, for all $\sigma \in G$, using the isomorphism from formula (62), the unitaries $\pi(\sigma)$ correspond to the unitary elements $u(\sigma)$ given by the formula:

$$\sum_i (t_i^* \otimes 1)(\sigma \otimes \sigma^{-1}) e_{\Gamma_{\sigma^{-1}}}(s_i \otimes 1) \in \chi_{\overline{\Gamma}}(\mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)))\chi_{\overline{\Gamma}}. \tag{64}$$

The above formula gives, for every $\sigma \in G$,

$$u(\sigma) = \sum_i \chi_{\overline{\Gamma}}(t_i^* \otimes 1)(\sigma \otimes \sigma^{-1})(s_i \otimes 1)\chi_{\overline{\Gamma}} = \sum_i \chi_{\overline{\Gamma}}(t_i^* \sigma s_i) \otimes \sigma^{-1} \chi_{\overline{\Gamma}}.$$

We denote by $t^{\Gamma\sigma\Gamma} \in \mathcal{L}(G)$ the sum $\sum_i t_i^* \sigma s_i$. Note that this is exactly the expression used in formula (30). Consequently, we have the following expression for the unitary $u(\sigma)$:

$$u(\sigma) = \chi_{\overline{\Gamma}}(t^{\Gamma\sigma\Gamma} \otimes 1)(1 \otimes \sigma^{-1})\chi_{\overline{\Gamma}}, \quad \sigma \in G. \tag{65}$$

Recall that $\sigma \to \pi(\sigma)$ is a unitary (projective) representation of G. Hence, by using the product operation introduced in formula (60) from Lemma 25, it follows that formula (65) defines a unitary representation u of G^{op} into

$$\mathcal{L}((G \times G) \rtimes L^\infty(K, \mu)) = \chi_{\overline{\Gamma}} \mathcal{L}((G \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu))\chi_{\overline{\Gamma}}.$$

We note that, a priori, in the above construction we are initially working with a larger C^*-norm on the crossed product C^*-algebra, coming from the full crossed product C^*-algebra

$$C^*((G \times G) \rtimes L^\infty(K, \mu)).$$

The initial choice of this C^*-algebra crossed product norm corresponds to the fact that we are working initially with a representation of the above groupoid crossed product C^*-algebra into $B(\ell^2(\Gamma))$. This representation is obtained by letting $G \times G^{\mathrm{op}}$ act by left and right multiplication operators on $\ell^2(\Gamma)$. This is due to the fact that the values of the unitary $u(\sigma)$ belong in the first identification to bimodules of the form

$$L^2(\mathcal{L}(\Gamma)(W_\sigma)e_{\Gamma_{\sigma^{-1}}}\mathcal{L}(\Gamma)).$$

On this bimodules, we have a left and a right action of G by multiplication.

On the other hand, the support of $u(\sigma)$ on the component corresponding to G^{op} is a singleton, so with respect to the component of G^{op} we may use the reduced C^*-norm topology. Similarly, corresponding to the G-component of the crossed product, the values of the unitary $u(\sigma)$ are already in $l^2(\Gamma\sigma\Gamma)$.

Hence the unitary representation u of G takes values in the reduced von Neumann algebra crossed product. Consequently, we may use indeed for the algebra $\mathcal{B}_\infty$ the reduced, groupoid, skewed, von Neumann algebra norm. Thus the representation u has values in the von Neumann reduced (skewed) groupoid crossed product ([31]). □

6. The classification of the representations π up to unitary conjugacy. Proof of Theorem 4

In this section we prove that the unitary representations π of the group G with the property that the restriction of π to Γ is unitary equivalent to the left regular representation ($D_\pi = 1$) are classified, up to unitary conjugacy, by the first group of cohomology of the group Γ with values in the unitary group of the factor P.

Remark 27. Let π and π_1 be two unitary representations of G such that $\pi|_\Gamma = \pi_1|_\Gamma = \lambda_\Gamma$. Assume there exists a unitary $\widetilde{U}$ conjugating π with π_1. Then necessary $\widetilde{U} \in \mathcal{R}(\Gamma)$. Using the canonical anti-isomorphism $\mathcal{R}(\Gamma) \cong \mathcal{L}(\Gamma)$, the unitary $\widetilde{U}$ corresponds to a unitary operator $U \in \mathcal{L}(\Gamma)$, with the properties introduced bellow, in formula (66). Let t, respectively t_1, be the representations of the Hecke algebra constructed in formula (30) that are respectively associated with the representations π, respectively π_1.

Then:

$$t_1(\Gamma\sigma\Gamma) = U[t(\Gamma\sigma\Gamma)]U^*, \quad \sigma \in G. \tag{66}$$

In the above equality we replaced $\widetilde{U}$ by U because we are passing, using the canonical anti-isomorphism, from right convolutors to left convolutors. The above identity is a consequence of the fact that in order to compute the coefficients for the representation t_1 as in formula (30), one has to substitute the cyclic and separating trace vector 1, used to compute the coefficients of t, by the vector $U \in \mathcal{L}(\Gamma) \subseteq \ell^2(\Gamma) \cong H$.

The converse also holds true: Let π, π_1 be two unitary representations of G, whose restriction to Γ is the left regular representation. Let t, t_1 be the associated representations of the Hecke algebra introduced in formula (30). Assume that the equation (66) holds true for U a unitary operator in $\mathcal{L}(\Gamma)$. Using the canonical anti-isomorphism $\mathcal{L}(\Gamma) \cong \mathcal{R}(\Gamma)$ we obtain a unitary $\widetilde{U} \in \mathcal{R}(\Gamma) = \pi(\Gamma)' = \pi_1(\Gamma)'$. Then the unitary $\widetilde{U}$ conjugates the unitary representation π and π_1 of G.

Indeed the equality in formula (66) implies that the matrix coefficients of the two representations coincide.

We prove in this section that the homomorphism $\alpha : G \to \mathrm{Aut}(P)$, constructed in Theorem 6, is uniquely determined up to cocycle conjugacy. Moreover, the representations π are classified, up to unitary conjugacy, by the cohomology group $H^1_\alpha(\Gamma, \mathcal{U}(P))$.

Recall that $u : G \to \mathcal{U}(P)$ is a 1-cocycle with respect to the action α if

$$u_{g_1 g_2} = u_{g_1}\alpha_{g_1}(u_{g_2}), \quad g_1, g_2 \in G.$$

We have:

Proposition 28. *Let π be a (projective) unitary representation of the group G as above. Let $\alpha : G \to \mathrm{Aut}(P)$ be the homomorphism constructed in Corollary 7. Then*
(i) Any other homomorphism $\widetilde{\alpha} : G \to \mathrm{Aut}(P)$ obtained from a similar splitting data as in Theorem 6, is of the form $\widetilde{\alpha}_g = \mathrm{Ad}(u_g)\alpha_g$, where $u_g \in \mathcal{U}(P)$ is a 1-cocycle of G with respect to α_g, with values in the unitary group $\mathcal{U}(P)$.
(ii) The unitary representations π' of G, whose restrictions to Γ are unitarily equivalent to the left regular representation of Γ are classified, up to unitary conjugacy, by the first cohomology group $H^1_\alpha(G,\mathcal{U}(P))$.
(iii) A cocycle c in the cohomology group $H^1_\alpha(G,\mathcal{U}(P))$ is trivial if and only if c is the coboundary of a unitary element in $\mathcal{L}(\Gamma) \subseteq P = \mathcal{L}(\Gamma \rtimes L^\infty(K,\mu))$.

Proof. Let $\alpha = (\alpha_g)_{g\in G}$ be as in Theorem 6. Then $\mathcal{M} = \mathcal{L}(G \rtimes L^\infty(\mathcal{G},\mu))$ has the G^{op}-equivariant tensor decomposition $P \otimes B(l^2(\Gamma\backslash G))$. The canonical action of G^{op} onto $\mathcal{M}$ is

$$\alpha_g \otimes \mathrm{Ad}\rho_{\Gamma/G}(g), \quad g \in G.$$

Elements in $\mathcal{M}$ are consequently identified with infinite matrices

$$(p_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G},$$

where the entries $p_{\Gamma\sigma_1,\Gamma\sigma_2}$ belong to the algebra P. Because any two matrix units are unitarily equivalent in a type II_∞ factor, it follows that any other G^{op}-equivariant matrix unit will be of the form

$$(u(\Gamma\sigma_1)u(\Gamma\sigma_2)^*)_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G},$$

where $u(\Gamma\sigma)$ are unitaries in P for all $\Gamma\sigma \in \Gamma \setminus G$.

For every $\sigma \in G$, we identify the coset $\Gamma\sigma$ with the characteristic function $\chi_{K\sigma}$. Then the diagonal algebra

$$D = l^\infty(\Gamma \setminus G) \subseteq B(l^2(\Gamma \setminus G)) \subseteq B(L^2(G \rtimes L^\infty(\mathcal{G},\mu))$$

is independent of the choice of an equivariant matrix unit.

The family $(u(\Gamma\sigma))_{\Gamma\sigma\in\Gamma\backslash G}$ defines the unitary

$$w = \sum_{\Gamma\sigma\in\Gamma\backslash G} u(\Gamma\sigma) \otimes \chi_{\overline{\Gamma\sigma}}. \tag{67}$$

Then the unitary w belongs to the unitary group of the algebra

$$D' \cap \mathcal{L}(G \rtimes L^\infty(\mathcal{G},\mu)).$$

We denote the initial matrix unit associated to the homomorphism α, splitting the action of $G^{\rm op}$, by

$$(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}.$$

We impose the $G^{\rm op}$-equivariance condition on the perturbed matrix unit

$$\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2} = u(\Gamma\sigma_1)^* u(\Gamma\sigma_2)\otimes v_{\Gamma\sigma_1,\Gamma\sigma_2}, \quad \Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G.$$

We consider the type I_∞ von Neumann algebra generated by the matrix units introduced above:

$$B=\{v_{\Gamma\sigma_1,\Gamma\sigma_2} |\ \Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G\}'', \quad \widetilde{B}=\{\widetilde{v}_{\Gamma\sigma_1\Gamma\sigma_2} |\ \Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G\}''.$$

For every double coset $\Gamma\sigma\Gamma$ in G, let

$$X^{\Gamma\sigma\Gamma}=\sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma} v_{\Gamma\sigma_1,\Gamma\sigma_2}, \quad \widetilde{X}^{\Gamma\sigma\Gamma}=\sum_{\Gamma\sigma_1\sigma_2^{-1}\Gamma=\Gamma\sigma\Gamma}\widetilde{v}_{\Gamma\sigma_1,\sigma_2\Gamma}. \tag{68}$$

These are the elements constructed in formula (37) in the proof of the implication (iv) $\Rightarrow$ (ii) in Theorem 6.

Consider the correspondence mapping $[\Gamma\sigma\Gamma]\to X^{\Gamma\sigma\Gamma}$ and the correspondence mapping $[\Gamma\sigma\Gamma]\to\widetilde{X}^{\Gamma\sigma\Gamma}$, where $[\Gamma\sigma\Gamma]$ runs over double cosets. Then both correspondences extend by linearity to $*$-representations of the algebra $\mathcal{H}_0=\mathbb{C}(\Gamma\setminus G/\Gamma)$. As explained in the proof of Theorem 6, this representations extend to representations of the reduced Hecke C^*-algebra $\mathcal{H}$, with values into the algebra $\mathcal{L}(G)$. Then we have:

$$w\big[(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}\big]w^*=(\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}. \tag{69}$$

Consequently

$$wBw^*=\widetilde{B},$$

and

$$wX^{\Gamma\sigma\Gamma}w^*=\widetilde{X}^{\Gamma\sigma\Gamma},$$

for all double cosets $[\Gamma\sigma\Gamma]$.

The $G^{\rm op}$-invariance of the perturbed matrix unit $(\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}$ implies that the unitary $\beta_g(w)$ has the same properties as w. Consequently $\beta_g(w)^*w$ belongs to $B'=P\otimes I$.

It follows that

$$c(g)=\beta_g(w)^*w, \quad g\in G,$$

is a 1-cocycle for the group G with respect to α_g, with values in $\mathcal{U}(P)$.

Let $\widetilde{\alpha}: G\to \mathrm{Aut}(P)$ be the homomorphism associated in Theorem 6 to the $G^{\rm op}$-matrix unit $(\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\backslash G}$. Then

$$\widetilde{\alpha}_g=\mathrm{Ad}c(g)\alpha_g, \quad g\in G.$$

This completes the proof of part (i) in the statement.

If, in the first cohomology group $H^1_\alpha(G, P)$, the 1-cocycle c with values in $P \cong P \otimes 1 = B' \subseteq D'$ vanishes, then it follows that there exists a unitary $p \in P$ such that

$$c(g) = \beta_g(p^*)p, \quad g \in G. \tag{70}$$

Let w be the unitary introduced in formula (67). Then

$$\beta_g(w^*)w = \beta_g(p^*)p, \tag{71}$$

for all g in G. Hence

$$\beta_g(wp^*) = wp^*, \ g \in G.$$

Consequently the unitary operator wp^* belongs to the subspace of G^{op}-invariant elements of $\mathcal{L}(G \rtimes L^\infty(\mathcal{G}, \mu))$. Hence

$$wp^* \in \mathcal{L}(G).$$

Since both w, p belong to D' it follows that

$$wp^* \in \mathcal{L}(G) \cap l^\infty(\Gamma \setminus G)' = \mathcal{L}(\Gamma).$$

We obtain that there exists a unitary operator $x \in \mathcal{L}(\Gamma)$ such that $w = xp$. Recall that, by definition, the unitary $p \in P$ commutes with the matrix unit $(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G}$. Using equation (69) we obtain that

$$\begin{aligned} w\big[(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G}\big]w^* &= x\big[(v_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G}\big]x^* \\ &= (\widetilde{v}_{\Gamma\sigma_1,\Gamma\sigma_2})_{\Gamma\sigma_1,\Gamma\sigma_2\in\Gamma\setminus G}, \end{aligned}$$

and

$$wBw^* = xBx^* = \widetilde{B}.$$

Using equations (68) we obtain that

$$\widetilde{X}^{\Gamma\sigma\Gamma} = xX^{\Gamma\sigma\Gamma}x^*, \quad \sigma \in G.$$

Recall that in Theorem 6 we proved that the elements constructed as in the formulae (68) are the same as the elements $t(\Gamma\sigma\Gamma)$, $t'(\Gamma\sigma\Gamma)$, $\sigma \in G$, associated to the (projective) unitary representations π and π' in part (ii) of the above mentioned theorem (formula (30)).

Using Remark 27, it follows that the unitary representations π and π' are conjugated by a unitary. The last statement is done in two steps: first we may assume, up to conjugation by a unitary operator, that the unitary representations π and π', when restricted to Γ, are equal the left regular representation of the group Γ. In a second step, we use the unitary x constructed above to verify the hypothesis of the converse statement in the above mentioned remark. This completes the proof of part (ii) in the statement.

To prove part (iii) we use the above arguments. Assume that c is the cocycle relating two homomorphisms α and $\widetilde{\alpha}$ of G as above. The cocycle $c(g)$ relating the homomorphisms α and $\widetilde{\alpha}$ is trivial if and only if there exists a unitary $p \in P$ such that equation (70) holds true.

From the previous part it follows that $\widetilde{\alpha}_g$ differs from α_g by conjugation with Ad(x), where x is a unitary element in $\mathcal{L}(\Gamma)$.

The homomorphisms α and $\widetilde{\alpha}$ of G into Aut$(\mathcal{L}(\Gamma \rtimes L^\infty(K,\mu)))$ are associated to the representations π and respectively π' by the correspondence introduced in Corollary 7. The equivalent construction for the homomorphisms α and $\widetilde{\alpha}$ described in the introduction, is realized by conjugating first the representations π, π' by a unitary so that $\pi|_\Gamma$, $\pi'|_\Gamma$ are the left regular representation of Γ.

We have therefore proved that if c is trivial, then the unitary p in the above mentioned formula may be replaced by a unitary x in $\mathcal{L}(\Gamma) \subseteq P$. Thus

$$c(g) = \beta_g(x^*)x, \quad g \in G.$$

□

In the next proposition we prove that the value at $\sigma \in G$ of the 1-cocycles relating two unitary representations as in part (iii) of Theorem 6, have an expression that depends only on the double coset $\Gamma\sigma\Gamma$.

Proposition 29. *Given a representation π as in Theorem 6 (i), consider as in part (iii) of the above mentioned theorem, the associated (projective) unitary representation $u : G \to \mathcal{U}(\mathcal{B}_\infty)$. Let u' be the (projective) unitary representation of G into $\mathcal{B}_\infty$ corresponding to another representation π' of G, as above. Let $v : G \to \mathcal{U}(\mathcal{A}_\infty)$ in $Z^1_\alpha(G, \mathcal{U}(\mathcal{A}_\infty))$ be the 1-cocycle defined by*

$$u'(\sigma) = v(\sigma)u(\sigma), \quad \sigma \in G. \tag{72}$$

Then:
(i) The 1-cocycle v has the following form: For each double coset $\Gamma\sigma\Gamma$, there exist Pimsner-Popa bases $(x_i^{[\Gamma\sigma\Gamma]})$, $(y_i^{[\Gamma\sigma\Gamma]})$ in $\mathcal{L}(\Gamma)$, $i = 1, 2, ..., [\Gamma : \Gamma_\sigma]$, for the inclusions $\mathcal{L}(\Gamma_\sigma) \subseteq \mathcal{L}(\Gamma)$ and respectively $\mathcal{L}(\Gamma_{\sigma^{-1}}) \subseteq \mathcal{L}(\Gamma)$, such that

$$v(\sigma) = \sum_{i=1}^{[\Gamma:\Gamma_\sigma]} x_i^{[\Gamma\sigma\Gamma]} \chi_{K_\sigma} y_i^{[\Gamma\sigma\Gamma]} \otimes 1 \in \mathcal{A}_\infty \subseteq \mathcal{B}_\infty. \tag{73}$$

(ii) The (projective) unitary representation π is unitarily equivalent to π' if and only if v is a coboundary. In this case, there exist a unitary $w \in \mathcal{L}(\Gamma)$ such that

$$v(g) = w^*[u(g)(w)u(g)^*], \quad g \in G.$$

Proof. We may assume that the unitary representations π, π' act on the same Hilbert space and that $\pi|_\Gamma = \pi'|_\Gamma$. The formula (72) is a consequence of the previous statement and of Proposition 13, since by the ergodicity assumptions, the relative commutant of $\mathcal{A}_\infty$ in $\mathcal{B}_\infty$ is trivial.

For $\sigma \in G$, let

$$V(\sigma) = \pi'(\sigma)\pi(\sigma)^*.$$

By definition, we have that

$$V(\sigma) \in \pi(\Gamma_\sigma)'. \tag{74}$$

It follows that $V(\sigma)$ depends only on the coset $\sigma\Gamma$ for $\sigma \in G$. We have

$$V(\gamma\sigma) = \pi(\gamma)V(\sigma)\pi(\gamma)^*, \quad \gamma \in \Gamma, \sigma \in G. \tag{75}$$

Using the identification in Section 4 of intertwiners with elements in the algebra $\mathcal{B}_\infty$ it follows that $V(\sigma)$, as a self-intertwiner of the von Neumann algebra $\pi(\Gamma_\sigma)'$ corresponds to a 1-cocycle v as in formula (72).

Using the identification in formula (11), formula (75) translates into a similar formula corresponding to the canonical right action of $\gamma \in \Gamma$, mapping $\mathcal{A}_{\Gamma\sigma}$ into $\mathcal{A}_{\Gamma\sigma\gamma}$. This fact and equation (75) are translated, using the terminology from the previous section, into the content of statement (i).

The statement (ii) is a direct consequence of Proposition 28. $\square$

Proof of Theorem 4. The proof of the implication (i) $\Rightarrow$ (iii) in Theorem 6 proves that the homomorphism α constructed in Corollary 7 coincides with the homomorphism $\alpha : G \to \mathrm{Aut}(P)$ introduced in formula (14). The latter is automatically a lifting of the map Φ to a homomorphism from G into Aut(P). This completes the proof of part (i).

Part (ii) is a direct consequence of formula (74). This, combined with the result in Proposition 28, proves part (iii). $\square$

7. Generalizing to the situation $D_\pi \neq 1$

In this section we construct the technical tools necessary to generalize the previous results to the case when the unitary representation π that restricted to Γ is a multiple (not necessary an integer) of the left regular representation of Γ. In this section we assume that π is a proper unitary representation (we are assuming that the cocycle ε is trivial).

Recall that the Schlichting completion $\mathcal{G}$ of the group G is a locally compact group, that is totally disconnected and contains G as a dense subgroup. The Haar measure on $\mathcal{G}$ is denoted as before by μ. We denote by $\mathcal{S}$ the lattice of finite index subgroups of K generated by subgroups of the form $K \cap \sigma K \sigma^{-1}, \sigma \in G$.

We consider the following reduced crossed product C^* algebras:

$$\mathcal{A} = C^*_{\mathrm{red}}(\mathcal{G} \rtimes L^\infty(\mathcal{G}, \mu)), \tag{76}$$

$$\mathcal{B} = C^*_{\mathrm{red}}((\mathcal{G} \times G^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)). \tag{77}$$

For a measurable subset A of $\mathcal{G}$, we identify the characteristic function with the corresponding projection $\chi_A \in L^\infty(\mathcal{G}, \mu)$.

By $L(\chi_A)$ we denote the corresponding left convolutor in $C^*_{\mathrm{red}}(\mathcal{G})$. By $R(\chi_A)$ we denote the corresponding right convolutor, viewed as an element in $C^*_{\mathrm{red}}(\mathcal{G}^{\mathrm{op}})$.

We also consider the C^*-algebras:

$$\mathcal{C} = \chi_K \mathcal{B} \chi_K \cong C^*_{\mathrm{red}}((\mathcal{G} \times G^{\mathrm{op}}) \rtimes L^\infty(K, \mu)) \quad \text{and} \tag{78}$$

$$\begin{aligned} \mathcal{D} &= \chi_K C^*_{\mathrm{red}}((\mathcal{G} \times \mathcal{G}^{\mathrm{op}}) \rtimes L^\infty(\mathcal{G}, \mu)) \chi_K \\ &\cong C^*_{\mathrm{red}}((\mathcal{G} \times \mathcal{G}^{\mathrm{op}}) \rtimes L^\infty(K, \mu)). \end{aligned} \tag{79}$$

In the above equations, the groups $\mathcal{G} \times G^{\mathrm{op}}$ and $\mathcal{G} \times \mathcal{G}^{\mathrm{op}}$ have an obvious action by partial measure preserving isomorphisms on K, and the second algebra in both equations is a reduced crossed product groupoid C^*-algebra. Let $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$, $\mathcal{D}_0$ be the corresponding involutive unital subalgebras of the C^*-algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ that are generated by the characteristic functions of cosets in $\mathcal{G}$ of subgroups in $\mathcal{S}$, by convolutors with such characteristic functions and, in the case of the algebra $\mathcal{B}_0$, by the elements of the group algebra, over $\mathbb{C}$, of the group G.

Remark 30. The multiplication rule in the C^*-crossed product algebra $\mathcal{A} = C^*_{\mathrm{red}}(\mathcal{G} \rtimes L^\infty(\mathcal{G}, \mu))$ is determined as follows:

(i) For every $\Gamma_0 \in \mathcal{S}$, let χ_{K_0} be the characteristic function of a subgroup $\mathcal{K}_0 = \overline{\Gamma_0}$. Then for all $g \in G$ we have

$$L(\chi_{K_0})\chi_{K_0 g} = \chi_{K_0 g} L(\chi_{K_0}).$$

(ii) If A, B are measurable subsets of $\mathcal{G}$, such that there exists a subgroup K_0 as above and finite families (s_i), (t_j) in G with

$$A = \bigcup_i s_i K_0, \quad B = \bigcup_j K_0 t_j,$$

then

$$L(\chi_A)\chi_B = \sum_{i,j} L(\chi_{s_i K_0})\chi_{K_0 t_j} = \sum_{i,j} \chi_{s_i^{-1} K_0 t_j} L(\chi_{s_i K_0}). \tag{80}$$

Proof. Property (i) is obvious since K_0 is a subgroup. Property (ii) is a direct consequence of Property (i). □

First we prove that the unitary representations u and Ψ from Theorems 6 and 8 have an abstract counterpart, taking values in the algebras $\mathcal{C}$ and $\mathcal{D}$.

Theorem 31. *For $\sigma \in G$, let $\chi_{K\sigma K}$ be the characteristic function of the double coset $K\sigma K$. Define*

$$U(\sigma) = \chi_K (L(\chi_{K\sigma K}) \otimes \sigma^{-1})\chi_K \in \mathcal{C}, \tag{81}$$

$$\Psi_0(\Gamma\sigma\Gamma) = \chi_K (L(\chi_{K\sigma K}) \otimes R(\chi_{K\sigma^{-1}K}))\chi_K \in \mathcal{D}. \tag{82}$$

Then:
(i) U is a unitary representation of G into $\mathcal{U}(\mathcal{C})$.
(ii) The correspondence associating to a double coset $\Gamma\sigma\Gamma$ of Γ in G the element $\Psi_0(\Gamma\sigma\Gamma)$ extends by linearity to a $*$-representation of the Hecke algebra $\mathcal{H}_0$ into $\mathcal{D}$.

Proof. Fix an element $\sigma_1 \in G$. Let (s_i) be a family of coset representatives for $\Gamma_{\sigma_1} = \sigma_1 \Gamma \sigma_1^{-1} \cap \Gamma$ in Γ. For $\theta \in G$, let K_θ be the subgroup $\theta K \theta^{-1} \cap K$. It is then obvious that

$$\begin{aligned} U(\sigma_1) &= \sum_i (L(\chi_{K\sigma_1 s_i}) \otimes \sigma_1^{-1})\chi_{K \cap s_i^{-1}\sigma_1^{-1} K \sigma_1} \\ &= \sum_i (L(\chi_{K\sigma_1 s_i}) \otimes \sigma_1^{-1})\chi_{s_i^{-1} K_{\sigma_1^{-1}}}. \end{aligned} \tag{83}$$

Let σ_2 be another element in G. Let (r_j) be a system of coset representatives for Γ_{σ_2} in Γ. Then

$$U(\sigma_2)U(\sigma_1) = \chi_K\big[\sum_{i,j}(L(\chi_{K\sigma_2 r_j \sigma_1 s_i}) \otimes \sigma_1^{-1}\sigma_2^{-1})\chi_{K\cap s_i^{-1}\sigma_1^{-1}K\sigma_1}\big].$$

This is further equal to

$$\sum_{i,j}(L(\chi_{K\sigma_2 r_j \sigma_1 s_i}) \otimes \sigma_1^{-1}\sigma_2^{-1})\chi_{B_{i,j}}, \tag{84}$$

where

$$B_{i,j} = \{k \in K | \sigma_1 s_i k \sigma_1^{-1} \in K, \sigma_2 r_j[\sigma_1 s_i k \sigma_1^{-1}]\sigma_2^{-1} \in K\}.$$

Thus, for a fixed i, $B_{i,j} \subseteq s_i^{-1}K_{\sigma_1^{-1}}$ is the preimage, through the map

$$\sigma_1 s_i \ \cdot \ \sigma_1^{-1},$$

of the partition of the subgroup K_{σ_1} given by

$$(r_j^{-1}K_{\sigma_2} \cap K_{\sigma_1})_{i,j}.$$

Since the cosets $s_i^{-1}K_{\sigma_1^{-1}}$ are disjoint, $(B_{i,j})_{i,j}$ is a partition of K.

On the other hand a term of the form

$$\chi_K(L(\chi_{K\alpha} \otimes \sigma_1^{-1}\sigma_2^{-1})\chi_K, \tag{85}$$

is different from zero if and only if $K\alpha$ is contained in $K\sigma_1\sigma_2 K$. Hence

$$\alpha = \sigma_2\sigma_1 v,$$

where v is a coset representative for $\Gamma_{\sigma_2\sigma_1}$ in Γ. Thus a non-zero term in formula (85) is necessarily of the form:

$$L(\chi_{Kv} \otimes \sigma_1^{-1}\sigma_2^{-1})\chi_{v^{-1}K_{(\sigma_2\sigma_1)^{-1}}}. \tag{86}$$

When v runs over the set of coset representatives for $\Gamma_{\sigma_2\sigma_1}$ in Γ, the sum of the terms in formula (86) is

$$U(\sigma_2\sigma_1) = \chi_K(L(\chi_{K\sigma_1\sigma_2 K} \otimes \sigma_1^{-1}\sigma_2^{-1})\chi_K.$$

Since both families $(v^{-1}K_{(\sigma_2\sigma_1)^{-1}})_v$ and $(B_{i,j})_{i,j}$ are partitions of K, it follows that sum of the terms in formula (84) is also equal to $U(\sigma_2\sigma_1)$, and hence U is a unitary representation of G.

The second formula is proven by a similar argument (see also [30]) noting that,in the above notation, one has that

$$\chi_K(L(\chi_{K\sigma K})\otimes R(\chi_{K\sigma^{-1}K}))\chi_K = \sum_{i,j} L(\chi_{K\sigma s_i})\otimes R(\chi_{s_j^{-1}\sigma^{-1}K})\chi_{K\cap s_i^{-1}\sigma^{-1}K\sigma s_j}.$$

□

Definition 32. Consider the following operator system inside $C^*_{\text{red}}(\mathcal{G})$:

$$\mathcal{E} = \ Sp\, \{L(\chi_{\sigma_1 K})L(\chi_{K\sigma_2})|\sigma_1, \sigma_2\} \subseteq C^*_{\text{red}}(\mathcal{G}).$$

Obviously the Hecke algebra of double cosets

$$\mathcal{H}_0 = \ Sp\, \{L(\chi_{K\sigma K})|\sigma \in G\},$$

is a subspace of $\mathcal{E}$.

Using the product in the algebra $C^*_{\text{red}}(\mathcal{G})$, it follows that every double coset $\chi_{K\sigma K}$ acts on $\mathcal{E}$, extending by linearity the correspondence

$$L(\chi_{\sigma_1 K})L(\chi_{K\sigma_2}) \to L(\chi_{\sigma_1 K})L(\chi_{K\sigma K})L(\chi_{K\sigma_2}) \in \mathcal{E}, \quad \sigma_1, \sigma_2 \in G. \tag{87}$$

Definition 33. Let $\mathcal{U}$ be an abstract involutive algebra over $\mathbb{C}$. Assume that $\mathcal{U}$ contains $L^\infty(\mathcal{G}, \mu)$ and an involutive subalgebra $\mathcal{V}$ such that

$$\mathcal{U} = \ Sp\, [\mathcal{V} \cdot L^\infty(\mathcal{G}, \mu)] = \ Sp\, [L^\infty(\mathcal{G}, \mu) \cdot \mathcal{V}].$$

We say that a linear map Φ, defined on $\mathcal{A}_0$ with values into $\mathcal{U}$, is a $\mathcal{SO}$-system map if the following properties hold true:
(i) The restriction of Φ preserves the operation described in Definition 32. Thus we assume that for all $\sigma, \sigma_1, \sigma_2 \in G$ we have
(88)
$$\Phi(L(\chi_{\sigma_1 K})L(\chi_{K\sigma K})L(\chi_{K\sigma_2})) = \Phi(L(\chi_{\sigma_1 K}))\Phi(L(\chi_{K\sigma K}))\Phi(L(\chi_{\sigma_2 K})).$$
(ii) The map Φ is $*$-preserving, that is, for all $\sigma \in G$, we have

$$\Phi(\chi_{K\sigma})^* = \Phi(\chi_{\sigma K}).$$

(iii) The map Φ is preserving the support in $L^\infty(\mathcal{G}, \mu)$, that is:

$$\Phi(f_1 X f_2) = f_1 \Phi(X) f_2, \quad f_1, f_2 \in L^\infty(\mathcal{G}, \mu), X \in \mathcal{E}. \tag{89}$$

We recall the context from the paper [27]. Let π be a unitary representation of G into a Hilbert space H. Assume that $\pi|_\Gamma$ admits a closed subspace $L \subseteq H$ such that

$$\pi(\gamma)L \perp L, \quad \gamma \in \Gamma \setminus \{e\},$$

and

$$\overline{Sp\, \{\pi(\gamma)L|\gamma \in \Gamma\}} = H.$$

We will call such a subspace a Γ-wandering generating subspace of H. In particular $\pi|_\Gamma$ is a multiple of the left regular representation λ_Γ with multiplicity equal to $\dim_{\mathbb{C}} L$. Assuming the above conditions and assuming that $\dim_{\mathbb{C}} L$ is a finite integer, the following proposition was proved in [27]. Let $\rho_\Gamma : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be the left regular representation of Γ. Let ρ_G be the right regular representation of G. Let $\mathcal{R}(G)$, respectively $\mathcal{R}(\Gamma)$, be the II_1 von Neumann algebras generated by the right regular representations of G and Γ respectively.

Proposition 34. *(see [27], [30]). Let P_L be the orthogonal projection from H onto L. For A a coset of a subgroup in $\mathcal{S}$ define*

$$\Phi_\pi(L(\chi_A)) = \sum_{\theta \in A} \rho(\theta) \otimes P_L \pi(\theta) P_L \tag{90}$$

Then Φ takes values in $\mathcal{R}(G) \otimes B(L)$ and $\Phi|_{\mathcal{E}}$ is a representation of the operator system $\mathcal{E}$, that verifies properties (i) and (ii) from Definition 33.

Using this, we can easily show that one can construct a representation into $\mathcal{R}(G) \otimes B(L) \cong \mathcal{L}(G) \otimes B(L)$ of the unitary representation and the Hecke algebra representation constructed in Theorem 31. With the notation from the previous proposition we can prove the following:

Theorem 35. *(i) For every $\sigma \in G$, the formula*

$$U(\sigma) = \chi_K(\Phi_\pi(\chi_{\Gamma\sigma\Gamma}) \otimes \sigma^{-1})\chi_K \in \mathcal{L}((G \otimes G^{\mathrm{op}}) \rtimes K) \otimes B(L)$$

defines a unitary representation U of G into $\mathcal{L}((G \otimes G^{\mathrm{op}}) \rtimes K) \otimes B(L)$.
(ii) The mapping

$$T(\chi_{\Gamma\sigma\Gamma}) = \chi_K(\Phi(\chi_{\Gamma\sigma\Gamma}) \otimes \Phi(\chi_{\Gamma\sigma^{-1}\Gamma}))\chi_K \in \mathcal{L}((G \otimes G^{\mathrm{op}}) \rtimes K) \otimes B(L)$$

extends by linearity to a representation of the Hecke algebra $\mathcal{H}_0$ into $\mathcal{L}((G \otimes G^{\mathrm{op}}) \rtimes K) \otimes B(L)$.

Proof. We extend Φ to the algebra generated by $\mathcal{E}$ and $L^\infty(\mathcal{G}, \mu)$ in $\mathcal{A}_0$. It is obvious that this can be done by letting Φ be equal to the identity on $L^\infty(\mathcal{G}, \mu)$. Because of the formula (90) and using the results from Proposition 34, it follows that the extension of Φ verifies the conditions from Definition 33. Since the verifications of the statement of Theorem 31 involve only identities of the type considered in Definition 33, it follows that the extension of Φ to the corresponding algebras translates the content of Theorem 31 into the statement of the present theorem. $\square$

Remark 36. Having constructed the unitary representation U as in part (iii) of Theorem 6, one may repeat ad-litteram, in this more general setting, the proof of the statements from the previous sections.

The case where $\pi|_\Gamma$ is a finite, but not necessary an integer multiple of the left regular representation, may be treated as in [27]. One assumes that the representation π admits a G-invariant subspace H_0 such that in the commutant of $\pi(\Gamma)$, the projection P_0 onto H_0 has trace t, a positive real number. Then one considers the unitary representation $\pi_0(g) = P_0\pi(g)P_0$, $g \in G$, which corresponds to the case where the multiplicity of the left regular representation λ_G in $\pi_0|_\Gamma$ is equal to t (in the sense of the Murray-von Neumann dimension theory). In this case all the above arguments may be repeated, replacing formula (90) by the formula

$$\Phi_\pi(L(\chi_A)) = \sum_{\theta\in A} \rho(\theta) \otimes P_L\pi_0(\theta)P_L. \tag{91}$$

The case where the representation π (and hence π_0) is unitary, projective is treated as in [26] (see also [28]).

We note a few remarks regarding the relation of this paper with the results in [30].

Remark 37. Consider a general representation π_0 on a Hilbert space H_0 such that $\pi_0|_\Gamma$ is a multiple of the left regular representation λ_Γ. In [30] we proved that there is a corresponding representation $\overline{\pi_0}^{\rm ad}$ on a Hilbert space $\overline{H_0}^{\rm ad}$, encoding all the information on the action of the representation π_0 on the extended vector space of vectors that are invariant to subgroups in $\mathcal{S}$.

The representation $\overline{\pi_0}^{\rm ad}$ is obtained through an adelic completion procedure, and therefore we will use the symbol $\overline{\cdot}^{\rm ad}$ to denote an object obtained through such a construction. For the conjugate Hilbert spaces and representation, we will use as usual the bar symbol. The essential property of the representation $\overline{\pi_0}^{\rm ad}$ is the fact that it extends to a C^*-representation, also denoted by $\overline{\pi_0}^{\rm ad}$, of the amalgamated free product C^*-algebra

$$C^*(\mathcal{G}) *_{C^*(K)} \left[C^*(K \rtimes L^\infty(K,\mu))\right] \tag{92}$$

into $B(\overline{H_0}^{\rm ad})$. It is proved in [30] that the character of the representation $\overline{\pi_0}^{\rm ad}$ is determined completely by the character of the representation π_0.

In general, if L is a Hilbert subspace such that

$$H_0 \cong L \otimes \ell^2(\Gamma)$$

and

$$\pi_0|_\Gamma \cong \mathrm{Id} \otimes \lambda_\Gamma,$$

then L is identified with the subspace of Γ-invariant vectors. For $\Gamma_0 \in \mathcal{S}$, the space $H_0^{\Gamma_0}$ of Γ_0-invariant vectors is

$$H_0^{\Gamma_0} \cong L \otimes \ell^2(\Gamma_0\backslash\Gamma). \tag{93}$$

The space $\overline{H_0}^{\mathrm{ad}}$ is then defined by

$$\overline{H_0}^{\mathrm{ad}} \cong L \otimes L^2(K, \mu).$$

The factor $L^\infty(K, \mu)$ in formula (92) acts in this representation by multiplication on the factor $L^2(K, \mu)$ in formula (93).

In the case presented in this work, for a unitary representation π as in the statement of Theorem 6, the unitary representation π_0 is $\mathrm{Ad}(\pi(g))_{g\in G}$ acting on the space of Hilbert-Schmidt operators on the Hilbert space H_π. Equivalently, π_0 is the unitary representation $\pi \otimes \overline{\pi}$ acting on $H_\pi \otimes \overline{H_\pi}$. The hypothesis that $D_\pi = 1$ corresponds to the fact that

$$H_0 = H_\pi \otimes \overline{H_\pi} \cong \ell^2(\Gamma) \otimes \ell^2(\Gamma).$$

Then

$$\overline{H_0}^{\mathrm{ad}} \cong \ell^2(\Gamma) \otimes L^2(K, \mu).$$

Clearly, since

$$L^2(P, \tau) = L^2(\mathcal{L}(\Gamma \rtimes L^\infty(K, \mu))) \cong \ell^2(\Gamma) \otimes L^2(K, \mu),$$

we obtain, using the algebras introduced in formula (11), that

$$H_0^{\Gamma_\sigma} \cong L^2(\mathcal{A}_{\sigma\Gamma}, \tau) \cong \ell^2(\Gamma) \otimes \ell^2(\Gamma_\sigma \backslash \Gamma), \quad \sigma \in G,$$

and

$$\overline{H_0}^{\mathrm{ad}} \cong \ell^2(\Gamma) \otimes L^2(K, \mu) \cong L^2(P, \tau).$$

The homomorphism $\alpha : G \to \mathrm{Aut}(P)$, which extends to $C^*(\mathcal{G})$ using conditional expectation on the algebras $\mathcal{A}_{\sigma\Gamma}, \sigma \in G$, yields a unitary representation U_α of $\mathcal{G}$ into $L^2(P, \tau)$.

Then, using the above isomorphism, the representation U_α is unitarily equivalent to the unitary representation $\overline{\pi_0}^{\mathrm{ad}}$.

We use the context from the Remark 37 to state the following conjecture:

Remark 38. The statement of the Ramanujan-Petersson conjectures may be generalized ([26]) to the question regarding weak containment of the unitary representation $\overline{\pi_0}^{\mathrm{ad}}$ of G (equivalently of $\mathcal{G}$) in the left regular representation of $\mathcal{G}$ (restricted to G if we analyze only representations of G). Thus an equivalent form of the above conjecture is the question whether the unitary representation $U_\alpha|_{L^2(P)\ominus\mathbb{C}1}$ is weakly contained in the restriction of the left regular representation of $\mathcal{G}$ to G.

Recall, as observed in Theorem 4, that factorizing the homomorphism α to the group $\mathrm{Out}(P) = \mathrm{Aut}(P)/\mathrm{Int}(P)$, one obtains a canonical homomorphism $G \to \mathrm{Out}(P)$ that is independent of choices. We also recall that the canonical Cartan subalgebra in the II_1 factor $\mathcal{L}(\Gamma \rtimes L^{\infty}(K, \mu))$ is unique, up to inner automorphisms ([21], see also [14], [24]).

We conjecture that the property of weak containment in the left regular representation introduced above is independent of a cocycle perturbation of the representation U_{α}, as in Theorem 28.

Acknowledgment. The author is deeply indebted to Ryszard Nest, Nicolas Monod, Stephan Vaes and to the anonymous referee for several comments on this paper. The author is deeply indebted to Florin Boca for several comments and for help with reorganizing the exposition of the paper. The author is deeply indebted to Narutaka Ozawa for putting out his notes ([28]) on the paper [26], and to allow their publication.

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