SOME TRANSFINITE NATURAL SUMS

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Abstract. We study a transfinite iteration of the ordinal Hessenberg natural sum obtained by taking suprema at limit stages and show that such an iterated natural sum differs from the more usual transfinite ordinal sum only for a finite number of iteration steps. The iterated natural sum of a sequence of ordinals can be obtained as a "mixed sum" (in an order-theoretical sense) of the ordinals in the sequence; in fact, it is the largest mixed sum which satisfies a finiteness condition, relative to the ordering of the sequence. We introduce other infinite natural sums which are invariant under permutations and show that they all coincide in the countable case. Finally, in the last section we use the above infinitary natural sums in order to provide a definition of size for a well-founded tree, together with an order-theoretical characterization in the countable case. The proof of this order-theoretical characterization is mostly independent from the rest of this paper.

1. Introduction

The *(Hessenberg) natural sum* $\alpha \# \beta$ of two ordinals α and β can be defined by expressing α and β in Cantor normal form and summing linearly. Further details shall be given below. In contrast with the more usual ordinal sum, the resulting natural operation $\#$ is commutative, associative and cancellative. The definition can be obviously extended to deal with a finite sum of ordinals; otherwise, using the fact that the binary operation $\#$ is commutative and associative, we get no ambiguity in defining the natural sum of a finite sequence of ordinals.

An infinitary version of the natural sum has appeared in Wang [W] and Väänänen and Wang [VW] in the countable case and we have extensively studied it in [L1], in particular, providing some order-theoretical characterization.

Here we extend the above countable natural sum to the transfinite. Expanding on the countable case, we take suprema at limit stages

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and the natural sum at successor steps. There are similarities with the countable case: the transfinitely iterated natural sum can be computed in a way similar to the more usual transfinite ordinal sum, except just for a finite number of steps, in which we should take the finite natural sum in place of the ordinal sum. Moreover, in the same spirit of [L1], the iterated natural sum has an order-theoretical characterization: it is the largest mixed sum which satisfies a finiteness condition.

On the other hand, in the uncountable case, the iterated natural sum turns out to be not invariant under permutations. In particular, the above mentioned order theoretical characterization depends on the ordering of the ordinals in the sequence. Some possible definitions of invariant infinitary operations will be given in Section 5. Significantly, all these operations coincide in the case of ω-indexed sequences of ordinals; this fact is not completely trivial, indeed the corresponding result would be false in the case of infinitary operations associated with the more usual ordinal sum.

Finally, we use the above infinitary natural sums in order to provide some definitions of size for well-founded trees, together with an order-theoretical characterization in the countable case. This is presented in Section 6. For the most part, Section 6 can be read independently from the rest of this paper; some familiarity with [L1] is sufficient.

As a final comment, it should be mentioned that, quite remarkably, the special case of the iterated natural sum in which all summands are equal has been introduced more than a century ago by Jacobsthal [Ja] in 1909. See Altman [Al] for further details, references and generalizations. To the best of our knowledge, the general case of arbitrary summands has never been considered before [VW, W], but it should be mentioned that the literature on the subject is so vast that a thorough check is virtually impossible.

2. Preliminaries

Recall that every nonzero ordinal α can be expressed in *Cantor normal form* in a unique way as follows.

$$\alpha = \omega^{\xi_k} r_k + \omega^{\xi_{k-1}} r_{k-1} + \cdots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0$$

for some integers $k \geqslant 0$, $r_k, \dots, r_0 > 0$ and ordinals $\xi_k > \xi_{k-1} > \cdots > \xi_1 > \xi_0$. The ordinal 0 can be considered as an "empty" sum. Here sums, products and exponentiations are always considered in the ordinal sense.

Definition 2.1. The natural sum $\alpha \# \beta$ of two ordinals α and β is the only operation satisfying

$$\alpha \# \beta = \omega^{\xi_k}(r_k + s_k) + \dots + \omega^{\xi_1}(r_1 + s_1) + \omega^{\xi_0}(r_0 + s_0)$$

whenever

$$\alpha = \omega^{\xi_k} r_k + \dots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0$$
$$\beta = \omega^{\xi_k} s_k + \dots + \omega^{\xi_1} s_1 + \omega^{\xi_0} s_0$$

and $k, r_k, \dots, r_0, s_k, \dots, s_0 < \omega$, $\xi_k > \dots > \xi_1 > \xi_0$.

The definition is justified by the fact that we can represent every nonzero α and β in Cantor normal form and then insert some more null coefficients for convenience in order to make the indices match. The null coefficients do not affect α and β, hence the definition is well-posed. See, e. g., Bachmann [B] and Sierpiński [Sier2] for further details.

Let α and η be ordinals, and express α in Cantor normal form as $\omega^{\xi_k} r_k + \dots + \omega^{\xi_0} r_0$. The ordinal $\alpha^{\restriction \eta}$, in words, α *truncated at the* η^{th} *exponent of* ω, is $\omega^{\xi_k} r_k + \dots + \omega^{\xi_\ell} r_\ell$, where ℓ is the smallest index such that $\xi_\ell \geqslant \eta$. We set $\alpha^{\restriction \eta} = 0$ in case that $\alpha < \omega^\eta$.

Proposition 2.2. *Let α, β, γ, η be ordinals and $r < \omega$.*

(1) The operation $\#$ is commutative, associative, both left and right cancellative and strictly monotone in both arguments.
(2) $\sup\{\alpha, \beta\} \leqslant \alpha + \beta \leqslant \alpha \# \beta$.
(3) If $\beta < \omega^\eta$, then $\alpha \# \beta < \alpha + \omega^\eta$.
(4) If $\alpha \# \beta \geqslant \omega^\eta r$, then there are $r_1, r_2 < \omega$ such that $r_1 + r_2 = r$, $\alpha \geqslant \omega^\eta r_1$ and $\beta \geqslant \omega^\eta r_2$.
(5) $\alpha + \beta = \alpha^{\restriction \eta} + \beta = \alpha^{\restriction \eta} \# \beta$, where η is the leading exponent in the Cantor normal expression of β.
(6) $(\alpha + \beta) \# \gamma \geqslant \alpha + (\beta \# \gamma)$.
(7) $\alpha \# (\beta + \gamma) \geqslant (\alpha \# \beta) + \gamma$.

Proof. Everything follows easily by expressing the relevant ordinals in Cantor normal form and applying the definitions. See [B, Sier2] for details. A proof of (3) can be found in [L1, Proposition 2.2(4)].

In order to prove (4), express both α and β in Cantor normal form. If the largest power of ω in the expression of, say, α is $> \eta$, then $\alpha \geqslant \omega^\eta r$ and we are done, taking $r_2 = 0$. The same holds for β. Otherwise, $\alpha = \omega^\eta s + \dots$ and $\beta = \omega^\eta t + \dots$ (possibly, $s = 0$ or $t = 0$). By the definition of $\alpha \# \beta$, and since $\alpha \# \beta \geqslant \omega^\eta r$, we get $s + t \geqslant r$, hence we can find r_1 and r_2 as desired.

(5) When computing $\alpha + \beta$, all the summands $< \omega^\eta$ in the Cantor normal expression of α are absorbed by the leading term of the normal expression of β.

(6) By (5) and associativity of $\#$, we get $(\alpha + \beta) \# \gamma = \alpha^{\restriction \eta} \# \beta \# \gamma$, where η is the leading exponent in the normal expression of β. In the same way, $\alpha + (\beta \# \gamma) = \alpha^{\restriction \xi} \# \beta \# \gamma$, where ξ is the leading exponent in the normal expression of $\beta \# \gamma$. Now trivially $\eta \leqslant \xi$, hence $\alpha^{\restriction \eta} \geqslant \alpha^{\restriction \xi}$, from which (6) follows.

(7) By (5) and associativity of $\#$, we get $\alpha \# (\beta + \gamma) = \alpha \# \beta^{\restriction \eta} \# \gamma$, where η is the leading exponent in the normal expression of γ. On the other hand, $(\alpha \# \beta) + \gamma = (\alpha \# \beta)^{\restriction \eta} \# \gamma = \alpha^{\restriction \eta} \# \beta^{\restriction \eta} \# \gamma$, thus the inequality follows from monotonicity of $\#$. $\square$

3. A transfinite iteration of the natural sum

Definition and notations. A countable iteration of the natural sum has been considered in Väänänen and Wang [VW], by taking the supremum at the limit stage. This countable operation has been extensively studied in [L1]. The construction can be iterated without special adjustments through the transfinite.

Definition 3.1. If $(\alpha_\gamma)_{\gamma < \bar{\varepsilon}}$ is a sequence of ordinals, we define the *iterated natural sum* $\sum^{\#}_{\gamma<\delta} \alpha_\gamma$, for every $\delta \leqslant \bar{\varepsilon}$, inductively as follows.

$$\sum^{\#}_{\gamma<0} \alpha_\gamma = 0 \tag{1}$$

$$\sum^{\#}_{\gamma<\delta+1} \alpha_\gamma = \left(\sum^{\#}_{\gamma<\delta} \alpha_\gamma \right) \# \alpha_\delta \tag{2}$$

$$\sum^{\#}_{\gamma<\delta} \alpha_\gamma = \sup_{\delta'<\delta} \sum^{\#}_{\gamma<\delta'} \alpha_\gamma \qquad \text{for } \delta \text{ limit} \tag{3}$$

Clearly, as far as $\bar{\varepsilon} \geqslant \delta$, the definition of $\sum^{\#}_{\gamma<\delta} \alpha_\gamma$ does not depend on $\bar{\varepsilon}$. Moreover, it does not depend on the values assumed by the α_γ's, for $\gamma \geqslant \delta$. This shows that there is no ambiguity in the notation.

In the particular case when $\delta = \omega$, the operation $\sum^{\#}_{\gamma<\omega} \alpha_\gamma$ has been considered in [VW] under the same notation and studied in [L1] under the notation $\#_{\gamma<\omega} \alpha_\gamma$. We are using here a different notation since we want to reserve the symbol $\#$ for some operation which is invariant under permutations. In fact, if $\delta = \omega$, then $\sum^{\#}$ is invariant while, if $\delta > \omega$, then $\sum^{\#}$ is not invariant. See Section 5 for further details;

there we shall also introduce some related operations which are indeed invariant under permutations.

Notation 3.2. If the sequence of the α_γ's in Definition 3.1 is understood, we shall sometimes simply write S_δ in place of $\sum^{\#}_{\gamma<\delta} \alpha_\gamma$. In a few cases, we shall also need a shorthand for the partial sums of the usual iterated ordinal sum $\sum_{\gamma<\delta} \alpha_\gamma$. This will be abbreviated as S^{Σ}_δ. I. e., in the recursive definition of S^{Σ}_δ, we use $S^{\Sigma}_{\delta+1} = S^{\Sigma}_\delta + \alpha_\delta$ in place of equation (2) above.

As an additional shorthand, particularly useful to simplify notations in proofs, for $\delta' \leqslant \delta$, we let $S_{[\delta',\delta)}$ denote $\sum^{\#}_{\delta' \leqslant \gamma<\delta} \alpha_\gamma$, and, similarly, $S^{\Sigma}_{[\delta',\delta)} = \sum_{\delta' \leqslant \gamma<\delta} \alpha_\gamma$. As usual, the notation $\sum^{\#}_{\delta' \leqslant \gamma<\delta} \alpha_\gamma$ is justified by the fact that, if $\delta' \leqslant \delta$, then there is a unique ordinal ε such that $\delta' + \varepsilon = \delta$, and then the formal definition of $\sum^{\#}_{\delta' \leqslant \gamma<\delta} \alpha_\gamma$ is $\sum^{\#}_{\varepsilon' < \varepsilon} \alpha_{\delta'+\varepsilon'}$. All this is standard; see the mentioned books [B, Sier2] for further details, e. g., in the case of the ordinary transfinite sum.

Some preliminary lemmas. Throughout the present subsection we fix a sequence $(\alpha_\gamma)_{\gamma<\bar{\varepsilon}}$ of ordinals, where $\bar{\varepsilon}$ is some sufficiently large ordinal.

Proposition 3.3. *Let δ, α_γ, β_γ ($\gamma < \delta$) be ordinals. Then the following statements hold.*

(1) $\sum_{\gamma<\delta} \alpha_\gamma \leqslant \sum^{\#}_{\gamma<\delta} \alpha_\gamma$
(2) If $\beta_\gamma \leqslant \alpha_\gamma$, for every $\gamma < \delta$, then $\sum^{\#}_{\gamma<\delta} \beta_\gamma \leqslant \sum^{\#}_{\gamma<\delta} \alpha_\gamma$
(3) If $\delta' < \delta$, then $\sum^{\#}_{\gamma<\delta'} \alpha_\gamma \leqslant \sum^{\#}_{\gamma<\delta} \alpha_\gamma$; equality holds if and only if $\alpha_\gamma = 0$, for every γ with $\delta' \leqslant \gamma < \delta$.
(4) If $(\alpha_{f(\varepsilon)})_{\varepsilon<\delta'}$ is the subsequence of $(\alpha_\gamma)_{\gamma<\delta}$ consisting exactly of the nonzero elements of the sequence, then $\sum^{\#}_{\gamma<\delta} \alpha_\gamma = \sum^{\#}_{\varepsilon<\delta'} \alpha_{f(\varepsilon)}$.
(5) If $\delta'' < \delta$, k is finite and $\delta'' \leqslant \gamma_0, \gamma_1, \dots, \gamma_{k-1} < \delta$, then $\sum^{\#}_{\gamma<\delta} \alpha_\gamma \geqslant \left(\sum^{\#}_{\gamma<\delta''} \alpha_\gamma\right) \mathbin{\#} \alpha_{\gamma_0} \mathbin{\#} \alpha_{\gamma_1} \mathbin{\#} \dots \mathbin{\#} \alpha_{\gamma_{k-1}}$.

Proof. (1)-(3) are immediate from Proposition 2.2 and Definition 3.1. (4) is proved by induction on δ, using (3).

In order to prove (5), let $(\alpha'_\gamma)_{\gamma<\delta}$ be the sequence obtained from $(\alpha_\gamma)_{\gamma<\delta}$, by changing to 0 all elements except for those elements which either have index $< \delta''$ or have index in the set $\{\gamma_0, \dots, \gamma_{k-1}\}$. Then, by (2), and applying (4) to the sequence $(\alpha'_\gamma)_{\gamma<\delta}$, we get $\sum^{\#}_{\gamma<\delta} \alpha_\gamma \geqslant \sum^{\#}_{\gamma<\delta} \alpha'_\gamma = \sum^{\#}_{\varepsilon<\delta'} \alpha_{f(\varepsilon)} = \left(\sum^{\#}_{\varepsilon<\delta''} \alpha_\gamma\right) \mathbin{\#} \alpha_{\gamma_0} \mathbin{\#} \alpha_{\gamma_1} \mathbin{\#} \dots \mathbin{\#} \alpha_{\gamma_{k-1}}$, where δ' is given by (4), we iterate clause (2) in Definition 3.1 a finite number of times, and we are using the fact that $\delta' = \delta'' + k$. ∎

Lemma 3.4. *Suppose that $\delta' \leqslant \delta$. Then*

$$\sum_{\gamma<\delta'}^{\#} \alpha_\gamma \mathbin{\#} \sum_{\delta' \leqslant \gamma<\delta}^{\#} \alpha_\gamma \geqslant \sum_{\gamma<\delta}^{\#} \alpha_\gamma \geqslant \sum_{\gamma<\delta'}^{\#} \alpha_\gamma + \sum_{\delta' \leqslant \gamma<\delta}^{\#} \alpha_\gamma$$

Proof. The proof is obtained by leaving δ' fixed and performing an induction on $\delta \geqslant \delta'$, using Proposition 2.2. Notice that, in the shorthand introduced in 3.2, the conclusion of the lemma reads $S_{\delta'} \mathbin{\#} S_{[\delta',\delta)} \geqslant S_\delta \geqslant S_{\delta'} + S_{[\delta',\delta)}$.

We first prove the left-hand inequality. If $\delta = \delta'$, then $S_{[\delta',\delta)}$ is an empty sum with value 0 and the result is trivial.

If δ is successor, say, $\delta = \delta'' + 1$, then, by the inductive hypothesis, $S_{\delta'} \mathbin{\#} S_{[\delta',\delta'')} \geqslant S_{\delta''}$. Then, by definition, $S_{\delta'} \mathbin{\#} S_{[\delta',\delta)} = S_{\delta'} \mathbin{\#} (S_{[\delta',\delta'')} \mathbin{\#} \alpha_{\delta''}) = (S_{\delta'} \mathbin{\#} S_{[\delta',\delta'')}) \mathbin{\#} \alpha_{\delta''} \geqslant S_{\delta''} \mathbin{\#} \alpha_{\delta''} = S_\delta$, by associativity 2.2(1).

If δ is limit, then, by Proposition 3.3(3) and the inductive hypothesis, $S_{\delta'} \mathbin{\#} S_{[\delta',\delta)} \geqslant S_{\delta'} \mathbin{\#} S_{[\delta',\delta^*)} \geqslant S_{\delta^*}$, for every δ^* with $\delta' \leqslant \delta^* < \delta$. Hence $S_{\delta'} \mathbin{\#} S_{[\delta',\delta)} \geqslant \sup_{\delta^*<\delta} S_{\delta^*} = S_\delta$.

Now let us prove the right-hand inequality. Again, this is trivial if $\delta = \delta'$.

If δ is successor, say, $\delta = \delta'' + 1$, then, by the inductive hypothesis, $S_{\delta''} \geqslant S_{\delta'} + S_{[\delta',\delta'')}$. Then, by definition and using Proposition 2.2(6), $S_\delta = S_{\delta''} \mathbin{\#} \alpha_{\delta''} \geqslant (S_{\delta'} + S_{[\delta',\delta'')}) \mathbin{\#} \alpha_{\delta''} \geqslant S_{\delta'} + (S_{[\delta',\delta'')} \mathbin{\#} \alpha_{\delta''}) = S_{\delta'} + S_{[\delta',\delta)}$.

If δ is limit, then, by Proposition 3.3(3) and the inductive hypothesis, $S_\delta \geqslant S_{\delta^*} \geqslant S_{\delta'} + S_{[\delta',\delta^*)}$, for every δ^* with $\delta' \leqslant \delta^* < \delta$. Hence $S_\delta \geqslant \sup_{\delta^*<\delta}(S_{\delta'} + S_{[\delta',\delta^*)}) = S_{\delta'} + \sup_{\delta^*<\delta} S_{[\delta',\delta^*)} = S_{\delta'} + S_{[\delta',\delta)}$, by right continuity of $+$. $\square$

Remark 3.5. Notice that the inequalities in Lemma 3.4 might be strict. For example, taking $\delta = \omega$, $\delta' = 1$ and $\alpha_\gamma = 1$, for every $\gamma < \omega$, we get $1 \mathbin{\#} \sum^{\#}_{1 \leqslant \gamma<\delta} \alpha_\gamma = \omega + 1 > \omega = \sum^{\#}_{\gamma<\delta} \alpha_\gamma$, showing that the left-hand inequality might be strict.

On the other hand, take $\delta = 2$, $\delta' = 1$, $\alpha_0 = 1$ and $\alpha_1 = \omega$, getting $\sum^{\#}_{\gamma<\delta} \alpha_\gamma = 1 \mathbin{\#} \omega = \omega + 1 > \omega = 1 + \omega = 1 + \sum^{\#}_{1 \leqslant \gamma<\delta} \alpha_\gamma$.

Lemma 3.6. *If $n < \omega$ and $\sum^{\#}_{\gamma<\delta} \alpha_\gamma \geqslant \omega^\xi n$, then $\sum_{\gamma<\delta} \alpha_\gamma \geqslant \omega^\xi n$.*

Proof. By induction on ordinals ι of the form $\omega^\xi n$.

The lemma is trivial if ι is either 0 or 1.

Suppose that $\iota = \omega^\xi n > 1$, and that the lemma is true for all $\iota' < \iota$ of the form $\omega^{\xi'} n'$. Suppose by contradiction that the lemma is false for ι, and choose a counterexample for which δ is minimal. Recalling the notation from 3.2, we are assuming that $S_\delta \geqslant \omega^\xi n$, but $S^\Sigma_\delta < \omega^\xi n$, and that δ is minimal with the above properties. Then $S_{\delta'} < \omega^\xi n$, for every $\delta' < \delta$, since, otherwise, $S_{\delta'} \geqslant \omega^\xi n$, and either δ' would give a counterexample, contradicting the minimality of δ, or $S^\Sigma_\delta \geqslant S^\Sigma_{\delta'} \geqslant \omega^\xi n$, contradicting the assumption that δ gives a counterexample.

Towards a contradiction, we shall exclude all the possibilities for δ. Necessarily, $\delta > 0$, since $\iota > 0$.

We shall now exclude the case when δ is successor. Suppose that $\delta = \varepsilon + 1$. Since, by definition, $S_\varepsilon \# \alpha_\varepsilon = S_\delta \geqslant \omega^\xi n$, we get by Proposition 2.2(4) that there are n_1 and n_2 such that $n_1 + n_2 = n$ and $S_\varepsilon \geqslant \omega^\xi n_1$, $\alpha_\varepsilon \geqslant \omega^\xi n_2$. Since we know that $S_{\delta'} < \omega^\xi n$, for every $\delta' < \delta$, we have $S_\varepsilon < \omega^\xi n$, so that $n_1 < n$, hence we can apply the inductive hypothesis with $\iota' = \omega^\xi n_1$, getting $S^\Sigma_\varepsilon \geqslant \omega^\xi n_1$. Then $S^\Sigma_\delta = S^\Sigma_\varepsilon + \alpha_\varepsilon \geqslant \omega^\xi n_1 + \omega^\xi n_2 = \omega^\xi n$, a contradiction. Thus we cannot get a counterexample at δ successor.

The remaining case is when δ is limit. The argument shall be split into two cases. First assume that $n = 1$, that is, $\iota = \omega^\xi > 1$. Then $\iota = \sup_{\iota' < \iota} \iota'$, with, as above, the ι's varying on ordinals of the form $\omega^{\xi'} n'$. Since $S_\delta \geqslant \omega^\xi > \iota'$, for every $\iota' < \iota$, we can apply the inductive hypothesis, getting $S^\Sigma_\delta \geqslant \iota'$, for every $\iota' < \iota$, hence $S^\Sigma_\delta \geqslant \sup_{\iota' < \iota} \iota' = \iota = \omega^\xi$, contradicting the assumption that δ gives a counterexample to the statement of the lemma.

The only case left is when δ is limit and $\iota = \omega^\xi n$, for some $n > 1$. We know that $S_{\delta'} < \omega^\xi n$, for every $\delta' < \delta$. But also $\sup_{\delta' < \delta} S_{\delta'} = S_\delta \geqslant \omega^\xi n$, hence there is some $\delta' < \delta$ such that $S_{\delta'} \geqslant \omega^\xi (n-1)$. Fix some δ' satisfying the above inequality. By Lemma 3.4, $S_{\delta'} \# S_{[\delta', \delta)} \geqslant S_\delta$. Since $S_\delta \geqslant \omega^\xi n$ and $S_{\delta'} < \omega^\xi n$, we necessarily get $S_{[\delta', \delta)} \geqslant \omega^\xi$, by Proposition 2.2(4). We can now apply the inductive hypothesis twice, getting $S^\Sigma_{\delta'} \geqslant \omega^\xi(n-1)$ from $S_{\delta'} \geqslant \omega^\xi(n-1)$ and $S^\Sigma_{[\delta', \delta)} \geqslant \omega^\xi$ from $S_{[\delta', \delta)} \geqslant \omega^\xi$, since $n > 1$, hence $\omega^\xi < \omega^\xi n = \iota$. Then $S^\Sigma_\delta = S^\Sigma_{\delta'} + S^\Sigma_{[\delta', \delta)} \geqslant \omega^\xi(n-1) + \omega^\xi = \omega^\xi n$. Again, this contradicts the assumption that δ gives a counterexample.

We have showed that the lemma holds for ι, assuming that it holds for every $\iota' < \iota$ of the form $\omega^\xi n$. By induction, the lemma holds for every ι of the form $\omega^\xi n$. □

The sums differ only for a finite number of inductive steps. If $\alpha > 0$ is an ordinal expressed in Cantor normal form as $\alpha = \omega^{\xi_k} r_k + \cdots + \omega^{\xi_0} r_0$ (hence $r_0 > 0$), we call ξ_0 the *smallest exponent* of α.

Theorem 3.7. *Suppose that ζ is a limit ordinal, $(\alpha_\gamma)_{\gamma < \zeta}$ is a sequence of ordinals which are not eventually zero, and let ξ be the smallest exponent of (the non zero ordinal) $\sum^{\#}_{\gamma < \zeta} \alpha_\gamma$.*

Then there is $\bar\gamma < \zeta$ such that, for every $\varepsilon \geqslant \bar\gamma$

$$\sum^{\#}_{\varepsilon \leqslant \gamma < \zeta} \alpha_\gamma = \sum_{\varepsilon \leqslant \gamma < \zeta} \alpha_\gamma = \omega^\xi \quad \textit{and} \tag{4}$$

$$\sum^{\#}_{\gamma < \zeta} \alpha_\gamma = \sum^{\#}_{\gamma < \varepsilon} \alpha_\gamma + \omega^\xi \tag{5}$$

Proof. Let $S_\zeta = \sum^{\#}_{\gamma<\zeta} \alpha_\gamma = \omega^{\xi_k} r_k + \dots + \omega^\xi r$ be expressed in Cantor normal form. Let $\varepsilon < \zeta$. Since the sequence $(\alpha_\gamma)_{\gamma<\zeta}$ is not eventually zero, then $S_\varepsilon < S_\zeta$, by Proposition 3.3(3), hence $S_\varepsilon + \omega^\xi \leqslant S_\zeta$, expressing S_ε in Cantor normal form and using Proposition 2.2(5). Since, by Lemma 3.4, $S_\zeta \leqslant S_\varepsilon \# S_{[\varepsilon,\zeta)}$, we get $S_{[\varepsilon,\zeta)} \geqslant \omega^\xi$, by Proposition 2.2(3) with $\alpha = S_\varepsilon$ and $\beta = S_{[\varepsilon,\zeta)}$ (were $S_{[\varepsilon,\zeta)} < \omega^\xi$, we would get $S_\zeta \leqslant S_\varepsilon \# S_{[\varepsilon,\zeta)} < S_\varepsilon + \omega^\xi \leqslant S_\zeta$, a contradiction). By Lemma 3.6, also $S^\Sigma_{[\varepsilon,\zeta)} \geqslant \omega^\xi$. All this holds for every $\varepsilon < \zeta$.

On the other hand, since ζ is limit, thus $S_\zeta = \sup_{\varepsilon<\zeta} S_\varepsilon$, and since $r > 0$, then there is some $\bar\gamma < \zeta$ such that $S_{\bar\gamma} \geqslant \omega^{\xi_k} r_k + \dots + \omega^\xi (r-1)$. Then $S_\varepsilon \geqslant S_{\bar\gamma} \geqslant \omega^{\xi_k} r_k + \dots + \omega^\xi (r-1)$, for every ε such that $\zeta > \varepsilon \geqslant \bar\gamma$. By Lemma 3.4 and the previous paragraph we get that if $\zeta > \varepsilon \geqslant \bar\gamma$, then $\omega^{\xi_k} r_k + \dots + \omega^\xi r = S_\zeta \geqslant S_\varepsilon + S_{[\varepsilon,\zeta)} \geqslant \omega^{\xi_k} r_k + \dots + \omega^\xi (r-1) + S_{[\varepsilon,\zeta)} \geqslant \omega^{\xi_k} r_k + \dots + \omega^\xi r$, hence all are equal, and we obtain at once $S_{[\varepsilon,\zeta)} = \omega^\xi$ (e. g., by Proposition 2.2(5)), as well as (5). Hence, recalling the above paragraph, we get also $\omega^\xi \leqslant S^\Sigma_{[\varepsilon,\zeta)} \leqslant S_{[\varepsilon,\zeta)} = \omega^\xi$, thus the proof of (4) is complete. □

We can introduce intermediate variants between $\sum_{\gamma<\zeta} \alpha_\gamma$ and $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ by taking the natural sum at certain successor stages and the usual ordinal sum at the remaining successor stages. As a simple consequence of Theorem 3.7, we will get in Corollary 3.11 below that $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ can be computed by taking natural sums at just a finite number of stages. In more details, we are dealing with the notion introduced in the following definition.

Definition 3.8. Suppose that ζ is an ordinal, $(\alpha_\gamma)_{\gamma<\zeta}$ is a sequence of ordinals, and G is a set of ordinals. For every $\delta \leqslant \zeta$, we define the *partial natural sum (relative to G)* $\sum^{G}_{\gamma<\delta} \alpha_\gamma$ inductively as follows.

$$\sum^{G}_{\gamma<0} \alpha_\gamma = 0$$

$$\sum_{\gamma<\delta+1}^{G} \alpha_\gamma = \left(\sum_{\gamma<\delta}^{G} \alpha_\gamma\right) \mathbin{\#} \alpha_\delta \quad \text{if } \delta \in G$$
$$\sum_{\gamma<\delta+1}^{G} \alpha_\gamma = \left(\sum_{\gamma<\delta}^{G} \alpha_\gamma\right) + \alpha_\delta \quad \text{if } \delta \notin G$$
$$\sum_{\gamma<\delta}^{G} \alpha_\gamma = \sup_{\delta'<\delta} \sum_{\gamma<\delta'}^{G} \alpha_\gamma \quad \text{for } \delta \text{ limit}$$

Notice that, in particular, if $G \supseteq \delta$, then $\sum_{\gamma<\delta}^{G} \alpha_\gamma = \sum_{\gamma<\delta}^{\#} \alpha_\gamma$, while if $G \cap \delta = \emptyset$, then $\sum_{\gamma<\delta}^{G} \alpha_\gamma = \sum_{\gamma<\delta} \alpha_\gamma$.

Of course, the definition of $\sum_{\gamma<\delta}^{G} \alpha_\gamma$ depends only on $G \cap \delta$, that is, we could have assumed that $G \subseteq \zeta$. However, since we shall have frequent occasion to deal with partial sums, it will be notationally convenient to take G as an arbitrary set of ordinals.

As custom by now, let S^G_ε abbreviate $\sum_{\gamma<\varepsilon}^{G} \alpha_\gamma$, and similarly when considering other sets of ordinals in place of G. Also $S^G_{[\delta,\zeta)}$ has the usual meaning, that is, $S^G_{[\delta,\zeta)} = \sum_{\delta\leqslant\gamma<\zeta}^{G} \alpha_\gamma = \sum_{\varepsilon'<\varepsilon}^{H} \alpha_{\delta+\varepsilon'}$, where ε is the only ordinal such that $\delta + \varepsilon = \zeta$, and $\varepsilon' \in H$ if and only if $\delta + \varepsilon' \in G$.

Lemma 3.9. *Under the assumptions in Definition 3.8, if $\delta \leqslant \zeta$, then*

$$\sum_{\gamma<\delta}^{G} \alpha_\gamma \mathbin{\#} \sum_{\delta\leqslant\gamma<\zeta}^{G} \alpha_\gamma \geqslant \sum_{\gamma<\zeta}^{G} \alpha_\gamma \geqslant \sum_{\gamma<\delta}^{G} \alpha_\gamma + \sum_{\delta\leqslant\gamma<\zeta}^{G} \alpha_\gamma$$

Proof. Similar to the proof of Lemma 3.4. At certain points in the successor induction step one needs Proposition 2.2(7). □

As a consequence of Theorem 3.7, we get a way for computing $\sum_{\gamma<\zeta}^{G} \alpha_\gamma$ in the case when ζ is limit.

Corollary 3.10. *Under the assumptions in Theorem 3.7 and if $G \subseteq \zeta$, there is $\bar\gamma < \zeta$ such that, for every $\varepsilon \geqslant \bar\gamma$,*

$$\sum_{\gamma<\zeta}^{G} \alpha_\gamma = \sum_{\gamma<\varepsilon}^{G} \alpha_\gamma + \omega^\xi$$

Proof. Let $\bar\gamma$ be given by Theorem 3.7. First, using the case $\varepsilon = \bar\gamma$ in equation (4) in Theorem 3.7, compute $\omega^\xi = S_{[\bar\gamma,\zeta)} \geqslant S^G_{[\bar\gamma,\zeta)} \geqslant S^\Sigma_{[\bar\gamma,\zeta)} = \omega^\xi$, hence $S^G_{[\bar\gamma,\zeta)} = \omega^\xi$. Using Lemma 3.9, we get $S^G_\zeta \geqslant S^G_{\bar\gamma} + S^G_{[\bar\gamma,\zeta)} = S^G_{\bar\gamma} + \omega^\xi$. To prove the reverse inequality, notice that, since $(\alpha_\gamma)_{\gamma<\zeta}$ is not eventually zero, we have that $S^G_{[\bar\gamma,\varepsilon)} < S^G_{[\bar\gamma,\zeta)} = \omega^\xi$, for every

ε such that $\bar{\gamma} \leqslant \varepsilon < \zeta$. Then $S^G_\varepsilon \leqslant S^G_{\bar{\gamma}} \# S^G_{[\bar{\gamma},\varepsilon)} < S^G_{\bar{\gamma}} + \omega^\xi$, for ε with $\bar{\gamma} \leqslant \varepsilon < \zeta$, by Lemma 3.9 and Proposition 2.2(3). Hence $S^G_\zeta = \sup_{\varepsilon<\zeta} S^G_\varepsilon \leqslant S^G_{\bar{\gamma}} + \omega^\xi$. □

Corollary 3.11. *Suppose that ζ is an ordinal and $(\alpha_\gamma)_{\gamma<\zeta}$ is a sequence of ordinals. Then the following hold.*

(1) There is a finite set F (depending on the sequence) such that $F \subseteq \zeta$ and $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma = \sum^F_{\gamma<\zeta} \alpha_\gamma$.
(2) More generally, we can choose a finite $F \subseteq \zeta$ in such a way that $\sum^G_{\gamma<\zeta} \alpha_\gamma = \sum^H_{\gamma<\zeta} \alpha_\gamma$, whenever G, H are sets of ordinals such that $G \cap F = H \cap F$.
(3) In particular, for every set G of ordinals, there is some finite $H \subseteq \zeta$ such that $\sum^G_{\gamma<\zeta} \alpha_\gamma = \sum^H_{\gamma<\zeta} \alpha_\gamma$. Letting G vary among all subsets of ζ, we get only a finite number of values for $\sum^G_{\gamma<\zeta} \alpha_\gamma$.

Proof. We first notice that (3) follow from (2): just take $H = G \cap F$, where F is given by (2). Moreover, the set $\{G \cap F \mid G \subseteq \zeta\}$ is finite.

We shall prove (1) and (2) by induction on ζ.

The result is trivial when $\zeta = 0$. The step from ζ to $\zeta + 1$ is trivial, too: if F works for ζ, then surely $F \cup \{\zeta\}$ works for $\zeta + 1$.

Hence let us assume that ζ is limit. If the sequence $(\alpha_\gamma)_{\gamma<\zeta}$ is constantly zero from some point on, say $\alpha_\gamma = 0$ for $\gamma \geqslant \zeta'$, then we can apply the inductive hypothesis for ζ', getting some F working for $(\alpha_\gamma)_{\gamma<\zeta'}$. But then trivially F works for the original sequence $(\alpha_\gamma)_{\gamma<\zeta}$, too. Hence we can suppose that $(\alpha_\gamma)_{\gamma<\zeta}$ is not eventually zero and apply Theorem 3.7, getting some $\bar{\gamma}$ for which equations (4) and (5) there holds.

Now (1) is easy. By the inductive hypothesis, there is some finite $F \subseteq \bar{\gamma}$ such that $S_{\bar{\gamma}} = S^F_{\bar{\gamma}}$. Then equations (5) and (4) with $\varepsilon = \bar{\gamma}$ give $S_\zeta = S_{\bar{\gamma}} + \omega^\xi = S^F_{\bar{\gamma}} + S^\Sigma_{[\bar{\gamma},\zeta)} = S^F_\zeta$, where the last identity is proved by induction on ζ', with $\bar{\gamma} \leqslant \zeta' \leqslant \zeta$, and where the successor step uses the assumption $F \subseteq \bar{\gamma}$, and the limit step uses (right) continuity of $+$ at limits.

To prove (2), use again the inductive hypothesis to get some finite $F \subseteq \bar{\gamma}$ such that $S^G_{\bar{\gamma}} = S^H_{\bar{\gamma}}$, whenever $G \cap F = H \cap F$. We claim that F works for ζ, too, that is, $S^G_\zeta = S^H_\zeta$, whenever $G \cap F = H \cap F$. Indeed, by Corollary 3.10, $S^G_\zeta = S^G_{\bar{\gamma}} + \omega^\xi$, for every $G \subseteq \zeta$ (notice that in Corollary 3.10 we can choose the same $\bar{\gamma}$ as the one given by Theorem 3.7). This is enough, since if $G \cap F = H \cap F$, then, by the inductive assumption $S^G_{\bar{\gamma}} = S^H_{\bar{\gamma}}$, and then $S^G_\zeta = S^G_{\bar{\gamma}} + \omega^\xi = S^H_{\bar{\gamma}} + \omega^\xi = S^H_\zeta$. □

4. Various kinds of mixed sums

Definition 4.1. If $(\alpha_i)_{i \in I}$ is a sequence of ordinals, we say that an ordinal β is a *mixed sum* of $(\alpha_i)_{i \in I}$ if there are pairwise disjoint subsets $(A_i)_{i \in I}$ of β such that $\bigcup_{i \in I} A_i = \beta$ and, for every $i \in I$, A_i has order type α_i, with respect to the order induced on A_i by β.

In the above situation, we say that β is a mixed sum of $(\alpha_i)_{i \in I}$ *realized by* $(A_i)_{i \in I}$, or simply that β is realized by $(A_i)_{i \in I}$ (notice that the α_i's can be retrieved from the A_i's, hence the terminology is not ambiguous).

Given a realization $(A_i)_{i \in I}$ and $\bar{\imath} \in I$, we say that $A_{\bar{\imath}}$ is *convex (in the realization $(A_i)_{i \in I}$)* if $(a, a')_\beta = \{\beta' < \beta \mid a < \beta' < a'\} \subseteq A_{\bar{\imath}}$, for every $a < a' \in A_{\bar{\imath}}$.

Carruth [Ca], in different terminology, showed that $\alpha_1 \# \alpha_2$ is the largest mixed sum of α_1 and α_2. In general, when I is infinite, there is no largest mixed sum of a sequence of ordinals; see the comment after Theorem 4.2 in [L1]. Hence Carruth theorem can be generalized only if one restricts to mixed sums satisfying particular properties. This has been done in [L1] in the case of mixed sums related to the countably infinite natural sum. We shall treat here the case of arbitrary iterated natural sums, a case which is slightly more involved.

Theorem 4.2. *Suppose that ζ is an ordinal, $(\alpha_\gamma)_{\gamma < \zeta}$ is a sequence of ordinals and $G \subseteq \zeta$. Put $\beta = \sum^{G}_{\gamma < \zeta} \alpha_\gamma$. Then β is a mixed sum of $(\alpha_\gamma)_{\gamma < \zeta}$. Moreover, β can be realized by $(A_\gamma)_{\gamma < \zeta}$ in such a way that*

(1) For every $\varepsilon < \zeta$, the set $\Gamma_\varepsilon = \{\gamma < \zeta \mid \gamma > \varepsilon \text{ and } b < a, \text{ for some } b \in A_\gamma \text{ and } a \in A_\varepsilon\}$ is finite, and
(2) all but a finite number of the A_ε's are convex in the realization.

Corollary 4.3. *In particular, $\sum^{\#}_{\gamma < \zeta} \alpha_\gamma$ is a mixed sum of $(\alpha_\gamma)_{\gamma < \zeta}$ and it can be realized in such a way that (1) and (2) above hold. Moreover, $\sum^{\#}_{\gamma < \zeta} \alpha_\gamma$ is the largest mixed sum of $(\alpha_\gamma)_{\gamma < \zeta}$ that can be realized in such a way that the following condition (weaker than (1) above) is satisfied.*

(3) For every $\varepsilon < \zeta$ and $a \in A_\varepsilon$, the set $\{\gamma < \zeta \mid \gamma > \varepsilon \text{ and } b < a, \text{ for some } b \in A_\gamma\}$ is finite.

In particular, $\sum^{\#}_{\gamma < \zeta} \alpha_\gamma$ is also the largest mixed sum of $(\alpha_\gamma)_{\gamma < \zeta}$ that can be realized in such a way that both (1) and (2) hold.

Before we can prove Theorem 4.2 and Corollary 4.3 we need to introduce some auxiliary definitions. These will be needed in order to state a stronger version of condition (2) above. The stronger condition will make the inductive proof easier.

Definition 4.4. Suppose that $\delta > 0$ is an ordinal. Express δ in Cantor normal form as $\delta = \omega^{\xi_k} r_k + \omega^{\xi_{k-1}} r_{k-1} + \dots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0$. We say that a subset B of δ is a *block of* δ if B has the form $[\beta_1, \beta_2) = \{\beta' \mid \beta_1 \leqslant \beta' < \beta_2\}$, for some β_1 and β_2 of the form $\beta_1 = \omega^{\xi_k} r_k + \dots + \omega^{\xi_\ell} s$ and $\beta_2 = \omega^{\xi_k} r_k + \dots + \omega^{\xi_\ell}(s+1)$, where ω^{ξ_ℓ} actually appears in the normal expression of δ and $s + 1 \leqslant r_\ell$ (we allow $s = 0$, and we allow $\ell = k$, thus β_1 is allowed to be 0). Notice that a block has order type ω^ξ, for some ξ (the block in the previous statement has order type ω^{ξ_ℓ}).

Essentially, in the above terminology, the Cantor normal form of δ provides the way of realizing δ as a finite sequence of blocks, one put after the other in decreasing order with respect to length (here and below, *decreasing* is intended in the broader sense, not necessarily in the sense of *strictly decreasing*).

Definition 4.5. Suppose that β is a mixed sum of α_1 and α_2, realized by A_1, A_2. By a slight abuse of terminology, we call $B \subseteq A_1$ a *block of* A_1 if B is the image of some block of α_1 under the order preserving bijection (recall that, by definition, A_1 has order type α_1), and the same for $B \subseteq A_2$. Notice that a block of A_1 is not necessarily also a block of β (see below for details).

We say that the realization A_1, A_2 is *pure* if, given a block B_1 of A_1 and a block B_2 of A_2, either all elements of B_1 precede all elements of B_2, or conversely. In other words, a realization is pure if all blocks from A_1 and all blocks from A_2 are convex subsets of $\beta = A_1 \cup A_2$.

We say that β is a *pure mixed sum* of α_1 and α_2 if it has some pure realization.

What will be relevant here is that if $\beta = \alpha_1 \# \alpha_2$, then β has a pure realization. This is the standard way to show that $\alpha_1 \# \alpha_2$ is a mixed sum of α_1 and α_2: just take all the blocks from both α_1 and α_2 and "put them together" ordered by decreasing length. In this case, the blocks of β are exactly the (images of the) blocks of α_1 and of α_2.

Notice that not every mixed sum is pure. For example, ω is a mixed sum of $\alpha_1 = \omega$ and $\alpha_2 = \omega$, but the only pure mixed sum of α_1 and α_2 is $\omega + \omega$. Notice also that, for every α_1 and α_2, both $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_1$ are pure mixed sums of α_1 and α_2. This shows that if A_1, A_2 is a pure realization of some β, it is not necessarily the case that every block of A_1 (or of A_2) is also a block of β. For example, if $\alpha_1 = 1$, $\alpha_2 = \omega$ and $\beta = \alpha_1 + \alpha_2 = \omega$, we get the only pure realization of β by putting all the elements of α_2 after the element of α_1. In this example, β is a single block, which is the union of a block from α_1 and a block from α_2.

On the other hand, we do have that, in a pure realization, each block of the components is a *subset* of some block of the mixed sum.

Lemma 4.6. *Suppose that A_1, A_2 is a pure realization of β. Then each block of A_1 is contained in some block of β, and similarly each block of A_2 is contained in some block of β.*

Proof. Indeed, suppose that B_1 is a block of A_1 and, say, B_1 has type ω^ξ. There are two cases. Either B_1 precedes some block from A_2 of type ω^η, for some $\eta > \xi$, or all blocks from A_2 which are after B_1 have type $\leqslant \omega^\xi$. We can speak of ordering between blocks since the realization is pure.

In the first case, let B_2 be the first block from A_2 which lies after B_1. Notice that, by definition, all blocks of A_2 are disposed in decreasing length, hence if some block of A_2 after B_1 is strictly longer than B_1, this is also true for the first block of A_2 after B_1. Hence B_2 has type, say, $\omega^\eta > \omega^\xi$. Then B_1 is "absorbed" by B_2, that is, they lie in the same block of β (if there are further blocks of A_1 between B_1 and B_2, they are either as long as or shorter than B_1, hence they, too, are absorbed by B_2).

By the same reason, in the second case, B_1 is contained in a block of β, since all blocks after B_1, either from A_1 or from A_2 are shorter than or as long as B_1 (B_1 might absorb some block of A_2 which lies before it, hence B_1 is not necessarily a block of β).

A symmetrical argument works for each block of A_2, thus the lemma is proved. □

Probably the notion of a pure mixed sum (both in the case of a finite number and of an infinite number of summands) deserves further study, but we shall not pursue it here.

Lemma 4.7. *Theorem 4.2 holds when clause (2) there is strengthened to*

(2′) There is some finite $F \subseteq \zeta$ such that if $\varepsilon \in \zeta \setminus F$, then A_ε is convex and is contained in some block of β.

Proof. By induction on ζ.

The result is trivial for $\zeta = 0$.

Suppose that $\zeta = \delta + 1$. If $\delta \notin G$, then the recursive definition 3.8 gives $\beta = S^G_\zeta = S^G_\delta + \alpha_\delta$. By the inductive hypothesis, S^G_δ is a mixed sum of $(\alpha_\gamma)_{\gamma<\delta}$, and can be realized by $(A_\gamma)_{\gamma<\delta}$ in such a way that (1) from 4.2 and (2′) from the present lemma are satisfied. But then, letting A_δ be a copy of α_δ, and adding A_ζ "at the top", we get that $(A_\gamma)_{\gamma<\zeta}$ realizes $\beta = S^G_\zeta$ and trivially satisfies (1) and (2′).

Next suppose that $\zeta = \delta + 1$ and $\delta \in G$. Then Definition 3.8 gives $\beta = S^G_\zeta = S^G_\delta \# \alpha_\delta$. Again by the inductive hypothesis, S^G_δ can be realized by some sequence $(A_\gamma)_{\gamma<\delta}$ which satisfies (1) and (2′). In particular, $\bigcup_{\gamma<\delta} A_\gamma$ is the ordinal S^G_δ. By the remark after Definition 4.5, $\beta = S^G_\delta \# \alpha_\delta$ can be realized as a pure mixed sum by A, A_δ, where A has order type S^G_δ and A_δ has order type α_δ. Letting $\varphi : S^G_\delta \to A$. be the order preserving bijection and setting $A'_\gamma = \varphi(A_\gamma)$, for $\gamma < \delta$, and $A'_\delta = A_\delta$, then clearly $(A'_\gamma)_{\gamma<\zeta}$ is a realization of $\beta = S^G_\zeta$. Now condition (1) in 4.2 is satisfied, since, for each $\varepsilon < \delta$, the construction adds at most one element to Γ_ε; indeed, the only element which perhaps should be added is δ. On the other hand, Γ_δ is empty, thus $(A'_\gamma)_{\gamma<\zeta}$ satisfies (1). As far as (2′) is concerned, we have by the inductive hypothesis that $(A_\gamma)_{\gamma<\delta}$ gives a realization of S^G_δ such that (2′) is satisfied, thus there is a finite $F \subseteq \delta$ such that if $\varepsilon \in \delta \backslash F$, then A_ε is convex and contained in some block of S^G_δ. By construction, blocks of S^G_δ are sent by φ to blocks of A; moreover, since $S^G_\zeta = S^G_\delta \# \alpha_\delta$ is realized as a pure mixed sum by A, A_δ, then every block of A is contained in some block of S^G_ζ, by Lemma 4.6. Thus if $\varepsilon \in \delta \backslash F$, then A'_ε is contained in some block of S^G_ζ. Moreover, if $\varepsilon \in \delta \backslash F$, then A'_ε is convex as a subset of A; but then A'_ε is also convex in S^G_ζ, since A'_ε is convex in A, A'_ε is contained in some block of A, and all the blocks of A are convex in S^G_ζ, the realization of S^G_ζ being pure. Hence (2′) holds at step ζ by taking $F' = F \cup \{\delta\}$.

Suppose now that ζ is a limit ordinal. If the sequence $(\alpha_\gamma)_{\gamma<\zeta}$ is constantly zero from some point on, then the result is immediate from the inductive hypothesis. Otherwise, let $\bar{\gamma}$ be given by Theorem 3.7. By the inductive hypothesis, $S^G_{\bar{\gamma}}$ can be realized by some sequence $(A_\gamma)_{\gamma<\bar{\gamma}}$ which satisfies (1) and (2′). By equation (4) in Theorem 3.7, $\sum_{\bar{\gamma}\leqslant\gamma<\zeta} \alpha_\gamma = \omega^\xi$. Then the order-theoretical characterization of $\sum$ shows that ω^ξ can be represented as a mixed sum of $(\alpha_\gamma)_{\bar{\gamma}\leqslant\gamma<\zeta}$ in such a way that all the pieces realizing the mixed sum are convex in the realization and, moreover, they are disposed in the same order as the corresponding α_γ's. If we join the two representations by putting all the elements representing ω^ξ above the elements representing $S^G_{\bar{\gamma}}$, we get the ordinal $S^G_{\bar{\gamma}} + \omega^\xi$, which is equal to $\beta = S^G_\zeta$, by Corollary 3.10. This new representation clearly satisfies (1); indeed, $\Gamma_\varepsilon = \varnothing$, if $\varepsilon \geqslant \bar{\gamma}$, and Γ_ε remains the same of the sequence $(A_\gamma)_{\gamma<\bar{\gamma}}$, if $\varepsilon < \bar{\gamma}$. Also (2′) is satisfied, since the new elements of the representation (those with $\varepsilon \geqslant \bar{\gamma}$) are all contained in the single block corresponding to ω^ξ, hence F does not become larger. Notice that ω^ξ could absorb some other blocks (of $S^G_{\bar\gamma}$) below it, but, even in case this happens, ω^ξ absorbs the whole of such blocks, hence each A_γ, for $\gamma < \bar\gamma$, is contained in a single block anyway, in the representation of β.

We have finished the proof of Lemma 4.7, hence of Theorem 4.2, as well. $\square$

Proof of 4.3. Since, as we have noticed before, $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ is the particular case of $\sum^{G}_{\gamma<\zeta} \alpha_\gamma$ when $G = \zeta$, we get from Theorem 4.2 that $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ is a mixed sum of $(\alpha_\gamma)_{\gamma<\zeta}$ and can be realized in such a way that (1) and (2) from 4.2 are satisfied, hence (3), too, is satisfied. It remains to show that $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ is the largest one among those mixed sums that satisfy (3). Again, this is proved by induction on ζ.

The result is trivial if $\zeta = 0$.

Let $\zeta = \delta + 1$ be a successor ordinal and let β' be any mixed sum of $(\alpha_\gamma)_{\gamma<\zeta}$ realized by $(A_\gamma)_{\gamma<\zeta}$ in such a way that (3) is satisfied. Set $A = \bigcup_{\gamma<\delta} A_\gamma$ and suppose that A, as a subset of β', has order type β''. Then, through a suitable bijection, $(A_\gamma)_{\gamma<\delta}$ gives a realization of β'', and this realization trivially satisfies (3), since the original realization $(A_\gamma)_{\gamma<\zeta}$ satisfies (3). By the inductive hypothesis, $\beta'' \leqslant S_\delta$, hence, since the pair A, A_δ gives a representation of β' as a mixed sum of β'' and α_δ, we get, by Carruth Theorem and Definition 3.1, that $\beta' \leqslant \beta'' \# \alpha_\delta \leqslant S_\delta \# \alpha_\delta = S_\zeta$. The successor step has thus been proved.

Let ζ be a limit ordinal. If the sequence $(\alpha_\gamma)_{\gamma<\zeta}$ is constantly zero from some point on, then the result follows trivially from the inductive hypothesis, hence we can suppose that $(\alpha_\gamma)_{\gamma<\zeta}$ is not eventually zero. Let again β' be any mixed sum of $(\alpha_\gamma)_{\gamma<\zeta}$ realized by $(A_\gamma)_{\gamma<\zeta}$ in such a way that (3) is satisfied. Since the sequence is not eventually zero and ζ is a limit ordinal, then, by (3), β', too, is a limit ordinal. Hence it is enough to show that, for every $\xi < \beta'$, we have $\xi \leqslant S_\zeta$.

So let $\xi < \beta'$, say, $\xi \in A_\varepsilon$. By (3), the set $\{\gamma < \zeta \mid \gamma > \varepsilon$ and $b < \xi$, for some $b \in A_\gamma\}$ is finite; enumerate it as $\gamma_0, \dots, \gamma$ and let $C_i = A_{\gamma_i} \cap [0, \xi)$, for $i = 0, \dots, n$. Setting $C = \bigcup_{\gamma<\varepsilon} A_\gamma$, we have that ξ is a mixed sum of C, C_0, $\dots C_n$. For each i, if β_i is the order type of C_i, then $\beta_i \leqslant \alpha_{\gamma_i}$, since the latter is the order type of A_{γ_i} and $C_i \subseteq A_{\gamma_i}$. Moreover, by the inductive hypothesis, if β'' is the order type of C, then $\beta'' \leqslant S_\varepsilon$. Since ξ is a mixed sum of C, C_0, $\dots C_n$, then, again by Carruth Theorem, $\xi \leqslant \beta'' \# \beta_0 \# \dots \# \beta_n \leqslant S_\varepsilon \# \alpha_{\gamma_0} \# \dots \# \alpha_{\gamma_n} \leqslant S_\zeta$, where, in order to get the last inequality, notice that, by construction, $\gamma_0, \dots, \gamma_n > \varepsilon$, hence the inequality follows from Proposition 3.3(5). $\square$

Notice that the proof also shows that $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ can be realized in such a way that (1) from 4.2 and (2$'$) from 4.7 are satisfied.

5. Invariant infinite natural sums

The problem of invariance under permutations. All the previous notions and results are dependent on the order in which the α_γ's appear in the sequence $(\alpha_\gamma)_{\gamma<\zeta}$. In particular, for $\zeta > \omega$, the value of $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ is generally *not* invariant under permutations (invariance holds, however, for a very special class of permutations, as we shall mention in Proposition 5.5 below). To see that a transfinite sum is not necessarily invariant, just take $\zeta = \omega + 1$, $\alpha_0 = 0$ and $\alpha_\gamma = 1$, for $0 < \gamma < \omega+1$, thus $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma = \omega+1$. On the other hand, if we permute α_0 and α_ω, that is, we reindex the α_γ's as $(\beta_\gamma)_{\gamma<\omega+1}$ by letting $\beta_\gamma = 1$, for $0 \leqslant \gamma < \omega$ and $\beta_\omega = 0$, then we get $\sum^{\#}_{\gamma<\zeta} \beta_\gamma = \omega \neq \omega + 1$.

Noninvariance of $\sum^{\#}$ under permutations strongly contrasts with [L1], that is, with the case of ω-indexed sequences. In fact, essentially all the results in [L1] are independent from the chosen ordering of the α_i's and some results there do not even mention the ordering. For example, in [L1, Theorem 4.7] we proved that the natural sum of an ω-indexed sequence of ordinals is the maximum of all the left-finite mixed sums of the ordinals in the sequence. Recall the definition of a mixed sum from 4.1. The definition of left-finiteness is recalled in the next definition.

Definition 5.1. A mixed sum β of $(\alpha_i)_{i\in I}$ is *left-finite* if it can be realized by $(A_i)_{i\in I}$ in such a way that, for every $a \in \beta$, the set of all the elements smaller than a is contained in the union of a finite number of A_i's.

Notice that, in the specific case of an ω-indexed sequence, left-finiteness is equivalent to condition (3) in Corollary 4.3; moreover, since we are assuming that $\zeta = \omega$, the condition is independent from the ordering of the sequence. Hence Theorem 6.4 can be obtained as a special case of Corollary 4.3. Notice also that it is by no means trivial that the set of all the left-finite mixed sums of some given ω-indexed sequence has a maximum, not simply a supremum.

A naïve approach in search of a generalization of the above mentioned Theorem 4.7 from [L1] for, say, a sequence indexed by a set of cardinality ω_1, would be to restrict oneself to *left-countable* mixed sums, that is, asking that, for every element $a \in \beta$, the set of all the elements smaller than a is contained in the union of countably many A_γ's. Here and below, by *countable* we mean either finite or denumerably infinite. However, fixed an ω_1-sequence of ordinals, the set of the left-countable mixed sums of the sequence might not have a maximum. Just take $\alpha_\gamma = \omega_1$, for $\gamma < \omega$, and $\alpha_\gamma = 1$, for $\omega \leqslant \gamma < \omega_1$. Every ordinal of the form $\omega_1\varepsilon$, for $0 \neq \varepsilon < \omega_1$, is a left-countable mixed sum of $(\alpha_\gamma)_{\gamma<\omega_1}$ but this is not the case for the supremum of the above values, i. e., ω_1^2. If $\varepsilon > \omega$, then in the above example we can even realize the left-countable mixed sum in such a way that all pieces are convex.

The above example suggests that it will be difficult, or perhaps impossible, to find some natural infinitary generalization of the Hessenberg sum for sequences indexed by a set which is not supposed to be (well-)ordered; or, put in another way, that the countable case of the infinitary natural sum is very special and, usually, results do not generalize to uncountable cardinals. In this respect, see also the remarks on [VW, p. 370] and our review [L2] of [VW]. In particular, it seems difficult to find some infinitary operation (on uncountably many arguments) which has some good purely order-theoretical characterization and which does not rely on the ordering of the sequence. See, however, [L4] for a possible alternative approach to the problem. Notice that, on the other hand, when the sequence is well-ordered, Corollary 4.3 provides such an order-theoretical characterization for the iterated natural sum of Definition 3.1.

Some invariant sums. Since, in general, by the above example, it is probably not always possible to find some kind of "maximal sum", we can at least define some minimal ones (which, by the very definition, will turn out to be automatically invariant under permutations).

Definitions 5.2. Suppose that ζ is an ordinal and $(\alpha_\gamma)_{\gamma<\zeta}$ is a sequence of ordinals. Define

$$\mathop{\#^{o}}_{\gamma<\zeta} \alpha_\gamma = \inf_{\pi} \sum_{\gamma<\zeta}^{\#} \alpha_{\pi(\gamma)}$$

where π varies among all the permutations of ζ. By a *permutation of* ζ we mean a bijection from ζ to ζ. Notice that in the above definition we are keeping ζ fixed. Allowing ζ to change, we generally obtain different results; for example, if $\alpha_\gamma = 1$, for every γ, then $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma = \sum_{\gamma<\zeta} \alpha_\gamma = \zeta$, for every ζ. In the next definition, on the contrary, we let the ordinal vary.

Suppose that I is *any* set and $(\alpha_i)_{i\in I}$ is a sequence of ordinals. Define

$$\mathop{\#}_{i\in I} \alpha_i = \inf_{\zeta, f} \sum_{\gamma<\zeta}^{\#} \alpha_{f(\gamma)}$$

where ζ varies among all the ordinals having cardinality $|I|$ and f varies among all the bijections from ζ to I. In this situation, we shall call f a *rearrangement* of the sequence.

Furthermore, let $\lambda = |I|$ and define

$$\mathop{\#^{\bullet}}_{i \in I} \alpha_i = \inf_f \sum_{\gamma < \lambda}^{\#} \alpha_{f(\gamma)}$$

where f varies among all the bijections from λ to I.

The difference between $\#$ and $\#^{\bullet}$ is that in $\#$ we consider rearrangements into a sequence of arbitrary length, while in $\#^{\bullet}$ we consider only rearrangements into a sequence of length $|I|$. On the other hand, in $\#^{o}$ the length of the sequences is assumed to be fixed.

For every I, trivially, $\#_{i \in I} \alpha_i \leqslant \#^{\bullet}_{i \in I} \alpha_i$ and if $|I| = |\zeta|$ and $(\alpha_i)_{i \in I}$ is a rearrangement of $(\alpha_\gamma)_{\gamma < \zeta}$, then $\#_{i \in I} \alpha_i \leqslant \#^{o}_{\gamma < \zeta} \alpha_\gamma$. Of course, we could have written the above inequality simply as $\#_{\gamma < \zeta} \alpha_\gamma \leqslant \#^{o}_{\gamma < \zeta} \alpha_\gamma$, with no need of introducing rearrangements, since an ordinal is, in particular, a set. However, it seems clearer to use a letter such as I for a set on which no particular order is defined, and we shall usually obey this convention. Though, as just mentioned, $\#$ is always $\leqslant$ than both $\#^{\bullet}$ and $\#^{o}$, on the other hand, in general, there is no provable inequality between $\#^{\bullet}$ and $\#^{o}$. Indeed, if $\alpha_\gamma = 1$, for every $\gamma < \omega + 1$, then $\#^{\bullet}_{\gamma < \omega+1} \alpha_\gamma = \omega < \omega + 1 = \#^{o}_{\gamma < \omega+1} \alpha_\gamma$. In the other direction, we shall show in the last sentence in Example 5.3 that $\#^{o}$ can be strictly smaller than $\#^{\bullet}$.

In any case, all the above operations coincide for ω-indexed sequences. If $\zeta = |I| = \omega$ and $(\alpha_i)_{i \in I}$ is a rearrangement of $(\alpha_\gamma)_{\gamma < \omega}$, then

$$\mathop{\#^{o}}_{\gamma < \omega} \alpha_\gamma = \sum_{\gamma < \omega}^{\#} \alpha_\gamma = \mathop{\#^{\bullet}}_{i \in I} \alpha_i = \mathop{\#}_{i \in I} \alpha_i \tag{6}$$

This shows that the notation in Definitions 5.2 is consistent both with [VW] and with [L1]. The first two identities in (6) follow from [VW, p. 362] or [L1, Proposition 2.4(5)], to the effect that the ω-indexed natural sum is invariant under permutations, a fact which shall be generalized in Proposition 5.5 below. Notice that invariance of $\sum^{\#}_{\gamma < \omega} \alpha_\gamma$ under permutations is also an immediate consequence of the order-theoretical characterization given in [L1, Theorem 4.7] and mentioned in the preceding subsection. The last identity in (6) shall be proved in Proposition 5.4 below. Before giving the proof, we provide a counterexample that shows the reason why the proof is not entirely trivial.

Example 5.3. It is somewhat surprising that it is possible to have the strict inequality $\#_{i \in I} \alpha_i < \#^{\bullet}_{i \in I} \alpha_i$ in case I is uncountable. This is somewhat counterintuitive, since, allowing a longer sequence, we might get a smaller outcome. Let $|I| = \omega_1$, $\alpha_i \in \{1, \omega_1\}$, for $i \in I$, $|\{i \in I \mid \alpha_i = 1\}| = \omega_1$ and $|\{i \in I \mid \alpha_i = \omega_1\}| = \omega$. For every bijection $f : \lambda \to I$, we have $\sum^{\#}_{\gamma<\lambda} \alpha_{f(\gamma)} \geqslant \omega_1(\omega + 1)$, in fact, we already have $\sum_{\gamma<\lambda} \alpha_{f(\gamma)} \geqslant \omega_1(\omega + 1)$. Recall that sums and products are always intended in the ordinal sense. Thus $\#^{\bullet}_{i\in I} \alpha_i \geqslant \omega_1(\omega + 1)$, in fact, $\#^{\bullet}_{i\in I} \alpha_i = \omega_1(\omega + 1)$, the example giving the reverse inequality being easy.

However, $\#_{i\in I} \alpha_i = \omega_1\omega$. Indeed, let $\zeta = \omega_1 + \omega$, $\beta_\gamma = 1$, for $\gamma < \omega_1$, and $\beta_\gamma = \omega_1$, for $\omega_1 \leqslant \gamma < \omega_1 + \omega$. Then $\sum^{\#}_{\gamma<\omega_1+\omega} \beta_\gamma = \sum_{\gamma<\omega_1+\omega} \beta_\gamma = \omega_1\omega$. Since the β_γ's are a rearrangement of the α_i's, we get $\#_{i\in I} \alpha_i \leqslant \omega_1\omega$. The reverse inequality is obvious.

Notice that we have also showed that $\#^{o}_{\gamma<\omega_1+\omega} \beta_\gamma = \omega_1\omega < \omega_1(\omega + 1) = \#^{\bullet}_{i\in I} \alpha_i$.

Similar counterexamples are well-known, when corresponding definitions are considered relative to the usual ordinal sum $\sum$ in place of $\sum^{\#}$. See Rado [R], in particular, p. 219 therein, where a counterexample is given even in the countable case. It is quite interesting that, on the other hand, the natural sum is immune to such counterexamples, as far as countable sums are taken into account.

Proposition 5.4. *If $|I| \leqslant \omega$, then $\#^{\bullet}_{i\in I} \alpha_i = \#_{i\in I} \alpha_i$.*

Proof. If I is finite, the definitions are clearly the same, so let $|I| = \omega$. As we mentioned above, the inequality $\#_{i\in I} \alpha_i \leqslant \#^{\bullet}_{i\in I} \alpha_i$ is trivial; moreover, $\#^{\bullet}_{i\in I} \alpha_i = \sum^{\#}_{\gamma<\omega} \alpha_\gamma$, for every rearrangement of the α_i's into a sequence of length ω. We have to show that if ζ is a countably infinite ordinal and $(\beta_\delta)_{\delta<\zeta}$ is another rearrangement of the α_i's, this time into a sequence of length ζ, then $\sum^{\#}_{\gamma<\omega} \alpha_\gamma \leqslant \sum^{\#}_{\delta<\zeta} \beta_\delta$.

Let ξ be the smallest ordinal such that $\{i \in I \mid \alpha_i \geqslant \omega^\xi\}$ is finite. Enumerate those α_i's such that $\alpha_i \geqslant \omega^\xi$ as $\alpha_{i_0}, \dots, \alpha_{i_h}$ (the sequence might be empty). If $\xi = 0$, then all but a finite number of the α_i's are zero and the result follows easily from the finite case (e. g., use Proposition 3.3(4)). If $\xi > 0$, then equation (6) in [L1, Corollary 5.1] gives $\#^{\bullet}_{i<\omega} \alpha_i = \sum^{\#}_{\gamma<\omega} \alpha_\gamma = \alpha_{i_0}^{\restriction\xi} \# \dots \# \alpha_{i_h}^{\restriction\xi} \# \omega^\xi$ (recall the definition of $\alpha^{\restriction\xi}$ given right before Proposition 2.2).

Turning to the rearrangement $(\beta_\delta)_{\delta<\zeta}$, let $\beta_{\delta_0}, \dots, \beta_{\delta_h}$, with $\delta_0 < \dots < \delta_h$, be an enumeration of those β_δ's such that $\beta_\delta \geqslant \omega^\xi$ (of course, $\beta_{\delta_0}, \dots, \beta_{\delta_h}$ is a rearrangement of $\alpha_{i_0}, \dots, \alpha_{i_h}$). Now the proof splits into two cases. First, suppose that ξ is a successor ordinal, say, $\xi = \xi' + 1$. Then, by the very definition of ξ, there are infinitely many β_δ's such that $\omega^\xi > \beta_\delta \geqslant \omega^{\xi'}$. Hence we can choose a subsequence of

$(\beta_\delta)_{\delta<\zeta}$ of order type ω and consisting of elements $\geqslant \omega^{\xi'}$ and $< \omega^\xi$. Define another sequence $(\beta'_\delta)_{\delta<\zeta}$ obtained from $(\beta_\delta)_{\delta<\zeta}$ by leaving unchanged the values of the elements of the above subsequence, by leaving unchanged the values of $\beta_{\delta_0}, \dots, \beta_{\delta_h}$, as well, and turning to 0 all the other values. By Proposition 3.3(2), $\sum^{\#}_{\delta<\zeta} \beta_\delta \geqslant \sum^{\#}_{\delta<\zeta} \beta'_\delta$. By Proposition 3.3(4), $\sum^{\#}_{\delta<\zeta} \beta'_\delta = \sum^{\#}_{\varepsilon<\omega+k} \beta''_\varepsilon$, where $(\beta''_\varepsilon)_{\varepsilon<\omega+k}$ is the subsequence of the nonzero β'_δ's, thus k is finite. Then, applying again [L1, Corollary 5.1], we get, for some $j \leqslant h$ (in fact, j is such that $j+k=h$), $\sum^{\#}_{\delta<\zeta} \beta_\delta \geqslant \sum^{\#}_{\varepsilon<\omega+k} \beta''_\varepsilon = \beta_{\delta_0}^{\restriction\xi} \mathbin{\#} \dots \mathbin{\#} \beta_{\delta_j}^{\restriction\xi} \mathbin{\#} \omega^\xi \mathbin{\#} \beta_{\delta_{j+1}} \mathbin{\#} \dots \mathbin{\#} \beta_{\delta_h} \geqslant \beta_{\delta_0}^{\restriction\xi} \mathbin{\#} \dots \mathbin{\#} \beta_{\delta_j}^{\restriction\xi} \mathbin{\#} \omega^\xi \mathbin{\#} \beta_{\delta_{j+1}}^{\restriction\xi} \mathbin{\#} \dots \mathbin{\#} \beta_{\delta_h}^{\restriction\xi} = \alpha_{i_0}^{\restriction\xi} \mathbin{\#} \dots \mathbin{\#} \alpha_{i_h}^{\restriction\xi} \mathbin{\#} \omega^\xi = \sum^{\#}_{\gamma<\omega} \alpha_\gamma$, what we had to show.

The case when ξ is limit is similar. This time, choose some subsequence of $(\beta_\delta)_{\delta<\zeta}$ of type ω in such a way that, for every $\xi' < \xi$, there is some element of the subsequence which is $\geqslant \omega^{\xi'}$ and $< \omega^\xi$ (notice that if ξ is limit, then necessarily ξ has cofinality ω).

All the rest goes the same way. ☐

Notice that the counterexample in 5.3 shows also that $\#^\bullet$ is not invariant under extending a sequence by adding further 0's, while $\#$ is indeed invariant in this sense. More formally, if $(\alpha_i)_{i\in I}$ is a sequence of ordinals, $J \supseteq I$ and we set $\alpha_i = 0$, for $i \in J \setminus I$, then $\#_{i\in I} \alpha_i = \#_{i\in J} \alpha_i$, as a consequence of Proposition 3.3(4). The analogous identity fails for $\#^\bullet$. Just consider the sequence $(\alpha_i)_{i\in I}$ from 5.3 and let $|J| \geqslant \omega_2$. Then the arguments in 5.3 show that $\#^\bullet_{i\in I} \alpha_i = \omega_1(\omega+1) \neq \omega_1\omega = \#^\bullet_{i\in J} \alpha_i$.

The above remark suggests that $\#$ is perhaps a more natural operation than $\#^\bullet$.

Invariance in special cases. As promised, we now show that $\sum^{\#}$ is invariant under a special class of permutations. First, a definition is in order. If ζ is an ordinal, let us call a subset A of ζ a *component* of ζ if A has either the form $[\alpha, \alpha+\omega)$ or $[\alpha, \zeta)$, where in both cases either $\alpha = 0$ or α is a limit ordinal. Thus the components partition ζ. Of course, there is just one component of the kind $[\alpha, \zeta)$, all the others have length ω. With the above definition, we can show that $\sum^{\#}$ is invariant under the (somewhat special kind of) permutations which act on each component. Moreover, a form of the general associative-commutative law holds in some special cases, to the effect that, besides performing the above kinds of permutations, we can associate sets of *finitely many* elements *inside the same component.*

Proposition 5.5. *(1) If π is a permutation of ζ such that $\pi(C) = C$, for every component C of ζ, then $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma = \sum^{\#}_{\gamma<\zeta} \alpha_{\pi(\gamma)}$.*

(2) More generally, suppose that $(F_h)_{h<\zeta'}$ is a partition of ζ such that
(a) each F_h is finite, say, $F_h = \{\delta_1, \dots, \delta_{r(h)}\}$,
(b) each F_h is a subset of some component C_h of ζ,
(c) the F_h's are ordered in such a way that if C_h occurs before $C_{h'}$ in ζ, then $h < h'$; that is, the ordering of the F_h's respects the ordering of the components, but, inside a component, the ordering of the F_h's can be arbitrary.

Then

$$\sum_{\gamma<\zeta}^{\#} \alpha_\gamma = \sum_{h<\zeta'}^{\#} \mathop{\#}_{\delta\in F_h} \alpha_\delta = \sum_{h<\zeta'}^{\#} (\alpha_{\delta_1} \# \alpha_{\delta_2} \# \dots \# \alpha_{\delta_{r(h)}})$$

Proof. The proposition has an elementary proof similar to [L1, Proposition 2.4(5)(6)].

The proposition can be given also an order-theoretical proof, using Corollary 4.3. As far as (1) here is concerned, just notice that finiteness of the sets in 4.3(3) is preserved under the permutations at hand.

As for (2), let $\beta_h = \#_{\delta\in F_h} \alpha_\delta$, for $h < \zeta'$. Then, applying Corollary 4.3 to $\sum_{h<\zeta'}^{\#} \beta_h$, we get a mixed sum of the β_h's which satisfies condition 4.3(3). Expanding the β_h's using Carruth's theorem (this is possible by (a)), we get a mixed sum of the α_γ's, and this sum satisfies 4.3(3), by the assumptions (b) and (c). Thus, by Corollary 4.3, $\sum_{\gamma<\zeta}^{\#} \alpha_\gamma \geqslant \sum_{h<\zeta'}^{\#} \beta_h$.

Conversely, apply Corollary 4.3 to $\sum_{\gamma<\zeta}^{\#} \alpha_\gamma$. In the mixed sum given by 4.3, for each $\delta \in F_h$, join together $A_{\delta_1}, A_{\delta_2}, \dots A_{\delta_{r(h)}}$, and call B_δ this union. Again by Carruth theorem and (a), the order type β'_δ of B_δ is $\leqslant \beta_h = \#_{\delta\in F_h} \alpha_\delta$. Since the B_δ's realize a mixed sum of the β'_δ's, and this realization satisfies 4.3(3), by (b) and (c), we get $\sum_{\gamma<\zeta}^{\#} \alpha_\gamma \leqslant \sum_{h<\zeta'}^{\#} \beta'_h \leqslant \sum_{h<\zeta'}^{\#} \beta_h$, by Corollary 4.3 and Proposition 3.3(2). □

Problems 5.6. (a) The iterated natural sum can be extended to the surreal numbers, in a way we are going to explain soon. See Conway [Co] for details about surreal numbers and, e. g., Siegel [Sieg] for an updated list of references. A surreal number s can be thought of as an ordinal-indexed string consisting of + and −'s; this is called the *sign expansion* of s. The ordinals can be considered as a substructure of the surreals; in this sense, an ordinal is a surreal having only +'s in its sign expansion. The surreal sum, when restricted to the ordinals, does correspond to the ordinal natural sum. One can also define the limit of a transfinite sequence of surreals; see Mező [M] and [L3]. Roughly, the limit of an ordinal-indexed sequence of surreals is the longest string s such that every initial segment of s is eventually coincident with the corresponding (possibly improper) initial segments of the members of the sequence (we are allowing the length of s to be a successor ordinal, in which case s is required to be eventually an initial segment of the members of the sequence). Notice that the limit s might be much shorter than the superior limit of the lengths of the members of the sequence, actually, s can be the empty sequence! Then Definition 3.1 extends to the surreals. See [L3] for full details.

Which results from the present paper and from [L1] generalize to this surreal iterated sum?

(b) Conversely, an ordinal sum can be defined within the surreals. In the sense of string expansions, it corresponds to string concatenation; see Conway [Co, Chapter 15, p. 193]. It can be obviously iterated through the transfinite. Which results about transfinite ordinal sums (of ordinals) do generalize to the surreals?

Most of the problems which follow can be extended to the surreals, too.

(c) Though, in general, the iterated natural sum $\sum^{\#}$ from 3.1 is not invariant under permutations, one might ask for which sequences $(\alpha_\gamma)_{\gamma<\zeta}$ of ordinals the sum $\sum^{\#}_{\gamma<\zeta} \alpha_\gamma$ turns out to be indeed invariant under permutations. The corresponding problem for the usual transfinite ordinal sum has been studied, see Hickman [H2] and further references there. Of course, for the iterated natural sum this kind of "generalized commutativity" is a much more frequent phenomenon, since it holds for all finite and ω-indexed sequences.

(d) In particular, under which conditions (on an ordinal-indexed sequence of ordinals) do some of the operations $\sum^{\#}$, $\#^{o}$, $\#$, $\#^{\bullet}$ and $\sum$ give the same outcome? One can also take into account the operations $\sum^{o}$, $\sum^{*}$ and $\sum^{\bullet}$, which are defined as in Definitions 5.2, by replacing everywhere $\sum^{\#}$ by $\sum$ ($\sum^{*}$ corresponds to $\#$). The operation $\sum^{*}$ has been studied quite thoroughly, see, e. g., Rado [R], Anderson [An2]. Notice that sometimes in the literature the word *permutation* is used to mean what we call here a *rearrangement*. As we mentioned, Rado [R, p. 219] shows that $\sum^{\bullet}$ and $\sum^{*}$ might give different outcomes. On the other hand, the operation $\sum^{o}$, when the index set is not a cardinal, seems to have received less attention.

(e) Of course, there is a more general formulation of Problem (c) above, asking how many values $\sum^{\#}$ assumes when we permute (or, more generally, rearrange) the elements of some given sequence. In the case of $\sum$ the corresponding problem has been studied; see, e. g., Sierpiński [Sier1], Ginsburg [G], Hickman [H1], Komjáth [K] and further references in these papers.

(f) The fact that we do not always have "maximal sums" for $\sum^{\#}$, i. e., that some suprema are not necessarily attained, leaves out the possibility of the existence of maximal sums for special kinds of sequences. For $\sum$ this has been studied; see Dushnik [D] and Anderson [An1].

(g) Study transfinite natural products defined in the same vein as of Definitions 3.1, 3.8 and 5.2. Ideas from Altman [Al] might be relevant to the problem. Here order-theoretical characterizations will be probably much harder to come by. All the problems above can be asked for infinite natural products, too.

For some properties of ordinary (not "natural") transfinite products see [Sier2, XIV, 17] and [B, III, & 10].

(h) By applying the characterization of $\sum^{\#}_{\gamma<\zeta}\alpha_\gamma$ given in Corollary 4.3, one can surely rephrase Definitions 5.2 in order to provide order-theoretical characterizations of $\#^o$, $\#$ and $\#^\bullet$. Such characterizations appear muddled, complicated and far from being useful. Are there simpler and more useful order-theoretical characterizations of $\#^o$, $\#$, $\#^\bullet$?

6. Notions of size for well-founded trees

Wang [W] and Väänänen and Wang [VW] defined notions of size for an $\mathcal{L}_{\omega_1,\omega}$-formula in negation normal form. Recall that $\mathcal{L}_{\omega_1,\omega}$ is the extension of first-order logic in which countable disjunctions and conjunctions are allowed. Since a (possibly infinitary) formula can be viewed as a labeled well-founded tree and Väänänen and Wang's definition depends only on the tree structure, not on the labels, they implicitly give definitions of size for countable well-founded trees (to be pedantic, for those trees arising from $\mathcal{L}_{\omega_1,\omega}$-formulas in negation normal form; notice also that, in the definition of size from [VW], negating an atomic formula does not augment size). By extending their ideas and using Definitions 5.2, we can provide notions of size which apply to every well-founded tree, not only to countable ones.

Here we intend a tree in the classical set-theoretical sense but we shall describe it in terms of the reversed order. A *(reversed) well-founded tree* is a well-founded partially ordered set $(T, \leqslant)$ such that, for every $t \in T$, the set of all successors of t is finite and linearly ordered. By, e. g., [Je, Theorem 2.27], every element t of a well-founded partially ordered set has a well-defined *rank* $\rho(t)$; the rank of t is the smallest ordinal which is strictly larger than all the ranks of the predecessors of t. This justifies inductive definitions on ranks. Ranks go the other direction with respect to *levels*; maximal elements are at level 0 but if the tree has just one maximal element (the *root*) this is the element of largest rank. When the order $\leqslant$ is understood, we shall simply write T in place of $(T, \leqslant)$.

Definition 6.1. If T is a well-founded (reversed) tree, then, for every $t \in T$, we define the *size* $\sigma(t)$ of t by induction on the rank of t as $\sigma(t) = \left(\#_{u \in P(t)} \sigma(u) \right) + 1$, where $P(t)$ is the set of all the immediate predecessors of t. In particular, minimal elements of T have size 1.

The *size* $\sigma(T)$ of T is $\sigma(T) = \#_{t \in M} \sigma(t)$, where M is the set of the maximal elements of T. In particular, if T has a unique root, then $\sigma(T)$ is the size of the root of T.

Notice that when T is finite the above defined size gives the cardinality of T (the number of its nodes).

Similar definitions can be given using $\#^o$ or $\#^\bullet$ in place of $\#$.

In particular, since a formula of a (possibly infinitary) logic can be seen as a (labeled) well-founded tree, the above definition furnishes a possible definition of size for a formula.

The size of a countable well-founded tree can be given an order-theoretical characterization, as we are going to show, after some preliminary definitions.

If $(T, \leqslant)$ is a partially ordered set and $t \in T$, we let $\downarrow t = \{u \in T \mid u \leqslant t\}$. We shall frequently consider another order $\leqslant'$ on T; usually $\leqslant'$ will be an *extension* of $\leqslant$, that is, $u \leqslant t$ implies $u \leqslant' t$ for every $u, t \in T$. In the above situation, we shall denote the set $\{u \in T \mid u \leqslant' t\}$ by $\downarrow' t$.

Definition 6.2. If $(T, \leqslant)$ is a partially ordered set, we say that $\leqslant'$ is *a downward-finite extension of* $\leqslant$ if $\leqslant'$ is an extension of $\leqslant$ and, for every $v \in T$, there are a finite number $u_0, \dots u_n$ of elements of T which are $\leqslant$-incomparable with v and such that $\downarrow' v \subseteq \downarrow v \cup \downarrow u_0 \cup \dots \cup \downarrow u_n$.

If $\leqslant$ is understood, we shall simply say that $\leqslant'$ is downward finite.

It is well-known that every well-founded partial order can be extended to a well-order. However, even in the case of a well-founded tree, the order-types of extensions might be unbounded; just consider an infinite antichain C. It has well-ordered extensions of every order-type having cardinality $|C|$, and the supremum of these order-types is $|C|^+$, which is not attained.

On the other hand, we are going to show that a maximum extension exists if we restrict ourselves to downward-finite well-ordered extensions of countable well-founded trees; moreover, the order-type of this extension is exactly the size of the tree, as introduced in Definition 6.1. In our opinion, this result shows the naturalness (at least in the countable case) both of the definition of $\#$ and of the above notion of size for a well-founded tree.

Theorem 6.3. *If $(T, \leqslant)$ is a countable well-founded (reversed) tree, then $\leqslant$ has a downward-finite extension which is a well-order of type $\sigma(T, \leqslant)$.*

Moreover, every downward-finite well-order extending $\leqslant$ has order-type less than or equal to $\sigma(T, \leqslant)$.

The proof of Theorem 6.3 proceeds through several lemmas. We first recall a result from [L1, Theorem 4.7], which has been mentioned in the previous section and which we shall repeatedly use here. Recall the definition of a mixed sum from 4.1 and the definition of left-finiteness from 5.1. Recall from equation (6) that, for ω-indexed sequences, $\#$ has many equivalent reformulations.

Theorem 6.4. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, then $\#_{i<\omega} \alpha_i$ is the largest left-finite mixed sum of $(\alpha_i)_{i<\omega}$.*

As we mentioned in the previous section, Theorem 6.4 can be obtained also a consequence of Corollary 4.3.

Lemma 6.5. *Suppose that $(T, \leqslant)$ is a countable well-founded tree and M is the set of the maximal elements of T. Furthermore, suppose that, for every $u \in M$, $\leqslant'_u$ is a well-ordered downward-finite extension of $\leqslant_{\restriction \downarrow u}$ in $\downarrow u$, and let α_u be the order-type of $\leqslant'_u$. Then $\leqslant$ has a well-ordered downward-finite extension $\leqslant''$ of order-type $\#_{u \in M} \alpha_u$. Moreover, $\leqslant''$ is such that $\leqslant''_{\restriction \downarrow u}$ is equal to $\leqslant'_u$, for every $u \in M$.*

Proof. By Theorem 6.4, $\#_{u \in M} \alpha_u$ is a left-finite mixed sum of $(\alpha_u)_{u \in M}$. Through the bijections from $\downarrow u$ to α_u given by each of the orders $\leqslant'_u$, we can use a realization of $\#_{u \in M} \alpha_u$ as a left-finite mixed sum of $(\alpha_u)_{u \in M}$ to construct a well-order $\leqslant''$ on T of type $\#_{u \in M} \alpha_u$ (notice that $T = \bigcup_{u \in M} \downarrow u$, since the set of successors of each element of T is finite, hence each element of T is $\leqslant u$, for some $u \in M$). The order $\leqslant''$ is such that $\leqslant''_{\restriction \downarrow u}$ is equal to $\leqslant'_u$, for every $u \in M$. Moreover, $\leqslant''$ has the following property.

(*) For every $v \in T$, the set $\{u \in M \mid w \leqslant'' v, \text{ for some } w \leqslant u\}$ is finite.

(this is the "translation" of left-finiteness to the new situation, since $w \in \downarrow u$ if and only if $w \leqslant u$).

Moreover, $\leqslant''$ extends $\leqslant$, since, by assumption, for every $u \in M$, the order $\leqslant'_u$ extends $\leqslant_{\restriction \downarrow u}$ and since, for $u \neq u^* \in M$, all the elements from $\downarrow u$ are $\leqslant$-incomparable with all the elements from $\downarrow u^*$.

Hence it remains to show that $\leqslant''$ is a downward-finite extension of $\leqslant$.

So let $v \in T$, hence $v \in {\downarrow}u$, for some $u \in M$. Since $\leqslant'_u$ is a downward-finite extension of $\leqslant_{\restriction{\downarrow}u}$, there are elements $v_0, \dots v_n \in {\downarrow}u$ which are $\leqslant$-incomparable with v and such that ${\downarrow}'_u v \subseteq {\downarrow}v \cup {\downarrow}v_0 \cup \dots \cup {\downarrow}v_n$, where, obviously, ${\downarrow}'_u v$ is computed using $\leqslant'_u$ in ${\downarrow}u$, and, since $v, v_0, \dots v_n \in {\downarrow}u$, then applying ${\downarrow}$ to v, v_0, ... gives the same result whether computed in $({\downarrow}u, \leqslant_{\restriction{\downarrow}u})$ or in $(T, \leqslant)$, hence the notation is not ambiguous. Similarly, the incomparabilities of v and v_0, etc., are equivalently evaluated using $\leqslant_{\restriction{\downarrow}u}$ or $\leqslant$.

By (*), and now working in T, there are finitely many elements $u_0, \dots, u_m$ in M such that ${\downarrow}''v \subseteq {\downarrow}u \cup {\downarrow}u_0 \cup \dots \cup {\downarrow}u_m$. Of course, we can assume that $u_0 \neq u$, ..., $u_m \neq u$, hence the u_h's are $\leqslant$-incomparable with v, since $v \leqslant u$, since the set of all the successors of v is linearly ordered and since, for each index h, we have that u and u_h's are distinct maximal elements of T, hence incomparable. Now, ${\downarrow}''v \cap {\downarrow}u = {\downarrow}'_u v$, since $\leqslant''_{\restriction{\downarrow}u}$ is equal to $\leqslant'_u$, which extends $\leqslant_{\restriction{\downarrow}u}$ in ${\downarrow}u$. In conclusion, ${\downarrow}''v \subseteq {\downarrow}'_u v \cup {\downarrow}u_0 \cup \dots \cup {\downarrow}u_m \subseteq {\downarrow}v \cup {\downarrow}v_0 \cup \dots \cup {\downarrow}v_n \cup {\downarrow}u_0 \cup \dots \cup {\downarrow}u_m$, with $v_0, \dots, v_n, u_0, \dots, u_m$ all $\leqslant$-incomparable with v, what we had to show. $\square$

Lemma 6.6. *If $(T, \leqslant)$ is a countable well-founded tree, then, for every $t \in T$, the restriction $\leqslant_{\restriction{\downarrow}t}$ of $\leqslant$ to ${\downarrow}t$ has a well-ordered downward-finite extension of order-type $\sigma(t)$.*

Proof. The proof is by induction of $\rho(t)$.

The base step $\rho(t) = 0$ is trivial, since in this case $|{\downarrow}t| = 1 = \sigma(t)$.

Suppose that $\rho(t) > 0$ and that the lemma holds for every $u \in T$ with $\rho(u) < \rho(t)$. In particular, the lemma holds for every $u \in P(t)$, where $P(t)$ denotes the set of all the immediate predecessors of t. Thus, for every $u \in P(t)$, $\leqslant_{\restriction{\downarrow}u}$ has a well-ordered downward-finite extension $\leqslant'_u$ on ${\downarrow}u$ of order-type $\sigma(u)$.

Let $T^* = \{v \in T \mid v < t\}$. Notice that $T^* = \bigcup_{u \in P(t)} {\downarrow}u$, since the elements of $P(t)$ are the immediate predecessors of t and, if $v < t$, then $v \leqslant u$, for some $u \in P(t)$, since, by the definition of a well-founded tree, the successors of v form a finite linearly ordered set.

We can now apply Lemma 6.5 to T^*, getting a well-ordered downward-finite extension $\leqslant''$ of $\leqslant_{\restriction T^*}$ on T^* in such a way that $\leqslant''$ has order-type $\#_{u \in P(t)} \sigma(u)$.

But then $\leqslant''$ can be obviously extended to an order $\leqslant'''$ on the whole of ${\downarrow}t$ by putting t on the top. Trivially $\leqslant'''$ is a well-ordered downward-finite extension of $\leqslant$, and $\leqslant'''$ has order-type $\left(\#_{u \in P(t)} \sigma(u)\right) + 1 = \sigma(t)$. $\square$

Proof of the first sentence in Theorem 6.3. If T has only one root, then the result is immediate by applying Lemma 6.6 to this unique root.

The general case follows from the previous case and Lemma 6.5. $\square$

Lemma 6.7. *Suppose that $(T, \leqslant)$ is a countable well-founded tree, M is the set of the maximal elements of T and $\leqslant'$ is a well-ordered downward-finite extension of $\leqslant$.*

If α is the order-type of $\leqslant'$ and, for $u \in M$, α_u is the order-type of $\leqslant'_{\restriction \downarrow u}$ in $\downarrow u$, then α is a left-finite mixed sum of the α_u's.

Proof. Since $T = \bigcup_{u \in M} \downarrow u$, then α is obviously a mixed sum of the α_u's. Indeed, if φ is the bijection from T onto α induced by $\leqslant'$, then, defining $A_u = \varphi(\downarrow u)$, for $u \in M$, we get that $(A_u)_{u \in M}$ is an appropriate realization of α.

It remains to show that $(A_u)_{u \in M}$ is a left-finite realization, but this follows easily from the assumption that $\leqslant'$ is a downward-finite extension of $\leqslant$. Indeed, for every $v \in T$, there are elements $v_0, \dots v_n \in T$ such that $\downarrow' v \subseteq \downarrow v \cup \downarrow v_0 \cup \dots \cup \downarrow v_n$. But $v \leqslant u$, for some (actually, a unique) $u \in M$; similarly, $v_0 \leqslant u_0$, for some $u_0 \in M$, etc. Hence $\downarrow' v \subseteq \downarrow v \cup \downarrow v_0 \cup \dots \cup \downarrow v_n \subseteq \downarrow u \cup \downarrow u_0 \cup \dots \cup \downarrow u_n$ (repetitions are possible, but they cause no trouble).

Thus if $a \in \alpha$ and $a = \varphi(v)$, then the set of the elements smaller than a in α is contained in the finite union $A_u \cup A_{u_0} \cup \dots \cup A_{u_n}$. Since φ is surjective, this holds for every $a \in \alpha$, that is, the realization is left-finite.

Notice that in the above proof we do not need the assumption that the v_h's are $\leqslant$-incomparable with v. $\square$

Lemma 6.8. *Suppose that $(T, \leqslant)$ is a countable well-founded tree and $\leqslant'$ is a well-ordered downward-finite extension of $\leqslant$. Then, for every $t \in T$, the restriction $\leqslant'_{\restriction \downarrow t}$ of $\leqslant'$ to $\downarrow t$ has of order-type $\leqslant \sigma(t)$.*

Proof. By induction of $\rho(t)$.

The base step $\rho(t) = 0$ is trivial, since in this case $|\downarrow t| = 1 = \sigma(t)$.

Suppose that $\rho(t) > 0$ and that the lemma holds for every $u \in T$ with $\rho(u) < \rho(t)$. In particular, the lemma holds for every $u \in P(t)$, where $P(t)$ is the set of all the immediate predecessors of t. Thus, for every $u \in P(t)$, if α_u is the order-type of $\leqslant'_{\restriction \downarrow u}$, then $\alpha_u \leqslant \sigma(u)$.

Letting $T^* = \{v \in T \mid v < t\}$, we have that $\leqslant'_{\restriction T^*}$ is a well-ordered downward-finite extension of $\leqslant_{\restriction T^*}$, since, by the very definition of $\downarrow t$, t is comparable with every element of $\downarrow t$, and comparable elements are not allowed in the definition of a downward-finite extension, Definition 6.2.

Since, as we noticed in the proof of 6.6, $T^* = \bigcup_{u \in P(t)} {\downarrow} u$, then we can apply Lemma 6.7. Hence, if α is the order-type of $\leqslant'_{\restriction T^*}$, then α is a left-finite mixed sum of $(\alpha_u)_{u \in P(t)}$. Since $\alpha_u \leqslant \sigma(u)$, for every $u \in P(t)$, and using Theorem 6.4 and Proposition 3.3(2), we get $\alpha \leqslant \#_{u \in P(t)} \alpha_u \leqslant \#_{u \in P(t)} \sigma(u)$. Clearly, the order-type of $\leqslant'_{\restriction \downarrow t}$ is $\alpha + 1$ and we are done, since, by above, $\alpha + 1 \leqslant \left(\#_{u \in P(t)} \sigma(u) \right) + 1 = \sigma(t)$. □

Proof of the last sentence in Theorem 6.3. If T has only one root, the result is immediate from Lemma 6.8.

The general case follows from the previous case, Lemma 6.5 and again Theorem 6.4. □

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This is a preliminary version, it might contain inaccuraccies (to be precise, it is more likely to contain inaccuracies than planned subsequent versions). We have not yet performed a completely accurate search in order to check whether some of the results presented here are already known. Credits for already known results should go to the original discoverers. The above applies in particular with respect to Section 6.

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