ON THE JÓNSSON DISTRIBUTIVITY SPECTRUM

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Abstract. Suppose throughout that $\mathcal{V}$ is a congruence distributive variety. If $m \geq 1$, let $J_{\mathcal{V}}(m)$ be the smallest natural number k such that the congruence identity $\alpha(\beta \circ \gamma \circ \beta \dots) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \dots$ holds in $\mathcal{V}$, with m occurrences of $\circ$ on the left and k occurrences on the right. We show that if $J_{\mathcal{V}}(m) = k$, then $J_{\mathcal{V}}(m\ell) \leq k\ell$, for every natural number ℓ. The key to the proof is an identity which, through a variety, is equivalent to the above congruence identity, but involves also reflexive and admissible relations. If $J_{\mathcal{V}}(1) = 2$, that is, $\mathcal{V}$ is 3-distributive, then $J_{\mathcal{V}}(m) = m$, for every $m \geq 3$. If $\mathcal{V}$ is m-modular, that is, congruence modularity of $\mathcal{V}$ is witnessed by $m+1$ Day terms, then $J_{\mathcal{V}}(2) \leq J_{\mathcal{V}}(1) + 2m^2 - 2m - 1$. Various problems are stated at various places.

1. The Jónsson distributivity spectrum

Obviously, an algebra $\mathbf{A}$ is congruence distributive if and only if, for every natural number $m \geq 2$, the congruence identity $\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta + \alpha\gamma$ holds in $\mathbf{Con}(\mathbf{A})$ (more precisely, in the algebra of reflexive and admissible relations on $\mathbf{A}$). Here α, β, ... are intended to vary among congruences of $\mathbf{A}$, juxtaposition denotes intersection, $+$ is join in the congruence lattice and $\beta \circ_m \gamma$ denotes $\beta \circ \gamma \circ \beta \dots$ with m factors, that is, with $m-1$ occurrences of $\circ$.

Let us say that a congruence identity holds in some variety $\mathcal{V}$ if it holds in every algebra in $\mathcal{V}$. By a celebrated theorem by Jónsson [10], a milestone both in the theory of Maltsev conditions and in the theory of congruence distributive varieties, a variety $\mathcal{V}$ is congruence distributive if and only if there is some n such that the congruence identity

$$\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_n \alpha\gamma \tag{1}$$

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holds in $\mathcal{V}$. In other words, for varieties, taking $m = 2$ in the above paragraph is already enough. Jónsson actual statement in [10] is about a set of terms naturally arising from identity (1), rather than about the identity itself. Jónsson terms shall be recalled later. Stating results with regard to congruence identities rather than terms is simpler and easier to understand, while proofs usually require the corresponding terms. Compare the perspicuous discussion in Tschantz [26].

Jónsson proof in [10] goes on by showing that if some variety $\mathcal{V}$ has terms witnessing (1), then, for every m, the inclusion $\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta + \alpha\gamma$ holds in $\mathcal{V}$. (By the way, let us mention that Jónsson paper [10] contains a big deal of other fundamental results about distributive varieties with significant and unexpected applications to lattices, among other.) It follows easily from Jónsson proof that, for every m, there is some k (which depends only on m and on the n given by (1), but otherwise not on the variety) such that

$$\alpha(\beta \circ_m \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma \tag{(m,k)-dist}$$

A variety is Δ_k in the sense of [10] if and only if it satisfies $(2, k)$-dist. Such varieties are sometimes called *k-distributive*, or are said to *have $k + 1$ Jónsson terms*. If k is minimal with the above property (with $m = 2$), $\mathcal{V}$ is said to be of *Jónsson level k* in Freese and Valeriote [5].

If we slightly modify the proof of Jónsson theorem as presented in Burris and Sankappanavar [2, Theorem 12.6] or in McKenzie, McNulty, and Taylor [20, Theorem 4.144], we see that if a variety $\mathcal{V}$ satisfies $(2, k+1)$-dist, then $\mathcal{V}$ satisfies $(\ell+1, k\ell+1)$-dist, for every $\ell \geq 1$. This result is also a special case of Corollary 2.2 below. To formulate this and other results in a more concise way, it is natural to introduce the following *Jónsson distributivity function* $J_{\mathcal{V}}$ of a congruence distributive variety $\mathcal{V}$. For every positive natural number m, we set $J_{\mathcal{V}}(m)$ to be the least k such that $\mathcal{V}$ satisfies the identity $(m+1, k+1)$-dist. The "shift by 1" in the above notation will greatly simplify subsequent statements. For example, the above remark is more neatly stated by asserting that if $J_{\mathcal{V}}(1) = k$, then $J_{\mathcal{V}}(\ell) \leq k\ell$, for every positive ℓ. We can now ask the following problem.

The Jónsson distributivity spectrum problem. *Which functions (with domain the set of positive natural numbers) can be realized as $J_{\mathcal{V}}$, for some congruence distributive variety $\mathcal{V}$?*

Obviously, $J_{\mathcal{V}}$ is a monotone function. By the above comments, if $J_{\mathcal{V}}(1) = 1$, then $J_{\mathcal{V}}(m) \leq m$, for every positive m. Moreover if, for some k, $J_{\mathcal{V}}(k) < k$, then $\mathcal{V}$ is k-permutable: just take $\alpha = 1$ in equation $(m \cancel{+1}, k \cancel{+1})$-dist. If $\mathcal{V}$ is k-permutable, then $J_{\mathcal{V}}(m) < k$, for every m.

As a consequence of the above observations, if $J_{\mathcal{V}}(1) = 1$, then $J_{\mathcal{V}}$ is either the identity function, or is the identity up to some point and then it is a constant function. An example of the first eventuality is the variety of lattices; on the other hand, in the variety of n-*Boolean algebras* from Schmidt [24] we have $J_{\mathcal{V}}(m) = \min\{m, n\}$. Indeed, n-Boolean algebras have a lattice operation, are $n+1$-permutable but, in general, not n-permutable; see also Hagemann and Mitschke [9] and [11, Example 2.8]. For convenience, we shall use Jónsson paper [11] as a reference for this and other examples. The author believes that this is the appropriate place to mention that Jónsson [11] has had a profound influence in his mathematical formation.

Mitschke [21] shows that the variety of *implication algebras* is 3-permutable, not permutable, Δ_3 and not Δ_2. See also [9, Example 1] and [11, Example 2.6]. Hence in the variety of implication algebras we have $J_{\mathcal{V}}(m) = 2$, for every m. Freese and Valeriote [5], using the reduct of an algebra formerly constructed by Kearnes [13], show that, for every n, there is an n-permutable variety which is Δ_n and not Δ_{n-1}. See [5, p. 70–71]. Thus $J_{\mathcal{V}}$ is constantly $n-1$ in this variety.

We now observe that the set of those functions which can be represented as $J_{\mathcal{V}}$, for some variety, is closed under pointwise maximum. This is immediate from the result that the non-indexed product of two varieties $\mathcal{V}$ and $\mathcal{V}'$ satisfies exactly the same strong Maltsev conditions satisfied both by $\mathcal{V}$ and $\mathcal{V}'$. See Neumann [22], Taylor [25] or [11, p. 368–369]. We also need the easy fact that, for every m and k, the condition $J_{\mathcal{V}}(m) \leq k$ is equivalent to a strong Maltsev condition; for example, this is a consequence of the equivalence of (A) and (B) in Theorem 2.1 below.

Proposition 1.1. *If $\mathcal{V}$ and $\mathcal{V}'$ are congruence distributive varieties, then their non-indexed product $\mathcal{V}''$ is such that $J_{\mathcal{V}''}(m) = \max\{J_{\mathcal{V}}(m), J_{\mathcal{V}'}(m)\}$, for every positive natural number m.*

We do not know whether, for every pair $\mathcal{V}$, $\mathcal{V}'$, we always have some $\mathcal{V}''$ such that $J_{\mathcal{V}''}(m) = \min\{J_{\mathcal{V}}(m), J_{\mathcal{V}'}(m)\}$, for every m.

If in the above proposition we consider the variety of lattices and the mentioned variety from [5, p. 70–71], we get $J_{\mathcal{V}''}(m) = \max\{n-1, m\}$. By taking the non-indexed product of the variety of n'-Boolean algebras and again the variety from [5, p. 70–71], with $n \leq n'$, we have $J_{\mathcal{V}''}(m) = n-1$, for $m \leq n-1$ and $J_{\mathcal{V}''}(m) = \min\{m, n'\}$, for $m > n-1$.

The above examples suggest that $J_{\mathcal{V}}(m)$ has little influence on the values of $J_{\mathcal{V}}(m')$, for $m' < m$. On the other hand, we are going to show that $J_{\mathcal{V}}(m)$ puts some quite restrictive bounds on $J_{\mathcal{V}}(m')$, for $m' > m$, as we already mentioned for the easier case $m = 1$.

2. Bounds on higher levels of the spectrum

Let R, S, ... be variables intended to be interpreted as reflexive and admissible (binary) relations on some algebra. If R is such a relation, let $R^\smile$ denote the *converse* of R, that is, $b \mathrel{R^\smile} a$ ~~holds~~ if and only if $a \mathrel{R} b$. In the next theorem we show that, for a variety, the congruence identity $(m+1, k+1)$-dist is equivalent to the relation identity $\alpha(R \circ_m R^\smile) \subseteq \alpha R \circ_k \alpha R^\smile$, with the further provision that if R can be expressed as a composition, then αR and $\alpha R^\smile$ factor out. See condition (3) in the next theorem for a formal statement. The latter provision is necessary, since, without it, the case $m = 1 \leq k$ would be trivially true in every variety and, for every $m > 1$ and using Day terms, it can be shown that a variety is congruence modular if and only if there is some k such that the relation identity $\alpha(R \circ_m R^\smile) \subseteq \alpha R \circ_k \alpha R^\smile$ holds. Hence this identity alone is too weak for our purposes. See [17].

Recall that a *tolerance* is a symmetric and reflexive admissible relations. We shall prove Part (C) in the next theorem in the general case when α is a tolerance, rather than a congruence. In particular, we get that, through a variety, the identity (m,k)-dist is equivalent to the same identity in which α is only assumed to be a tolerance. However this stronger version shall not be used in what follows, hence the reader might always assume to be in the simpler case in which α is a congruence.

Theorem 2.1. *For every variety $\mathcal{V}$ and integers $m, k \geq 1$, the following conditions are equivalent.*

(A) $J_{\mathcal{V}}(m) \leq k$*, that is, $\mathcal{V}$ satisfies the congruence identity $(m+1, k+1)$-dist*

$$\alpha(\beta \circ_{m+1} \gamma) \subseteq \alpha\beta \circ_{k+1} \alpha\gamma$$

(equivalently, we can ask that the free algebra in $\mathcal{V}$ generated by $m+2$ elements satisfies the above identity.)

(B) $\mathcal{V}$ has $m+2$-ary terms $t_0, \dots, t_{k+1}$ such that the following identities hold in $\mathcal{V}$:

$$x = t_0(x, x_1, x_2, x_3, \dots, x_m, x_{m+1}), \tag{B1}$$

$$x = t_i(x, x_1, x_2, x_3, \dots, x_m, x), \qquad \textit{for } 0 \leq i \leq k+1, \tag{B2}$$

(B3)

$$\begin{cases} t_i(x_0, x_0, x_2, x_2, x_4, x_4, \dots) = t_{i+1}(x_0, x_0, x_2, x_2, x_4, x_4, \dots), \\ \qquad\qquad \textit{for even } i,\ 0 \leq i \leq k, \\ t_i(x_0, x_1, x_1, x_3, x_3, \dots) = t_{i+1}(x_0, x_1, x_1, x_3, x_3, \dots), \\ \qquad\qquad \textit{for odd } i,\ 0 \leq i \leq k, \end{cases}$$

(B4) $$t_{k+1}(x_0, x_1, x_2, x_3, \dots, x_m, z) = z$$

(C) For every algebra $\mathbf{A} \in \mathcal{V}$, every positive integer ℓ (equivalently, for $\ell = 1$), every tolerance α of $\mathbf{A}$ and all reflexive and admissible relations R, S_0, $\dots$, S_ℓ on $\mathbf{A}$, if $R = S_0 \circ S_1 \circ \dots \circ S_\ell$ and $\Theta = \alpha S_0 \circ \alpha S_1 \circ \dots \circ \alpha S_\ell$, then

(C1) $$\alpha(R \circ_m R^\smile) \subseteq \Theta \circ_k \Theta^\smile$$

Proof. The equivalence of (A) and (B) is an instance of the Pixley-Wille algorithm [23, 27] (actually, it can be seen as a good exercise to check a student's understanding of the algorithm). We shall need here only (A) $\Rightarrow$ (B), which can be proved as follows. Consider the free algebra $\mathbf{F}_\mathcal{V}(m+2)$ in $\mathcal{V}$ over $m+2$ generators $y_0, \dots, y_{m+1}$ and let α be the congruence generated by (y_0, y_{m+1}), β be the congruence generated by $\{(y_0, y_1), (y_2, y_3), (y_4, y_5), \dots\}$ and γ be the congruence generated by $\{(y_1, y_2), (y_3, y_4), \dots\}$. Thus $(y_0, y_{m+1}) \in \alpha(\beta \circ_{m+1} \gamma)$ hence, by (A), $(y_0, y_{m+1}) \in \alpha\beta \circ_{k+1} \alpha\gamma$. This latter relation is witnessed by $k+2$ elements of $\mathbf{F}_\mathcal{V}(m+2)$ which give rise to terms witnessing (B).

(B) $\Rightarrow$ (C) Let $(a, c) \in \alpha(R \circ_m R^\smile)$ in some algebra $\mathbf{A} \in \mathcal{V}$. This is witnessed by elements $a = b_0, b_1, b_2, \dots, b_m = c$ such that $b_0 \ R \ b_1 \ R^\smile$ $b_2 \ R \ b_3 \ R^\smile \ b_4 \dots$ Furthermore $a \ \alpha \ c$.

First suppose that m is even. For $0 \leq i \leq k$, we shall consider the elements

$$\begin{aligned} e_i = t_i(b_0, b_0, b_2, b_2, \dots, b_{m-2}, b_{m-2}, b_m, b_m) = \\ t_{i+1}(b_0, b_0, b_2, b_2, \dots, b_{m-2}, b_{m-2}, b_m, b_m) \quad \text{for } i \text{ even}, \\ e_i = t_i(b_0, b_1, b_1, b_3, \dots, b_{m-3}, b_{m-1}, b_{m-1}, b_m) = \\ t_{i+1}(b_0, b_1, b_1, b_3, \dots, b_{m-3}, b_{m-1}, b_{m-1}, b_m) \quad \text{for } i \text{ odd}, \end{aligned}$$

where the identities follow from (B3). In writing the above formula, we are supposing that m is large enough. Otherwise, say, for $m = 2$ and i odd, e_i should be set equal to $t_i(b_0, b_1.b_1, b_2)$.

If we had to show only $\alpha(R \circ_m R^\smile) \subseteq \alpha R \circ_k \alpha R^\smile$, for α a congruence, it would be enough to consider the above elements, since, $e_0 = b_0 = a$ by (B1), $e_k = b_m = c$ by (B4) and, say, for i even, $e_i = t_{i+1}(b_0, b_0, b_2, b_2, \dots,$ $b_{m-2}, b_{m-2}, b_m, b_m) \ R \ t_{i+1}(b_0, b_1, b_1, b_3, \dots, b_{m-3}, b_{m-1}, b_{m-1}, b_m) = e_{i+1}$, since R is admissible. Similarly $e_i \ R^\smile \ e_{i+1}$, for i odd. Notice that $b_h \ R \ b_{h+1}$ for h even and $b_h \ R^\smile \ b_{h+1}$ for h odd, hence $b_{h+1} \ R^\smile \ b_h$ for h even and $b_{h+1} \ R \ b_h$ for h odd. Moreover, for i even, if α is a congruence, then $e_i = t_i(a, a, b_2, \dots, c) \ \alpha \ t_i(a, a, b_2, \dots, a) = a = t_{i+1}(a, b_1, b_1, \dots, a) \ \alpha \ t_{i+1}(a, b_1, b_1, \dots, c) = e_{i+1}$, by (B2), hence $e_i \ \alpha$ e_{i+1}. Similarly, $e_i \ \alpha \ e_{i+1}$, for i odd. Hence the elements e_i, for $0 \leq i \leq k$, witness $(a, c) \in \alpha R \circ_k \alpha R^\smile$. However, as we mentioned, the identity $\alpha(R \circ_m R^\smile) \subseteq \alpha R \circ_k \alpha R^\smile$ is too weak to chain back to the other conditions.

Hence we need to use the assumption $R = S_0 \circ S_1 \circ \dots \circ S_\ell$ to prove the stronger conclusion $\alpha(R \circ_m R^\smile) \subseteq \Theta \circ_k \Theta^\smile$. Moreover, we shall also extend some ideas from Czédli and Horváth [3] in order to prove the case of (C) in which α is only assumed to be a tolerance.

By the assumption $R = S_0 \circ S_1 \circ \dots \circ S_\ell$ and given the elements $b_0, b_1, b_2, \dots, b_m$ introduced at the beginning, we have that, for every h with $0 \leq h < m$, there are elements $b_{h,0}, b_{h,1}, \dots, b_{h,\ell+1}$ such that $b_h = b_{h,0}\ S_0\ b_{h,1}\ S_1\ \dots\ S_\ell\ b_{h,\ell+1} = b_{h+1}$, for h even, and $b_{h+1} = b_{h,0}\ S_0\ b_{h,1}\ S_1\ \dots\ S_\ell\ b_{h,\ell+1} = b_h$, for h odd, since in this latter case $b_h\ R^\smile\ b_{h+1}$, that is, $b_{h+1}\ R\ b_h$. Hence, with the e_i's defined as in the above-displayed formula, we have, for i even,

$$
\begin{aligned}
e_i = t_{i+1}(b_0, b_0, b_2, \dots, b_{m-2}, b_m, b_m) = \\
t_{i+1}(b_0, b_{0,0}, b_{1,0}, \dots, b_{m-2,0}, b_{m-1,0}, b_m)\ S_0 \\
t_{i+1}(b_0, b_{0,1}, b_{1,1}, \dots, b_{m-2,1}, b_{m-1,1}, b_m)\ S_1 \\
t_{i+1}(b_0, b_{0,2}, b_{1,2}, \dots, b_{m-2,2}, b_{m-1,2}, b_m)\ S_2 \dots \\
S_\ell\ t_{i+1}(b_0, b_{0,\ell+1}, b_{1,\ell+1}, \dots, b_{m-2,\ell+1}, b_{m-1,\ell+1}, b_m) = \\
t_{i+1}(b_0, b_1, b_1, \dots, b_{m-1}, b_{m-1}, b_m) = e_{i+1}
\end{aligned}
$$

Moreover, for $0 \leq q \leq \ell$ and $0 \leq i \leq k$, we have, by (B2):

$$
\begin{aligned}
& t_{i+1}(a, b_{0,q}, b_{1,q}, \dots, c) = \\
& t_{i+1}(t_{i+1}(a, b_{0,q}, b_{1,q}, \dots, \boldsymbol{c}), b_{0,q+1}, b_{1,q+1}, \dots, t_{i+1}(\boldsymbol{a}, b_{0,q}, b_{1,q}, \dots, c))\ \alpha \\
& t_{i+1}(t_{i+1}(a, b_{0,q}, b_{1,q}, \dots, \boldsymbol{a}), b_{0,q+1}, b_{1,q+1}, \dots, t_{i+1}(\boldsymbol{c}, b_{0,q}, b_{1,q}, \dots, c)) = \\
& \qquad t_{i+1}(a, b_{0,q+1}, b_{1,q+1}, \dots, c)
\end{aligned}
$$

(elements in bold are those moved by α). Hence, for i even, the elements $t_{i+1}(a, b_{0,q}, b_{1,q}, b_{2,q}, \dots, c)$, for $0 \leq q \leq \ell$, witness $e_i\ \Theta\ e_{i+1}$, recalling that $\Theta = \alpha S_0 \circ \alpha S_1 \circ \dots \circ \alpha S_\ell$ and that $a = b_0$, $b_m = c$. Similarly $e_i\ \Theta^\smile\ e_{i+1}$, for i odd. After the above considerations, we see that the elements e_i, for $0 \leq i \leq k$, witness $(a, c) \in \alpha\Theta \circ_k \alpha\Theta^\smile$, what we had to show.

The case m odd is similar. This time, the e_i's are defined as follows, again for $0 \leq i \leq k$.

$$
\begin{aligned}
e_i &= t_i(b_0, b_0, b_2, b_2, \dots, b_{m-3}, b_{m-1}, b_{m-1}, b_m) && \text{for } i \text{ even,} \\
e_i &= t_i(b_0, b_1, b_1, b_3, \dots, b_{m-2}, b_{m-2}, b_m, b_m) && \text{for } i \text{ odd.}
\end{aligned}
$$

The rest is similar.

(C) $\Rightarrow$ (A) Take $\ell = 1$, $S_0 = \beta$ and $S_1 = \gamma$ in (C). Apparently, computing $R \circ_m R^\smile = (\beta \circ \gamma) \circ_m (\gamma \circ \beta)$ gives $2m$ factors, but we have $m-1$ adjacent pairs of the same congruence, hence $R \circ_m R^\smile = \beta \circ_{m+1} \gamma$. Similarly, $\Theta \circ_k \Theta^\smile = \alpha\beta \circ_{k+1} \alpha\gamma$ and we get (A). $\square$

Corollary 2.2. *If $J_\mathcal{V}(m) = k$ and $\ell > 0$, then $J_\mathcal{V}(m\ell) \leq k\ell$.*

Proof. We assume $J_\mathcal{V}(m) = k$ and we have to show that $\alpha(\beta \circ_{m\ell+1} \gamma) \subseteq \alpha\beta \circ_{k\ell+1} \alpha\gamma$. We apply Theorem 2.1(A) $\Rightarrow$ (C), taking $S_0 = S_2 = \cdots = \beta$ and $S_1 = S_3 = \cdots = \gamma$. We have $R \circ_m R^\smile = \beta \circ_{m\ell+1} \gamma$, since, as in the proof of (B) $\Rightarrow$ (C) above, we apparently have $m(\ell+1)$ factors, but $m-1$ many of them are absorbed, hence we end up with $m(\ell+1)-(m-1) = m\ell+1$ factors. The definition of Θ in (C) becomes $\Theta = \alpha\beta \circ_{\ell+1} \alpha\gamma$, hence, arguing as above, $\Theta \circ_k \Theta^\smile$ has $k\ell+1$ actual factors, that is, $\Theta \circ \Theta^\smile = \alpha\beta \circ_{k\ell+1} \alpha\gamma$. The inclusion (C1) thus gives the corollary. $\square$

Stronger results can be proved in the case of 3-distributivity.

Lemma 2.3. *If $J_\mathcal{V}(1) = 2$, $\mathbf{A} \in \mathcal{V}$, α is a congruence, R, S, T are reflexive and admissible relations on $\mathbf{A}$, $R \subseteq T$ and $S \subseteq T^\smile$, then*

$$\alpha(R \circ S \circ T \circ T^\smile) = \alpha R \circ \alpha S \circ \alpha(T \circ T^\smile) \tag{2}$$

Proof. One inclusion is trivial. To prove the nontrivial inclusion, recall that if $J_\mathcal{V}(1) = 2$, then by Jónsson [10] $\mathcal{V}$ has terms t and s such that the identities $x = t(x,x,y)$, $t(x,y,y) = s(x,y,y)$, $s(x,x,y) = y$ $x = t(x,y,x)$ and $x = s(x,y,x)$ hold in $\mathcal{V}$. The general form of Jónsson result shall be recalled at the beginning of the next section. Notice that the existence of t and s as above is also a consequence of Theorem 2.1, taking $m = 1$ and $k = 2$.

Let $(a,e) \in \alpha(R \circ S \circ T \circ T^\smile)$, with $a \ \alpha \ e$ and $a \ R \ b \ S \ c \ T \ d \ T^\smile$ e. We have $a = t(a,a,e) \ R \ t(a,b,e)$ and $a = t(a,b,a) \ \alpha \ t(a,b,e)$, thus $a \ \alpha R \ t(a,b,e)$. Similarly, $t(a,b,e) \ S \ t(a,c,e)$ and $t(a,b,e) \ \alpha$ $a = t(a,c,a) \ \alpha \ t(a,c,e)$, hence $t(a,b,e) \ \alpha S \ t(a,c,e)$. Furthermore, $t(a,c,e) \ T \ t(b,d,d) = s(b,d,d) \ T^\smile \ s(c,c,e) = e$, since $R \subseteq T$ and $S \subseteq T^\smile$. Hence $t(a,c,e) \ T \circ T^\smile \ e$ Moreover, $t(a,c,e) \ \alpha \ a \ \alpha \ e$, hence $t(a,c,e) \ \alpha(T \circ T^\smile) \ e$. In conclusion, the elements $t(a,b,e)$ and $t(a,c,e)$ witness $(a,e) \in \alpha R \circ \alpha S \circ \alpha(T \circ T^\smile)$. $\square$

Theorem 2.4. *If $J_\mathcal{V}(1) = 2$, that is, $\mathcal{V}$ is 3-distributive, then $J_\mathcal{V}(m) = m$, for every $m \geq 3$. Moreover the following congruence identity holds in $\mathcal{V}$*

$$\alpha\gamma \circ \alpha(\beta \circ \gamma \circ \beta) = \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \alpha\beta \tag{3}$$

Proof. We first prove (3). One inclusion is trivial and, in order to prove the nontrivial inclusion, it is enough to prove $\alpha(\beta \circ \gamma \circ \beta) \subseteq \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \alpha\beta$, since $\alpha\gamma$ is a congruence, hence transitive. So let $(a, d) \in \alpha(\beta \circ \gamma \circ \beta)$, hence $a \ \alpha \ d$ and $a \ \beta \ b \ \gamma \ c \ \beta \ d$, for some b and c. Recalling the terms t and s mentioned in the proof of Lemma 2.3, let us compute $a = s(c, c, a) \ \gamma \ s(c, b, a) \ \beta \ s(c, b, b) = t(c, b, b) \ \beta \ t(d, b, a) \ \gamma \ t(d, c, a) \ \beta \ t(d, d, a) = d$. Moreover, $a = t(a, b, a) \ \alpha \ t(d, b, a)$, hence $a \ \alpha(\gamma \circ \beta) \ t(d, b, a)$. Furthermore, $t(d, c, a) \ \alpha \ t(d, c, d) = d$ hence also $t(d, b, a) \ \alpha \ a \ \alpha \ d \ \alpha \ t(d, c, a)$. Hence the elements $t(d, b, a)$ and $t(d, c, a)$ witness $(a, d) \in \alpha(\gamma \circ \beta) \circ \alpha\gamma \circ \alpha\beta \subseteq \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \alpha\gamma \circ \alpha\beta = \alpha\gamma \circ \alpha\beta \circ \alpha\gamma \circ \alpha\beta$, where the inclusion follows from 3-distributivity.

Next, we use Lemma 2.3 to prove $J_{\mathcal{V}}(3) = 3$. By taking $R = \beta$, $S = \gamma$ and $T = \gamma \circ \beta$ in equation (2), we get $\alpha(\beta \circ \gamma \circ \beta \circ \gamma) = \alpha(\beta \circ \gamma \circ \gamma \circ \beta \circ \beta \circ \gamma) = \alpha(R \circ S \circ T \circ T^{\smile}) \subseteq \alpha R \circ \alpha S \circ \alpha(T \circ T^{\smile}) = \alpha\beta \circ \alpha\gamma \circ \alpha(\gamma \circ \beta \circ \beta \circ \gamma) = \alpha\beta \circ \alpha(\gamma \circ \beta \circ \gamma) = \alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \alpha\gamma$, where we used the obvious fact that $\alpha\gamma \circ \alpha(\gamma \circ \beta \circ \gamma) = \alpha(\gamma \circ \beta \circ \gamma)$ and then we used (3) with β and γ exchanged.

The rest of the proof follows now quite easily by induction. Suppose that $m \geq 4$ and that we have proved $J_{\mathcal{V}}(n) = n$, for every n with $3 \leq n < m$. If m is even, use equation (2) in Lemma 2.3 with $R = \beta$, $S = \gamma$ and $T = \beta \circ_{\frac{m}{2}} \gamma$, getting $\alpha(\beta \circ_{m+1} \gamma) = \alpha(R \circ S \circ T \circ T^{\smile}) = \alpha R \circ \alpha S \circ \alpha(T \circ T^{\smile}) = \alpha\beta \circ \alpha\gamma \circ \alpha(\beta \circ_{m-1} \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ (\alpha\beta \circ_{m-1} \alpha\gamma) = \alpha\beta \circ_{m+1} \alpha\gamma$, where the inclusion follows from the inductive hypothesis, except in case $m = 4$, where we need (3).

If m is odd, use equation (2) in Lemma 2.3 with $R = \beta$, $S = 0$ and $T = \gamma \circ_{\frac{m+1}{2}} \beta$, getting $\alpha(\beta \circ_{m+1} \gamma) = \alpha(R \circ S \circ T \circ T^{\smile}) = \alpha R \circ \alpha S \circ \alpha(T \circ T^{\smile}) = \alpha\beta \circ \alpha(\gamma \circ_m \beta) \subseteq \alpha\beta \circ (\alpha\gamma \circ_m \alpha\beta) = \alpha\beta \circ_{m+1} \alpha\gamma$, by the inductive hypothesis. We have proved $J_{\mathcal{V}}(m) = m$ in any case and the induction is thus complete. $\square$

3. Variants of the spectrum

One can consider an alternative function $J_{\mathcal{V}}^{\smile}$ which is defined in such a way that $J_{\mathcal{V}}^{\smile}(m)$ is the smallest k such that the identity

$$(m+1, k+1)\text{-dist}^{\smile} \qquad \qquad \alpha(\beta \circ_{m+1} \gamma) \subseteq \alpha\gamma \circ_{k+1} \alpha\beta$$

holds in $\mathcal{V}$. Here γ and β are exchanged on the right-hand side, in comparison with $(m+1, k+1)$-dist. It is easy to see that $J_{\mathcal{V}}^{\smile}$ and $J_{\mathcal{V}}$ are different functions; indeed, $J_{\mathcal{V}}^{\smile}(m) = m$, for some m, implies $m+1$-permutabilty, while $J_{\mathcal{V}}(m) = m$ holds in lattices, for every m. Obviously, however, $J_{\mathcal{V}}^{\smile}(m)$ and $J_{\mathcal{V}}(m)$ differ at most by 1. There are some further simple relations connecting $J_{\mathcal{V}}^{\smile}$ and $J_{\mathcal{V}}$. For example, if

m and $J_{\mathcal{V}}(m)$ have the same parity, then $J^{\smile}_{\mathcal{V}}(m) \geq J_{\mathcal{V}}(m)$. Compare the parallel discussion (corresponding to the case $m = 1$ here) in [5, p. 63].

A theorem analogous to Theorem 2.1 holds for $J^{\smile}_{\mathcal{V}}$: just exchange even and odd in (B3) and replace identity (C1) by $\alpha(R \circ_m R^{\smile}) \subseteq \Theta^{\smile} \circ_k \Theta$. Arguing as in Corollary 2.2, we then get that if $J^{\smile}_{\mathcal{V}}(m) = k$ and ℓ is odd, then $J^{\smile}_{\mathcal{V}}(m\ell) \leq k\ell$. If ℓ is even, then $J^{\smile}_{\mathcal{V}}(m) = k$ implies $J_{\mathcal{V}}(m\ell) \leq k\ell$.

A probably more significant variant is suggested by the use of directed Jónsson terms. See Zádori [28] and Kazda, Kozik, McKenzie and Moore [12]. Let us recall the definitions. *Jónsson terms* [10] are terms $j_0, \dots, j_k$ satisfying

$$\text{(J1)} \qquad x = j_0(x, y, z),$$

$$\text{(J2)} \qquad x = j_i(x, y, x), \qquad \text{for } 0 \leq i \leq k,$$

$$\text{(J3)} \qquad \begin{aligned} j_i(x, x, z) &= j_{i+1}(x, x, z), && \text{for even } i,\ 0 \leq i < k, \\ j_i(x, z, z) &= j_{i+1}(x, z, z), && \text{for odd } i,\ 0 \leq i < k, \end{aligned}$$

$$\text{(J4)} \qquad j_k(x, y, z) = z$$

Notice that this is exactly condition (B) in Theorem 2.1 in the particular case $m = 1$ and with k in place of $k + 1$. We get *directed Jónsson terms*, or *Zádori terms* [28, 12] if in the above set of identities we replace condition (J3) by

$$\text{(D)} \qquad j_i(x, z, z) = j_{i+1}(x, x, z) \quad \text{for } \ 0 \leq i < k$$

Seemingly, directed Jónsson terms first appeared (unnamed) in [28, Theorem 4.1], whose proof relies on McKenzie [19]. In [28] it is shown, among other, that a finite bounded poset admits Jónsson operations for some k if and only if it admits directed Jónsson operations for some k'. Kazda, Kozik, McKenzie and Moore [12] proved the equivalence for terms in an arbitrary variety, thus, by [10], a variety is congruence distributive if and only if it has directed Jónsson terms.

For a binary relation R, let $R^k = R \circ R \circ R \dots$ with k factors. In other words, $R^k = R \circ_k R$.

Proposition 3.1. *If some variety $\mathcal{V}$ has $k + 1$ directed Jónsson terms $d_0, \dots, d_k$, with $k \geq 1$, then, for every $\ell \geq 1$, $\mathcal{V}$ satisfies the identity*

$$\alpha(S_1 \circ S_2 \circ \dots \circ S_\ell) \subseteq (\alpha S_1 \circ \alpha S_2 \circ \dots \circ \alpha S_\ell)^{k-1}$$

where α varies among tolerances and $S_1, S_2, \dots$ vary among reflexive and admissible relations on some algebra in $\mathcal{V}$. In particular, for ℓ

even, $\mathcal{V}$ satisfies

$$\alpha(S \circ_\ell T) \subseteq \alpha S \circ_{\ell(k-1)} \alpha T \tag{4}$$

and, for ℓ odd,

$$\alpha(\beta \circ_\ell T) \subseteq \alpha\beta \circ_{k'} \alpha T,$$

where β varies among congruences and $k' = \ell(k-1) - k + 2$.

Proof. Let us work in some fixed algebra belonging to $\mathcal{V}$. Let $(a, c) \in \alpha(S_1 \circ S_2 \circ \dots \circ S_\ell)$. Thus $a \ \alpha \ c$ and there are elements $b_0, b_1, b_2, \dots, b_\ell$ such that $a = b_0 \ S_1 \ b_1 \ S_2 \ b_2 \dots b_{\ell-1} \ S_\ell \ b_\ell = c$. First suppose that α is a congruence. For every i and h, we have $d_i(a, b_h, c) \ \alpha \ d_i(a, b_h, a) = a$, hence all such elements are α-related. Moreover, for every $i < k$, $d_i(a, a, c) \ S_1 \ d_i(a, b_1, c) \ S_2 \ d_i(a, b_2, c) \ S_3 \ \dots \ S_\ell \ d_i(a, c, c) = d_{i+1}(a, a, c)$. This shows that, for every $i < k$, $(d_i(a, a, c), d_{i+1}(a, a, c)) \in \alpha S_1 \circ \alpha S_2 \circ \dots \circ \alpha S_\ell$.

Since $a = d_0(a, c, c) = d_1(a, a, c)$ and $d_k(a, a, c) = c$, then the elements $d_i(a, a, c)$, for $1 \leq i \leq k$, witness $(a, c) \in (\alpha S_1 \circ \alpha S_2 \circ \dots \circ \alpha S_\ell)^{k-1}$.

The last statement follows immediately. We just mention that in the last equation $k - 2$ factors are absorbed, since we assume that β is transitive.

The case when α is just a tolerance is treated as in Czédli and Horváth [3]. Indeed, for every i, j, h, h', we have $d_i(a, b_h, c) = d_i(d_j(a, b_{h'}, a), b_h, d_j(c, b_{h'}, c)) \ \alpha \ d_i(d_j(a, b_{h'}, c), b_h, d_j(a, b_{h'}, c)) = d_j(a, b_{h'}, c)$. The rest is the same. □

Since congruences and tolerances are, in particular, reflexive and admissible, we obtain corresponding congruence/tolerance identities from the above identities about relations. In particular, by [10, 12] and Proposition 3.1, a variety is congruence distributive if and only if equation (4) holds for some $\ell \geq 2$ and k, equivalently, for $\ell = 2$ and some k. If * denotes transitive closure, we then get that a variety $\mathcal{V}$ is congruence distributive if and only if $(\alpha(S \circ T))^* = (\alpha S \circ \alpha T)^*$ holds in $\mathcal{V}$, if and only if $\alpha^*(S \circ T)^* = (\alpha S \circ \alpha T)^*$ holds in $\mathcal{V}$. Notice that, on the other hand, neither the identity $(\alpha(S \circ S))^* = (\alpha S)^*$ nor the identity $\alpha^* S^* = (\alpha S)^*$ imply congruence distributivity, since the identities hold, e. g., in permutable varieties (in fact, we have a proof that both identities are equivalent to congruence modularity [17]). See however [18] for a variation actually equivalent to congruence distributivity. In all the above statements α can be taken equivalently as a tolerance or as a congruence, while S and T vary among reflexive and admissible relations.

We do not know whether, for every congruence distributive variety $\mathcal{V}$, there is k such that $\mathcal{V}$ satisfies the relation identity $R(S \circ T) \subseteq RS \circ_k RT$.

The existence of some k as above is equivalent to $R^*(S \circ T)^* = (RS \circ RT)^*$. The arguments from Gyenizse and Maróti [8] can be adapted to show that if $\mathcal{V}$ is congruence distributive and the free algebra $\mathbf{F}_{\mathcal{V}}(3)$ in $\mathcal{V}$ over 3 elements is finite, then $R^*(S \circ T)^* = (R^*S \circ R^*T)^*$. It is easy to see that 2-distributivity does imply $R(S \circ T) \subseteq RS \circ RT$. Cf. [15, Remark 17]. We have a proof that if $\mathcal{V}$ has 4 directed Jónsson terms d_0, d_1, d_2, d_3 and $\mathbf{F}_{\mathcal{V}}(2)$ is finite, then $R(S \circ T) \subseteq RS \circ_k RT$, for some k which depends on the variety.

Notice also that, since the composition of reflexive and admissible relations is still reflexive and admissible, we can get new identities by substitution, without recurring to terms. E. g., from $\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha S$, replacing T by $T \circ S$, we get $\alpha(S \circ T \circ S) \subseteq \alpha S \circ \alpha(T \circ S) \circ \alpha S \subseteq \alpha S \circ \alpha T \circ \alpha S \circ \alpha T \circ \alpha S$. We leave similar computations to the interested reader, observing that arguments using terms, though more involved, generally produce stronger results.

By the above considerations, one would be tempted to believe that it is always more convenient to use directed Jónsson terms, rather than Jónsson terms. However, the exact relation between the (minimal) number of directed Jónsson terms and of Jónsson terms in a variety is not clear; see [12, Observation 1.2 and Section 7]. Even in the case when we have the same number of Jónsson and of directed Jónsson terms, there are situations in which it is more convenient to use the former terms. For example, with 5 Jónsson terms we have $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_4 \alpha\gamma$, while from 5 Jónsson directed terms we obtain only $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_6 \alpha\gamma$ from equation (4). Notice that here we count terms including the initial and final projections (that is, 5 terms correspond to the case $k = 4$).

On the other hand, an example in which it is more convenient to use directed terms is Baker example from [1], the variety $\mathcal{V}$ generated by the polynomial reducts of lattices in which only the ternary operation $f(a, b, c) = a \wedge (b \vee c)$ is considered. See also [11, Example 2.12]. Baker showed that $\mathcal{V}$ is Δ_4 but not Δ_3, that is, in the present terminology, $J_{\mathcal{V}}(1) = 3$, from which we get from Corollary 2.2 that $J_{\mathcal{V}}(2) \leq 6$ and $J_{\mathcal{V}}(3) \leq 9$. However, taking $d_1(x, y, z) = x \wedge (y \vee z)$, $d_2(x, y, z) = z \wedge (x \vee y)$ and d_0 and d_3 the first and last projections, we get a sequence of directed Jónsson terms, hence we can apply Proposition 3.1 with $k = 3$ in order to get $J_{\mathcal{V}}(\ell - 1) \leq 2\ell - 2$, for ℓ odd and $J_{\mathcal{V}}(\ell - 1) \leq 2\ell - 1$, for ℓ even, hence, say, $J_{\mathcal{V}}(2) \leq 4$ and $J_{\mathcal{V}}(3) \leq 7$, which give improvements on the previous bounds.

We can introduce a version of $J_{\mathcal{V}}$ which takes into account reflexive and admissible relations, in place of congruences. For $m \geq 1$, we let $J^r_{\mathcal{V}}(m)$ be the least k such that $\alpha(S \circ_{m+1} T) \subseteq \alpha S \circ_{k+1} \alpha T$; similarly,

$J_{\mathcal{V}}^{r\smile}$ is defined by considering $\alpha T \circ_{k+1} \alpha S$ on the right-hand side. The definitions are justified in view of [10, 12] and Proposition 3.1. We can thus formulate the following general problem.

The generalized Jónsson distributivity spectrum problem. *Which quadruplets of functions can be realized as* $(J_{\mathcal{V}}, J_{\mathcal{V}}^{\smile}, J_{\mathcal{V}}^{r}, J_{\mathcal{V}}^{r\smile})$, *for some congruence distributive variety* $\mathcal{V}$*?*

In the above problem we could take into account additional conditions. For example, consider the formula $\alpha(\beta \circ \gamma \circ \delta \circ \beta \dots) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\delta \circ \alpha\beta \dots$, with m factors on the left-hand side. Which is the smallest number of factors on the right-hand side such that the formula holds in a specific congruence distributive variety? Similarly, which is the best bound in $(\alpha_1 \circ_m \beta_1)(\alpha_2 \circ_m \beta_2) \subseteq \alpha_1\alpha_2 \circ \alpha_1\beta_2 \circ \beta_1\alpha_2 \circ \beta_1\beta_2 \circ \alpha_1\alpha_2 \dots$? Or, more generally, for $(\alpha_1 \circ_m \beta_1)(\alpha_2 \circ_m \beta_2) \dots (\alpha_h \circ_m \beta_h)$? What about $(\alpha \circ_m \beta)(\alpha \circ_m \gamma)(\beta \circ_m \gamma) \subseteq \alpha\beta \circ \beta\gamma \circ \alpha\gamma \circ \alpha\beta \dots$?

Of course, the analogue of Proposition 1.1 holds for $J_{\mathcal{V}}^{\smile}$, $J_{\mathcal{V}}^{r}$ and $J_{\mathcal{V}}^{r\smile}$, too, using similar arguments.

4. An unexpected connection with the modularity spectra

One can also define functions analogous to $J_{\mathcal{V}}$ in the case of congruence modular varieties (here we temporarily give up the convention that every variety at hand is congruence distributive). By a fundamental theorem by A. Day [4], a variety $\mathcal{V}$ is congruence modular if and only there is some k such that the congruence identity

$$(\mathrm{D}_k) \qquad \alpha(\beta \circ \alpha\gamma \circ \beta) \subseteq \alpha\beta \circ_k \alpha\gamma$$

holds in $\mathcal{V}$. Again, Day result is stated in a form involving a certain number of terms, but we shall not need the explicit Day terms here. A variety $\mathcal{V}$ is *k-modular* if $\mathcal{V}$ satisfies equation (D_k), and $\mathcal{V}$ has *Day level k* if such a k is minimal. For a congruence modular variety $\mathcal{V}$, we can define the *Day modularity function $D_{\mathcal{V}}$* as follows. For $m \geq 3$, $D_{\mathcal{V}}(m)$ is the least k such that $\alpha(\beta \circ_m \alpha\gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$ holds in $\mathcal{V}$. The arguments from [4] show that $D_{\mathcal{V}}(m)$ is defined for every m and every congruence modular variety $\mathcal{V}$, but the methods from [4] do not furnish the best value. See [16].

The case of congruence modularity is substantially more involved than the distributivity case treated here. Gumm [6, 7] provided another characterization of congruence modular varieties by considering terms which "compose permutability with distributivity". In detail, *Gumm terms* are terms $p, j_1, \dots, j_k$ satisfying the above conditions (J2)-(J4) (the distributivity part, involving only the j's), together with the following permutability part:

$$\text{(P)} \qquad x = p(x, z, z), \quad \text{and} \quad p(x, x, z) = j_1(x, x, z)$$

Notice that the definition given here is slightly different from Gumm original conditions in [6, Theorem 1.4], where odd and even are exchanged and where p is considered after the j_i's (the q_i's in the notation from [6]). This is not simply a matter of symmetry: in the formulation by Gumm, when n is odd, one gets an unnecessary term which can be discarded, hence Gumm actual condition is interesting only for n even. Of course, the above remark is interesting only when one is concerned with the smallest k (or n) for which there are Gumm terms. If one is concerned just with the existence of some k for which such terms exist, then the remark is irrelevant. As far as we know, the above formulation first appeared in Lakser, Taylor and Tschantz [14].

Using Gumm terms, Tschantz [26] showed that a variety $\mathcal{V}$ is congruence modular if and only if the congruence identity $\alpha(\beta + \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma + \alpha\beta)$ holds in $\mathcal{V}$. Hence it is also natural to introduce the *Tschantz modularity function* $T_{\mathcal{V}}$ for a congruence modular variety $\mathcal{V}$ in such a way that, for $m \geq 2$, $T_{\mathcal{V}}(m)$ is the least k such that the following congruence identity holds in $\mathcal{V}$

$$\alpha(\beta \circ_m \gamma) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_k \alpha\beta)$$

Notice that $T_{\mathcal{V}}(2) = 0$ is equivalent to congruence permutability, just take $\alpha = 1$ (by convention we let $\beta \circ_0 \gamma = 0$). More generally, $T_{\mathcal{V}}(2) = k$ if and only if $\mathcal{V}$ has $k + 2$ Gumm terms $p, j_1, \dots, j_{k+1}$, but not $k - 1$ Gumm terms.

The relationships between $D_{\mathcal{V}}$ and $T_{\mathcal{V}}$ appear rather involved. A detailed study of their connection goes beyond the scope of the present note; we refer to [16] for further details. We just mention that we can define the $\smile$-variants of the above notions. Moreover, Kazda, Kozik, McKenzie and Moore [12] introduced the directed variants of Gumm
too terms. Using their result we can see that a variety is congruence modular if and only if, for every $m \geq 1$, there is some h such that $\alpha(S \circ_{m+1} T) \subseteq \alpha(T \circ S) \circ (\alpha T \circ_h \alpha S)$, where, as above, S and T vary among reflexive and admissible relations. See [17].

We finally note an intriguing connection among the Jónsson and the modularity spectra. We first need a preliminary lemma.

Lemma 4.1. *If a variety $\mathcal{V}$ has $k+2$ Gumm terms $p, j_1, \dots, j_{k+1}$, then $\mathcal{V}$ satisfies the following congruence identity (as above, α can be taken* *as a tolerance).*

$$\alpha(\beta \circ \gamma \circ \beta) \subseteq \alpha(\gamma \circ \beta) \circ (\alpha\gamma \circ_{2k} \alpha\beta)$$

Proof. If we do not care about the exact number of factors on the right-hand side, the proposition follows from [26]. The arguments from [26] seem to produce a much larger number of factors, anyway.

Suppose that $(a, d) \in \alpha(\beta \circ \gamma \circ \beta)$, thus $a \; \alpha \; d$ and $a \; \beta \; b \; \gamma \; c \; \beta \; d$, for some b and c. Using equations (J2)-(J4), no essentially new argument is needed to show that $(j_1(a, a, d), d) \in (\alpha\beta \circ \alpha\gamma \circ \alpha\beta)^k$.

Indeed, by (J2), $j_i(a, e, d) = j_i(j_{i'}(a, e', \boldsymbol{a}), e, j_{i'}(\boldsymbol{d}, e', d)) \; \alpha \; j_i(j_{i'}(a, e', \boldsymbol{d}), e, j_{i'}(\boldsymbol{a}, e', d)) = j_{i'}(a, e', d)$, for all indices i, i' and elements e, e', hence all the elements of the above form are α-related. For i odd, we have $j_i(a, a, d) \; \beta \; j_i(a, b, d) \; \gamma \; j_i(a, c, d) \; \beta \; j_i(a, d, d)$, thus $(j_i(a, a, d), j_i(a, d, d)) \in \alpha\beta \circ \alpha\gamma \circ \alpha\beta$. Similarly, $(j_i(a, d, d), j_i(a, a, d)) \in \alpha\beta \circ \alpha\gamma \circ \alpha\beta$, for i even. Hence $j_1(a, a, d)$, $j_1(a, d, d) = j_2(a, d, d)$, $j_2(a, a, d) = j_3(a, a, d)$, ... witness $(j_1(a, a, d), d) \in (\alpha\beta \circ \alpha\gamma \circ \alpha\beta)^k = \alpha\beta \circ_{2k+1} \alpha\gamma$.

Moreover, $a = p(a, b, \boldsymbol{b}) \; \gamma \; p(a, \boldsymbol{b}, \boldsymbol{c}) \; \beta \; p(a, \boldsymbol{a}, \boldsymbol{d}) = j_1(a, a, d)$, by (P), hence $(a, j_1(a, a, d)) \in \alpha(\gamma \circ \beta)$, thus $(a, d) \in \alpha(\gamma \circ \beta) \circ (\alpha\beta \circ_{2k+1} \alpha\gamma)$. Finally, $\alpha(\gamma \circ \beta) \circ \alpha\beta = \alpha(\gamma \circ \beta)$ (or, better, use $j_1(a, b, d)$ in place of $j_1(a, a, d)$), hence one more factor is absorbed and we get the conclusion. □

Theorem 4.2. *Suppose that V is a congruence distributive variety.*

1. *If $m \geq 2$ and $\mathcal{V}$ is m-modular, then $J_{\mathcal{V}}(2) \leq J_{\mathcal{V}}^{\smile}(1) + 2m^2 - 2m - 2$ and $J_{\mathcal{V}}^{\smile}(2) \leq J_{\mathcal{V}}(1) + 2m^2 - 2m - 2$.*
2. *If $\mathcal{V}$ has $k + 2$ Gumm terms, then $J_{\mathcal{V}}(2) \leq J_{\mathcal{V}}^{\smile}(1) + 2k$ and $J_{\mathcal{V}}^{\smile}(2) \leq J_{\mathcal{V}}(1) + 2k$. The bounds can be improved by 1 if $J_{\mathcal{V}}^{\smile}(1)$ is odd, respectively, if $J_{\mathcal{V}}(1)$ is even.*

Proof. Part (2) is immediate from Lemma 4.1.

In [14, Theorem 2] it is proved that if a variety is m-modular, then $\mathcal{V}$ has $\leq m^2 - m + 1$ Gumm terms. Hence (1) follows from (2). □

There are versions of Lemma 4.1 and of Theorem 4.2 obtained using Gumm directed terms [12] in place of Gumm terms. See [17].

As far as we know, there is the possibility that Theorem 4.2 is an empty result, namely that, say, for $m \geq 3$, every congruence distributive variety with Day level m has Jónsson level $< 2m^2 - 2m - 2$ (for, were this the case, then Corollary 2.2 would give a better evaluation of $J_{\mathcal{V}}(2)$). However, this would be a quite unexpected result. What Theorem 4.2 and the above comment do show is that *in any case* the Day level of a congruence distributive variety has an influence on the very low levels of the Jónsson spectrum. This connection appears rather surprising, whichever of the above cases occurs.

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