A VARIETY $\mathcal{V}$ IS CONGRUENCE MODULAR IF AND ONLY IF $\mathcal{V}$ SATISFIES $\Theta(R \circ R) \subseteq (\Theta R)^h$, FOR SOME h

PAOLO LIPPARINI

ABSTRACT. We present a characterization of congruence modularity by means of an identity involving a tolerance Θ and a reflexive and admissible relation R.

More than forty years ago Nation [N] proved a result which is still intriguing today: there are non-equivalent lattice identities which nevertheless are equivalent as congruence identities in varieties. Many results of this kind followed, for example, Freese and Jónsson [FJ] proved that modularity is equivalent to the Arguesian identity for congruence lattices in varieties. As another example of a slightly different nature, it is an almost immediate consequence of the arguments in the proof of Lampe's Lemma [FLT] and of the construction of an affine-modulo-abelian term in Taylor [Ta] that every m-permutable variety satisfies a non-trivial congruence lattice identity. See [L1] for a short history of the result and for another proof. See also Kearnes and Nation [KN]. More results about congruence identities and further references can be found in Day and Freese [DF], Freese and McKenzie [FM], Gumm [G], Hobby and McKenzie [HM], Jónsson [J] and Kearnes and Kiss [KK].

It turns out that frequently tolerances, and sometimes just reflexive and admissible relations, are at work behind the scene even when results about congruences are considered. This is particularly evident, for example, in Czédli, Horváth and Lipparini [CHL], Jónsson [J, p. 370] or Tschantz [Ts]. Here we present a proof that a variety $\mathcal{V}$ is congruence modular if and only if there is some natural number h such that $\Theta(R \circ R) \subseteq (\Theta R)^h$ holds in every algebra in $\mathcal{V}$, for every tolerance Θ and every reflexive and admissible relation R. The proof given here is slightly simpler (but less general) than the proof presented in [L2].

Our notation is as follows. Juxtaposition denotes intersection. If T is a binary relation, we let T^h denote the relational composition

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$T \circ T \circ T \circ \cdots \circ T$ with h factors, that is, with $h-1$ occurrences of $\circ$. Moreover, T^* denotes the transitive closure of T.

Theorem 1. *For every variety $\mathcal{V}$, the following conditions are equivalent.*

(1) $\mathcal{V}$ is congruence modular.
(2) There is some natural number h such that $\Theta(R \circ R) \subseteq (\Theta R)^h$ holds in every algebra in $\mathcal{V}$, for every tolerance Θ and every reflexive and admissible relation R.
(3) The inclusion $\Theta R^* \subseteq (\Theta R)^*$ holds in every algebra in $\mathcal{V}$, for every tolerance Θ and every reflexive and admissible relation R.
(4) The identity $\alpha\Phi^* = (\alpha\Phi)^*$ holds in every algebra in $\mathcal{V}$, for every congruence α and every tolerance Φ.

Proof. The proof that (1) implies (2) relies heavily on a recent result by Kazda, Kozik, McKenzie and Moore [KKMM]. There the authors showed that a variety $\mathcal{V}$ is congruence modular if and only if, for some k, $\mathcal{V}$ has $k+1$ *directed Gumm terms*, that is, terms $p, j_1, \dots, j_k$ satisfying the following set of equations.

(DG1) $$x = p(x,z,z)$$

(DG2) $$p(x,x,z) = j_1(x,x,z)$$

(DG3) $$x = j_i(x,y,x), \qquad \text{for } 1 \leq i \leq k,$$

(DG4) $$j_i(x,z,z) = j_{i+1}(x,x,z) \qquad \text{for } 1 \leq i < k$$

(DG5) $$j_k(x,y,z) = z$$

Notice that we have given the definition of directed Gumm terms in the reversed order, in comparison with [KKMM]. Now suppose that $\mathbf{A}$ is an algebra in $\mathcal{V}$, Θ is a tolerance of $\mathbf{A}$ and R is a reflexive and admissible relation on $\mathbf{A}$. If $a, c \in A$ and $(a,c) \in \Theta(R \circ R)$, then $a \mathrel{\Theta} c$ and $a \mathrel{R} b \mathrel{R} c$, for some $b \in A$. By [KKMM] and (1) we have terms satisfying (DG1)-(DG5) above. Let us compute

$$a = p(a, p(a\boldsymbol{a}b), p(aa\boldsymbol{b})) \mathrel{R} p(a, p(a\boldsymbol{b}b), p(aa\boldsymbol{c})) = p(a,a,p(aac)),$$
$$a = p(a,a,a) = p(a,a,p(aa\boldsymbol{a})) \mathrel{\Theta} p(a,a,p(aa\boldsymbol{c})),$$

where elements in bold are those moved by R or Θ and we have used (DG1). Moreover, $p(a,a,p(aac)) = j_1(a,a,j_1(aac))$, by (DG2), hence

(5) $$a \mathrel{\Theta R} j_1(a,a,j_1(aac)).$$

Next, for $\ell = 1, \dots, k-1$, we have

$$j_\ell(a, \boldsymbol{a}, c) \; R \; j_\ell(a, \boldsymbol{b}, c) \; R \; j_\ell(a, \boldsymbol{c}, c) =^{(\mathrm{DG4})} j_{\ell+1}(a, a, c),$$

$$j_\ell(a, a, c) = j_\ell(j_\ell(aa\boldsymbol{c}), b, j_\ell(\boldsymbol{a}ac)) \; \Theta \; j_\ell(j_\ell(aa\boldsymbol{a}), b, j_\ell(\boldsymbol{c}ac)) = j_\ell(a, b, c)$$

$$j_\ell(a, b, c) = j_\ell(j_\ell(ab\boldsymbol{c}), c, j_\ell(\boldsymbol{a}bc)) \; \Theta \; j_\ell(j_\ell(ab\boldsymbol{a}), c, j_\ell(\boldsymbol{c}bc)) = j_\ell(a, c, c),$$

where in the last two equations we have repeatedly used (DG3). Compare Czédli and Horváth [CH]. Hence

$$j_\ell(a, a, c) \; \Theta R \; j_\ell(a, b, c) \; \Theta R \; j_\ell(a, c, c) = j_{\ell+1}(a, a, c).$$

Concatenating, we get $j_1(a, a, c) \; (\Theta R)^{2(k-1)} \; j_k(a, a, c) =^{(\mathrm{DG5})} c$. Then

$$j_1(a, a, j_1(aac)) \; (\Theta R)^{2(k-1)} \; j_1(a, a, c) \; (\Theta R)^{2(k-1)} \; c.$$

Finally, using (5), we have $a \; (\Theta R)^{1+4(k-1)} \; c$, hence (2) holds with $h = 1 + 4(k-1) = 4k - 3$. Let us remark that a slightly better value for h is provided in [L2].

(2) $\Rightarrow$ (3) First we shall show that if

$$\Theta(R \circ R) \subseteq (\Theta R)^h \tag{6}$$

holds in some algebra $\mathbf{A}$, for every reflexive and admissible relation R, then, for every $n \geq 1$, also

$$\Theta R^{2^n} \subseteq (\Theta R)^{h^n} \tag{7}$$

holds in $\mathbf{A}$, for every reflexive and admissible relation R. This is proved by induction on n. The basis $n = 1$ is the assumption (6). If (7) holds for some n, then

$$\Theta R^{2^{n+1}} = \Theta(R^{2^n} \circ R^{2^n}) \subseteq^{(6)} (\Theta R^{2^n})^h \subseteq^{(7)} ((\Theta R)^{h^n})^h = (\Theta R)^{h^{n+1}},$$

where we have used the fact that if R is reflexive and admissible, then R^i is reflexive and admissible, for every $i \geq 1$.

Now we can prove (3). If $(a, c) \in \Theta R^*$, then $(a, c) \in \Theta R^i$, for some i, hence $(a, c) \in \Theta R^{2^n}$, for some sufficiently large n, since R is reflexive. Then by (7), $(a, c) \in (\Theta R)^{h^n} \subseteq (\Theta R)^*$. Again, notice that, given $k+1$ directed Gumm terms, [L2] provides a much better bound for ΘR^{2^n}, in comparison with the bound $(\Theta R)^{(4k-3)^n}$ given by the present proof.

(3) $\Rightarrow$ (4) The inclusion $\alpha R^* \supseteq (\alpha R)^*$ is trivial, since α is a congruence, hence transitive. The reverse inclusion follows trivially from (3).

(4) $\Rightarrow$ (1) Let α, β, γ be congruences and let Φ be the tolerance $\alpha\gamma \circ \beta \circ \alpha\gamma$. Then $\Phi^* = \beta + \alpha\gamma$, where $+$ denotes join in the lattice of congruences. Moreover, $\alpha\Phi = \alpha\gamma \circ \alpha\beta \circ \alpha\gamma$. Indeed, if $a \; \alpha \; d$ and $a \; \Phi \; d$, then there are elements b and c such that $a \; \alpha\gamma \; b \; \beta \; c \; \alpha\gamma \; d$. Since $b \; \alpha \; a \; \alpha \; d \; \alpha \; c$, we get $b \; \alpha \; c$, hence $(a, d) \in \alpha\gamma \circ \alpha\beta \circ \alpha\gamma$. From

$\alpha\Phi = \alpha\gamma \circ \alpha\beta \circ \alpha\gamma$ we obtain $(\alpha\Phi)^* = \alpha\beta + \alpha\gamma$, hence from (4) and $\Phi^* = \beta + \alpha\gamma$ we get $\alpha(\beta + \alpha\gamma) = \alpha\beta + \alpha\gamma$. □

We do not know whether we still get a condition equivalent to congruence modularity if in Condition (2) in Theorem 1 we consider a reflexive and admissible relation S in place of a tolerance Θ. In any case, the varieties for which Condition (2) in Theorem 1 holds with S in place of Θ satisfy a much cleaner identity.

Proposition 2. *For every variety $\mathcal{V}$, the following conditions are equivalent.*

(1) $S(R \circ R) \subseteq (SR)^h$, for some h;
(2) $S(R \circ R) \subseteq (SR)^*$;
(3) $S^*R^* = (SR)^*$,

where each condition is intended to hold in every algebra in $\mathcal{V}$ for all reflexive and admissible relations S and R.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) is standard, though not completely usual. Suppose that (2) holds in the free algebra generated by 3 elements x, y and z and let S, R, respectively, be the smallest reflexive and admissible relations containing (x, z), respectively, (x, y) and (y, z). Then $(x, z) \in S(R \circ R)$, hence, by (2), $(x, z) \in (SR)^*$, thus $(x, z) \in (SR)^h$, for some h. This is witnessed by appropriate terms, which also witness that $S(R \circ R) \subseteq (SR)^h$ holds throughout $\mathcal{V}$. See, e. g., [L3] for many similar arguments.

(3) $\Rightarrow$ (2) is trivial.

If (2) holds, then an argument similar to the proof of (2) $\Rightarrow$ (3) in Theorem 1 shows

$$SR^* \subseteq (SR)^*, \tag{8}$$

for all R and S. Applying (8) twice, we have

$$S^*R^* \subseteq^{(8)} (S^*R)^* = (RS^*)^* \subseteq^{(8)} (RS)^{**} = (SR)^*.$$

The inclusion $S^*R^* \supseteq (SR)^*$ is trivial. □

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Dipartimento di Matematica, Viale dell'Eccellenza Scientifica, Università di Roma "Tor Vergata", I-00133 ROME ITALY

URL: `http://www.mat.uniroma2.it/~lipparin`