ASYMPTOTIC BLOW-UP ANALYSIS FOR SINGULAR LIOUVILLE TYPE EQUATIONS WITH APPLICATIONS.

D. BARTOLUCCI$^{(1,\dagger)}$ - G. TARANTELLO$^{(2,\dagger)}$

Abstract. We generalize the pointwise estimates obtained in [2], [17] and [30] concerning blow-up solutions of the Liouville type equation:

$$-\Delta u = |x|^{2\alpha} W(x)\, e^{u} \quad \text{in} \quad \Omega,$$

with $\Omega \subset \mathbb{R}^2$ open and bounded, $\alpha \in (-1, +\infty)$ and W any Lipschitz continuous function which satisfies $0 < a \leq W \leq b < \infty$. Mainly, we focus to the case where the parameter $\alpha \in \mathbb{N}$, whose analysis is much more involved due to the non radial behaviour of blow-up solutions. In the worst situation there is no chance (in general) to resolve the profile in the form of a solution of a Liouville equation in $\mathbb{R}^2$, instead we need to adopt iterated blow-up arguments.
Moreover, we refine our blow up analysis to a class of planar Liouville type equation arising from the study of Cosmic Strings (cfr. [24],[31]). In this context, we are able to distinguish between a single blow-up radial profile and the case of multiple blow-up profiles, typical of non radial solutions. As a consequence we obtain a (radial) symmetry result which is interesting in itself but also relates to the "sharp" solvability issue concerning the planar problem (1.27)-(1.28) discussed below.

Keywords: Blow up solutions, Non radial bubbling, singular Liouville type equations, Cosmic Strings.

1. Introduction

The analysis of Liouville-type equations [20] with singular data [23], has attracted much attention in recent years due to their application to several issues of interest in Mathematics and Physics, such as Chern-Simons and Electroweak self-dual vortices [27], [30], [34], conformal metrics on surfaces with conical singularities [1], [4], [6], [32], statistical mechanics of two-dimensional turbulence [10] and of self-gravitating systems [33] and cosmic strings [24], [31], and more recently the theory of hyperelliptic curves [11].
Existence (and multiplicity) results for such class of equations (see for example [5], [16], [19], [21] and references therein) are based on topological and variational methods that require in an essential way the compactness property of the solution set. To this purpose, refined blow up techniques have been developed for the description of the "local" asymptotic profile of solution sequences around a (blow-up) point, where "bubbling" phenomena develop, see [8], [9], [17], [18], and more recently [2], [15], [28], [35].
In this note, we shall contribute in this direction and provide pointwise estimates for the profile

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(1) Daniele Bartolucci, Departmento di Matematica, Universita' di Roma *"Tor Vergata"*, Via della ricerca scientifica n.1, 00133 Rome, Italy. e-mail: bartoluc@mat.uniroma2.it.
(2) Gabriella Tarantello, Dipartimento di Matematica, Universita' di Roma *"Tor Vergata"*, Via della ricerca scientifica n.1, 00133 Rome, Italy. e-mail: tarantel@mat.uniroma2.it.
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of a solution sequence in the situation where "multiple-bubbles" occur. To be more precise, let us mention that, after introducing suitable coordinates, one can "localize" the problem around the origin and so analyse the behavior of a solution sequence $u_n \in C^0(\overline{B_r})$ satisfying:

$$\begin{cases} -\Delta u_n = |x|^{2\alpha_n} W_n(x)\, e^{u_n} \ \text{ in } \ B_r, \\ |u_n(x) - u_n(y)| \le c_0, \quad \text{for } |x| = |y| = r, \\ \int\limits_{B_r} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \le c_1, \end{cases} \tag{1.1}$$

with suitable constants $c_0 \ge 0$, $c_1 > 0$ and $B_r := \{x \in \mathbb{R}^2 \,:\, |x| < r\}$, $r > 0$.
We shall assume that:

$$0 < a \le \min_{\overline{B_r}} W_n \le \max_{\overline{B_r}} W_n \le b < +\infty, \ \ \max_{\overline{B_r}} \|\nabla W_n\| \le A, \tag{1.2}$$

and let

$$\alpha_n \to \alpha \in (-1, +\infty), \text{ as } \ n \to +\infty; \tag{1.3}$$

and (by taking r smaller if necessary) we also suppose that u_n admits the origin as its only blow up point in B_r. In other words,
$\forall\ 0 < \varepsilon < r$ there exist $C_\varepsilon > 0$ (depending on ε) such that:

$$\max_{\overline{B_r}} u_n \to +\infty, \quad \limsup_{n\to+\infty} \left(\max_{\overline{B_r}\setminus B_\varepsilon} u_n \right) \le C_\varepsilon. \tag{1.4}$$

By well known results, we have the following:

Theorem 1.1 ([9],[18],[8],[7])**.** *If u_n satisfies (1.1) and (1.4) and if (1.2)-(1.3) hold, then along a subsequence we have:*

$$|x|^{2\alpha_n} W_n(x) e^{u_n} \rightharpoonup 8\pi(1+\alpha)\delta_0, \, as \ \ n \to +\infty, \tag{1.5}$$

weakly in the sense of measures in B_r, where δ_0 denotes the Dirac delta with pole $x = 0$. In particular, $\forall\ \varepsilon \in (0, r)$ we have:

$$\int\limits_{\varepsilon \le |x| \le r} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \to 0, \quad \int_{B_\varepsilon} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \to 8\pi(1+\alpha), \, as \ \ n \to +\infty. \tag{1.6}$$

Furthermore, in the situation where $\alpha \notin \mathbb{N}$, it was shown in [2], [17] that after scaling the sequence u_n behaves as a "single" bubble around the origin. More precisely, by setting:

$$x_n \in \overline{B}_r \,:\, u_n(x_n) = \max_{\overline{B_r}} u_n, \ \ \tau_n = e^{u_n(x_n)} \ \text{ and } \ (\varepsilon_n)^{2(1+\alpha_n)} := {\tau_n}^{-1}, \tag{1.7}$$

so that,

$$|x_n| \to 0, \quad \tau_n \to +\infty, \quad \varepsilon_n \to 0, \ \text{ as } \ n \to +\infty,$$

the following holds:

Theorem 1.2 ([17],[2])**.** *Assume (1.2), (1.3) and let u_n satisfy (1.1) and (1.5). Then with the notation in (1.7) we have:*
(1) if $\alpha_n \equiv \alpha = 0$, then

$$\left| u_n(x) - \log \frac{\tau_n}{\left(1 + \frac{\tau_n}{8} W_n(0)|x - x_n|^2\right)^2} \right| \le C, \quad \forall\ |x| \le r, \quad \forall\ n \in \mathbb{N}; \tag{1.8}$$

(2) if $\alpha_n \equiv \alpha > 0$ *and* $\alpha \notin \mathbb{N}$*, then,*

$$i)\ \frac{|x_n|}{\varepsilon_n} \to 0, \text{ as } n \to +\infty, \tag{1.9}$$

$$ii)\ \left| u_n(x) - \log \frac{\tau_n}{\left(1 + \frac{\tau_n}{8(1+\alpha)^2} W_n(0)|x - x_n|^{2(1+\alpha)}\right)^2} \right| \leq C, \quad \forall\ |x| \leq r, \quad \forall\ n \in \mathbb{N}; \tag{1.10}$$

for a suitable $C > 0$.

Remark 1.3. *The assumption that* u_n *admits uniformly bounded oscillation on* ∂B_r*, is crucial for the validity of (1.5) in Theorem 1.1 as well as (1.8) and (1.10) in Theorem 1.2, as discussed in* [28].

Our first contribution will be to show that (1.10) remains valid also when,

$$\alpha_n \to \alpha \in (-1, 0), \tag{1.11}$$

see Theorem 4.1. Although this fact is not surprising, its proof still requires some care in view of the weaker regularity of the solutions in this case, and it has been included in section 4.

While, for $\alpha > 0$, we observe that (1.10) implies that:

$$u_n(0) = \max_{\overline{B}_r} u_n + \mathrm{O}(1), \quad \text{as } n \to +\infty, \tag{1.12}$$

$$\sup_{\overline{B}_r} \left(u_n(x) + 2(1 + \alpha_n)\log(|x|)\right) \leq C, \quad \forall\ n \in \mathbb{N}. \tag{1.13}$$

In fact, it follows from the analysis in [30] (see Corollary 5.2.24) that (1.13) implies (1.12) and moreover (1.13) always holds when $\alpha_n = \alpha \notin \mathbb{N}$. On the other hand, for $\alpha \in \mathbb{N}$, (1.12) may not be satisfied (together with (1.13) and (1.10)) due to the richer structure of the solution set of the planar "singular" Liouville equation:

$$\begin{cases} -\Delta U = |x|^{2\alpha} e^U & \text{in } \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} |x|^{2\alpha} e^U < +\infty, \end{cases} \tag{1.14}$$

which occurs as the so called "limiting" problem satisfied by the profile descibing u_n after blow up. Indeed, in view of the classification result in [26] (see also [13],[14]) we know that, for $\underline{\alpha \in \mathbb{N}}$, every solution of (1.14) admits (in complex notations) exactly the following form:

$$U(z) = \log \frac{\tau}{\left(1 + \frac{\tau}{8(1+\alpha)^2} |z^{\alpha+1} - \xi|^2\right)^2}, \quad z \in \mathbb{C}, \tag{1.15}$$

for every $\tau > 0$, $\xi \in \mathbb{C}$.
On the other hand, when $\underline{\alpha \notin \mathbb{N}}$, then (1.15) holds only with $\xi = 0$ (to avoid multiple valued solutions) and every solution of (1.14) is radially symmetric in this case. In other words (1.14) admits non-radial solutions only if $\alpha \in \mathbb{N}$.
In addition we observe that, for $\alpha > -1$, <u>every</u> solution of (1.14) must satisfy:

$$\int\limits_{\mathbb{R}^2} |x|^{2\alpha} e^U = 8\pi(1 + \alpha), \tag{1.16}$$

which justifies (1.5). Furthermore, we can use (1.15) with,

$$\alpha = N \in \mathbb{N}, \quad \tau = \tau_n \to +\infty, \quad 0 \neq \xi_n \to 0, \text{ and } \tau_n |\xi_n|^2 \to +\infty, \tag{1.17}$$

to obtain a solution sequence u_n of (1.14) with $\alpha = N$ (or equivalently of (1.1), with $\alpha_n = N \in \mathbb{N}$ and $W_n \equiv 1$), for which there holds:
if $x_n : x_n^{N+1} = \xi_n$ then, as $n \to \infty$,

$$x_n \to 0, \quad \text{and} \quad u_n(x_n) = \max_{B_r} u_n = \tau_n \to +\infty, \tag{1.18}$$

$$u_n(0) - u_n(x_n) = -\log\left(\tau_n |x_n|^{2(N+1)}\right) + \mathrm{O}(1) \to -\infty \tag{1.19}$$

$$u_n(x_n) + 2(N+1)\log(|x_n|) = \log\left(\tau_n |x_n|^{2(N+1)}\right) + \mathrm{O}(1) \to \infty \tag{1.20}$$

and therefore both (1.12) and (1.13) fail for u_n, together with the pointwise estimates (1.10). In addition, if we consider,

$$v_n(x) = u_n(|x_n|x) + 2(N+1)\log(|x_n|), \tag{1.21}$$

then we check easily that (along a subsequence) v_n admits $N+1$-blow up points (in the sense of Brezis-Merle [9]) which (up to a rotation) coincide with the distinct $N+1$-roots of the identity. This will be our guiding example in the investigation of the blow-up behaviour of a sequence u_n satisfying (1.1) and (1.5) beyond (1.10) To this purpose, we recall that in Theorem 5.6.51 of [30] it has been shown that actually (1.12) is a necessary and sufficient condition for the validity of (1.10), whenever $\alpha_n \equiv \alpha \geq 0$, hence also when $\alpha \in \mathbb{N}$. Thus, (1.12) and (1.13) are equivalent (and always hold for $0 \leq \alpha_n = \alpha \notin \mathbb{N}$), and by virtue of (1.19) and (1.20), we expect such conditions to be formulated only in terms of the quantity: $\tau_n \xi_n = \tau_n x_n^{N+1}$. Indeed,we show in Corollary 2.4 below, that (1.12) (or equivalently (1.13)) fails, if and only if $\tau_n |x_n|^{2(\alpha_n)} \to +\infty$, as $n \to +\infty$, or in terms of the notations in (1.7),

$$\frac{|x_n|}{\varepsilon_n} \to +\infty, \quad \text{as } n \to +\infty. \tag{1.22}$$

In this situation, we prove that the following pointwise estimates hold:

Theorem 1.4. *Let u_n satisfy (1.1), (1.5) and suppose that (1.2), (1.3) hold with $\alpha = N \in \mathbb{N}$. Assume (1.22), with x_n and ε_n defined in (1.7). There exist $(N+1)$ sequences $\{\xi_{j,n}\}_{j=1,\cdots,N+1}$ such that*

$$\xi_{n,j} \to \xi_j, \text{ as } n \to +\infty, \forall\, j = 1, \cdots, N+1, \tag{1.23}$$

and up to a rotation $\{\xi_1, \cdots, \xi_{N+1}\}$ define the distinct $(N+1)$-roots of unity, i.e. $\xi_j \neq \xi_\ell$, $j \neq \ell$ and $\xi_j^{N+1} = 1$, for any $j = 1, \cdots, N+1$. Furthermore, by setting:

$$z_{n,j} = |x_n|\xi_{j,n}, \quad j = 1, \cdots, N+1,$$

there holds:

$$\left| u_n(x) - U_{n,\infty}^{(z_{n,1},\cdots,z_{n,N+1})}(x) \right| = \mathrm{O}\left(1 + 4|\alpha_n - N| \log\left(1 + \frac{|x|}{|x_n|}\right)\right), \quad \forall\, |x| \leq r, \quad \forall\, n \in \mathbb{N}, \tag{1.24}$$

where

$$U_{n,\infty}^{(z_1,\cdots,z_{N+1})}(x) := \log \frac{\tau_n}{\left(1 + \frac{W_n(0)\tau_n}{8(1+N)^2} \prod_{j=1}^{N+1} |x - z_j|^2\right)^2}, \quad |x| \leq r. \tag{1.25}$$

Remark 1.5. *Notice that (1.24)-(1.25) imply for u_n an asymptotic behaviour analogous to that of the explicit example illustrated above, in particular (1.19) and (1.20) hold for u_n as well. When $\alpha_n \neq \alpha = N$ (see assumption in (ii) of Theorem 1.2) then the pointwise estimate (1.24) involves the error term: $|\alpha_n - N| \log\left(\frac{|x|}{|x_n|}\right)$, which concerns the behaviour of u_n in the "neck" region obtained after scaling the domain at a rate of lower order than $|x_n|$. In general it cannot be avoided, although we may wonder if: $|\alpha_n - N| \log\left(\frac{1}{|x_n|}\right) = \mathrm{O}(1)$, as actually it occurs for blow-up sequences satisfying the cosmic string equation (1.27) discussed below , see Theorem 5.10 and Theorem 5.11 for details.*

Remark 1.6. *In account of the example discussed above, it is natural to ask whether or not in (1.25) it is possible to replace the product term: $\prod_{j=1}^{N+1} |x - z_{n,j}|^2$ with (in complex notations) $|x^{N+1} - x_n^{N+1}|^2$. Clearly, when $\frac{|x_n|}{\varepsilon_n} \leq C$, then we can certainly use such a replacement in (1.10). On the other hand, when (1.22) holds, to operate such a replacement in (1.25), it would require to show that the sequence $\xi_{n,j}$ in (1.23) should be (almost) $\mathbb{Z}_{N+1}$-symmetric itself, in the sense that, the well known identities valid for the k-th symmetric product of the $(N+1)$-roots of the unity, would remain valid for the $(N+1)$-sequences $\{\xi_{n,1}, \cdots, \xi_{n,N+1}\}$ up to an error term of order $\left(\frac{|x_n|}{\varepsilon_n}\right)^{-\frac{1}{k+1}}$ (n large), for any $k = 1, \cdots, N$. For example for $N = 1$, this would require that,*

$$\sqrt{\frac{|x_n|}{\varepsilon_n}} |\xi_{n,1} + \xi_{n,2}| \leq C, \quad \forall\, n \in \mathbb{N}, \tag{1.26}$$

for a suitable constant $C > 0$.
Such an estimate has been established in [3] *for blow-up solutions of a planar elliptic problem obtained (after stereographic projection) from solutions of a mean field equation on $\mathbb{S}^2$. Actually, in* [3] *the authors show a much stronger property, namely that $\xi_{1,n} + \xi_{2,n} = 0$, for any large $n \in \mathbb{N}$, a fact which we believe true also for blow-up solutions of the cosmic string equation (1.27) considered below. In this respect, we refer to Remark 7.5 in* [31] *for more details,.*

More precisely, in the construction of cosmic strings, one is interested to characterise the set of $\beta > 0$ such that the equation:

$$-\Delta u = |x|^{2N} e^u + e^{au} \quad \text{in} \quad \mathbb{R}^2, \tag{1.27}$$

admits a solution u satisfying:

$$\beta := \frac{1}{2\pi} \int_{\mathbb{R}^2} \left(|x|^{2N} e^u + e^{au}\right), \tag{1.28}$$

see [24], [31]. From the analysis in [24], [31] we know that, except for the exceptional value $a = \frac{1}{N+1}$, where (1.27)-(1.28) is conformally invariant and $\beta = 4(N+1)$ (see also [12]) , for $a \neq \frac{1}{N+1}$ this is no longer the case, and so far the solvability issue for (1.27)-(1.28) is fully understood only for a restricted range of parameters $a > 0$ and $N > -1$, or in the radial setting. On the other hand, it is known that (1.27)-(1.28) admits non-radial solutions (for suitable choices of the parameters: $a > 0, N > 0$ and β) which also exhibit a blow up behaviour, see [31]. Thus, to understand the solvability of (1.27)-(1.28) beyond the radial framework, it is necessary to grasp the blow-up behaviour of (possibly non-radial) solution sequences of (1.27)-(1.28). This task has been recently carried out in [31] according to different choices of the parameters $a > 0$ and $N > -1$. In particular, the analysis of [31] indicates possible ranges of β's where one can exclude the presence of non-radial solutions, in the same spirit of [3]. In this direction we mention that, for a sequence u_n satisfying:

$$\begin{cases} -\Delta u_n = |x|^{2N} e^{u_n} + e^{a u_n} \quad \text{in } \mathbb{R}^2, \\ \beta_n = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(|x|^{2N} e^{u_n} + e^{a u_n} \right) \to 4(N+1), \text{as } n \to +\infty, \end{cases} \tag{1.29}$$

from [31] we know that necessarily $a \geq \frac{1}{2(N+1)}$ and u_n must blow-up, possibly at infinity. To be more precise, let us start to describe what happens when in (1.29) we take $N \notin \mathbb{N}$.

Lemma 1.7 ([31]). *If $-1 < N \notin \mathbb{N}$, then (1.29) holds if and only if either*

$$(i)\ \frac{1}{2(N+1)} < a < \frac{1}{N+1} \quad \sup_{|x|<\varepsilon} u_n \to +\infty \text{ and } \sup_{|x|\geq\varepsilon} \{u_n + \beta_n \log(|x|)\} \to -\infty, \forall\, \varepsilon > 0, \tag{1.30}$$

as $n \to +\infty$, or,

$$(ii)\ \frac{1}{N+1} < a < 1 \quad \sup_{|x|<R} u_n \to -\infty \text{ and } \sup_{|x|\geq R} \{u_n + \beta_n \log(|x|)\} \to +\infty, \forall\, R > 0, \tag{1.31}$$

as $n \to +\infty$.

We can better interpret (1.31) by setting:

$$\widehat{u}_n(x) = u_n\left(\frac{x}{|x|^2}\right) + \beta_n \log\left(\frac{1}{|x|}\right), \tag{1.32}$$

so that $\widehat{u}_n$ extends smoothly at the origin and satisfies:

$$\begin{cases} -\Delta \widehat{u}_n = |x|^{\lambda_n} e^{\widehat{u}_n} + |x|^{\gamma_n} e^{a\widehat{u}_n} \quad \text{in } \mathbb{R}^2, \\ \beta_n = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(|x|^{\lambda_n} e^{\widehat{u}_n} + |x|^{\gamma_n} e^{a\widehat{u}_n} \right) \to 4(N+1), \text{ as } n \to +\infty, \end{cases} \tag{1.33}$$

where $\lambda_n = \beta_n - 2(N+2)$ and $\gamma_n = a\beta_n - 4$. Furthermore, (1.31) can be stated equivalently as follows:

$$\begin{aligned} &\max_{|x|\leq\varepsilon} \widehat{u}_n \to +\infty \\ &\sup_{|x|\geq\varepsilon} \{\widehat{u}_n + \beta_n \log(|x|)\} \to -\infty, \text{as } n \to +\infty, \end{aligned} \tag{1.34}$$

$\forall\, \varepsilon > 0$.

In other words, for $0 \leq N \notin \mathbb{N}$, Lemma 1.7 states that: either u_n (when $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$) or $\widehat{u}_n$ (when $\frac{1}{N+1} < a < 1$) admits the origin as its only blow-up point. See [31] for details. Such blow-up behaviour agrees exactly with that described in [24] for radial solutions, and it has motivated us to establish the following:

Theorem 1.8. *Let $0 < N \notin \mathbb{N}$, there exists $\varepsilon_0 > 0$ such that for $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$ and $\beta \in (4(N+1), 4(N+1)+\varepsilon_0)$ or for $\frac{1}{N+1} < a < 1$ and $\beta \in (4(N+1)-\varepsilon_0, 4(N+1))$, problem (1.27)-(1.28) admits a unique solution which is also radially symmetric.*

The symmetry result of Theorem 1.8 is sharp, since from [24] it is possible to exhibit values of $a \in (\frac{1}{4}, \frac{3}{4})$, $0 < N \notin \mathbb{N}$ and $\beta = \frac{2N}{1-a}$ corresponding to which problem (1.27)-(1.28) admits both radial and non radial solutions.
More precisely, we show (also when $N \in \mathbb{N}$) that any sequence u_n satisfying (1.29) is necessarily radially symmetric provided that,

$$\text{for} \quad x_n \in \mathbb{R}^2 \,:\, u_n(x_n) = \max_{\mathbb{R}^2} u_n \quad \text{there holds: } |x_n|^{2(N+1)} e^{u_n(x_n)} \to 0, \text{ as } n \to +\infty. \tag{1.35}$$

See Theorem 5.8.
As before, we shall show that, when $N \notin \mathbb{N}$, then (1.35) is automatically satisfied. More generally, we shall provide a profile estimate for u_n of the type (1.10) under the more general assumption that $|x_n|^{2(N+1)} e^{u_n(x_n)} \leq C$ (please compare with (1.13)). We refer to Theorem 5.10 for details.

Again we are left to analyse what happens when

$$|x_n|^{2(N+1)} e^{u_n(x_n)} \to +\infty, \tag{1.36}$$

which can occur only if $N \in \mathbb{N}$. To this purpose, we start by observing that when $N \in \mathbb{N}$ then (contrary to Lemma 1.7) we can only claim that the blow-up set S of u_n and the blow-up set $\widehat{S}$ of $\widehat{u}_n$ satisfy: $S \neq \emptyset$ if and only if $0 \notin \widehat{S}$, and so in principle the origin could be a blow up point for u_n or for $\widehat{u}_n$ regardless on whether $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$ or $\frac{1}{N+1} < a < 1$. Thus, to provide pointwise estimates analogous to (1.24) we need to be more specific about this possibility, and so we assume:

$$\begin{cases} \text{if } \frac{1}{2(N+1)} < a < \frac{1}{N+1}, & \text{then} \quad 0 \notin \widehat{S}, \\ \text{if } \frac{1}{N+1} < a < 1, & \text{then} \quad 0 \notin S. \end{cases} \tag{1.37}$$

We prove the following:

Theorem 1.9. *Let u_n satisfy (1.29) and assume that (1.36) holds with $x_n \in \mathbb{R}^2 : u_n(x_n) = \max_{\mathbb{R}^2} u_n$. Then $N \in \mathbb{N}$ and there exists $(N+1)$-sequences $\{\xi_{j,n}\}_{j=1,\cdots,N+1}$ such that*

$$\xi_{n,j} \to \xi_j, \text{ as } n \to +\infty, \, \forall \, j = 1, \cdots, N+1, \tag{1.38}$$

and (after a rotation) $\{\xi_j\}_{j=1,\cdots,N+1}$ define the distinct $N+1$-roots of unity.
Moreover, for $\varepsilon > 0$ sufficiently small and $v_n(x) = u_n(|x_n|x) + 2(N+1)\log(|x_n|)$ we have:

$$v_n(\xi_{n,j}) = \max_{B_\varepsilon(\xi_j)} v_n \to +\infty, \text{ as } n \to +\infty, \, j = 1, \cdots, N+1.$$

Furthermore, if (1.37) holds, then for $z_{n,j} = |x_n|\xi_{n,j}$ we have:

$$(\beta_n - 4(N+1)) \log\left(\frac{1}{|x_n|}\right) \to 0, \text{ as } n \to +\infty, \tag{1.39}$$

and,

$$\left| u_n(x) - \log \frac{e^{u_n(x_n)}}{(1+\frac{e^{u_n(x_n)}}{8(N+1)^2}\prod_{j=1}^{N+1}|x-z_{n,j}|^2)^2} + (\beta_n - 4(N+1))(\log\left(1+\min_{j=1,\cdots,N+1}|x-z_{n,j}|\right)) \right| \le C,\ x\in\mathbb{R}^2. \tag{1.40}$$

In concluding, we mention that the information about the $\mathbb{Z}_{N+1}$-symmetry of the blow-up points: ξ_j $j=1,\cdots,N+1$, of v_n in (1.23) and (1.38), relies upon a Pohožaev type identity, which allows one to obtain (in complex notation) the following $(N+1)$-identities:

$$2\xi_i \sum_{j\neq i,\, j=1}^{N+1} \frac{1}{\xi_i - \xi_j} = N, \quad \forall\ i = 1,\cdots,N+1. \tag{1.41}$$

At this point, one can show that,

(1.41) holds if and only if $\{\xi_1,\cdots,\xi_{N+1}\}$ define the $(N+1)$-roots of the value $(-1)^N\xi_1\xi_2\cdots\xi_{N+1}\neq 0$. (1.42)

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2. Preliminaries and first estimates

We will denote by $C>0$ a suitable positive constant possibly depending on the given data and whose value may change even from line to line. Furthermore, we use the notation of O(1) to denote a quantity uniformly bounded from above and below, with respect to the variables considered (e.g. $n\in\mathbb{N}$, $x\in\mathbb{R}^2$ etc..). As usual, we let $B_r(x_0)=\{x\in\mathbb{R}^2\,|\,|x-x_0|<r\}$ the ball with center $x_0\in\mathbb{R}^2$ and of radius $r>0$.

We recall from [9] that, for a sequence u_n defined in a domain $\Omega\subset\mathbb{R}^2$, a point $x_0\in\Omega$ is called a blow up point for u_n in Ω, if there exists a sequence of points $\{y_k\}\subset\Omega\,:\,y_k\to x_0$ and a subsequence u_{n_k} such that,

$$u_{n_k}(y_k)\to+\infty, \quad \text{as } k\to+\infty.$$

The following Lemma is essentially proved in [2].

Lemma 2.1. *Let u_n be a sequence of solutions of (1.1),(1.2),(1.3). For $r_n>0$, define*

$$v_n(z) = u_n(r_n z) + 2(1+\alpha)\log(r_n), \quad |z|\le r r_n^{-1}, \tag{2.1}$$

satisfying:

$$-\Delta v_n = |z|^{2\alpha_n} W_n(r_n z)\, e^{v_n}, \ \ in \ \ \{|z|<r r_n^{-1}\}. \tag{2.2}$$

By passing to a subsequence if necessary, for any $0<R<r r_n^{-1}$ the blow up set S_R of v_n in B_R is finite. Moreover for any compact subset $\Omega_0\subset B_R\setminus S_R$, there exists a constant C_0 (depending on Ω_0, R, c_0, c_1) such that,

$$|v_n(z_1)-v_n(z_2)|\le C_0, \quad \forall\ \{z_1,z_2\}\subset\Omega_0.$$

To simplify notations, and without loss of generality, we let

$$W_n(0) = 1. \tag{2.3}$$

We start our discussion with the following important information about the blow-up behavior of u_n in the situation where (1.22) holds.

Lemma 2.2. *Let u_n satisfy (1.1) and assume that (1.2), (1.5) and (1.3) holds with $\alpha = N \in \mathbb{N}$. If there exists $\{y_n\} \subset \overline{B_r}\,:\, u(y_n)+2(\alpha_n+1)\log(|y_n|) \to +\infty$, then there exists a constant $d_1 > 1$ such that:*

$$\frac{1}{d_1} \leq \frac{|y_n|}{|x_n|} \leq d_1, \quad \forall\ n \in \mathbb{N}. \tag{2.4}$$

In particular, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$:

$$\sup_{(1+\varepsilon)d_1|x_n|\leq|x|\leq r} \{u_n(x) + 2(\alpha_n + 1)\log(|x|)\} \leq C_\varepsilon,\ \forall\ n \in \mathbb{N}, \tag{2.5}$$

and,

$$\int\limits_{2d_1|x_n|\leq|x|\leq r} |x|^{2\alpha_n} W_n(x) e^{u_n} \to 0,\ as\ \ n \to +\infty. \tag{2.6}$$

Proof. We start with the following:
Claim: along a subsequence:

$$u_n(x_n) + 2(\alpha_n + 1)\log(|x_n|) \to +\infty. \tag{2.7}$$

To establish (2.7) we argue by contradiction and assume that,

$$u_n(x_n) + 2(\alpha_n + 1)\log(|x_n|) \leq C, \tag{2.8}$$

with suitable $C > 0$. From (1.5) we know that necessarily $|y_n| \to 0$ and so $u_n(y_n) \to +\infty$. As a consequence, also $u_n(x_n) \to +\infty$. We define,

$$U_n(x) = u_n(\varepsilon_n x) - u_n(x_n),$$

which satisfies,

$$\begin{cases} -\Delta U_n = |x|^{2\alpha_n} W_n(\varepsilon_n x) e^{U_n}, \ \text{ in } \ \{|x| < r\varepsilon_n^{-1}\}, \\ U(\frac{x_n}{\varepsilon_n}) = 0 = \max U_n, \\ \frac{1}{2\pi} \int\limits_{\{|x|<r\varepsilon_n^{-1}\}} |x|^{2\alpha_n} W_n(\varepsilon_n x) e^{U_n} \to 4(N+1), \text{as } \ n \to +\infty, \end{cases}$$

and from (2.8) it follows that $\left|\dfrac{x_n}{\varepsilon_n}\right| \leq C$. As a consequence (along a subsequence): $U_n \to U$, uniformly in C^2_{loc}, with U satisfying (1.14). Therefore, from (1.16), we see that for any $\delta > 0$ we find $R = R(\delta) > 0$ such that for large $n \in \mathbb{N}$ we have:

$$\frac{1}{2\pi} \int\limits_{\{|x|\leq R\varepsilon_n\}} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \geq 4(\alpha + 1) - \delta. \tag{2.9}$$

On the other hand, by considering v_n in (2.1) with $r_n = |y_n|$ we find that

$$v_n\left(\frac{y_n}{|y_n|}\right) \to +\infty, \text{ as } n \to +\infty;$$

and so, along a subsequence, v_n admits a blow up point $y_0 = \lim_{n\to+\infty} \frac{y_n}{|y_n|}$ in the unit circle. Therefore we can apply Theorem 1.1 to the sequence $v_n(x+y_0)$ with $\alpha_n \equiv 0$ and

$$W_{1,n}(x) = W_n\left(\varepsilon_n\left(x + \frac{y_n}{|y_n|}\right)\right)\left|x + \frac{y_n}{|y_n|}\right|^{2\alpha_n} \text{ in } B_{\frac{1}{2}},$$

and conclude that, $\forall\, \varepsilon > 0$:

$$\frac{1}{2\pi}\int\limits_{\{|x-y_n|\le\varepsilon_n|y_n|\}} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \to 4, \text{ as } n \to +\infty. \tag{2.10}$$

Since $\frac{|y_n|}{\varepsilon_n} = |y_n| e^{\frac{u_n(x_n)}{2(1+\alpha_n)}} \ge |y_n| e^{\frac{u_n(y_n)}{2(1+\alpha_n)}} \to +\infty$, as $n \to +\infty$, we see that, for large $n \in N$, $B_{\varepsilon_n|y_n|}(y_n) \cap B_{R\varepsilon_n} = \emptyset$, and in view of (2.9) and (2.10) we get a contradiction to (1.5). So (2.7) is established.

Next, to establish (2.4), we argue by contradiction and on the basis of (2.7), assume without loss of generality that, along a subsequence,

$$\frac{|y_n|}{|x_n|} \to 0, \text{ as } n \to +\infty. \tag{2.11}$$

Now we consider v_n in (2.1) with $r = |x_n|$, namely

$$v_n(z) := u_n(|x_n|z) + 2(1+\alpha_n)\log(|x_n|), \quad z \in D_n := \{|z| < r|x_n|^{-1}\}.$$

satisfying:

$$\begin{cases} -\Delta v_n = |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n}, \text{ in } D_n, \\ \int\limits_{D_n} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \to 4(N+1), \text{as } n \to +\infty, \end{cases} \tag{2.12}$$

with $W_{1,n}(z) = W_n(|x_n|z)$. Since by assumptions $u(y_n) + 2(\alpha_n+1)\log(|y_n|) \to +\infty$, in view of (2.11) we find that,

$$v_n\left(\frac{y_n}{|x_n|}\right) = u_n(y_n) + 2(\alpha_n+1)\log(|y_n|) + 2(\alpha_n+1)\log\left(\frac{|x_n|}{|y_n|}\right) \to +\infty, \text{ as } n \to +\infty,$$

and $\frac{y_n}{|x_n|} \to 0$. In other words v_n admits a blow up point at the origin. Thus, in view of Lemma 2.1, we are in position to apply Theorem 1.1 to v_n, and for $\varepsilon > 0$ sufficiently small, conclude that:

$$\frac{1}{2\pi}\int\limits_{B_\varepsilon} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \to 4(N+1), \text{as } n \to +\infty.$$

On the other hand, $v_n\left(\frac{x_n}{|x_n|}\right) \to +\infty$, and therefore (up to a subsequence) v_n admits a blow up point also at $z_0 := \lim_{n\to+\infty} \frac{x_n}{|x_n|}$ and $|z_0| = 1$. Therefore, as above, for $\varepsilon > 0$ sufficiently small, we conclude that:

$$\frac{1}{2\pi}\int\limits_{B_\varepsilon(z_0)} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \to 4, \text{as } \ n\to+\infty,$$

and this is impossible, by virtue of (2.12). As a consequence (2.4) and (2.5) holds. In other words, we have verified that, for $\varepsilon>0$ sufficiently small, the following hold for u_n:

$$u_n(x_n)+2(\alpha_n+1)\log(|x_n|)\to+\infty, \text{as } \ n\to+\infty,$$

$$\sup_{|x|\leq 2\varepsilon|x_n|}\{u_n(x)+2(\alpha_n+1)\log(|x|)\}\leq C,\ \forall\ n\in\mathbb{N},$$

$$\sup_{(1+\varepsilon)|x_n|d_1\leq|x|\leq r}\{u_n(x)+2(\alpha_n+1)\log(|x|)\}\leq C,\ \forall\ n\in\mathbb{N},$$

and so we can apply Proposition 5.5.45 of [30] to conclude that,

$$\int\limits_{2d_1|x_n|\leq|x|\leq r} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \to 0, \text{as } \ n\to+\infty,$$

and we obtain (ii). □

Remark 2.3. *As we know from* [28] *(see also* [30]*), in general properties (2.4)-(2.5) fail when we remove the assumption of uniformly bounded oscillation of u_n on ∂B_r.*

As already discussed in the Introduction, from Lemma 2.2 we readily derive the following:

Corollary 2.4. *Under the assumption above there holds:*

$$\frac{|x_n|}{\varepsilon_n}\to+\infty \iff \sup_{0<|x|\leq r}\{u_n(x)+2(\alpha_n+1)\log(|x|)\}\to+\infty,\ \textit{as}\ \ n\to+\infty. \tag{2.13}$$

Proof. Clearly if (1.22) holds then necessarily

$$\sup_{0<|x|\leq r}\{u_n(x)+2(\alpha_n+1)\log(|x|)\}\geq u_n(x_n)+2(\alpha_n+1)\log(|x_n|)\to+\infty, \text{as } \ n\to+\infty.$$

Viceversa, if there exists $y_n\in\overline{B_r}: u_n(y_n)+2(\alpha_n+1)\log(|y_n|)\to+\infty$, then we can use Lemma 2.2 to see that necessarily,

$$u_n(x_n)+2(\alpha_n+1)\log(|x_n|)\to+\infty, \text{as } \ n\to+\infty,$$

or equivalently $\frac{|x_n|}{\varepsilon_n}\to+\infty$, as claimed. □

3. The proof of Theorem 1.4

Throughout this section we let u_n satisfy (1.1) with,

$$\alpha_n \to \alpha = N \in \mathbb{N}, \tag{3.1}$$

and assume that (1.2), (1.5) and (1.7) hold. Furthermore, by replacing u_n with $u_n\left(\frac{r}{4}x\right)+2(\alpha_n+1)\log\left(\frac{r}{4}\right)$ we can assume for simplicity that,

$$r = 4. \tag{3.2}$$

As above we define,

$$v_n(z) = u_n(|x_n|z) + 2(1+\alpha_n)\log(|x_n|\,), \quad z \in D_n := \left\{|z| \le 4|x_n|^{-1}\right\},$$

which satisfies (2.12) with $W_{1,n}(z) = W_n(|x_n|z)$. In view of Lemma 2.1 and Lemma 2.2 the following holds for v_n.

Proposition 3.1. *There exist $(N+1)$ sequences $\{\xi_{j,n}\}_{n\in\mathbb{N}}$, $j \in \{1,\cdots,N+1\}$ such that (along a subsequence) we have:*

$$i)\ \xi_{j,n} \to \xi_j, \quad v_n(\xi_{j,n}) \to +\infty, \text{ as } n \to +\infty, \tag{3.3}$$
$$\text{and} \quad v(\xi_{j,n}) = v(\xi_{1,n}) + \mathrm{O}(1), \text{ as } n \to +\infty, \quad \forall\, j \in \{1,\cdots,N+1\}$$

$$ii)\ |z|^{2\alpha_n} W_n(|x_n|z)e^{v_n} \rightharpoonup 8\pi \sum_{j=1}^{N+1} \delta_{\xi_j}, \tag{3.4}$$

weakly in the sense of measures on compact subsets;

iii) For $\varepsilon > 0$ sufficiently small and a suitable constant $C > 0$ there holds:

$$\left| v_n(z) - \log \frac{e^{v_n(\xi_{j,n})}}{\left(1 + \frac{e^{v_n(\xi_{j,n})}}{8} W_n(|x_n|\xi_{j,n})|z-\xi_{j,n}|^2\right)^2} \right| \le C_r, \tag{3.5}$$

$\forall\, z\,:\,|z-\xi_j| \le \varepsilon$, $\forall\, j \in \{1,\cdots,N+1\}$.

Proof. According to Lemma 2.2, we know that,

$$\sup_{2d_1 \le |z| \le \frac{4}{|x_n|}} \{v_n(z) + 2(\alpha_n+1)\log(|z|)\} \le C, \tag{3.6}$$

and the blow-up set S of v_n is not empty and it is contained in the annulus $\{\frac{1}{2d_1} < |z| < 2d_1\}$. Thus, by setting $S = \{\xi_1,\cdots,\xi_m\}$, $m \in \mathbb{N}$, for $\varepsilon_0 > 0$ sufficiently small, we can consider:

$$\xi_{j,n} \in \overline{B_{\varepsilon_0}(\xi_j)}\ :\ v(\xi_{j,n}) = \max_{\overline{B_{\varepsilon_0}(\xi_j)}} v(z), \tag{3.7}$$

and obtain that (along a subsequence):

$$\xi_{j,n} \to \xi_j, \quad v_n(\xi_{j,n}) \to +\infty, \text{ as } n \to +\infty. \tag{3.8}$$

Furthermore, as already observed in the proof of Lemma 2.2, we can apply Theorem 1.1 and Theorem 1.2 to v_n with $\alpha_n \equiv 0$ and conclude that,

$$|z|^{2\alpha_n} W_n(|x_n|z)e^{v_n} \rightharpoonup 8\pi\delta_{\xi_j}, \tag{3.9}$$

weakly in the sense of measure in $B_{\varepsilon_0}(\xi_j)$. In particular (3.5) holds in $B_{\varepsilon_0}(\xi_j)$ for any $j \in \{1, \cdots, m\}$. Moreover, from (2.6) of Lemma 2.2, we know that,

$$\int\limits_{2d_1 \le |z| \le 4|x_n|^{-1}} |z|^{2\alpha_n} W_n(|x_n|z)\, e^{v_n} = \int\limits_{2d_1|x_n| \le |x| \le 4} |x|^{2\alpha_n} W_n(x)\, e^{u_n} \to 0, \text{ as } n \to +\infty,$$

and therefore

$$\frac{1}{2\pi} \int\limits_{|z| \le 2d_1} |z|^{2\alpha_n} W_n(|x_n|z)\, e^{v_n} \to 4(N+1), \text{ as } n \to +\infty. \tag{3.10}$$

Thus, by simply using Lemma 2.1 for v_n in $\overline{\Omega_0} = \overline{B_{2d_1}} \setminus \bigcup_{j=1}^{N+1} B_{\varepsilon_0}(\xi_j)$, by (3.5) we find that

$$\max_{\Omega_0} v_n = -v_n(\xi_{1,n}) + \mathrm{O}(1), \text{ as } n \to +\infty, \tag{3.11}$$

and in particular,

$$\frac{e^{v_n(\xi_{j,n})}}{e^{v_n(\xi_{1,n})}} = \mathrm{O}(1), \text{ as } n \to +\infty, \ \forall\, j \in \{1, \cdots, N+1\}. \tag{3.12}$$

Consequently, by (3.9) and (3.11), we find:

$$\frac{1}{2\pi} \int\limits_{|z| \le d_1} |z|^{2\alpha_n} W_n(|x_n|z)\, e^{v_n} = \frac{1}{2\pi} \sum_{j=1}^{m} \left(\int\limits_{|x-\xi_j| \le \varepsilon_0} |z|^{2\alpha_n} W_n(|x_n|z)\, e^{v_n} \right) + \mathrm{o}(1) \to 4m, \text{ as } n \to +\infty.$$

Therefore from (3.10), we get that $m = N+1$, and in this way also (3.4) is established. $\square$

Remark 3.2. *Since* $v_n\left(\frac{x_n}{|x_n|}\right) = \max_{D_n} v_n \to +\infty$, *without loss of generality, we can assume (along a subsequence)*

$$\xi_{1,n} \equiv \frac{x_n}{|x_n|} \to \xi_1, \text{ as } n \to +\infty. \tag{3.13}$$

Therefore, by setting

$$\omega_n = v_n\left(\frac{x_n}{|x_n|}\right) = u_n(x_n) + 2(\alpha_n + 1)\log(|x_n|), \tag{3.14}$$

and

$$U_N(z) := \log \frac{e^{\omega_n}}{\left(1 + \frac{e^{\omega_n}}{8(N+1)^2} W_n(0) \prod_{j=1}^{N+1} |z - \xi_{j,n}|^2\right)^2}, \tag{3.15}$$

as an immediate consequence of Proposition 3.1 and (3.11) we obtain

Corollary 3.3. *For every* $R > 2d_1$ *there exists a constant* $C_R > 0$*:*

$$|v_n(z) - U_N(z)| \le C_R, \quad \forall\, |z| \le R. \tag{3.16}$$

Our next task is to control the behavior of v_n in the region $\{x \in \mathbb{R}^2 : 2d_1 \leq |x| \leq \frac{4}{|x_n|}\}$, beyond the estimate (3.5).
To this purpose, we exploit the uniform bounded oscillations of u_n on ∂B_4, which enables us to obtain a Green's representation formula for u_n in B_4, as in [8], [2] and [28]. From [2] and [28] we know that a "sup+inf"-type estimate holds for u_n in B_4, in the sense that there exists a suitable constant $C > 0$ (depending only on the given data) such that,

$$\max_{B_1} u_n + \min_{\partial B_4} u_n = \max_{B_1} u_n + \min_{\overline{B_4}} u_n \leq C,$$

see [2], [28]. In other words,

$$\min_{\partial B_4} u_n = -u_n(x_n) + \mathrm{O}(1). \tag{3.17}$$

On the other hand, from (3.11) and (3.13) we also know that,

$$u_n(0) = -u_n(x_n) + \mathrm{O}(1), \tag{3.18}$$

and consequently $\xi_n(x) = u_n(x) - u_n(0)$ satisfies:

$$-\Delta \xi_n = |x|^{2\alpha_n} W_n(x) e^{u_n} \quad \text{in} \quad B_4$$

$$|\xi_n(x)| \leq C \quad \text{on} \quad \partial B_4.$$

Therefore, we can use Green's representation formula for ξ_n in B_4 and conclude that,

$$u_n(x) - u_n(0) = \frac{1}{2\pi} \int\limits_{B_4} \log\left(\frac{|y|}{|x-y|}\right) |y|^{2\alpha_n} W_n(y)\, e^{u_n(y)}\, dy + \phi_n(x) \tag{3.19}$$

with ϕ_n a smooth function uniformly bounded in $C^2_{\text{loc}}(B_4) \cap C^0(\overline{B_4})$. After scaling, an analogous representation formula can be derived for v_n, as follows:

$$v_n(z) = v_n(0) + \frac{1}{2\pi} \int\limits_{D_n} \log\left(\frac{|y|}{|z-y|}\right) |y|^{2\alpha_n} W_{1,n}(y)\, e^{v_n(y)}\, dy + \phi_n(|x_n| z) = \tag{3.20}$$

$$-\omega_n + \frac{1}{2\pi} \int\limits_{D_n} \log\left(\frac{|y|}{|z-y|}\right) |y|^{2\alpha_n} W_{1,n}(y)\, e^{v_n(y)}\, dy + \phi_n(|x_n| z) + \mathrm{O}(1), \tag{3.21}$$

and in particular,

$$\nabla v_n(z) = -\frac{1}{2\pi} \int_{D_n} \frac{z-y}{|z-y|^2} |z|^{2\alpha_n} W_{1,n}(y)\, e^{v_n(y)} dy + |x_n| \nabla \phi(|x_n| z). \tag{3.22}$$

Thus, by setting:

$$M_n := \int\limits_{B_4} |x|^{2\alpha_n} W_n(x) e^{u_n(x)} dx,$$

we can argue exactly as in [2] or [28] (see Lemma 5.6.54 in [30]), to show that

$$\left| v_n(z) + \omega_n + \frac{M_n}{2\pi} \log|z| \right| \leq C, \quad 2d_1 \leq |z| \leq \frac{4}{|x_n|}, \tag{3.23}$$

and

$$\left| \nabla v_n(z) + \frac{M_n}{2\pi} \frac{z}{|z|^2} \right| \leq \frac{C}{|z|^2}, \quad 2d_1 \leq |z| \leq \frac{4r_0}{|x_n|}, \tag{3.24}$$

for n large enough, $r_0 \in (0,1)$ and a suitable constant $C > 0$. Details about (3.23), (3.24) can be found in Lemma 4.4, which deals with the more delicate case where $\alpha_n \to \alpha \in (-1,0)$, see also Remark 4.3.

Next we provide the following useful estimate about the rate of convergence of M_n towards $8\pi(N+1)$.

Lemma 3.4.

$$|M_n - 8\pi(1+\alpha_n)| \leq C\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-1}, \quad as\ n \to +\infty. \tag{3.25}$$

Proof. In view of (3.23) and (3.24) we have

$$u_n(x) = \mathrm{O}(1) - \omega_n + \frac{M_n}{2\pi}\log\left(\frac{1}{|x|}\right) + \left(2(1+\alpha_n) - \frac{M_n}{2\pi}\right)\log\left(\frac{1}{|x_n|}\right) \tag{3.26}$$

and

$$\nabla u_n(x) = -\frac{M_n}{2\pi}\frac{x}{|x|^2} + \mathrm{O}\left(\frac{|x_n|}{|x|^2}\right), \tag{3.27}$$

in $R|x_n| \leq |x| \leq 3$ which we use to analyse the Pohožaev identity on the shrinking ball

$$B_{n,0} := \left\{|x| \leq |x_n|\log\left(\frac{1}{|x_n|}\right)\right\},$$

that is

$$\int\limits_{\partial B_{n,0}} \left((\nu, \nabla u_n)(x, \nabla u_n) - \frac{1}{2}(x,\nu)|\nabla u_n|^2\right) + \int\limits_{\partial B_{n,0}} |x|^{(2\alpha_n+1)} W_n\, e^{u_n} = \tag{3.28}$$

$$\int\limits_{B_{n,0}} \left(2|x|^{2\alpha_n} W_n + x\cdot\nabla(|x|^{2\alpha_n} W_n)\right)\, e^{u_n}. \tag{3.29}$$

By using (3.26) we can estimate each term in (3.29) as follows

$$I_{1,n} := \int\limits_{B_{n,0}} \left(2|x|^{2\alpha_n} W_n + x\cdot\nabla(|x|^{2\alpha_n} W_n)\right)\, e^{u_n} = \tag{3.30}$$

$$\int\limits_{B_1} (2+2\alpha_n)|x|^{2\alpha_n} W_n\, e^{u_n} + I_{1,n,0} = 2(1+\alpha_n)M_n + I_{1,n,0},$$

where

$$I_{1,n,0} := \int\limits_{B_1\setminus B_{n,0}} (2+2\alpha_n)|x|^{2\alpha_n} W_n\, e^{u_n} + \int\limits_{B_{n,0}} (x\cdot\nabla(\log(W_n)))|x|^{2\alpha_n} W_n\, e^{u_n},$$

satisfies

$$|I_{1,n,0}| \leq C\left(\frac{\varepsilon_n}{|x_n|}\right)^{2(1+\alpha_n)} |x_n|^{\frac{M_n}{2\pi}-2(1+\alpha_n)} \int\limits_{B_1\setminus B_{n,0}} \frac{|x|^{2\alpha_n}}{|x|^{\frac{M_n}{2\pi}}}\,dx +$$

$$|x_n|\log\left(\frac{1}{|x_n|}\right)\int\limits_{B_{n,0}}|x|^{2\alpha_n}W_n\,e^{u_n}\leq C\left(\frac{\varepsilon_n}{|x_n|}\right)^{2(1+\alpha_n)}\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-\frac{M_n}{2\pi}+2(1+\alpha_n)}+$$

$$C|x_n|\log\left(\frac{1}{|x_n|}\right)\leq C\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-(1+\alpha_n)},\text{ as } n\to+\infty.$$

Concerning the last term in (3.28) we find

$$\int\limits_{\partial B_{n,0}}|x|^{(2\alpha_n+1)}W_n\,e^{u_n}= \tag{3.31}$$

$$\mathrm{O}(1)\left(|x_n|\log\left(\frac{1}{|x_n|}\right)\right)^{2(1+\alpha_n)}\left(\frac{\varepsilon_n}{|x_n|}\right)^{2(1+\alpha_n)}\frac{|x_n|^{\frac{M_n}{2\pi}-2(1+\alpha_n)}}{\left(|x_n|\log\left(\frac{1}{|x_n|}\right)\right)^{\frac{M_n}{2\pi}}}\leq$$

$$C\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-(1+\alpha_n)},\text{ as } n\to+\infty.$$

Concerning the first term in (3.28) we can use (3.27) to find, after a lengthy but straightforward calculation that,

$$\int\limits_{\partial B_{n,0}}\left((\nu,\nabla u_n)(x,\nabla u_n)-\frac{1}{2}(x,\nu)|\nabla u_n|^2\right)= \tag{3.32}$$

$$2\pi\left(\frac{M_n}{2\pi}\right)^2-2\pi\frac{1}{2}\left(\frac{M_n}{2\pi}\right)^2+\mathrm{O}\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-1},\text{ as } n\to+\infty.$$

At this point, by using (3.28), (3.29) together with (3.30), (3.31), (3.32), we obtain

$$\pi\left(\frac{M_n}{2\pi}\right)^2+\mathrm{O}\left(\log\left(\frac{1}{|x_n|}\right)\right)^{-1}=2(1+\alpha_n)M_n,\text{ as } n\to+\infty,$$

and (3.25) is established. □

Proposition 3.5. *For any $R>2d_1$ there exists a constant $C_R>0$ such that*

$$|v_n(z)-U_N(z)|\leq C_R(1+4|\alpha_n-N||\log(|z|)|),\quad\forall\, R\leq|z|\leq 4|x_n|^{-1}. \tag{3.33}$$

Proof. At this point (3.33) is an easy consequence of (3.23) and (3.25). In fact we have

$$|v_n(z)+\omega_n+4(1+N)\log(|z|)\,|=\left|\left(v_n(z)+\omega_n+\frac{M_n}{2\pi}\log(|z|)\right)+\right.$$

$$\left.\left(\frac{M_n}{2\pi}\log(|z|)-4(1+\alpha_n)\log(|z|)\right)+(4(1+\alpha_n)\log(|z|)-4(1+N)\log(|z|))\,\right|=$$

$$\mathrm{O}(1+4|\alpha_n-N||\log(|z|)|),\quad\forall\, R\leq|z|\leq 4|x_n|^{-1},$$

and the proof of Proposition 3.5 is completed. □

Finally, we are left to show that the blow-up points of v_n given by: $\{\xi_1,\cdots,\xi_{N+1}\}$, define, after a rotation, the distinct $(N+1)$-roots of the unity. To this purpose, we start to establish the following,

Lemma 3.6. *For any $R > 2d_1$ and $\varepsilon > 0$ sufficiently small let $D_{R,\varepsilon} = \left\{ z \in \overline{B_R} \setminus \bigcup_{j=1}^{N+1} B_\varepsilon(\xi_j) \right\}$. There holds:*

$$\max_{D_{R,\varepsilon}} \left| \nabla v_n(z) + 4 \sum_{j=1}^{N+1} \frac{z-\xi_j}{|z-\xi_j|^2} \right| \to 0, \text{ as } n \to +\infty. \tag{3.34}$$

Proof. By using (3.22) we see that

$$\nabla v_n(z) = I_{n,1}(z) + I_{n,2}(z),$$

where

$$I_{n,1}(z) = -\frac{1}{2\pi} \int\limits_{D_n} \frac{z-y}{|z-y|^2} |y|^{2\alpha_n} W_{1,n}(y)\, e^{v_n}, \text{ and } |I_{n,2}(z)| \le C|x_n|, \quad \forall\, |z| \le R.$$

Concerning $I_{n,1}$ we can use (3.4) in order to estimate,

$$\left| I_{n,1}(z) + 4 \sum_{j=1}^{N+1} \frac{z-\xi_j}{|z-\xi_j|^2} \right| \le \frac{1}{2\pi} \left(\int\limits_{2R \le |y| \le 4|x_n|^{-1}} \frac{1}{|z-y|} |y|^{2\alpha_n} W_{1,n}(y)\, e^{v_n} dy \right) + \mathrm{o}(1) \le$$

$$C \left(\int\limits_{2R \le |y| \le 4|x_n|^{-1}} |y|^{2\alpha_n}\, e^{v_n}\, dy \right) + \mathrm{o}(1), \text{ as } n \to +\infty.$$

In view of (3.23) we easily check that, $\int\limits_{2R \le |y| \le 4|x_n|^{-1}} |y|^{2\alpha_n}\, e^{v_n}\, dy \to 0$, as $n \to +\infty$, and (3.34) is established. ☐

On the basis of (3.34) we have:

Lemma 3.7. *Let $\{\xi_1, \cdots, \xi_{N+1}\}$ be the blow up set of v_n in (3.3), then*

$$N \frac{\xi_i}{|\xi_i|^2} = 2 \sum_{j \ne i, j=1}^{N+1} \frac{\xi_i - \xi_j}{|\xi_i - \xi_j|^2}, \quad \forall\, i \in \{1, \cdots, N+1\}. \tag{3.35}$$

Proof. The proof is based on the following well known Pohožaev identity around each blow up point:

$$\int\limits_{\partial B_\varepsilon(\xi_j)} \left((\nu, \nabla v_n) \nabla v_n - \frac{1}{2} |\nabla v_n|^2 \nu \right) + \int\limits_{\partial B_\varepsilon(\xi_j)} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \nu = \tag{3.36}$$

$$\int\limits_{B_\varepsilon(\xi_j)} \nabla \left(|z|^{2\alpha_n} W_{1,n}(z) \right) e^{v_n}, \tag{3.37}$$

for any $\varepsilon > 0$ sufficiently small. There is no loss of generality in assuming $i = 1$ so that we are reduced to prove that,

$$N \frac{\xi_1}{|\xi_1|^2} = 2 \sum_{j=2}^{N+1} \frac{\xi_1 - \xi_j}{|\xi_1 - \xi_j|^2}. \tag{3.38}$$

We pass first to the limit in (3.36), (3.37) as $n \to +\infty$, and then as $\varepsilon \to 0^+$. Indeed, by using Lemma 3.6 and since $\nu = \frac{z-\xi_1}{|z-\xi_1|}$, we find:

$$\lim_{n\to+\infty} \int\limits_{\partial B_\varepsilon(\xi_1)} \left((\nu, \nabla v_n)\nabla v_n - \frac{1}{2}|\nabla v_n|^2 \nu \right) =$$

$$\int\limits_{\partial B_\varepsilon(\xi_1)} \left(\frac{4}{|z-\xi_1|} + 4\sum_{j=2}^{N+1} \frac{(z-\xi_1)(z-\xi_j)}{|z-\xi_1||z-\xi_j|^2} \right) \left(4\sum_{j=1}^{N+1} \frac{(z-\xi_j)}{|z-\xi_j|^2} \right) - \frac{1}{2} \left(4\sum_{j=1}^{N+1} \frac{(z-\xi_j)}{|z-\xi_j|^2} \right)^2 \frac{z-\xi_1}{|z-\xi_1|} =$$

$$\int\limits_{\partial B_\varepsilon(\xi_1)} \left(16\frac{z-\xi_1}{|z-\xi_1|^3} + 32\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_1||z-\xi_j|^2} + 16\frac{z-\xi_1}{|z-\xi_1|} \left(\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_j|^2} \right)^2 \right)$$

$$-\frac{1}{2} \int\limits_{\partial B_\varepsilon(\xi_1)} \left(16\frac{z-\xi_1}{|z-\xi_1|^3} + 32\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_1||z-\xi_j|^2} + 16\frac{z-\xi_1}{|z-\xi_1|} \left(\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_j|^2} \right)^2 \right) =$$

$$8\int\limits_{\partial B_\varepsilon(\xi_1)} \left(2\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_1||z-\xi_j|^2} + \frac{z-\xi_1}{|z-\xi_1|} \left(\sum_{j=2}^{N+1} \frac{(z-\xi_j)}{|z-\xi_j|^2} \right)^2 \right), \tag{3.39}$$

where we have used that,

$$\int\limits_{\partial B_\varepsilon(\xi_1)} \frac{z-\xi_1}{|z-\xi_1|^3} = 0. \tag{3.40}$$

At this point we pass to the limit as $\varepsilon \to 0^+$ and conclude, after a lengthy but straightforward calculation, that:

$$\lim_{\varepsilon\to0^+} \lim_{n\to+\infty} \int\limits_{\partial B_\varepsilon(\xi_1)} \left((\nu, \nabla v_n)\nabla v_n - \frac{1}{2}|\nabla v_n|^2 \nu \right) = 32\pi \sum_{j=2}^{N+1} \frac{\xi_1-\xi_j}{|\xi_1-\xi_j|^2}. \tag{3.41}$$

Moreover, by using (3.9) for the second term in (3.36), we have:

$$\int\limits_{\partial B_\varepsilon(\xi_1)} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \nu \to 0, \text{ as } n \to +\infty. \tag{3.42}$$

Finally, we write

$$\int\limits_{B_\varepsilon(\xi_1)} \nabla \left(|z|^{2\alpha_n} W_{1,n}(z)\right) e^{v_n} = 2\alpha_n \int\limits_{B_\varepsilon(\xi_1)} \frac{z}{|z|^2} |z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} + \tag{3.43}$$

$$|x_n| \int\limits_{B_\varepsilon(\xi_1)} (\nabla \log(W_{1,n})))|z|^{2\alpha_n} W_{1,n}(z)\, e^{v_n} \to 16\pi N \frac{\xi_1}{|\xi_1|^2}, \text{ as } n \to +\infty,$$

(where we have used (3.11)) and (3.38) follows. □

Corollary 3.8. *After a rotation, $\{\xi_1, \cdots, \xi_{N+1}\}$ define the $(N+1)$-roots of the unity.*

Proof. From (3.13), we know that $|\xi_1| = 1$ and so, after a rotation, we can assume that $\xi_1 = 1$. Moreover, in view of (3.38) and (1.42) (proved in the Appendix) we also know that the points $\{\xi_1, \cdots, \xi_{N+1}\}$ provide the distinct $(N+1)$-roots of $\eta = (-1)^N \xi_1 \xi_2 \cdots \xi_{N+1} \neq 0$. Since $\xi_1 = 1$, then necessarily $\eta = 1$ and the desired conclusion follows. □

At this point the conclusion of Theorem 1.4 easily follows simply by scaling back to u_n the results established above. □

4. The bubbling behavior of u_n: case $\alpha \in (-1, 0)$.

In this section we prove the following

Theorem 4.1. *Let u_n satisfy (1.1),(1.4),(1.2)-(1.3). If $\alpha \in (-1,0)$, then $\frac{|x_n|}{\varepsilon_n} \to 0$ and (1.10) hold.*

As mentioned in the introduction, the idea of the proof is the same as in [2] with some technical modifications where we take advantage of an improvement introduced in [30]. We will be very sketchy about those parts which can be recovered as a straightforward adaptations of the arguments provided in [2], [30].

Proof. In view of (1.7) we have $|x_n| \to 0$ and $\varepsilon_n \to 0$, as $n \to +\infty$. Moreover, in view of (1.5), for any fixed $\alpha \in (-1, 0)$ we can assume that

$$\int_{B_3} |x|^{2\alpha} V_n(x)\, e^{u_n} \leq 8\pi - \vartheta_\alpha, \tag{4.1}$$

for some $\vartheta_\alpha \in (0, 8\pi|\alpha|)$. Let $U_\alpha : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$U_\alpha(y) = \log \frac{1}{\left(1 + \frac{1}{8(1+\alpha)^2}|y|^{2(1+\alpha)}\right)^2},$$

The next proposition is the analogue of the results established in Step 1 in [2] in case $\alpha_n \equiv \alpha \in (0, +\infty) \setminus \mathbb{N}$. The blow up argument used there has to be modified in this situation where we take advantage of the more recent result established in [7].

Proposition 4.2. *It holds*

(*i*) $\lim\limits_{n \to +\infty} \frac{|x_n|}{\varepsilon_n} = 0$,

(*ii*) *$w_n(y)$ converges to U_α in $C^0_{\rm loc}(\mathbb{R}^2) \cap W^{2,q}_{\rm loc}(\mathbb{R}^2) \cap C^2_{\rm loc}(\mathbb{R}^2 \setminus \{0\})$, for any $q \in [1, \frac{1}{|\alpha|})$.*

where

$$w_n(y) = u_n(\varepsilon_n y) - u_n(x_n), \quad |y| \leq 4\varepsilon_n^{-1}. \tag{4.2}$$

Proof. To prove (*i*) we argue by contradiction and assume that, along a subsequence,

$$\frac{|x_{n_k}|}{\varepsilon_{n_k}} \geq C, \ \forall\, k \in \mathbb{N}. \tag{4.3}$$

In the rest of the proof of (*i*) we will freely pass to subsequences which possibly will not be relabeled. For any $z \in A_k := \{z \in \mathbb{R}^2 \,|\, x_{n_k} + |x_{n_k}| z \in B_2\}$ let us define

$$\tilde{u}_k(z) = u_{n_k}(x_{n_k} + |x_{n_k}| z) + 2(1+\alpha)\log|x_{n_k}|.$$

Then $\tilde{u}_k$ satisfies

$$-\Delta\tilde{u}_k = \left|\frac{x_{n_k}}{|x_{n_k}|}+z\right|^{2\alpha} W(x_{n_k}+|x_{n_k}|z)\, e^{\tilde{u}_k} \ \text{ in } \ A_k.$$

In particular, in view of (4.1) we have

$$\int\limits_{A_k} \left|\frac{x_{n_k}}{|x_{n_k}|}+z\right|^{2\alpha} V(x_{n_k}+|x_{n_k}|z)\, e^{\tilde{u}_k} \le 8\pi - \vartheta_\alpha < 8\pi, \ \ \forall\ k\in\mathbb{N}. \tag{4.4}$$

Hence, along a subsequence, the standard blow up analysis of Brezis-Merle and Li-Shafrir applies to $\tilde{u}_k$ in $\overline{B}_r$ for $r\in(0,1)$ and in view of (4.4) allows us to conclude that the origin cannot be a blow up point for $\tilde{u}_k$. In other words, there exists $C>0$ such that

$$\tilde{u}_k(0) = u_{n_k}(x_{n_k}) + 2(1+\alpha)\log|x_{n_k}| \le C, \ \ \forall\ k\in\mathbb{N}. \tag{4.5}$$

Thus (passing to a subsequence if necessary), in view of (4.3), we can assume that,

$$-\frac{x_{n_k}}{\varepsilon_{n_k}} \to y_0 \in \mathbb{R}^2\setminus\{0\}. \tag{4.6}$$

Let us define $\tilde{w}_k(z) = u_{n_k}(x_{n_k}+\varepsilon_{n_k}z) - u_{n_k}(x_{n_k})$ in $D_k = \{z\in\mathbb{R}^2 \,|\, x_{n_k}+\varepsilon_{n_k}z\in B_4\}$. Then

$$-\Delta\tilde{w}_k = \left|\frac{x_{n_k}}{|\varepsilon_{n_k}|}+z\right|^{2\alpha} V(x_{n_k}+\varepsilon_{n_k}z)\, e^{\tilde{w}_k} \ \text{ in } \ D_k,$$

and

$$\tilde{w}_k(z) \le \tilde{w}_k(0) = 0.$$

Since the above mentioned concentration-compactness results, and more specifically Theorem 2.1 in [7], applies to $\tilde{w}_k$ in $B_R(y_0)$, for any $R>|y_0|$, we conclude that $\tilde{w}_k$ is uniformly bounded in $B_R(y_0)$. At this point, by a diagonal argument, we find a subsequence of $\tilde{w}_k$ which converges to $\tilde{w}$ in $C^0_{\rm loc}(\mathbb{R}^2)\cap W^{2,q}_{\rm loc}(\mathbb{R}^2)\cap C^2_{\rm loc}(\mathbb{R}^2\setminus\{0\})$, for any $q\in[1,\frac{1}{|\alpha|})$, and $\tilde{w}$ satisfies

$$\begin{cases} -\Delta\tilde{w} = |z-y_0|^{2\alpha}\, e^{\tilde{w}} \ \text{ in } \ \mathbb{R}^2, \\[2mm] \int\limits_{\mathbb{R}^2} |z-y_0|^{2\alpha}\, e^{\tilde{w}} < +\infty, \\[2mm] \tilde{w}(z) \le \tilde{w}(0) = 0. \end{cases} \tag{4.7}$$

By the results in [26], and in view of (2.3), we know that

$$\tilde{w}(y+y_0) = \log\frac{1}{\left(1+\frac{1}{8(1+\alpha)^2}|y|^{2(1+\alpha)}\right)^2},$$

and we see that the last condition in (4.7) can be satisfied if and only if $y_0=0$, in contradiction to (4.6). Hence (i) holds, and we can use the arguments above for

$$w_n(y) = \tilde{w}_n\left(y - \frac{x_n}{\varepsilon_n}\right),$$

to establish (ii). □

Let

$$M_n := \int\limits_{B_1} |x|^{2\alpha_n} V_n(x) e^{u_n}$$

and recall that in view of (1.3) and (1.5)

$$M_n = 8\pi(1+\alpha_n) + \mathrm{o}(1), \quad \text{as } \ n \to +\infty. \tag{4.8}$$

For later use we also recall that

$$\int\limits_{\mathbb{R}^2} |y|^{2\alpha} e^{U} = 8\pi(1+\alpha). \tag{4.9}$$

Let γ_n be defined as

$$\gamma_n = \frac{M_n}{2\pi} - 2\alpha_n - 1 = 3 + 2\alpha_n + \mathrm{o}(1), \quad \text{as } \ n \to +\infty. \tag{4.10}$$

Remark 4.3. *Lemmas 4.4 and 4.5 below show the main differences with respect to the estimates in* [2]*. Since in that case* $\alpha > 0$*, then, in the the corresponding version of Lemma 4.4 (see (3.12) and (3.13) in* [2]*) one always finds the number* 2 *replacing* $\min\{\gamma_n, 2\}$*. The point is that* γ_n *gets close to* 1^+ *as* $\alpha \searrow (-1)^+$*. A similar problem shows up in the decay of the quantity* $|y|^{2\alpha_n}\, e^{w_n(y)}$*. As a consequence, some terms which was negligible for* $\alpha > 0$ *in the analysis of the first Pohožaev identity (see Lemma 4.5 below) cannot be neglected. Actually this point is crucial to the understanding of the rate of convergence of* $M_n - 8\pi(1+\alpha_n)$*. This is why the situation where* $\alpha \in (-1,0)$ *is slightly more subtle. However we will take advantage of a refined argument adopted in Lemma 5.6.54 in* [30] *which makes the proof simpler.*

Lemma 4.4. *For any* n *large enough, we have*

$$\left| w_n(y) + \frac{M_n}{2\pi} \log|y| \right| \le C, \quad 3 \le |y| \le \frac{4}{\varepsilon_n}, \tag{4.11}$$

and

$$\left| \nabla w_n(y) + \frac{M_n}{2\pi} \frac{y}{|y|^2} \right| \le \frac{C}{|y|^{\min\{\gamma_n, 2\}}} + \mathrm{O}(\varepsilon_n), \quad 3 \le |y| \le \frac{3}{\varepsilon_n}. \tag{4.12}$$

Proof. We will denote by O(1) various quantities uniformly bounded with respect to n and $x \in \overline{B}_4$ in C^0-norm and by C various uniform and positive constants which may vary even from line to line. By using the Green's formula, Proposition 4.2 and (4.9) as in (3.11), (3.17) in [2] or either as in Lemma 5.6.54 in [30] we find that,

$$w_n(y) = \frac{1}{2\pi} \int_{\{|z| \le \frac{1}{\varepsilon_n}\}} \log\left(\frac{|z|}{|y-z|} \right) |z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} dz + \mathrm{O}(1),\ \forall\ |y| \le 4\varepsilon_n^{-1}, \tag{4.13}$$

and in particular we obtain the following property for w_n:
for every small $\sigma > 0$, there exists $R_\sigma > 4$ and $\overline{n} = \overline{n}_\sigma \in \mathbb{N}$, such that,

$$w_n(y) \le -\left(\frac{M_n}{2\pi} - \sigma \right) \ln|y| + \mathrm{O}(1),\ \forall\ |y| \ge 2R_\sigma,\ \forall\ n \ge \overline{n}. \tag{4.14}$$

In view of (4.8), then (4.14) implies that

$$|z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} \le \frac{C}{|z|^{\frac{M_n}{2\pi} - 2\alpha_n - \sigma}} = \frac{C}{|z|^{\gamma_n + 1 - \sigma}}, \ \ \forall\ |z| \ge 2R_\sigma, \tag{4.15}$$

for any n large enough. We continue to use (4.14) to refine our estimate. In view of (4.10), for any $\alpha \in (-1,0)$ we can choose $\sigma = \sigma(\alpha) > 0$ sufficiently small in (4.14) to guarantee that

$$\gamma_n - 2\sigma \geq 1, \quad \forall\, n \in \mathbb{N}. \tag{4.16}$$

Thus, we may conclude that,

$$\int_{\{|z|\leq \frac{1}{\varepsilon_n}\}} |\log(|z|)|\, |z|^{2\alpha_n} W(\varepsilon_n z)\, e^{v_n(z)} dy \leq C,\ \forall\, n \in \mathbb{N}. \tag{4.17}$$

We will use the following inequality

$$\left|\log\left(\frac{|y||z|}{|y-z|}\right)\right| \leq 2\,|\log|z|| + \left(\log\left(\frac{1}{|y-z|}\right)\right)^{+} + \log(2),\ \forall\, |y|\geq 3,\ \forall\, |z| < \frac{1}{\varepsilon_n},\ z\neq y, \tag{4.18}$$

whose proof can be found in (5.6.34) in [30]. By using (4.13), (4.15), (4.17) and (4.18) we obtain

$$\left|w_n(y) + \frac{M_n}{2\pi}\log|y|\right| \leq \frac{1}{2\pi}\int_{\{|z|\leq \frac{1}{\varepsilon_n}\}} \left|\log\left(\frac{|y||z|}{|y-z|}\right)\right| |z|^{2\alpha_n} W(\varepsilon_n z)\ e^{w_n(z)} dz + C \leq$$

$$\frac{1}{\pi}\int_{\{|z|\leq \frac{1}{\varepsilon_n}\}} |\log|z||\, |z|^{2\alpha_n} W(\varepsilon_n z)\, e^{v_n(z)} dz + C\int_{\{|y-z|\leq 1\}} \log\left(\frac{1}{|y-z|}\right) dz + C \leq C,$$

for any $3 \leq |y| \leq \frac{1}{\varepsilon_n}$. In particular this proves (4.11) so we readily deduce that

$$|z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} = \frac{\mathrm{O}(1)}{|z|^{\gamma_n+1}},\ \ \forall\, 3 \leq |z| \leq \frac{1}{\varepsilon_n}, \tag{4.19}$$

for any n large enough. Concerning (4.12), we first use Green's representation formula to obtain,

$$\nabla w_n(y) + \frac{M_n}{2\pi}\frac{y}{|y|^2} = \frac{1}{2\pi}\int_{|z|\leq \frac{1}{\varepsilon_n}} \left\{\frac{y}{|y|^2} - \frac{y-z}{|y-z|^2}\right\} |z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} dz + \mathrm{O}(\varepsilon_n),$$

where $3 \leq |y| \leq \frac{3}{\varepsilon_n}$. Notice that

$$\left|\frac{y}{|y|^2} - \frac{y-z}{|y-z|^2}\right| \leq \frac{3}{|y-z|},\ \ \forall\, |y|\geq 3,\ \ \forall\, z \neq y \in D_1 := \left\{|y-z| \leq \frac{|y|}{2}\right\}.$$

Next, we will use the identity

$$\left|\frac{y}{|y|^2} - \frac{y-z}{|y-z|^2}\right| = \frac{|z|}{|y||y-z|},\ \ \forall\, |y|\geq 3,\ \ \forall z\ :\ \left\{|y-z| \geq \frac{|y|}{2}\right\}. \tag{4.20}$$

By using (4.20), for any $|y| \geq 3$ we find,

$$\left|\frac{y}{|y|^2} - \frac{y-z}{|y-z|^2}\right| = \frac{|z|}{|y||y-z|} \leq 2\frac{|z|}{|y|^2},\ \ \forall\, z \in D_2 := \left\{|y-z| \geq \frac{|y|}{2}\right\} \cap \{|z| \leq 2|y|\}.$$

On the other side, if $|y| \geq 3$, then $|z| \geq 2|y|$ implies $|y-z| \geq \frac{|z|}{2}$ and then (4.20) shows that

$$\left|\frac{y}{|y|^2} - \frac{y-z}{|y-z|^2}\right| = \frac{|z|}{|y||y-z|} \leq \frac{2}{|y|},\ \ \forall\, z \in D_3 = \left\{|y-z| \geq \frac{|y|}{2}\right\} \cap \{|z| \geq 2|y|\}.$$

Therefore, for any $3 \leq |y| \leq \frac{3}{\varepsilon_n}$, we have

$$\left|\nabla w_n(y) + \frac{M_n}{2\pi}\frac{y}{|y|^2}\right| \leq I_{n,1}(y) + I_{n,2}(y) + I_{n,3}(y) + \mathrm{O}(\varepsilon_n),$$

where

$$I_{n,1}(y) = \frac{3}{2\pi}\int_{D_1} \frac{1}{|y-z|}|z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} dz,$$
$$I_{n,2}(y) = \frac{1}{\pi}\int_{D_2} \frac{|z|}{|y|^2}|z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} dz,$$
$$I_{n,3}(y) = \frac{1}{\pi}\int_{D_3} \frac{1}{|y|}|z|^{2\alpha_n} W(\varepsilon_n z)\, e^{w_n(z)} dz.$$

By using (ii) in Proposition 4.2, (4.16) and (4.19) we obtain

$$I_{n,2}(y) \leq \frac{C}{|y|^2} + \frac{C}{|y|^2}\int\limits_{\{3\leq|z|\leq 2|y|\}} \frac{|z|}{|z|^{\gamma_n+1}}\, dz \leq \frac{C}{|y|^{\min\{\gamma_n,2\}}},\ \forall\, 3 \leq |y| \leq \frac{3}{\varepsilon_n},$$

and

$$I_{n,3}(y) \leq \frac{C}{|y|}\int\limits_{\{|z|\geq 2|y|\}} \frac{C}{|z|^{\gamma_n+1}}\, dz \leq \frac{C}{|y|^{\gamma_n}},\ \forall\, 3 \leq |y| \leq \frac{3}{\varepsilon_n}.$$

On the other side, since $z\in D_1$ implies $|z| \geq \frac{|y|}{2}$, then by using (ii) in Proposition 4.2 and (4.19), we see that

$$I_{n,1}(y) \leq \int_{D_1} \frac{1}{|y-z|}\frac{C}{|z|^{\gamma_n+1}}\, dz \leq \frac{C}{|y|^{\gamma_n+1}}\int_{D_1} \frac{1}{|y-z|}\, dz \leq \frac{C}{|y|^{\gamma_n}},$$

for all $3 \leq |y| \leq \frac{3}{\varepsilon_n}$, as claimed. □

Let us define

$$\varrho_n = \min\{\gamma_n, 2\},$$

and let τ_n be any sequence of strictly positive numbers. Then we have

Lemma 4.5. *There exists a constant $C>0$ such that*

$$|M_n - 8\pi(1+\alpha_n)| \leq C\left(\log\frac{1}{\varepsilon_n}\right)^{-\tau_n(\varrho_n-1)}. \tag{4.21}$$

Proof. Scaling back to u_n and using (4.11) and (4.12) we find

$$u_n(x) = \frac{M_n}{2\pi}\log\frac{1}{|x|} + \left[2(1+\alpha_n) - \frac{M_n}{2\pi}\right]\left(\log\left(\frac{1}{\varepsilon_n}\right)\right) + \mathrm{O}(1), \tag{4.22}$$

and

$$\nabla u_n(x) = -\frac{M_n}{2\pi}\frac{x}{|x|^2} + \mathrm{O}\left(\frac{(\varepsilon_n)^{\varrho_n-1}}{|x|^{\varrho_n}} + 1\right), \tag{4.23}$$

for any $3 \leq |x| \leq \frac{3}{\varepsilon_n}$. We use the second Pohožaev identity in the region

$$\tilde{B}_n := \left\{|x| \leq \varepsilon_n\left(\log\left(\frac{1}{\varepsilon_n}\right)\right)^{\tau_n}\right\},$$

that is

$$\int_{\tilde{B}_n}\left[2|x|^{2\alpha_n}V_n(x) + x\cdot\nabla\left(|x|^{2\alpha_n}V_n(x)\right)\right]\, e^{u_n(x)} dx - \int_{\partial\tilde{B}_n}|x||x|^{2\alpha_n}V_n(x)\, e^{u_n(x)} d\sigma = \int_{\partial\tilde{B}_n}|x|\left[\left(\frac{\partial u_n}{\partial\nu}\right)^2 - \frac{1}{2}|\nabla u_n|^2\right] d\sigma. \tag{4.24}$$

Since $\alpha \in (-1,0)$ then ∇u_n is not continuous near the origin. Therefore one should first show that such an identity, well known in the smooth setting, still holds in this case as well. The proof of this fact is rather standard and we sketch it in the Appendix, see section 7.

Substituting (4.22) and (4.23) in both sides of (4.24) and recalling (ii) in Proposition 4.2 and (4.10) we obtain

$$\int_{\tilde{B}_n} \left[2|x|^{2\alpha_n} V_n(x) + x\cdot\nabla\left(|x|^{2\alpha_n}V_n(x)\right)\right] e^{u_n(x)} - \int_{\partial\tilde{B}_n} |x||x|^{2\alpha_n}V_n(x)\, e^{u_n(x)} d\sigma =$$

$$(2+2\alpha_n)\beta_n + \mathrm{O}(1)\int_{B_1\setminus\tilde{B}_n} \frac{(\varepsilon_n)^{\gamma_n-1}}{|x|^{\gamma_n+1}}\,dx + \mathrm{O}\left(\left(\log\left(\frac{1}{\varepsilon_n}\right)\right)^{-\tau_n(\gamma_n-1)}\right) =$$

$$(2+2\alpha_n)\beta_n + \mathrm{O}(1)\left(\log\frac{1}{\varepsilon_n}\right)^{-\tau_n(\gamma_n-1)}, \text{ as } n\to+\infty,$$

and

$$\int_{\partial\tilde{B}_n} |x|\left[\left(\frac{\partial u_n}{\partial\nu}\right)^2 - \frac{1}{2}|\nabla u_n|^2\right] d\sigma = \int_{\partial\tilde{B}_n} \frac{1}{2}\left(\frac{M_n}{2\pi}\right)^2 \frac{1}{|x|} d\sigma +$$

$$\int_{\partial\tilde{B}_n} |x|\left[\mathrm{O}\left(\frac{(\varepsilon_n)^{\varrho_n-1}}{|x|^{\varrho_n+1}}\right) + \mathrm{O}\left(\frac{(\varepsilon_n)^{2(\varrho_n-1)}}{|x|^{2\varrho_n}}\right)\right] d\sigma =$$

$$\frac{M_n^2}{4\pi} + \mathrm{O}(1)\left(\log\left(\frac{1}{\varepsilon_n}\right)\right)^{-\tau_n(\varrho_n-1)} \text{ as } n\to+\infty.$$

Therefore we conclude that

$$M_n = 8\pi(1-\alpha_n) + \mathrm{O}(1)\left(\log\frac{1}{\varepsilon_n}\right)^{-\tau_n(\varrho_n-1)},$$

as claimed. □

At this point we can prove Theorem 4.1. Indeed, in view of (ii) in Proposition 4.2, we already know that (1.10) holds for $|x| \leq 4\varepsilon_n$, and we are left with showing that

$$|w_n(y) + 4(1+\alpha_n)\log|y|| \leq C, \quad 3 \leq |y| \leq \frac{4}{\varepsilon_n}. \tag{4.25}$$

Let us choose any sequence τ_n such that

$$\tau_n(\varrho_n - 1) \geq 1, \forall\, n\in\mathbb{N},$$

in Lemma 4.5. Then we can use (4.11) to conclude that

$$|w_n(y) + 4(1+\alpha_n)\log|y|| \leq$$

$$\left|w_n(y) + \frac{M_n}{2\pi}\log|y|\right| + \left|\frac{M_n}{2\pi} - 4(1+\alpha_n)\right| |\log|y|| \leq C + \mathrm{O}(1)\left(\log\frac{1}{\varepsilon_n}\right)^{-\tau_n(\varrho_n-1)+1} \leq C$$

which proves (4.25). □

5. Blow-up analysis for the cosmic string equation.

Throughout this section we assume that u_n satisfies (1.29) and we let $\widehat{u}_n(x) = u_n\left(\frac{x}{|x|^2}\right) + \beta_n \log\left(\frac{1}{|x|}\right)$, which satisfies (1.33).

We shall denote by S and $\widehat{S}$ respectively the (possibly empty) blow-up set of u_n and $\widehat{u}_n$.
Observe that if $z \in S$ and $z \neq 0$ then $\frac{z}{|z|^2} \in \widehat{S}$. Furthermore, as shown first by Chen-Li [14], we know that any solution of (1.27)-(1.28) satisfies the following:

$$\lim_{|x|\to+\infty}(u(x)+\beta\log(|x|)) < +\infty\ ,\ \lim_{|x|\to+\infty}(x\cdot\nabla u(x)+\beta) = 0 \ \text{ and } \lim_{|x|\to+\infty}\frac{\partial u}{\partial\theta}(x) = 0, \tag{5.1}$$

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}|x|^{2N}e^u = \frac{\beta(4-a\beta)}{4(1-a(N+1))},\quad \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}e^{au} = \frac{a\beta(\beta-4(N+1))}{4(1-a(N+1))}, \tag{5.2}$$

for $0 < a \neq \frac{1}{N+1}$, while $\beta = 4(N+1)$ for $a = \frac{1}{N+1}$.

Both (5.1) and (5.2) are based on the following Green's representation formula for u and ∇u, namely:

$$u(x) - u(\overline{x}) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\log\left(\frac{|\overline{x}-y|}{|x-y|}\right)\left(|y|^{2N}e^{u(y)} + e^{au(y)}\right)dy,\ \forall\ x,\overline{x}\in\mathbb{R}^2, \tag{5.3}$$

$$\nabla u(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\left(\frac{x-y}{|x-y|^2}\right)\left(|y|^{2N}e^{u(y)} + e^{au(y)}\right)dy, \tag{5.4}$$

and a suitable Pohozaev's type identity, see [12] and [31] for details.

As a consequence of (5.1) and (5.2), we obtain the following necessary condition for the solvability of (1.27)-(1.28):

$$\min\left\{\frac{4}{a},4(N+1)\right\} < \beta < \max\left\{\frac{4}{a},4(N+1)\right\},\ \text{and } \beta > \max\left\{\frac{2}{a},2(N+1)\right\}, \tag{5.5}$$

see [24] and [31] which include also a discussion on when (5.5) become also sufficient.

From (5.5) we see that a solution sequence u_n of (1.29) is allowed only for $\frac{1}{2(N+1)} \leq a$, and in particular, in view of (5.2), there holds:

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}|x|^{2N}e^{u_n} \to 4(N+1) \ \text{ and } \ \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}e^{au_n} \to 0, \text{ as } n\to+\infty. \tag{5.6}$$

Furthermore, for u_n satisfying (1.29), we must have:

$$S\cup\widehat{S} \neq \emptyset, \tag{5.7}$$

as otherwise, along a subsequence, we could pass to the limit (in any reasonable norm) and obtain a solution of (1.27)-(1.28) with $\beta = 4(N+1)$, in contradiction to (5.5). To avoid additional technicalities, we shall focus to the case:

$$\text{either} \quad \frac{1}{2(N+1)} < a < \frac{1}{N+1} \quad \text{or} \quad \frac{1}{N+1} < a < 1, \tag{5.8}$$

where, from [31], we can claim the following:

Proposition 5.1. *Assume (5.8) and let u_n satisfy (1.29). Then the following alternative holds:*

$$i) \text{ either } \exists\, R_0>0 : \sup_{|x|<R_0} u_n \to +\infty, \quad \text{and} \quad \sup_{|x|\geq R_0}\{u_n+\beta_n\log(|x|)\}\to -\infty, \text{ as } n\to+\infty \tag{5.9}$$

$$ii) \text{ or } \forall\, R>0 : \sup_{|x|<R} u_n \to -\infty, \quad \text{and} \quad \sup_{|x|\geq R}\{u_n+\beta_n\log(|x|)\}\to +\infty, \text{ as } n\to+\infty. \tag{5.10}$$

Namely, either: $S\neq\emptyset$ and $0\notin\widehat{S}$, or: $S=\emptyset$ and $\widehat{S}=\{0\}$. Furthermore, whenever (5.9) holds then necessarily $a<\min\left\{1,\frac{2}{N+1}\right\}$.

Proof. Properties (5.9) and (5.10) are direct consequences of Proposition 2.2 and Theorems 2.3 - Therem 2.4 (for $\frac{1}{2(N+1)}<a<\frac{1}{N+1}$) and Theorems 2.6 - Theorem 2.7 (for $\frac{1}{N+1}<a$) in [31], which we apply to our situation where (in the notations of [31]) we have: $\beta^\infty=4(N+1)$. Furthermore, when (5.9) holds, and so $S\neq\emptyset$, then we can use Lemmata 5.11-5.12 in [31] to find that, $a<\min\left\{1,\frac{2}{N+1}\right\}$ in this case. □

Actually, by following [31], we can be more specific about the behavior of u_n, by considering a suitable re-scaled sequence. To this purpose, for $\varepsilon>0$ fixed sufficiently small, we let:

$$r_n>0 \;:\; \frac{1}{2\pi}\int_{|x|\leq r_n}\left(|x|^{2N}e^{u_n}+e^{au_n}\right)=4(N+1)-\varepsilon. \tag{5.11}$$

Clearly we have:

Lemma 5.2.

$$i) \quad (5.9) \text{ holds} \iff \exists\, r_0>0 \;:\; 0<r_n\leq r_0, \forall\, n\in\mathbb{N};$$

$$ii) \quad (5.10) \text{ holds} \iff r_n\to+\infty, \text{ as } n\to+\infty.$$

At this point, we consider the following re-scaled sequence:
Definition:

$$\text{If (5.9) holds, then let } v_n(x)=u_n(r_nx)+2\min\left\{N+1,\frac{2}{a}\right\}\log(r_n); \tag{5.12}$$

$$\text{If (5.10) holds, then let } v_n(x)=u_n(r_nx)+2\max\left\{N+1,\frac{2}{a}\right\}\log(r_n). \tag{5.13}$$

The sequence v_n satisfies:

$$\begin{cases} -\Delta v_n=\varepsilon_{1,n}|x|^{2N}e^{v_n}+\varepsilon_{2,n}e^{av_n}=:g_n \quad \text{in } \mathbb{R}^2, \\ \beta_n=\frac{1}{2\pi}\int_{\mathbb{R}^2} g_n(x)\,dx\to 4(N+1), \text{as } n\to+\infty, \end{cases} \tag{5.14}$$

where the parameters $\varepsilon_{1,n}$ and $\varepsilon_{2,n}$ are specified as follows:
(i)

$$\text{if } \frac{1}{2(N+1)} < a < \frac{1}{N+1} \text{ and (5.9) holds, or } \frac{1}{N+1} < a < 1 \text{ and (5.10) holds,} \tag{5.15}$$

then,

$$\varepsilon_{1,n} \equiv 1 \quad \text{and} \quad \varepsilon_{2,n} = r_n^{2(1-a(N+1))}; \tag{5.16}$$

(ii)

$$\text{if } \frac{1}{2(N+1)} < a < \frac{1}{N+1} \text{ and (5.10) holds, or } \frac{1}{N+1} < a < 1 \text{ and (5.9) holds,}$$

then,

$$\varepsilon_{1,n} \equiv r_n^{\frac{2}{a}(a(N+1)-1)} \quad \text{and} \quad \varepsilon_{2,n} \equiv 1. \tag{5.17}$$

Observe that in any case $\varepsilon_{1,n} > 0$ and $\varepsilon_{2,n} > 0$ are uniformly bounded from above and at most one of them can collapse to zero, as $n \to +\infty$.
We denote by S_1 the (possibly empty) blow-up set of v_n, and by $\widehat{S}_1$ the (possibly empty) blow-up set of the sequence:

$$\widehat{v}_n(x) = v_n\left(\frac{x}{|x|^2}\right) + \beta_n \log\left(\frac{1}{|x|}\right),$$

which satisfies,

$$\begin{cases} -\Delta\widehat{v}_n = \dfrac{\varepsilon_{1,n}}{|x|^{2(N+2)-\beta_n}} e^{\widehat{v}_n} + \dfrac{\varepsilon_{2,n}}{|x|^{4-a\beta_n}} e^{a\widehat{v}_n} =: \widehat{g}_n \quad \text{in } \mathbb{R}^2, \\ \\ \beta_n = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \widehat{g}_n(x)\, dx \to 4(N+1), \text{as } n \to +\infty. \end{cases} \tag{5.18}$$

We can apply the blow-up analysis developed in [31] for sequences satisfying equations of the type (5.14) and (5.18) and in view of (5.6) conclude the following:

Lemma 5.3. *Let*

$$N(x_0) = \begin{cases} 0, & \textit{for } x_0 \neq 0, \\ N, & \textit{for } x_0 = 0. \end{cases}$$

Then,

$$i) \textit{ If } x_0 \in S_1 \textit{ then } \frac{1}{2\pi}\varepsilon_{1,n}|x|^{2N} e^{v_n} \rightharpoonup 4(1+N(x_0))\delta_{x_0}, \tag{5.19}$$

weakly in the sense of measures in $B_\rho(x_0)$*;*

$$ii) \textit{ If } 0 \in \widehat{S}_1 \textit{ then } \frac{\varepsilon_{1,n}}{|x|^{2(N+2)-\beta_n}} e^{\widehat{v}_n} \rightharpoonup 4(1+N)\delta_0, \tag{5.20}$$

weakly in the sense of measures in $B_\rho(0)$.

Proof. We just apply to v_n and $\widehat{v}_n$ the Proposition 4.15 or the Proposition 4.24 and the Proposition 4.25 (according to whether $\varepsilon_{1,n} \to 0$ or $\varepsilon_{2,n} \to 0$) of [31] which, in view of (5.6), immediately yield to (5.19) and (5.20). ☐

Since, from (5.11) we have:

$$\frac{1}{2\pi} \int\limits_{|x|\leq 1} g_n(x)\, dx = 4(N+1) - \varepsilon, \tag{5.21}$$

by virtue of Lemma 5.3 we see that,

$$0 \notin S_1 \quad \text{and} \quad 0 \notin \widehat{S}_1. \tag{5.22}$$

As above we set:

$$\varepsilon_n = e^{-\frac{u_n(x_n)}{2(N+1)}}, \quad \text{with } x_n \in \mathbb{R}^2 \, : \, u_n(x_n) = \max_{\mathbb{R}^2} u_n. \tag{5.23}$$

We derive the following:

Proposition 5.4. *Suppose that (5.9) holds. Then* $u_n(x_n) = \max_{\mathbb{R}^2} u_n \to +\infty$ *and we have:*

(1) *if* $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$ *then (along a subsequence) the following alternative holds:*

$$i) \text{ either } \frac{|x_n|}{\varepsilon_n} \le C, \; S_1 = \emptyset, \; v_n \to U, \text{ as } n \to +\infty, \tag{5.24}$$

uniformly in $C^2_{\text{loc}}(\mathbb{R}^2)$, *with* U *satisfying (1.14) (with* $\alpha = N$*);*

$$ii) \text{ or } \frac{|x_n|}{\varepsilon_n} \to +\infty, \text{ as } n \to +\infty, \; N \in \mathbb{N} \text{ and}$$

$S_1 = \{\xi_1, \cdots, \xi_{N+1}\}$ *is formed (up to a rotation) by the distinct* $(N+1)$*-roots of unity (i.e.* $\xi_j^{N+1} = 1, \; \forall \, j = 1, \cdots, N+1$*); moreover:*

$$\frac{1}{2\pi}\varepsilon_{1,n}|x|^{2N}e^{v_n} \rightharpoonup 4\sum_{j=1}^{N+1}\delta_{\xi_j}, \tag{5.25}$$

weakly in the sense of measures on compact sets.

(2) *if* $\frac{1}{N+1} < a < 1$ *then* $N \in \mathbb{N}$, *and alternative* $ii)$ *holds.*

Proof. We start with the following:

Claim:

$$S_1 \neq \emptyset \iff x_n \neq 0 \text{ and (along a subsequence) } u_n(x_n) + 2(N+1)\log(|x_n|) \to +\infty. \tag{5.26}$$

To establish the claim, suppose first that: $S_1 \neq \emptyset$ then, along a subsequence, $\exists \xi_n \to \xi_0 \neq 0$: $v_n(\xi_n) \to +\infty$. Since,

$$v_n\left(\frac{x_n}{r_n}\right) \ge v_n(\xi_n) \to +\infty, \tag{5.27}$$

and in view of (5.22), we find a constant $d > 1$:

$$\frac{1}{d} \le \left|\frac{x_n}{r_n}\right| \le d. \tag{5.28}$$

As a consequence $x_n \neq 0$, and since,

$$u_n(x_n) + 2(N+1)\log(|x_n|) = v_n\left(\frac{x_n}{r_n}\right) + 2(N+1)\log\left(\frac{|x_n|}{r_n}\right), \tag{5.29}$$

then, from (5.27) and (5.28) we derive that, $u_n(x_n) + 2(N+1)\log(|x_n|) \to +\infty$, as $n \to +\infty$.

To establish the viceversa, we use again (5.29) with $u_n(x_n) + 2(N+1)\log(|x_n|) \to +\infty$, as $n \to +\infty$. Therefore, if we assume by contradiction that $v_n\left(\frac{x_n}{r_n}\right) = \max_{\mathbb{R}^2} v_n \le C$, then necessarily

$\frac{|x_n|}{r_n} \to +\infty$. But $0 \notin \widehat{S}_1$, and so we have also that, $v_n\left(\frac{x_n}{r_n}\right)+\beta_n \log\left(\frac{|x_n|}{r_n}\right) \leq C$.As a consequence, $v_n\left(\frac{x_n}{r_n}\right) \to -\infty$, which is impossible, since from (5.21), it yields to the following contradiction:

$$4(N+1) - \varepsilon = \frac{1}{2\pi} \int\limits_{|x|\leq 1} g_n(x)\, dx \to 0, \text{ as } \ n \to +\infty.$$

In particular we have also verified that,

$$v_n\left(\frac{x_n}{r_n}\right) = \max_{\mathbb{R}^2} v_n \geq -C, \tag{5.30}$$

for suitable $C > 0$. In conclusion, $v_n\left(\frac{x_n}{r_n}\right) \to +\infty$ and (5.28) holds and thus, $S_1 \neq \emptyset$ and the Claim is established.

Next, we proceed to establish part (1) of our statement, where we take: $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$. To obtain $i)$, we observe that (by virtue of the claim) in this case we have: $S_1 = \emptyset$ and $x_n \to 0$. In particular, in view of (5.30), we obtain:

$$-C \leq v_n\left(\frac{x_n}{r_n}\right) = \max_{\mathbb{R}^2} v_n \leq C, \tag{5.31}$$

and, by well known elliptic estimates, we deduce that v_n is actually uniformly bounded in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$. Therefore, along a subsequence, we derive that,
$\varepsilon_{2,n} \to \varepsilon_2 \geq 0$ and $v_n \to U$, uniformly in $C^2_{\text{loc}}\left(\mathbb{R}^2\right)$ as $n \to +\infty$, with U satisfying:

$$\begin{cases} -\Delta U = |x|^{2N} e^U + \varepsilon_2 e^{aU} \quad \text{in } \mathbb{R}^2, \\ \beta = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(|x|^{2N} e^U + \varepsilon_2 e^{aU}\right) < +\infty. \end{cases} \tag{5.32}$$

To conclude that actually $\varepsilon_2 = 0$, we recall that $0 \notin \widehat{S}_1$, which implies:

$$\beta_n = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} g_n(x) \to \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(|x|^{2N} e^U + \varepsilon_2 e^{aU}\right) = 4(N+1), \text{ as } \ n \to +\infty.$$

On the other hand, for $\varepsilon_2 > 0$, we know that every solution of (5.32) must satisfy (5.5), and so $\beta > 4(N+1)$ in this case. As a consequence $\varepsilon_2 = 0$, and the proof of $i)$ is completed.
To complete also the proof of $ii)$, we use again the Claim above, (5.22) and Lemma 5.3, to see that necessarily S_1 must contain exactly $(N+1)$-points, and in particular $N \in \mathbb{N}$ in this case. Furthermore, to account for (5.21), we see that at least one element of S_1 must belong to the unit circle. At this point, as in section 3, we can further deduce that (up to a rotation) S_1 is formed by the distinct $(N+1)$-roots of the unity, and refer to [31] for details.

Finally, to conclude the proof, we need to consider $\frac{1}{N+1} < a < 1$, and rule out the possibility that $S_1 = \emptyset$ in this case. Indeed, if by contradiction, $S_1 = \emptyset$, then as above we would find:

$$\varepsilon_{1,n} \to \varepsilon_1 \geq 0 \quad v_n \to U, \text{ as } \ n \to +\infty, \ (\text{uniformly in } C^2_{\text{loc}}(\mathbb{R}^2)).$$

But now, in view of (5.12), we find that U satisfies:

$$\begin{cases} -\Delta U = \varepsilon_1 |x|^{2N} e^U + e^{aU} \quad \text{in } \mathbb{R}^2, \\ \beta = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(\varepsilon_1 |x|^{2N} e^U + e^{aU} \right) = 4(N+1), \end{cases} \tag{5.33}$$

which is impossible, since for $\varepsilon_1 > 0$, we know that every solution of (5.33) must satisfy (5.5), while in case $\varepsilon_1 = 0$, then $\beta = \frac{4}{a}$ (cfr. [13]). So in any case: $\beta < 4(N+1)$ and we arrive at the desired contradiction. So $S_1 \neq \emptyset$ in this case, and the proof is completed. □

Analogously, we obtain:

Proposition 5.5. *Assume that (5.10) holds.*

(1) if $\dfrac{1}{N+1} < a < 1$ *then (along a subsequence) alternative i) and ii) hold for* v_n.

(2) if $\dfrac{1}{2(N+1)} < a < \dfrac{1}{N+1}$ *then* $N \in \mathbb{N}$ *and alternative ii) hold for* v_n.

Proof. Since (5.22) holds for v_n, we see that (1) and (2) follow exactly as above, we omit the details. □

From Proposition 5.4 and Proposition 5.5 we can derive the following:

Corollary 5.6. *There exists a constant $d > 1$ such that,*

$$i) \text{ If } \frac{|x_n|}{\varepsilon_n} \leq C \text{ then } \quad \frac{1}{d} \leq \frac{r_n}{\varepsilon_n} \leq d; \tag{5.34}$$

$$ii) \text{ If } \frac{|x_n|}{\varepsilon_n} \to +\infty \text{ then } \quad \frac{1}{d} \leq \frac{r_n}{|x_n|} \leq d. \tag{5.35}$$

Proof. The inequality in (5.34) follows simply by observing that,

$$2(N+1)\log\left(\frac{r_n}{\varepsilon_n}\right) = v_n\left(\frac{x_n}{\varepsilon_n}\right),$$

and under the given assumption, then we know that (5.34) holds.
On the other hand, when *ii)* holds, then $S_1 \neq \emptyset$ and therefore, $\max\limits_{\mathbb{R}^2} v_n = v_n\left(\frac{x_n}{r_n}\right) \to +\infty$, and (5.35) follows from (5.22). □

Corollary 5.7. *If $N \notin \mathbb{N}$, then u_n is radially symmetric about the origin, for large $n \in \mathbb{N}$.*

Proof. As $N \notin \mathbb{N}$ only part (1)-*i)* of Proposition 5.4 and Proposition 5.5 can occur. Therefore, as stated in Lemma 1.7, we have:

$$\text{either } \frac{1}{2(N+1)} < a < \frac{1}{N+1} \text{ and (5.9) holds, or } \frac{1}{N+1} < a < 1 \text{ and (5.10) holds.} \tag{5.36}$$

Furthermore, $\frac{|x_n|}{\varepsilon_n} \leq C$, and from part *i)* of Corollary 5.6 also follows that $\frac{|x_n|}{r_n} \leq C$. Consequently, along a subsequence, we have:

$$\varepsilon_{2,n} \to 0, \quad \frac{x_n}{r_n} \to x_0, \quad v_n \to U, \text{ uniformly in } C^2_{\text{loc}}(\mathbb{R}^2), \text{ as } n \to +\infty,$$

with $U(z)$ satisfying (1.14) and $U(x_0) = \max_{\mathbb{R}^2} U$. Since $N \notin \mathbb{N}$, we know that U must take the expression:

$$U(z) = \log \frac{\tau}{\left(1 + \frac{\tau}{8(N+1)^2} |z|^{2(N+1)}\right)^2}, \quad \text{for some } \tau > 0,$$

(see [14], [26]), and consequently we find that, $x_0 = 0$. In other words, $\frac{x_n}{r_n} \to 0$ and by (5.34), we may conclude that,

$$\frac{x_n}{\varepsilon_n} \to 0, \text{ as } n \to +\infty. \tag{5.37}$$

At this point, the remaining part of the proof follows by means of the following (more general) result:

Theorem 5.8. *Suppose that u_n satisfies (1.29) and (5.37) holds. Then $x_n = 0$ and u_n is radially symmetric about the origin, for $n \in \mathbb{N}$ large.*

Proof. In view of (5.37) we are dealing with the situation described in (5.36). Furthermore, by part *i)* of Corollary 5.6, it is equivalent to consider the following re-scaled sequence of $\{u_n\}$:

$$v_n(x) := u_n(\varepsilon_n x) + 2(N+1)\log \varepsilon_n, \tag{5.38}$$

satisfying:

$$\begin{cases} -\Delta v_n = \varepsilon_n^{2(1-a(N+1))} e^{av_n} + |x|^{2N} e^{v_n} \quad \text{in } \mathbb{R}^2 \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} \varepsilon_n^{2(1-a(N+1))} e^{av_n} + |x|^{2N} e^{v_n} = \beta_n \to 4(N+1). \end{cases} \tag{5.39}$$

with $\varepsilon_n > 0$ and $x_n \in \mathbb{R}^2$ defined in (5.23), and

$$v_n\Big(\frac{x_n}{\varepsilon_n}\Big) = \max_{\mathbb{R}^2} v_n = 0, \quad \varepsilon_n^{2(1-a(N+1))} \to 0, \quad \text{as } n \to +\infty.$$

Therefore, along a subsequence, we have:

$$v_n(x) \to V(x) = \log \frac{1}{\left(1 + \dfrac{1}{8(N+1)^2} |x|^{2(N+1)}\right)^2} \quad \text{uniformly in } C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2), \tag{5.40}$$

as $n \to +\infty$. Moreover, since $0 \notin \widehat{S}_1$, we find a suitable constant $C > 0$:

$$\sup_{|x|\geq 1} |v_n(x) + \beta_n \log|x|| \leq C, \quad |x| \geq 1 \quad n \in \mathbb{N}. \tag{5.41}$$

To proceed further, we argue by contradiction and assume that u_n (and so v_n) is not radially symmetric. In other words, $\dfrac{\partial v_n}{\partial \vartheta}(x) \neq 0$. Since, $\dfrac{\partial v_n}{\partial \vartheta}(x) \to 0$, as $|x| \to +\infty$ (recall (5.1)), we find $y_n \in \mathbb{R}^2$:

$$\max_{\mathbb{R}^2} \left|\frac{\partial v_n}{\partial \vartheta}\right| = \left|\frac{\partial v_n}{\partial \vartheta}(y_n)\right| = M_n > 0.$$

By replacing if necessary, $\dfrac{\partial v_n}{\partial \vartheta}$ with $-\dfrac{\partial v_n}{\partial \vartheta}$, we can always assume that $M_n = \dfrac{\partial v_n}{\partial \vartheta}(y_n)$.
We consider:

$$w_n(x) = \frac{1}{M_n}\frac{\partial v_n}{\partial \vartheta}(x),$$

which satisfies:

$$\begin{cases} -\Delta w_n = \left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n} + |x|^{2N}\,e^{v_n}\right) w_n =: h_n(x) \\ w_n(x) \to 0 \quad \text{ as } \; |x| \to +\infty \\ \max_{\mathbb{R}^2} |w_n(x)| = w_n(y_n) = 1, \end{cases} \tag{5.42}$$

and moreover we have,

$$\int_0^{2\pi} w_n\left(r\cos\vartheta, r\sin\vartheta\right) d\vartheta = 0 \quad \forall\ r > 0, \tag{5.43}$$

$$\int_{\mathbb{R}^2} \left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n} + |x|^{2N}\,e^{v_n}\right) w_n\, dx = 0. \tag{5.44}$$

By the decay property of v_n in (5.41), we can use Green's representation formula for w_n and write:

$$w_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right)\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n} + |y|^{2N}\,e^{v_n}\right) w_n dy. \tag{5.45}$$

Actually, by using (5.44), we can express (5.45) equivalently as follows:

$$w_n(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n} + |y|^{2N}\,e^{v_n}\right) w_n dy \quad \text{ for } x \neq 0. \tag{5.46}$$

We are going to estimate the integral in the right hand side of (5.46) in order to provide uniform (decay) estimates on w_n as $|x| \to +\infty$. To this purpose, for given $R > 1$ sufficiently large and $|x| > 2R$, we split the integral in (5.46) over the following different regions: $D_1 := \{|y| < R\}$, $D_2 := \left\{R \leq |y| \leq \frac{|x|}{2}\right\}$, $D_3 := \left\{|x-y| < \frac{|x|}{2}\right\}$, $D_4 := \left\{\frac{|x|}{2} \leq |y| \leq 2\,|x|\right\} \cap \left\{|x-y| \geq \frac{|x|}{2}\right\}$ and $D_5 := \{|y| \geq 2\,|x|\}$, so that we have:

$$|w_n(x)| \leq \frac{1}{2\pi}\sum_{i=1}^{5}\int_{D_i}\left|\log\left(\frac{1}{\left|\frac{x}{|x|} - \frac{y}{|y|}\right|}\right)\right| |h_n(y)|\, dy =: \frac{1}{2\pi}\sum_{i=1}^{5} J_i \tag{5.47}$$

Next, we estimate each of the terms above as follows:

$$\begin{aligned} J_1 &= \frac{1}{2\pi}\int_{D_1}\left|\log\left(\frac{1}{\left|\frac{x}{|x|} - \frac{y}{|x|}\right|}\right)\right| |h_n(y)|\, dy \\ &\leq \frac{1}{2\pi}\log\left(1 + \frac{R}{|x|}\right)\int_{\mathbb{R}^2}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n} + |y|^{2N}\,e^{v_n}\right) dy \\ &\leq C_1 \log\left(1 + \frac{R}{|x|}\right), \end{aligned} \tag{5.48}$$

and

$$
\begin{aligned}
J_2 &= \frac{1}{2\pi}\int_{D_2}\left|\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)\right||h_n(y)|\,dy \\
&\le \frac{\log 2}{2\pi}\int_{\{|y|\ge R\}}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n}+|y|^{2N}\,e^{v_n}\right)dy \\
&\le C_2\left(\frac{\varepsilon_n^{2(1-a(N+1))}}{R^{2(2a(N+1)-1)}}+\frac{1}{R^{2(N+1)}}\right),
\end{aligned}
\tag{5.49}
$$

where in the last estimate, we have used (5.41) together with the fact that $\beta_n \searrow 4(N+1)$, as $n \to +\infty$.
Furthermore, since for $y \in \mathbb{R}^2$: $|x-y| < \frac{|x|}{2}$, we have: $|y| \ge \frac{|x|}{2}$, we can estimate

$$
\begin{aligned}
J_3 &= \frac{1}{2\pi}\int_{D_3}\left|\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)\right||h_n(y)|\,dy \\
&\le \sup_{|y|\ge\frac{|x|}{2}}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n}+|x|^{2N}\,e^{v_n}\right)\int_{\left\{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|<1/2\right\}}\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)dy \\
&\le \frac{1}{2\pi}|x|^2\sup_{|y|\ge\frac{|x|}{2}}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n}+|x|^{2N}\,e^{v_n}\right)\int_{\{|\xi|<1/2\}}\log\left(\frac{1}{|\xi|}\right)d\xi \\
&\le C_3\left(\frac{a\varepsilon_n^{2(1-a(N+1))}}{|x|^{2(2a(N+1)-1)}}+\frac{1}{|x|^{2(N+1)}}\right),
\end{aligned}
\tag{5.50}
$$

where again we have used (5.41).
Similarly, by setting $D_4^* := \left\{\frac{1}{2} \le \left|\frac{x}{|x|}-\frac{y}{|x|}\right| \le 2\right\} \cap \left\{|y| \ge \frac{|x|}{2}\right\}$ again by (5.41) we also get

$$
\begin{aligned}
J_4 &= \frac{1}{2\pi}\int_{D_4}\left|\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)\right||h_n(y)|\,dy \le \frac{1}{2\pi}\int_{D_4^*}\left|\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)\right||h_n(y)|\,dy \\
&\le \frac{1}{2\pi}|x|^2\sup_{|y|\ge\frac{|x|}{2}}\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n}+|x|^{2N}\,e^{v_n}\right)\int_{\left\{\frac{1}{2}\le|\xi|\le 2\right\}}\left|\log\frac{1}{|\xi|}\right|d\xi \\
&\le C_4\left(\frac{a\varepsilon_n^{2(1-a(N+1))}}{|x|^{2(2a(N+1)-1)}}+\frac{1}{|x|^{2(N+1)}}\right).
\end{aligned}
\tag{5.51}
$$

Finally, we observe that for $y \in \mathbb{R}^2$: $|y| \ge 2|x|$, there holds:

$$
\left|\frac{x}{|x|}-\frac{y}{|x|}\right| = \frac{|y|}{|x|}\left|\frac{y}{|y|}-\frac{x}{|y|}\right| \ge 2,
$$

and as a consequence we obtain the estimate:

$$
\begin{aligned}
J_5 &= \frac{1}{2\pi}\int_{D_5}\left|\log\left(\frac{1}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|}\right)\right| |h_n(y)|\,dy \\
&\le \frac{1}{2\pi}\int_{D_5}\log\left(1+\frac{|y|}{|x|}\right)\left(a\varepsilon_n^{2(1-a(N+1))}e^{av_n}+|y|^{2N}e^{v_n}\right)dy \\
&\le C_5|x|^2\left(\frac{a\varepsilon^{2(1-a(N+1))}}{|x|^{a\beta_n}}\int_{D_5}\log\left(1+\frac{|y|}{|x|}\right)\left(\frac{|x|}{|y|}\right)^{a\beta_n}\frac{dy}{|x|^2}\right) \\
&+ C_5|x|^2\left(\frac{1}{|x|^{\beta_n-2N}}\int_{D_5}\log\left(1+\frac{|y|}{|x|}\right)\left(\frac{|x|}{|y|}\right)^{\beta_n-2N}\frac{dy}{|x|^2}\right) \\
&\le C_5\left(\frac{a\varepsilon^{2(1-a(N+1))}}{|x|^{2(2a(N+1)-1)}}+\frac{1}{|x|^{2(N+1)}}\right)\int_{\{|\xi|\ge 2\}}\log\left(1+|\xi|\right)\left[\left(\frac{1}{|\xi|}\right)^{4a(N+1)}+\left(\frac{1}{|\xi|}\right)^{2(N+2)}\right]d\xi \\
&\le C_5\left(\frac{a\varepsilon_n^{2(1-a(N+1))}}{|x|^{2(2a(N+1)-1)}}+\frac{1}{|x|^{2(N+1)}}\right).
\end{aligned}
\tag{5.52}
$$

Since by assumption: $a > \frac{1}{2(N+1)}$, the above estimates implay that, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$, such that:

$$\sup_{|x|\ge R_\varepsilon}|w_n(x)| < \varepsilon. \tag{5.53}$$

Therefore, if we let $y_n \in \mathbb{R}^2$: $w_n(y_n) = 1\,(\equiv \max_{\mathbb{R}^2}|w_n|)$, then, $|y_n| \le R$, for suitable $R > 0$. Moreover, from (5.42) and standard elliptic estimates, we deduce that, along a suitable subsequence, the following holds:

$$w_n \to w \quad \text{uniformly in } C^{2,\alpha}_{\text{loc}}\left(\mathbb{R}^2\right) \quad \text{and } y_n \to y_0 \qquad \text{as } n\to+\infty,$$

with w satisfying:

$$\begin{cases} -\Delta w = |x|^{2N}\,e^{V}w \\ w(y_0) = 1 = \max_{\mathbb{R}^2}|w| \\ \int_0^{2\pi} w\left(r\cos\vartheta, r\sin\vartheta\right)d\vartheta = 0, \quad r>0. \end{cases} \tag{5.54}$$

with V given in (5.40). But this is impossible, since we know (see e.g. [3]) that for $N \notin \mathbb{N}$ the only bounded solution of equation in (5.54) is given by:

$$w_0(r) = \frac{1-\frac{1}{(2(N+1))^2}r^{2(N+1)}}{1+\left(\frac{1}{2(N+1)}\right)^2 r^{2(N+1)}}, \qquad \text{with } r=|x|,$$

which obviously cannot satisfy the integral condition in (5.54). In this way, we have reached a contradiction which allow us to conclude the Proof of Theorem 5.8 and of Corollary 5.7 as well. □

From Theorem 5.8 and the uniqueness of radial solutions of (1.27)-(1.28) established in [24], [25] we can conclude the following:

Corollary 5.9. *Let $N \notin \mathbb{N}$ and $\frac{1}{2(N+1)} < a \neq \frac{1}{N+1} < 1$. There exists $\varepsilon_0 > 0$, such that for every $\beta : |\beta - 4(N+1)| < \varepsilon_0$ satisfying* (5.5)*, problem* (1.27)*-*(1.28) *admits a unique solution and moreover such solution is radially symmetric about the origin.*

We observe that Corollary 5.9 is sharp, in the sense that for β away from $4(N+1)$, it is possible to exhibit non-radial solutions of (1.27)-(1.28). We refer to [24] for details.
As already observed by the arguments provided in [31] it is reasonable to believe that the statement of Corollary 5.9 remains valid also for $N = 1$, and $\frac{1}{(N+1)} < a < 1$.
The analysis above allows us to obtain the following extension to u_n of the pointwise estimates (1.10) (namely of Theorem 5.6.51 of [30]) :

Theorem 5.10. *Let u_n satisfy* (1.29) *and (by recalling* (5.23)*) assume that,*

$$\frac{|x_n|}{\varepsilon_n} \leq C, \tag{5.55}$$

then

$$(\beta_n - 4(N+1)) \log\left(\frac{1}{\varepsilon_n}\right) \searrow 0, \text{ as } n \to +\infty, \tag{5.56}$$

and the following estimate holds:

$$\left| u_n(x) - \log \frac{e^{u_n(x_n)}}{(1 + \frac{e^{u_n(x_n)}}{8(N+1)^2}|x - x_n|^{2(N+1)})} + (\beta_n - 4(N+1)) \log(1 + |x - x_n|) \right| \leq C, \; x \in \mathbb{R}^2, \tag{5.57}$$

with suitable $C > 0$.

Proof. As before, we may use the rescaled sequence v_n in (5.38) which satisfies (5.39) and (5.41). Furthermore, along a subsequence, the following holds:

$$\frac{x_n}{\varepsilon_n} \to z_0, \quad \varepsilon_n^{2(1-a(N+1))} \to 0, \text{ as } n \to +\infty, \tag{5.58}$$

and in complex notations,

$$v_n(z) \to U(z) = \log \frac{1}{\left(1 + \frac{1}{8(N+1)^2}|z^{N+1} - z_0^{N+1}|^2\right)^2}, \text{ as } n \to +\infty, \tag{5.59}$$

uniformly in $C^2_{\text{loc}}(\mathbb{R}^2)$. Notice, that we may have: $z_0 \neq 0$ only if $N \in \mathbb{N}$. Therefore, if we combine (5.59) with (5.41) we obtain that,

$$\left| v_n(z) - \log \frac{e^{u_n(x_n)}}{(1 + \frac{e^{u_n(x_n)}}{8(N+1)^2}|x - x_n|^{2(N+1)})^2} + (\beta_n - 4(N+1)) \log\left(1 + \frac{|x|}{\varepsilon_n}\right) \right| \leq C, \; \forall \; x \in \mathbb{R}^2. \tag{5.60}$$

Next, we establish (5.56). To this purpose, we recall (via (5.2)) that,

$$(\beta_n - 4(N+1)) = \frac{4(1 - a(N+1))}{a\beta_n} \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} e^{au_n} = \frac{4(1 - a(N+1))}{a\beta_n} \varepsilon_n^{2(1-a(N+1))} \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} e^{av_n},$$

and by means of (5.41) and (5.59), we know that,

$$\int\limits_{\mathbb{R}^2} e^{av_n} \to \int\limits_{\mathbb{R}^2} e^{aU}, \text{ as } n \to +\infty.$$

As a consequence, as $n \to +\infty$,

$$(\beta_n - 4(N+1))\log\left(\frac{1}{\varepsilon_n}\right) = \varepsilon_n^{2(1-a(N+1))}\log\left(\frac{1}{\varepsilon_n}\right)\left(\frac{4(1-a(N+1))}{a\beta_n}\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} e^{aU} + \mathrm{o}(1)\right) \quad (5.61)$$

Next, by combining Lemma 5.2 and Corollary 5.6 we know that, if $\frac{1}{2(N+1)} < a < \frac{1}{N+1}$ then $\beta_n > 4(N+1)$ and $\varepsilon_n \to 0$, while for $\frac{1}{N+1} < a < 1$ then $\beta_n < 4(N+1)$ and $\varepsilon_n \to +\infty$. Whence, (5.56) readily follows from (5.61). At this point, we can finally conclude (5.57). Indeed, if (along a subsequence) $\frac{|x_n|}{\varepsilon_n} \to 0$ then from Theorem 5.8 we know that $x_n = 0$, and in this case (5.57) follows as a consequence of (5.60) and (5.56). On the other hand, if (along a subsequence) $\frac{|x_n|}{\varepsilon_n} \to l \neq 0$ then $|x_n| = \mathrm{O}(\varepsilon_n)$ and again we deduce (5.57) from (5.60) and (5.56).

□

Theorem 5.11. *Let u_n satisfy* (1.29) *and suppose that,*

$$N \in \mathbb{N} \quad \textit{and} \quad \frac{|x_n|}{\varepsilon_n} \to +\infty, \textit{ as } n \to +\infty.$$

If (1.37) *holds, then there exist $(N+1)$-sequences $\{\xi_{j,n}\}_{j=1,\cdots,N+1}$ such that, for $\varepsilon > 0$ sufficiently small and $v_n(x) = u_n(|x_n|x) + 2(N+1)\log(|x_n|)$ (along a subsequence) we have:*

$$v_n(\xi_{n,j}) = \max_{B_\varepsilon(\xi_j)} v_n \to +\infty,$$

$$\xi_{n,j} \to \xi_j,$$

as $n \to +\infty$, and (after a rotation) $\{\xi_j\}_{j=1,\cdots,N+1}$ define the distinct $N+1$-roots of unity. Furthermore,

$$(\beta_n - 4(N+1))\log\left(\frac{1}{|x_n|}\right) \to 0, \textit{ as } n \to +\infty,$$

and for $z_{n,j} = |x_n|\xi_{n,j}$:

$$\left| u_n(x) - \log\frac{e^{u_n(x_n)}}{(1+\frac{e^{u_n(x_n)}}{8(N+1)^2}\prod\limits_{j=1}^{N+1}|x-z_{n,j}|^2)^2} + (\beta_n - 4(N+1))(\log\left(1 + \min_{j=1,\cdots,N+1}|x-z_{n,j}|\right)) \right| \leq C,$$

$\forall\, x \in \mathbb{R}^2$, with suitable $C > 0$.

Proof. According to (1.37) we see that,

$$\text{either } \frac{1}{2(N+1)} < a < \frac{1}{N+1} \text{ and (5.9) holds, or } \frac{1}{N+1} < a < 1 \text{ and } 0 \notin S_1 \text{ holds,} \quad (5.62)$$

and in particular, when $\frac{1}{N+1} < a < 1$, then necessarily $|x_n| \geq C$. Therefore, by taking into account Corollary 5.6 *ii*), it is convenient to consider (similarly to (5.14)-(5.16)) the following re-scaled sequence:

$$v_n(x) = u_n(|x_n|x) + 2(N+1)\log(|x_n|),$$

satisfying:

$$\begin{cases} -\Delta v_n = |x|^{2N} e^{v_n} + |x_n|^{2(1-a(N+1))} e^{a v_n} =: f_n \quad \text{in } \mathbb{R}^2, \\ \beta_n = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} f_n(x)\, dx \to 4(N+1), \text{as } \ n \to +\infty, \end{cases} \tag{5.63}$$

where we keep in mind that,

$$0 < |x_n|^{2(1-a(N+1))} \le C. \tag{5.64}$$

Furthermore, in view of Proposition 5.4 and Proposition 5.5, for the blow-up set S_1 of v_n we have (up to rotation):

$$S_1 = \{\xi_1, \cdots, \xi_{N+1}\} \ : \ \xi_i \neq \xi_j \quad \text{and} \quad \xi_i^{N+1} = \xi_j^{N+1} = 1, \text{ for } i \neq j \in \{1, \cdots, N+1\},$$

while, in view of (5.22), $0 \notin \widehat{S}_1$, with $\widehat{S}_1$ the blow up set of $\widehat{v}_n(x) = v_n\left(\frac{x}{|x|^2}\right) + \beta_n \log\left(\frac{1}{|x|}\right)$.
Since for v_n and ∇v_n is available the following Green's representation formula, given as follows:

$$v_n(x) - v_n(\overline{x}) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \log\left(\frac{|\overline{x} - y|}{|x - y|}\right) f_n(y)\, dy, \ \forall \ x, \overline{x} \in \mathbb{R}^2,$$

$$\nabla v_n(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \left(\frac{x - y}{|x - y|^2}\right) f_n(y)\, dy, \ \forall \ x \in \mathbb{R}^2,$$

which we easily obtain by scaling (5.3) and (5.4), (valid for u_n), we can argue exactly as in the previous section (see also [31]) to derive the necessary estimates and obtain the blow-up profile of v_n around each of its blow up point. More precisely, it is possible to establish the following: for $\varepsilon > 0$ sufficiently small and for $\xi_{j,n} \in B_\varepsilon(\xi_j)$:

$$v_n(\xi_{j,n}) = \max_{B_\varepsilon(\xi_j)} v_n, \ j = 1, \cdots, N+1,$$

there holds:

$$\xi_{n,j} \to \xi_j \quad \text{and} \quad \left| v_n(x) - \log \frac{e^{v_n(\xi_{j,n})}}{\left(1 + \frac{e^{v_n(\xi_{j,n})}}{8} |x - \xi_{j,n}|^2\right)^2} \right| \le C, \tag{5.65}$$

$\forall\, x \in B_\varepsilon(\xi_j)$, and suitable $C > 0$.
Furthermore, by recalling that $0 \notin \widehat{S}_1$, one can also show that,
$\forall\ R > 1$ sufficiently large and $\varepsilon > 0$ sufficiently small, there exists $C > 0$ (depending only on R and ε):

$$|v_n(x) - v_n(\overline{x})| \le C, \ \forall \ x, \overline{x} \in B_R \setminus \bigcup_{j=1}^{N+1} B_\varepsilon(\xi_j), \tag{5.66}$$

$$|\widehat{v}_n(x) - \widehat{v}_n(\overline{x})| \le C, \ \forall \ x, \overline{x} \in B_{1/R}. \tag{5.67}$$

Furthermore,

$$\max_{\mathbb{R}^2} v_n = v_n\left(\frac{x_n}{|x_n|}\right) = u_n(x_n) + 2(N+1)\log(|x_n|) \to +\infty, \text{ as } \ n \to +\infty,$$

so, without loss of generality, we can assume:

$$\xi_{1,n} = \frac{x_n}{|x_n|} \to \xi_1, \text{ as } \ n \to +\infty.$$

As above, by (5.65), (5.66) and (5.67), we may conclude that,

$$\frac{e^{v_n(\xi_{j,n})}}{e^{v_n(\xi_{i,n})}} = \mathrm{O}(1) \text{ and } |v_n(x) - v_n(\xi_{1,n}) + \beta_n \log(|x|)| \le C, \ \forall\ |x| > 2.$$

Thus, we can combine the above estimates as follows:

$$\left| v_n(x) - \log \frac{e^{v_n(\xi_{1,n})}}{(1 + \frac{e^{v_n(\xi_{1,n})}}{8(1+N)^2} \prod_{j=1}^{N+1} |x - \xi_{j,n}|^2)^2} + (\beta_n - 4(N+1)) \log (1 + |x|)) \right| \le C, \tag{5.68}$$

$\forall\ x \in \mathbb{R}^2$, which we easily scale back to u_n and obtain:

$$\left| u_n(x) - \log \frac{e^{u_n(x_n)}}{(1 + \frac{e^{u_n(x_n)}}{8(1+N)^2} \prod_{j=1}^{N+1} |x - z_{j,n}|^2)^2} + (\beta_n - 4(N+1)) \log \left(1 + \frac{|x|}{|x_n|}\right) \right| \le C, \tag{5.69}$$

$\forall\ x \in \mathbb{R}^2$.

At this point, to conclude (1.40), we need only to show (1.39), which clearly holds in case $|x_n|$ is uniformly bounded from above and from below away from zero.
So, according to our assumption, we need only to account for the following situation:

$$\begin{cases} i) \ \frac{1}{2(N+1)} < a < \frac{1}{N+1} \text{ and } |x_n| \to 0, \\ \\ ii) \ \frac{1}{N+1} < a < 1 \text{ and } |x_n| \to +\infty. \end{cases} \tag{5.70}$$

Notice that under (5.70) we have: $(\beta_n - 4(N+1)) \log\left(\frac{1}{|x_n|}\right) > 0$ for large n.
Furthermore, as above we find:

$$(\beta_n - 4(N+1)) = \frac{4(1 - a(N+1))}{a\beta_n} \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{au_n} =$$

$$|x_n|^{2(1-a(N+1))} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{av_n} \right) \left(\frac{1 - a(N+1)}{(N+1)a} + \mathrm{o}(1) \right), \text{as } \ n \to +\infty.$$

By virtue of (5.68) we can easily check that: $\int_{\mathbb{R}^2} e^{av_n} \to 0$, as $n \to +\infty$, and in view of (1.39) we conclude that,

$$(\beta_n - 4(N+1)) \log \left(\frac{1}{|x_n|} \right) =$$

$$|x_n|^{2(1-a(N+1))} \log \left(\frac{1}{|x_n|} \right) \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{av_n} \right) \left(\frac{1 - a(N+1)}{(N+1)a} + \mathrm{o}(1) \right) \to 0, \text{ as } \ n \to +\infty.$$

So (1.39) is established, and (1.40) follows from (5.69). □

It is clear that when (5.9) holds, then (5.57) and (1.40) express suitable extensions of (1.10) and (1.24) to the cosmic string solution-sequence u_n in B_R. More interestingly, when (5.10) holds, (i.e. the origin is a blow up point for $\widehat{u}_n$), then from (5.57) and (1.40), we can read pointwise estimates for $\widehat{u}_n$, around the origin, analogous to (1.10) and (1.24), which more precisely (see Remark 1.5) can be expressed as follows:

Corollary 5.12. *Assume (5.10), then, along a subsequence, we have,*

$$\left| \widehat{u}_n(x) - \log \frac{e^{\widehat{u}_n(x_n)}}{(1 + \frac{e^{\widehat{u}_n(x_n)}}{8(N+1)^2} \prod_{j=1}^{N+1} |x - \widehat{z}_{n,j}|^{2(N+1)})} \right| \le C, \; |x| \le 1,$$

with suitable $C > 0$*, and there holds:*

$$i) \textit{ If } \frac{|x_n|}{\varepsilon_n} \to 0, \textit{as } n \to +\infty, \textit{ then } x_n = 0 = \widehat{x}_n \textit{ and } \widehat{u}_n(0) = -u_n(0) + O(1),$$

$$ii) \textit{ If } \frac{|x_n|}{\varepsilon_n} \to L \neq 0, \textit{as } n \to +\infty, \textit{ then } \widehat{x}_n = \frac{x_n}{|x_n|^2} = \widehat{z}_{n,j}, \; \forall \, j = 1, \cdots, N+1,$$

$$iii) \textit{ If } \frac{|x_n|}{\varepsilon_n} \to +\infty, \textit{as } n \to +\infty, \textit{ then } \widehat{x}_n = \frac{x_n}{|x_n|^2} \textit{ and } \widehat{z}_{n,j} = \frac{\xi_{j,n}}{|x_n||\xi_{j,n}|}, \; \forall \, j = 1, \cdots, N+1,$$

with $\xi_{j,n}$ *as specified in Theorem 5.11.*

6. Appendix: The proof of (1.42).

In this Appendix we report the proof by R. Tauraso and C. Pagano [22] of (1.42).

The proof of Lemma (1.42).
Set

$$P(\mathrm{w}) = \prod_{j=1}^{N+1} (\mathrm{w} - \mathrm{w}_j) = \mathrm{w}^{N+1} + \sum_{k=0}^{N} a_k \mathrm{w}^k.$$

Then

$$P'(\mathrm{w}_1) = \prod_{j=2}^{N+1} (\mathrm{w}_1 - \mathrm{w}_j), \quad \text{and} \quad P''(\mathrm{w}_1) = 2 \sum_{\substack{|A|=N-1 \\ A \subset \{2, \cdots, N+1\}}} \prod_{j=2}^{N+1} (\mathrm{w}_1 - \mathrm{w}_j).$$

Hence, by means of our hypothesis we find:

$$NP'(\mathrm{w}_1) = 2\mathrm{w}_1 \prod_{j=2}^{N+1} (\mathrm{w}_1 - \mathrm{w}_j) \sum_{j=2}^{N+1} \frac{1}{\mathrm{w}_1 - \mathrm{w}_j} = \mathrm{w}_1 P''(\mathrm{w}_1),$$

and the same holds for all $i = 1, \cdots, N+1$, that is

$$NP'(\mathrm{w}_i) - \mathrm{w}_i P''(\mathrm{w}_i), \quad \forall \, i = 1, \cdots, N+1.$$

As a consequence we find that, the polynomial

$$Q(\mathrm{w}) := NP'(\mathrm{w}) - \mathrm{w}P''(\mathrm{w}) = \sum_{k=1}^{N} k(N+1-k) a_k \mathrm{w}^{k-1},$$

admits degree less then N and at the same time admits $N+1$ distinct roots, namely: $Q(\mathrm{w}_i) = 0$, $\forall\ i = 1, \cdots, N+1$. Whence $Q(\mathrm{w}) \equiv 0$, and therefore $a_k = 0$ for $k = 1, \cdots, N$, or in other word: $P(z) = z^{N+1} - a_0$, as claimed. □

7. Appendix: The proof of (4.24).

First of all, let us turn our attention to the Pohožaev identity (4.24). We have the following:
Claim: Let $n \in \mathbb{N}$ be fixed. Then:
If $\alpha \in (-\frac{1}{2}, 0)$ then $\nabla u_n \in L^\infty(B_1(0))$;
If $\alpha = -\frac{1}{2}$ then $|\nabla u_n| \leq C\left(\log\left(\frac{1}{|x|}\right)\right)$;
If $\alpha \in (-1, -\frac{1}{2})$ then $|\nabla u_n| \leq \left(\frac{C}{|x|^{2\alpha-1}}\right)$, for a suitable $C > 0$.
The proof of the claim uses the Green's representation formula (4.13) and well known potential estimates based on the Hölder's inequality. We skip it here since it is rather standard.
Since u_n is smooth away from the origin, we can apply the Pohožaev identity in $\varepsilon \leq |x| \leq r_n$ for any $0 < \varepsilon < r_n \leq 1$. By using the claim and the fact that (by standard elliptic regularity theory) u_n is continuous, we see that the terms which show up at the inner boundary $|x| = \varepsilon$ must vanish in the limit as $\varepsilon \to 0^+$, and so we obtain (4.24). □

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