Unions of admissible relations

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Abstract. We show that a variety $\mathcal{V}$ is congruence distributive if and only if there is some h such that the inclusion

$$\Theta \cap (\sigma \circ \sigma) \subseteq (\Theta \cap \sigma) \circ (\Theta \cap \sigma) \circ \dots \qquad (h \text{ factors}) \tag{1}$$

holds in every algebra in $\mathcal{V}$, for every tolerance Θ and every U-admissible relation σ. By a *U-admissible* relation we mean a binary relation which is the set-theoretical union of a set of reflexive and admissible relations. For any fixed h, a Maltsev-type characterization is given for the inclusion (1). It is an open problem whether (1) is still equivalent to congruence distributivity when Θ is assumed to be a U-admissible relation, rather than a tolerance. In both cases many equivalent formulations for (1) are presented. The results suggest that it might be interesting to study the structure of the set of U-admissible relations on an algebra, as well as identities dealing with such relations.

1. Introduction

Congruence identities have played an important role in universal algebra right from the beginning. The study of congruence permutable, distributive and modular varieties has led to a plethora of significant techniques and results. Jónsson [6] is a good introduction to the classical results. Research in the field is still very active; recent, advanced and sophisticated results appear, for example, in Kearnes and Kiss [9], where the reader can also find further references.

Already from the earlier arguments it appeared evident that tolerances and, more generally, admissible relations are fundamental tools even when the main focus are congruences. See, e.g., Gumm [4], Jónsson [6, p. 370] or Tschantz [15]. In a form or another and more or less explicitly, this aspect appears also in [2, 5, 9, 10, 11, 12, 16], just to limit ourselves to a few references. Recall that a *tolerance* is a reflexive, symmetric and admissible relation.

In [12] we have found a particularly simple characterization of congruence modularity. A variety $\mathcal{V}$ is congruence modular if and only if there is some k such that the inclusion

$$\Theta(R \circ R) \subseteq (\Theta R)^{\circ k} \tag{2}$$

holds in every algebra in $\mathcal{V}$, for every reflexive and admissible relation R and every congruence (equivalently, every tolerance) Θ.

2010 *Mathematics Subject Classification*: 08B10; 08B05.

Key words and phrases: Congruence distributive variety; (directed) Jónsson terms; tolerance; (unions of) reflexive and admissible relations.

Work performed under the auspices of G.N.S.A.G.A. Work partially supported by PRIN 2012 “Logica, Modelli e Insiemi”.

Here and below juxtaposition denotes intersection and, for a binary relation R, $R^{\circ h}$ denotes the relational composition $R \circ R \circ \dots$ with h factors, that is, with $h-1$ occurrences of $\circ$. The displayed formula in the abstract is then written as $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$. We shall also use the shorthand $S \circ_m T$ to denote $S \circ T \circ S \dots$ with m factors. Thus $R^{\circ h}$ is the same as $R \circ_h R$.

We say that an inclusion like (2) *holds* in some variety $\mathcal{V}$ if the inclusion holds when interpreted in the standard way in the set of binary relations on every algebra in $\mathcal{V}$. Similar expressions like "$\mathcal{V}$ *satisfies* an inclusion" shall be used with the same meaning. Notice that, since an inclusion $A \subseteq B$ is equivalent to the identity $A = AB$, we can equivalently—and usually shall—speak of *identities*. As a standard convention, we assume that juxtaposition ties more than any other binary operation symbol, namely, an expression like $(\alpha\beta) \circ (\alpha\gamma)$ will be simply written as $\alpha\beta \circ \alpha\gamma$. However, exponents bind more than anything else, e.g., $\Theta R^{\circ h}$ means $\Theta(R^{\circ h})$. If not explicitly stated otherwise, α, β and γ are variables for congruences, Θ can be equivalently taken to be a variable for congruences or tolerances and R, S and T are variables for reflexive and admissible relations. All the binary relations considered in this note are assumed to be reflexive, hence sometimes we shall simply say *admissible* in place of reflexive and admissible .

Identity (2) above is related to various similar identities. For example, Werner [16] showed, among other, that a variety $\mathcal{V}$ is congruence permutable if and only if $\mathcal{V}$ satisfies $R \circ R \subseteq R$. This corresponds to the special case $\Theta = 1$ in (2). Here 1 or $1_{\mathbf{A}}$ denotes the largest congruence on the algebra $\mathbf{A}$ under consideration. As another example, and an almost immediate consequence of results from Kazda, Kozik, McKenzie and Moore [8], we observed in [11, Proposition 3.1 and p.10] that a variety $\mathcal{V}$ is congruence distributive if and only if there is some k such that $\mathcal{V}$ satisfies $\Theta(R \circ S) \subseteq \Theta R \circ_k \Theta S$. Taking $R = S$ in this identity, we get (2) again. In this sense, identity (2) seems still another way to see that congruence modularity is some kind of a combination of congruence permutability with distributivity [4].

By the way, notice that congruence distributivity is equivalent also to $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_h \alpha\gamma$, for some h and for congruences α, β, γ, by a classical paper by Jónsson [5]. On the other hand, both identity (2) and Werner's identity $R \circ R = R$ become trivially true in every algebra, if we let R be a congruence rather than a reflexive and admissible relation. This shows that many (but not all) relation identities become trivial when relations are replaced by congruences. In particular, considering relation identities for their own sake seems to give a new perspective to the subject.

Motivated by the above considerations and, in particular, by the characterization (2) of congruence modularity, we looked for a characterization of congruence distributivity by means of some expression of the form $X(Y \circ Y) \subseteq$ *something*, where in the factor on the left-hand side we are taking the composition of some Y *with itself*. At the beginning this looked only like an odd curiosity; however, after a while, a rather clear and, at least in the author's opinion, interesting picture emerged.

We found that congruence distributivity is equivalent to equation (2), for some h, provided that R there is taken to be the set-theoretical union of some family of admissible relations, rather than an admissible relation. Equivalently, we can take R there to be the set-theoretical union of two congruences. Similar ideas are not completely new. Unions of congruences have been used in a proof of Jónsson's characterization of congruence distributivity, as presented in McKenzie, McNulty and Taylor [14, Theorem 1.144]. Unions of congruences have been used also in Kaarli and Pixley [7, Lemma 1.1.12], giving a proof that every algebra with a compatible near unanimity term is congruence distributive. The proof is credited to E. Fried.

More characterizations of congruence distributivity are possible in terms of unions of admissible relations, *U-admissible* relations, for short. For example, a variety $\mathcal{V}$ is congruence distributive if and only if $\mathcal{V}$ satisfies $(\Theta(\sigma \circ \sigma))^{\circledast} = (\Theta\sigma)^{\circledast}$. Here $^{\circledast}$ denotes transitive closure; σ, τ and υ shall be used as variables for U-admissible relations. Further characterizations shall be presented in Corollary 2.7. In most cases, σ, τ and υ can be equivalently interpreted in other ways, for example, as unions of two congruences. In each result we shall explicitly specify the possibilities for σ, τ and υ. In any case, both in the present discussion and in the rest of the paper, the reader might always assume that σ, τ and υ are U-admissible relations.

The question naturally arises whether in the above formulae the tolerance Θ, too, can be equivalently taken to be a U-admissible relation, still getting a condition equivalent to congruence distributivity. This seems to be the main problem left open by the present note. On the positive side, we show that Θ can be taken to be a U-admissible relation in some special cases, i.e., in varieties with a majority term and in a 4-distributive variety introduced by Baker [1]. This leads to some new characterizations of arithmetical varieties and of varieties with a majority term.

Dealing with general results, we show, among other, that a variety $\mathcal{V}$ satisfies $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ_h \sigma\upsilon$, for some h, if and only if $\mathcal{V}$ satisfies $\sigma^{\circledast}\tau^{\circledast} = (\sigma\tau)^{\circledast}$. When expressed in terms of tolerances, the latter identity is equivalent to congruence modularity and has found many applications. See, e.g., Czédli, Horváth and Lipparini [3]. We also show that we get equivalent conditions if we consider the identity $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ_h \sigma\upsilon$ for U-admissible relations and for admissible relations. Here the value of h remains the same in both cases. This is quite surprising; in this paper we shall find many equivalences of this kind, starting from Theorem 2.1 below, but usually the parameters are not constant.

Finally, for any fixed h, we shall find a characterization of the identity $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$. A variety satisfies such an identity for U-admissible relations if and only if there is some function $f : \{0, 1, \dots, h-1\} \to \{1, 2\}$ such that $\mathcal{V}$ satisfies the identity $\alpha(R_1 \circ R_2) \subseteq \alpha R_{f(0)} \circ \alpha R_{f(1)} \circ \dots \circ \alpha R_{f(h-1)}$

for admissible relations. This shows that the identity $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$ is equivalent to a finite union of strong Maltsev classes. It suggests that the study of relation identities satisfied in congruence distributive varieties might provide a finer classification of such varieties, in comparison with the study of congruence identities alone.

In conclusion, the equivalences we have found and the connections with other kinds of identities suggest that the study of U-admissible relations has an intrinsic interest and can be pursued further in the case of congruence distributive varieties and probably even in more general contexts.

2. Congruence distributivity is equivalent to the identity $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$, for some h

Recall that a binary relation σ on some algebra $\mathbf{A}$ is *U-admissible* if σ can be expressed as the set-theoretical union of some nonempty set of reflexive and admissible relations on $\mathbf{A}$ (we are not assuming σ itself to be admissible!). A relation is *U_2-admissible* if it can be expressed as the union of two reflexive and admissible relations. Let $\mathbf{F}_{\mathcal{V}}(3)$ denote the free algebra in $\mathcal{V}$ generated by 3 elements.

Theorem 2.1. *For every variety $\mathcal{V}$, the following conditions are equivalent. Each condition holds for $\mathcal{V}$ if and only if it holds for $\mathbf{F}_{\mathcal{V}}(3)$.*

(1) $\mathcal{V}$ is congruence distributive;

(2) for some h, $\mathcal{V}$ satisfies the identity $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$, for every tolerance Θ and every U-admissible relation σ.

(3) for some k, $\mathcal{V}$ satisfies the identity $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ k}$, for every congruence α and every binary relation σ expressible as the set-theoretical union of two congruences.

Proof. For $n = 1, 2, 3$, let $(n)_{\mathcal{V}}$ denote the respective condition supposed to hold for $\mathcal{V}$. Let $(n)_{\mathbf{F}_{\mathcal{V}}}$ denote the condition supposed to hold just for $\mathbf{F}_{\mathcal{V}}(3)$.

$(1)_{\mathcal{V}} \Rightarrow (2)_{\mathcal{V}}$ will follow from Proposition 2.4 below.

$(2) \Rightarrow (3)$ is trivial for both indices.

$(3)_{\mathbf{F}_{\mathcal{V}}} \Rightarrow (1)_{\mathcal{V}}$ will follow from Lemma 2.2 below.

Since $(n)_{\mathcal{V}} \Rightarrow (n)_{\mathbf{F}_{\mathcal{V}}}$ trivially, for $n = 1, 2, 3$, and $(1)_{\mathbf{F}_{\mathcal{V}}} \Rightarrow (1)_{\mathcal{V}}$ by Jónsson [5], we get that all the conditions are equivalent. □

To prove the implication (3) ⇒ (1) in Theorem 2.1 we shall use a classical result by Jónsson [5]. Recall that *Jónsson terms* are terms $j_0, \dots, j_k$, for $k \geq 1$, satisfying

$$x = j_0(x, y, z), \qquad j_k(x, y, z) = z \tag{J1}$$

$$x = j_i(x, y, x), \qquad \text{for } 0 \leq i \leq k, \tag{J2}$$

$$\begin{aligned} j_i(x, x, z) &= j_{i+1}(x, x, z), && \text{for even } i,\ 0 \leq i < k, \\ j_i(x, z, z) &= j_{i+1}(x, z, z), && \text{for odd } i,\ 0 \leq i < k, \end{aligned} \tag{J3}$$

Jónsson proved that a variety $\mathcal{V}$ is congruence distributive if and only if there is some k such that $\mathcal{V}$ has Jónsson terms $j_0, j_1, \dots j_k$. If this is the case, $\mathcal{V}$ is said to be *k-distributive* or to be Δ_k.

Lemma 2.2. *If a variety $\mathcal{V}$ (equivalently, $\mathbf{F}_{\mathcal{V}}(3)$) satisfies condition (3) in Theorem 2.1 for some specified k, then $\mathcal{V}$ is $(k+1)$-distributive.*

Proof. Let α, β, γ be congruences and $\sigma = \beta \cup \gamma$. From condition (3) we get

$$\alpha(\beta \circ \gamma) \subseteq \alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ k} = (\alpha\beta \cup \alpha\gamma)^{\circ k} \subseteq \\ (\alpha\beta \circ \alpha\gamma) \circ_k (\alpha\gamma \circ \alpha\beta) = \alpha\beta \circ_{k+1} \alpha\gamma, \quad \text{(C)}$$

since $\alpha\gamma$ and $\alpha\beta$ are congruences, hence transitive. It is a standard and well-known fact already implicit in [5] that, within a variety (equivalently, in $\mathbf{F}_{\mathcal{V}}(3)$), the identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_{k+1} \alpha\gamma$ is equivalent to $(k+1)$-distributivity. See, e.g., [11] for full details. Thus the lemma is proved. $\square$

In fact, slightly more can be proved.

Lemma 2.3. *If a variety $\mathcal{V}$ satisfies condition (3) in Theorem 2.1 for some k, then $\mathcal{V}$ satisfies either $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$ or $\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ_k \alpha\beta$ for congruences.*

Proof. Let us work in $\mathbf{F}_{\mathcal{V}}(3)$ with generators x, y and z. Let $\alpha = Cg(x, z)$, $\beta = Cg(x, y)$ and $\gamma = Cg(y, z)$, hence $(x, z) \in \alpha(\beta \circ \gamma)$. Since

$$(\alpha\beta \cup \alpha\gamma)^{\circ k} \subseteq (\alpha\beta \circ_k \alpha\gamma) \cup (\alpha\gamma \circ_k \alpha\beta),$$

then from the first line in equation (C) in the proof of 2.2 above, we get that either $(x, z) \in \alpha\beta \circ_k \alpha\gamma$ or $(x, z) \in \alpha\gamma \circ_k \alpha\beta$. By the standard homomorphism argument, we have that, correspondingly, either $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$ or $\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ_k \alpha\beta$ hold for arbitrary congruences of algebras in $\mathcal{V}$. $\square$

Compare Lemma 2.3 with Theorem 4.1 and Remark 4.2.

In order to prove (1) $\Rightarrow$ (2) in Theorem 2.1 we shall rely heavily on a recent result by Kazda, Kozik, McKenzie and Moore [8]. *Directed Jónsson terms*, or *Zádori terms* [8, 17] are obtained from Jónsson conditions (J1)-(J3) above by replacing condition (J3) with

$$j_i(x, z, z) = j_{i+1}(x, x, z), \qquad \text{for} \ \ 0 \leq i < k. \tag{D}$$

To the best of our knowledge, directed Jónsson terms first appeared unnamed in Zádori [17, Theorem 4.1]. Kazda, Kozik, McKenzie and Moore [8] proved that a variety $\mathcal{V}$ has directed Jónsson terms for some k if and only if $\mathcal{V}$ has Jónsson terms for some k'. In particular, they obtained that a variety $\mathcal{V}$ is congruence distributive if and only if $\mathcal{V}$ has directed Jónsson terms for some k. Notice that here we are using a slightly different indexing of the terms, in comparison with [8]. The argument below showing that we can switch from congruences to tolerances is due to Czédli and Horváth [2].

Proposition 2.4. *If $\mathcal{V}$ is congruence distributive with directed Jónsson terms $d_0, d_1, \dots, d_n$, then $\mathcal{V}$ satisfies condition (2) in Theorem 2.1 with $h = 2n-2$.*

Proof. Fix some algebra $\mathbf{A}$ in $\mathcal{V}$ and suppose that Θ is a tolerance on $\mathbf{A}$ and σ is a union of reflexive and admissible relations on $\mathbf{A}$. If $(a, c) \in \Theta(\sigma \circ \sigma)$, then $a \mathrel{\Theta} c$ and there is b such that $a \mathrel{\sigma} b \mathrel{\sigma} c$. By using directed Jónsson terms, we get

$$a = d_0(a, c, c) = d_1(a, a, c) \mathrel{\sigma} d_1(a, b, c) \mathrel{\sigma} d_1(a, c, c) =$$
$$d_2(a, a, c) \mathrel{\sigma} d_2(a, b, c) \mathrel{\sigma} \dots \mathrel{\sigma} d_{n-1}(a, b, c) \mathrel{\sigma} d_{n-1}(a, c, c) = d_n(a, a, c) = c.$$

Indeed, by assumption, $\sigma = \bigcup_{g \in G} \sigma_g$, for some family $\{\sigma_g \mid g \in G\}$ of reflexive and admissible relations. Since $a \mathrel{\sigma} b$, then $a \mathrel{\sigma_g} b$, for some $g \in G$, hence $d_i(a, a, c) \mathrel{\sigma_g} d_i(a, b, c)$, for every i, since σ_g is reflexive and admissible. Thus we get $d_i(a, a, c) \mathrel{\sigma} d_i(a, b, c)$ from $\sigma = \bigcup_{g \in G} \sigma_g$. Similarly, $d_i(a, b, c) \mathrel{\sigma} d_i(a, c, c)$. The point is that when we go, say, from $d_i(a, a, c)$ to $d_i(a, b, c)$, only *one* element is moved. In this case, the a in the second position is changed to b and all the other arguments are left unchanged. Moreover,

$$d_i(a, a, c) = d_i(d_i(a, b, a), a, d_i(c, b, c)) \mathrel{\Theta} d_i(d_i(a, b, c), a, d_i(a, b, c)) = d_i(a, b, c)$$

and similarly $d_i(a, b, c) \mathrel{\Theta} d_i(a, c, c)$. This is essentially an argument from [2]. □

By iterating the formula $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circ h}$, we get, say,

$$\Theta(\sigma \circ \sigma \circ \sigma \circ \sigma) \subseteq (\Theta(\sigma \circ \sigma))^{\circ h} \subseteq (\Theta\sigma)^{h^2},$$

where we have used the fact that if σ and τ are U-admissible, then $\sigma \circ \tau$ is U-admissible, in particular, $\sigma \circ \sigma$ is U-admissible. Indeed, if $\sigma = \bigcup_{g \in G} \sigma_g$ and $\tau = \bigcup_{f \in F} \tau_f$, then $\sigma \circ \tau = \bigcup_{g \in G, f \in F} (\sigma_g \circ \tau_f)$. However, we get better bounds by adapting the proof of Proposition 2.4, as we are going to show.

Proposition 2.5. *Suppose that $\mathcal{V}$ is a congruence distributive variety with directed Jónsson terms $d_0, d_1, \dots, d_n$.*

(1) For every m, $\mathcal{V}$ satisfies the identity $\Theta\sigma^{\circ m} \subseteq (\Theta\sigma)^{\circ(mn-m)}$.

(2) More generally, for m even, $\mathcal{V}$ satisfies $\Theta(\sigma \circ_m \tau) \subseteq \Theta\sigma \circ_{mn-m} \Theta\tau$.

In the above identities, Θ is a tolerance and σ is a U-admissible relation.

Proof. The proof is similar to the proof of Proposition 2.4. We shall prove (2). When m is even, (1) is the special case $\sigma = \tau$ of (2). When m is odd, (1) allows a similar proof.

Suppose that $(a, c) \in \Theta(\sigma \circ_m \tau)$. Then $a \mathrel{\Theta} c$ and $a = b_0 \mathrel{\sigma} b_1 \mathrel{\tau} b_2 \mathrel{\sigma} b_3 \mathrel{\tau} \dots \mathrel{\sigma} b_{m-1} \mathrel{\tau} b_m = c$, for certain $b_0, b_1, \dots$ We have $b_{m-1} \mathrel{\tau} b_m$ since m is even. For $\ell = 1, \dots, n-1$, arguing as in the proof of Proposition 2.4, we get

$$d_\ell(a, a, c) = d_\ell(a, b_0, c) \mathrel{\sigma} d_\ell(a, b_1, c) \mathrel{\tau} d_\ell(a, b_2, c) \mathrel{\sigma} \dots$$
$$\mathrel{\tau} d_\ell(a, b_m, c) = d_\ell(a, c, c) = d_{\ell+1}(a, a, c).$$

All the elements in the above chain are Θ-related, since

$$d_\ell(a, b_h, c) = d_\ell(d_\ell(a, b_k, a), b_h, d_\ell(c, b_k, c))\ \Theta\ d_\ell(d_\ell(a, b_k, c), b_h, d_\ell(a, b_k, c)) = d_\ell(a, b_k, c),$$

for all indices h, k. This shows that $(d_\ell(a, a, c), d_{\ell+1}(a, a, c)) \in \Theta\sigma \circ_m \Theta\tau$, for $\ell = 1, \dots, n-1$. Since $a = d_0(a, c, c) = d_1(a, a, c)$ and $d_n(a, a, c) = c$, then, by putting everything together, we get a chain of length $m(n-1)$ from a to c. Notice that the factors of the form $\Theta\sigma$ and $\Theta\tau$ do always alternate, since m is assumed to be even. Indeed, $d_\ell(a, b_{m-1}, c)\ \tau\ d_\ell(a, c, c) = d_{\ell+1}(a, a, c)\ \sigma\ d_{\ell+1}(a, b_1, c)$. ☐

If we restrict ourselves to relations σ which are unions of tolerances, the above arguments can be carried over just using Jónsson terms. Let $S\ {}_m\!\circ\ T$ denote $\dots T \circ S \circ T$ with m factors. Formally, $S\ {}_m\!\circ\ T = S \circ_m T$ if m is even and $S\ {}_m\!\circ\ T = T \circ_m S$ if m is odd. Let $\smile$ denotes *converse*.

Proposition 2.6. *If $\mathcal{V}$ is a k-distributive variety, then the following statements hold.*

(1) For every m, $\mathcal{V}$ satisfies the identity $\Theta\sigma^{\circ m} \subseteq (\Theta\sigma)^{\circ(mk-m)}$, for every tolerance Θ and every relation σ which is a union of tolerances.

(2) For every m, $\mathcal{V}$ satisfies

$$\Theta(\sigma \circ_m \tau) \subseteq (\Theta\sigma \circ_m \Theta\tau) \circ_{k-1} (\Theta\tau^\smile\ {}_m\!\circ\ \Theta\sigma^\smile),$$

for every tolerance Θ and all U-admissible relations σ and τ.

In turn, if a variety satisfies (2) above in the particular case $m = 2$, then $\mathcal{V}$ is k-distributive.

Proof. (1) is the particular case of (2) when $\sigma = \tau$, since if σ is a union of tolerances, then $\sigma = \sigma^\smile$. Hence it is enough to prove (2).

Let $j_0, \dots, j_k$ be Jónsson terms. Under the same assumptions as in the proof of 2.5, we get $(j_\ell(a, a, c), j_\ell(a, c, c)) \in \Theta\sigma \circ_m \Theta\tau$, for $\ell = 1, \dots, k-1$ and, taking converses, $(j_\ell(a, c, c), j_\ell(a, a, c)) \in \Theta\tau^\smile\ {}_m\!\circ\ \Theta\sigma^\smile$. In the present situation we have $j_\ell(a, a, c) = j_{\ell+1}(a, a, c)$, for ℓ even, and $j_\ell(a, c, c) = j_{\ell+1}(a, c, c)$, for ℓ odd. Hence, in order to join the partial chains, we have to consider $\Theta\sigma \circ_m \Theta\tau$ and $\Theta\tau^\smile\ {}_m\!\circ\ \Theta\sigma^\smile$, alternatively.

In order to prove the last sentence, take $\alpha = \Theta$, $\beta = \sigma$ and $\gamma = \tau$ congruences. In the displayed formula in (2) we then get adjacent occurrences of $\alpha\gamma$ and $\alpha\beta$ which pairwise absorb. We end up with $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_k \alpha\gamma$, an identity equivalent to k-distributivity. ☐

Recall that $^{\circledast}$ denote transitive closure.

Corollary 2.7. *Within a variety $\mathcal{V}$, each of the following identities is equivalent to congruence distributivity. Each identity holds in $\mathcal{V}$ if and only if it holds in $\mathbf{F}_{\mathcal{V}}(3)$.*

(1) $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma)^{\circledast}$, equivalently, $(\Theta(\sigma \circ \sigma))^{\circledast} = (\Theta\sigma)^{\circledast}$;

(1′) $\Theta(\sigma \circ \sigma) \subseteq (\Theta\sigma \circ \Theta\sigma^{\smile})^{\circledast}$;
(1″) $\Theta(\sigma \circ \sigma^{\smile}) \subseteq (\Theta\sigma \circ \Theta\sigma^{\smile})^{\circledast}$, *equivalently,* $(\Theta(\sigma \circ \sigma^{\smile}))^{\circledast} = (\Theta\sigma \circ \Theta\sigma^{\smile})^{\circledast}$;
(2) $\Theta\sigma^{\circledast} \subseteq (\Theta\sigma)^{\circledast}$, *equivalently,* $(\Theta\sigma^{\circledast})^{\circledast} = (\Theta\sigma)^{\circledast}$;
(3) $\Theta(\sigma \circ \tau) \subseteq (\Theta\sigma \circ \Theta\tau)^{\circledast}$, *equivalently,* $(\Theta(\sigma \circ \tau))^{\circledast} = (\Theta\sigma \circ \Theta\tau)^{\circledast}$;
(4) $\Theta(\sigma \circ \tau)^{\circledast} \subseteq (\Theta\sigma \circ \Theta\tau)^{\circledast}$, *equivalently,* $(\Theta(\sigma \circ \tau)^{\circledast})^{\circledast} = (\Theta\sigma \circ \Theta\tau)^{\circledast}$;
(4′) $\Theta(\sigma \circ \tau)^{\circledast} \subseteq (\Theta\sigma \circ \Theta\tau \circ \Theta\sigma^{\smile} \circ \Theta\tau^{\smile})^{\circledast}$, *equivalently,*
$(\Theta(\sigma \circ \tau \circ \sigma^{\smile} \circ \tau^{\smile})^{\circledast})^{\circledast} = (\Theta\sigma \circ \Theta\tau \circ \Theta\sigma^{\smile} \circ \Theta\tau^{\smile})^{\circledast}$.

In the above identities, Θ is a tolerance, equivalently, a congruence, and σ and τ are U-admissible relations, equivalently, relations which are expressible as the union of two congruences.

Proof. For each of the above conditions, let the *strong form* be the one in which Θ is supposed to be a tolerance and σ, τ are U-admissible relations. Let the *weak form* be the one in which Θ is supposed to be a congruence and σ and τ are relations which are the union of two congruences. Clearly, in each case, the strong form implies the weak form.

Inside each item, in each case the conditions are equivalent, since σ and τ are reflexive and since transitive closure is a monotone and idempotent operator. A further comment is necessary for (4′). To get the last condition, first apply the first condition with $\sigma \circ \tau$ in place of σ and with $\sigma^{\smile} \circ \tau^{\smile}$ in place of τ, then apply again the first condition.

Congruence distributivity implies the strong form of (4), as a consequence of Proposition 2.5(2). Trivially, (4) $\Rightarrow$ (4′), (4) $\Rightarrow$ (1″), (4) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (1′) and (4) $\Rightarrow$ (2) $\Rightarrow$ (1), either in the strong or in the weak case. Moreover, the weak forms of (4) and (4′) are equivalent, since if σ is a union of congruences, then $\sigma = \sigma^{\smile}$ and the same for τ. Similarly, the weak forms of (1), (1′) and (1″) are equivalent.

Hence, in order to close the cycles of implications, it is enough to prove that the weak form of (1) implies congruence distributivity. As in the proof of Lemma 2.2, take $\Theta = \alpha$ and $\sigma = \beta \cup \gamma$, with α, β and γ congruences. Then the weak form of (1) implies

$$\alpha(\beta \circ \gamma) \subseteq \alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circledast} = (\alpha\beta \cup \alpha\gamma)^{\circledast} = \alpha\beta + \alpha\gamma.$$

This condition (even only in $\mathbf{F}_{\mathcal{V}}(3)$) implies congruence distributivity by Jónsson [5]. □

By [8], Proposition 2.5 and the above proof we have that a variety $\mathcal{V}$ is congruence distributive if and only if, for some (equivalently every) $m \geq 2$, $\mathcal{V}$ satisfies the identity $\Theta(\sigma \circ_m \sigma) \subseteq (\Theta\sigma)^{\circledast}$. A similar remark applies to all the conditions in Corollary 2.7, except for (2).

3. Working with U-admissible relations only and a major problem

Notice that we do need to consider the possibility for σ to be a U-admissible (not just admissible) relation in Theorem 2.1 and in Corollary 2.7 (1)-(2), in order to obtain conditions equivalent to congruence distributivity. Of course, if an identity holds for U-admissible relations, then it holds for admissible relations, too, hence both versions of the identities we have introduced hold in congruence distributive varieties. However, when expressed in terms of admissible relations only, the identities might turn out to be too weak to imply back congruence distributivity. Indeed, as we mentioned in the introduction, $R \circ R = R$ holds in congruence permutable varieties, for every reflexive and admissible relation R. A fortiori, $\Theta(R \circ R) = \Theta R \subseteq (\Theta R)^{\circledast}$ hold, but there are congruence permutable varieties which are not congruence distributive. Hence the identities in 2.1(2)(3) and 2.7 (1)(1′)(2) do not imply congruence distributivity, if σ is taken to be an admissible relation. In congruence permutable varieties $R^{\smile} = R$ holds, too, again by Werner [16]. This shows that the identity in 2.7(1″) considered only for admissible relations fails to imply congruence distributivity, as well.

On the other hand, we do not know the exact possibilities for Θ in the identities we have discussed.

Problem 3.1. (a) Do we get conditions equivalent to congruence distributivity if we take a reflexive and admissible relation T (or even a U-admissible relation υ) in place of Θ in Theorem 2.1 and Corollary 2.7?

(b) Similarly, do we get a condition equivalent to congruence modularity if we take a reflexive and admissible relation T in place of Θ in equation (2) from the introduction?

The results of the present section show that in many cases the answer to Question (a) above is the same when asked for admissible relations or for U-admissible relations. In any case, the next three propositions provide examples of varieties in which corresponding identities hold true. In passing, we also get some further characterizations of varieties with a majority term and of arithmetical varieties.

Proposition 3.2. *For every variety $\mathcal{V}$, the following conditions are equivalent.*

(1) $\mathcal{V}$ has a majority term.

(2) $\mathcal{V}$ satisfies $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma$ for congruences.

(3) $\mathcal{V}$ satisfies $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ \sigma\upsilon$ for U-admissible relations, equivalently, for reflexive and admissible relations, equivalently, for tolerances.

Proof. The equivalence of (1) and (2) is standard and well-known. If $\mathcal{V}$ satisfies (3) for U-admissible relations, then $\mathcal{V}$ obviously satisfies (3) for admissible relations and for tolerances. In all cases this implies (2). In order to prove that (1) implies the strongest form of (3), let m be a majority term. If $(a, c) \in \sigma(\tau \circ \upsilon)$ with $a \ \sigma \ c$ and $a \ \tau \ b \ \upsilon \ c$, then $a = m(a, a, c) \ \tau \ m(a, b, c)$ and $a = m(a, b, a) \ \sigma \ m(a, b, c)$. Hence $a \ \sigma\tau \ m(a, b, c)$. Here, as in the proof of Proposition 2.4, we are using the fact that only one element is moved at a time. Symmetrically, $m(a, b, c) \ \sigma\upsilon \ c$. Hence the element $m(a, b, c)$ witnesses $(a, c) \in \sigma\tau \circ \sigma\upsilon$. □

We are now going to prove the result analogous to 3.2 for arithmetical varieties, but first a remark is in order.

Remark 3.3. Clearly, we can have equality in place of inclusion in 3.2(2); actually, *as far as* σ *is transitive*, we always have $\sigma(\tau \circ \upsilon) \supseteq \sigma\tau \circ \sigma\upsilon$. On the other hand, for U-admissible relations, the identity $\sigma(\tau \circ \upsilon) = \sigma\tau \circ \sigma\upsilon$ holds only in the trivial variety, as we are going to show. First observe that, by taking $\tau = \upsilon = 1$, we get $\sigma = \sigma \circ \sigma$.

We now show that, for U-admissible relations, $\sigma = \sigma \circ \sigma$ holds only in the trivial variety. Let $\mathbf{A}$ have at least two elements, consider the algebra $\mathbf{B} = \mathbf{A} \times \mathbf{A}$ and let σ be the union of the kernels of the two projections. Then $\sigma \neq 1_{\mathbf{B}} = \sigma \circ \sigma$. Actually, we have shown that $\sigma = \sigma \circ \sigma$ fails in any non-trivial variety already for σ a union of two congruences.

Turning to *admissible* relations, we are going to show that we can actually have $T(R \circ S) = TR \circ TS$. Curiously, this identity is stronger than having a majority term; indeed it is equivalent to arithmeticity. On the other hand, the inclusion $T(R \circ S) \subseteq TR \circ TS$ is equivalent to having a majority term, by 3.2(3).

Proposition 3.4. *For every variety* $\mathcal{V}$*, the following conditions are equivalent.*

(1) $\mathcal{V}$ *is arithmetical, that is,* $\mathcal{V}$ *has a Pixley term.*
(2) $\mathcal{V}$ *satisfies* $\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ \alpha\beta$ *for congruences.*
(3) $\mathcal{V}$ *satisfies* $\sigma(\tau \circ \upsilon) \subseteq \sigma\upsilon \circ \sigma\tau$ *for U-admissible relations, equivalently, for reflexive and admissible relations, equivalently, for tolerances.*
(4) $\mathcal{V}$ *satisfies* $T(R \circ S) = TR \circ TS$*, for reflexive and admissible relations, equivalently, for tolerances (notice that in the present identity we use equality and* R *and* S *are not shifted).*

Proof. Again, (1) $\Leftrightarrow$ (2) is well-known and standard. In each case, (3) $\Rightarrow$ (2).

In order to show that (1) implies (3) an additional argument is needed, in comparison with 3.2. Since (1) implies congruence permutability, we have $R = R^{\smile}$, for admissible relations, by [16]. This implies $\sigma = \sigma^{\smile}$, for *U*-admissible relations. Then we can proceed as usual employing the Pixley term p. If $a \ \sigma \ c$ and $a \ \tau \ b \ \upsilon \ c$, then, for example $a = p(a, c, c) \ \upsilon^{\smile} \ p(a, b, c)$. Since we have just proved that $\upsilon^{\smile} = \upsilon$, then $a \ \upsilon \ p(a, b, c)$. With similar computations we show that $a \ \sigma\upsilon \ p(a, b, c) \ \sigma\tau \ c$.

Finally, we show that arithmeticity is equivalent to (4). We shall use the fact that arithmeticity is equivalent to the existence of a majority term together with congruence permutabilty. In congruence permutable varieties any reflexive and admissible relation is a congruence, hence (4) follows from arithmeticity by 3.2(2) and the first sentence in Remark 3.3. Conversely, if (4) holds, then taking $R = S = 1$ we get $T = T \circ T$ and this identity implies congruence permutability by [16]. Trivially (4) implies 3.2(2), hence we get a majority term. □

Proposition 3.5. *The identity $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ \sigma\upsilon \circ \sigma\tau \circ \sigma\upsilon$ holds for U-admissible relations in the variety introduced by Baker* [1]*, the variety generated by polynomial reducts of lattices in which the ternary operation $f(a,b,c) = a \wedge (b \vee c)$ is the only fundamental operation.*

Proof. If, as usual, $(a,c) \in \sigma(\tau \circ \upsilon)$ with $a \; \tau \; b \; \upsilon \; c$, then the elements $f(a,b,c)$, $a \wedge c = f(a,c,c)$ and $f(c,b,a)$ witness $(a,c) \in \sigma\tau \circ \sigma\upsilon \circ \sigma\tau \circ \sigma\upsilon$. For example,

$$f(a,b,c) = f(f(a,b,c), f(a,b,c), a) \; \sigma \; f(f(c,b,c), f(a,b,c), a) = a \wedge c. \quad \square$$

Of course, Proposition 3.5 applies also to any expansion of Baker's variety.

We now show that equivalences similar to those in Propositions 3.2 and 3.4 hold in a more general context. In particular, we shall prove the quite surprising fact that the identity $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ_h \sigma\upsilon$, holds for U-admissible relations if and only if it holds for admissible relations, where h is the same in both cases.

Theorem 3.6. *Within a variety, the following identities are equivalent. Each identity holds in $\mathcal{V}$ if and only if it holds in $\mathbf{F}_{\mathcal{V}}(3)$.*

(1) $\sigma(\tau \circ \upsilon) \subseteq (\sigma\tau \circ \sigma\upsilon)^{\circledast}$, equivalently, $(\sigma(\tau \circ \upsilon))^{\circledast} = (\sigma\tau \circ \sigma\upsilon)^{\circledast}$;
(2) $\sigma(\tau \circ \tau) \subseteq (\sigma\tau)^{\circledast}$, equivalently, $(\sigma(\tau \circ \tau))^{\circledast} = (\sigma\tau)^{\circledast}$;
(3) $\sigma^{\circledast}\tau^{\circledast} = (\sigma\tau)^{\circledast}$, equivalently, $(\sigma^{\circledast}\tau^{\circledast})^{\circledast} = (\sigma\tau)^{\circledast}$.

In the above identities, σ, τ and υ are U-admissible relations, equivalently, U_2-admissible relations. Limited to item (1), we can equivalently take them to be reflexive and admissible relations.

In order to prove Theorem 3.6 we first need a Maltsev characterization of condition (1). In the formulae below we sometimes use a semicolon in place of a comma in order to improve readability.

Proposition 3.7. *Let σ, τ, υ denote U-admissible relations, equivalently, reflexive and admissible relations. A variety $\mathcal{V}$ satisfies $\sigma(\tau \circ \upsilon) \subseteq \sigma\tau \circ_h \sigma\upsilon$ if and only if $\mathcal{V}$ has ternary terms t_i, $i = 0, \dots, h$, and 4-ary terms s_i and u_i, $i = 0, \dots, h-1$, such that the following identities hold throughout $\mathcal{V}$.*

$$\begin{aligned}
x &= t_0(x,y,z), & t_h(x,y,z) &= z, & \\
t_i(x,y,z) &= u_i(x,y,z;x), & u_i(x,y,z;z) &= t_{i+1}(x,y,z), & &\text{for } i < h, \\
t_i(x,y,z) &= s_i(x,y,z;x), & s_i(x,y,z;y) &= t_{i+1}(x,y,z), & &\text{for even } i < h, \\
t_i(x,y,z) &= s_i(x,y,z;y), & s_i(x,y,z;z) &= t_{i+1}(x,y,z), & &\text{for odd } i < h.
\end{aligned}$$

All the above conditions hold in $\mathcal{V}$ if and only if they hold in $\mathbf{F}_{\mathcal{V}}(3)$.

Proof. Suppose that $\sigma(\tau \circ \upsilon) \subseteq (\sigma\tau \circ_h \sigma\upsilon)$ holds in $\mathcal{V}$, or just in $\mathbf{F}_{\mathcal{V}}(3)$, in the weak form, that is for reflexive and admissible relations. Let $\mathbf{F}_{\mathcal{V}}(3)$ be generated by the elements x, y and z and let σ, τ, υ be the smallest reflexive and admissible relations containing, respectively, (x,z), (x,y), (y,z), thus $(x,z) \in \sigma(\tau \circ \upsilon)$. By the assumption, $(x,z) \in \sigma\tau \circ_h \sigma\upsilon$. Since we are in the free algebra generated by x, y, z, the above relation is witnessed by ternary terms t_i, $i = 0, \dots, h$. The elements $t_i(x, y, z)$ of $\mathbf{F}_{\mathcal{V}}(3)$ have to satisfy $x = t_0(x, y, z)$, $t_i(x, y, z)\ \sigma\tau\ t_{i+1}(x, y, z)$, for i even, $t_i(x, y, z)\ \sigma\upsilon\ t_{i+1}(x, y, z)$, for i odd, and $t_h(x, y, z) = z$. It is easy to see that in $\mathbf{F}_{\mathcal{V}}(3)$ we have $\sigma = \{(u(x, y, z; x), u(x, y, z; z)) \mid u \text{ a 4-ary term of } \mathcal{V}\}$. Hence the relation $t_i(x, y, z)\ \sigma\ t_{i+1}(x, y, z)$ implies the existence of terms u_i satisfying the desired equations. The equations hold throughout $\mathcal{V}$ since we are in the free algebra generated by 3 elements and the terms depend on 3 variables. Arguing in the same way with τ and υ, we get all the claimed equations.

Now suppose that the displayed equations hold in $\mathcal{V}$. The arguments familiar by now show that $\sigma(\tau \circ \upsilon) \subseteq (\sigma\tau \circ_h \sigma\upsilon)$ holds in the strong form in which σ, τ and υ are taken to be U-admissible relations. Indeed, let $\mathbf{A} \in \mathcal{V}$, σ, τ and υ be U-admissible relations of $\mathbf{A}$ and $(a, c) \in \sigma(\tau \circ \upsilon)$, thus $a\ \tau\ b\ \upsilon\ c$, for some b. Then we have, say, $t_i(a, b, c) = u_i(a, b, c; a)\ \sigma\ u_i(a, b, c; c) = t_{i+1}(a, b, c)$. By using the other equations in a similar way we get that the elements $t_i(a, b, c)$, $i = 0, \dots, h$ witness $(a, c) \in \sigma\tau \circ_h \sigma\upsilon$.

Since the stronger form of $\sigma(\tau \circ \upsilon) \subseteq (\sigma\tau \circ_h \sigma\upsilon)$ implies the weaker form, we have that all the conditions are equivalent. $\square$

Of course, from the equations in Proposition 3.7 the t_i's are determined either by the u_i's or by the s_i's, hence, in principle, it is not even necessary to mention the t_i's explicitly, it would be enough to reformulate the equations as follows.

$$x = u_0(x, y, z; x) = s_0(x, y, z; x), \qquad u_{h-1}(x, y, z; z) = z,$$
$$s_{h-1}(x, y, z; z) = z \text{ if } h \text{ is even}, \quad s_{h-1}(x, y, z; y) = z \text{ if } h \text{ is odd},$$
$$u_i(x, y, z; z) = u_{i+1}(x, y, z; x) = s_i(x, y, z; y) = s_{i+1}(x, y, z; y) \text{ if } i < h - 1,\ i \text{ even},$$
$$u_i(x, y, z; z) = u_{i+1}(x, y, z; x) = s_i(x, y, z; z) = s_{i+1}(x, y, z; x) \text{ if } i < h - 1,\ i \text{ odd}.$$

However, both the statement and the proof seem clearer if we mention explicitly the t_i's.

Proof of Theorem 3.6. Clearly, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) either in the weak or in the strong case. If (2) holds for U-admissible relations, then, by an easy induction, we get

$$\sigma\tau^{\circledast} \subseteq (\sigma\tau)^{\circledast}. \tag{P}$$

Applying (2) again with $\tau^{\circledast}$ in place of σ, we get

$$\tau^{\circledast}(\sigma \circ \sigma) \subseteq (\tau^{\circledast}\sigma)^{\circledast} = (\sigma\tau^{\circledast})^{\circledast} \subseteq (\sigma\tau)^{\circledast\circledast} = (\sigma\tau)^{\circledast},$$

by (P) and idempotence of $^{\circledast}$. Then another induction shows $\sigma^{\circledast}\tau^{\circledast} = \tau^{\circledast}\sigma^{\circledast} \subseteq (\sigma\tau)^{\circledast}$. The reverse inclusion is trivial. Thus we have that (2) and (3) are equivalent for U-admissible relations.

Since, by Proposition 3.7, the weaker and the stronger forms of (1) are equivalent, it is enough to prove that the version of (2) for U_2-admissible relations implies the version of (1) for reflexive and admissible relations (for this in turn implies the stronger form of (1), hence the stronger form of (2), hence (3) follows also from the version of (2) only for U_2-admissible relations).

The proof that (2) for U_2-admissible relations implies (1) for reflexive and admissible relations is similar to the proof of Lemma 2.2. Indeed, for R, S, and T reflexive and admissible relations, letting $\sigma = S \cup T$ and applying (2), we have $R(S \circ T) \subseteq R(\sigma \circ \sigma) \subseteq (R\sigma)^{\circledast} = (RS \cup RT)^{\circledast} = (RS \circ RT)^{\circledast}$. $\square$

4. A Maltsev-like characterization of $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ h}$

In this section we present a Maltsev type characterization of the identity $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ h}$, for fixed h. This is somewhat similar to Theorem 3.6 and Proposition 3.7, but qualitatively different in that, for fixed h, the identity turns out to be equivalent to a finite disjunction of identities involving reflexive and admissible relations. In other words, for the reader who knows the terminology, while the identities in Theorem 3.6 are described by Proposition 3.7 as Maltsev classes, on the other hand, for fixed h, we describe $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ h}$ as a finite union of strong Maltsev classes.

Theorem 4.1. *For every natural number $h > 0$ and every variety $\mathcal{V}$, the following conditions are equivalent. Each condition holds for $\mathcal{V}$ if and only if it holds for $\mathbf{F}_{\mathcal{V}}(3)$.*

(1) $\mathcal{V}$ satisfies the identity $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ h}$, where σ is a U-admissible relation, equivalently, a U_2-admissible relation.

(2) There is a function $f : \{0, 1, \dots, h-1\} \to \{1, 2\}$ such that one (and hence all) of the following equivalent conditions hold.

- *(a) $\mathcal{V}$ satisfies the identity $\alpha(R_1 \circ R_2) \subseteq \alpha R_{f(0)} \circ \alpha R_{f(1)} \circ \dots \circ \alpha R_{f(h-1)}$, where R_1 and R_2 are reflexive and admissible relations.*
- *(b) $\mathcal{V}$ has 4-ary terms s_i, $i = 0, \dots, h-1$ such that the following identities hold throughout $\mathcal{V}$.*

$$x = s_0(x, y, z; w_{f(0)}), \quad s_{h-1}(x, y, z; w'_{f(h-1)}) = z, \quad \text{and}$$
$$\left.\begin{aligned} s_i(x, y, z; w'_{f(i)}) &= s_{i+1}(x, y, z; w_{f(i+1)}) \\ x &= s_{i+1}(x, y, x; w'_{f(i)}) \end{aligned}\right\} \quad \text{for } i < h-1,$$

where w_1, w_2, w'_1 and w'_2 denote, respectively, the variables x, y, y and z.

In the identities in (1) and (2)(a) above we can let α be equivalently a variable for congruences or tolerances.

Proof. That in each case we get an equivalent condition letting α be a congruence or tolerance is proved using the argument from [2] recalled at the end of the proof of Proposition 2.4. We shall deal here with the simpler case when α is a congruence.

Suppose that $\mathcal{V}$ satisfies the identity $\alpha(\sigma \circ \sigma) \subseteq (\alpha\sigma)^{\circ h}$, where σ is a U_2-admissible relation. Let us work in the free algebra $\mathbf{F}_{\mathcal{V}}(3)$ in $\mathcal{V}$ generated by the three elements x, y and z. Let α be the congruence generated by (x,z) and let R_1 and R_2 be, respectively, the smallest reflexive and admissible relations containing (x,y), respectively, (y,z). Let $\sigma = R_1 \cup R_2$. We have that $(x,z) \in \alpha(\sigma \circ \sigma)$, hence, by (1), $(x,z) \in (\alpha\sigma)^{\circ h}$. Hence there are ternary terms $t_0, t_1, \dots, t_h$ such that $x = t_0(x,y,z)$, $t_i(x,y,z)\ \alpha\sigma\ t_{i+1}(x,y,z)$, for $i = 0, \dots, h-1$ and $t_h(x,y,z) = z$. The usual arguments show that the identities $t_i(x,y,x) = x$ should hold throughout $\mathcal{V}$. Moreover, since $\sigma = R_1 \cup R_2$, then, for every $i = 0, \dots, h-1$, either $t_i(x,y,z)\ R_1\ t_{i+1}(x,y,z)$ or $t_i(x,y,z)\ R_2\ t_{i+1}(x,y,z)$. Let f be such that $t_i(x,y,z)\ R_{f(i)}\ t_{i+1}(x,y,z)$. Arguing as in the proof of Proposition 3.7, there are 4-ary terms s_i such that $t_i(x,y,z) = s_i(x,y,z;w_{f(i)})$ and $s_i(x,y,z;w'_{f(i)}) = t_{i+1}(x,y,z)$, for $i = 0, \dots, h-1$ and where w_1, w_2, w'_1 and w'_2 denote, respectively, the variables x, y, y and z. Eliminating the t_i's from the equations we have obtained, we get exactly the identities in (2)(b).

We have proved that if (1) holds for U_2-admissible relations, then there exists some function f such that (2)(b) holds.

It is by now standard to show that, given some function f, (2)(a) and (2)(b) are equivalent. Obviously, if (a) holds, then (a) holds in $\mathbf{F}_{\mathcal{V}}(3)$; moreover, the argument above can be easily reformulated in order to show that if (a) holds in $\mathbf{F}_{\mathcal{V}}(3)$, then (b) holds. On the other hand, the proof that (b) implies (a) is similar to the final part of the proof of Proposition 3.7. If $a\ \alpha\ c$ and $a\ R_1\ b\ R_2\ c$, then $(a,c) \in \alpha R_{f(0)} \circ \alpha R_{f(1)} \circ \dots \circ \alpha R_{f(h-1)}$ is witnessed by the elements $s_i(a,b,c;d_{f(i)})$, for $i < h-1$, where $d_1 = b$ and $d_2 = c$. Indeed, if, say, $f(i) = f(i+1) = 1$, then $s_i(a,b,c;b) = s_{i+1}(a,b,c;a)\ R_1\ s_{i+1}(a,b,c;b)$ and $s_i(a,b,c;a)\ \alpha\ s_i(a,b,a;a) = a = s_{i+1}(a,b,a;a)\ \alpha\ s_{i+1}(a,b,c;a)$.

It is now easy to show that (2)(a), for some f, implies the stronger form of (1) for U-admissible relations. Indeed, if $\sigma = \bigcup_{g \in G} R_g$, $a\ \alpha\ c$ and $a\ \sigma\ b\ \sigma\ c$, then $a\ R_{g_1}\ b\ R_{g_2}\ c$, for some $g_1, g_2 \in G$, hence (2)(a) implies $(a,c) \in \alpha R_{g_{f(0)}} \circ \alpha R_{g_{f(1)}} \circ \dots \circ \alpha R_{g_{f(h-1)}} \subseteq (\alpha\sigma)^{\circ h}$. ☐

Remark 4.2. Let $(\mathrm{a})_f$ denote the identity given by (a) in Theorem 4.1 when applied to some specific function f. It is probably an interesting problem to determine which implications hold among identities of the form $(\mathrm{a})_f$ and $(\mathrm{a})_{f'}$, letting f and f' vary, possibly with $h \neq h'$. Of course, if k is the number of "variations" in the sequence $f(0), f(1), \dots, f(h-1)$, then, letting R_1 and R_2 be congruences in $(\mathrm{a})_f$, we obtain $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ_{k+1} \alpha\gamma$ if $f(0) = 1$, and $\alpha(\beta \circ \gamma) \subseteq \alpha\gamma \circ_{k+1} \alpha\beta$ if $f(0) = 2$. By [8] and the last statement in Proposition 2.5, if one of the above congruence identities holds, then $\alpha(R \circ S) \subseteq \alpha R \circ_{k'} \alpha S$ holds, for some k', but the known proofs, so far, provide a relatively large k'.

As a related observation, we know that, in general, adjacent identical relations do not absorb in relation identities. For example, it follows from equation (10) in [13] that Baker's variety (cf. Proposition 3.5) satisfies $T(R \circ S \circ R) \subseteq TR \circ TS \circ TR \circ TR \circ TS \circ TR$ but fails to satisfy $T(R \circ S \circ R) \subseteq TR \circ TS \circ TR \circ TS \circ TR$. Though there are probably similar examples in which the left-hand side has the form $T(R \circ S)$ or $\Theta(R \circ S)$, at present we have none of them at hand.

Problem 4.3. As pointed out by several authors, e.g. Jónsson [6, p. 370] or Tschantz [15], it is interesting to study reflexive and admissible relations on some algebra, as well as the identities they satisfy within a variety. Here the identities can be constructed using the operations of intersection, converse, composition and transitive closure. The results presented in this note suggest that it might be interesting to study also identities satisfied by U-admissible relations. Notice that the set of U-admissible relations on some algebra is closed under the above-mentioned operations, as well as, obviously, under set-theoretical union and under taking admissible closure. Hence it might be interesting to study the structure $\mathscr{U}(\mathbf{A}) = (U, \cap, \cup, \circ, ^{\smile}, ^{\circledast}, ^{-})$ associated to any algebra $\mathbf{A}$, where U is the set of all U-admissible relations of $\mathbf{A}$ and $^{-}$ denotes the operation of taking the smallest reflexive and admissible relation containing the argument.

Obviously, the arguments from Propositions 3.7 and Theorem 4.1 can be merged in order to obtain a characterization of the identity $\sigma(\tau \circ \tau) \subseteq (\sigma\tau)^{\circ h}$, for U-admissible relations.

More generally, parts of the arguments in the proof of Proposition 4.1 can be inserted into a broader context, related to Problem 4.3.

If $\varepsilon(\sigma_1, \sigma_2, \dots) \subseteq \varepsilon'(\sigma_1, \sigma_2, \dots)$ is an inclusion depending on the variables σ_1, σ_2, …, we say that an inclusion $\delta(R_{1,1}, R_{1,2}, \dots, R_{1,j_1}, R_{2,1}, \dots) \subseteq \delta'(R_{1,1}, R_{1,2}, \dots, R_{1,j_1}, R_{2,1}, \dots)$ is an *expansion* of $\varepsilon \subseteq \varepsilon'$ if j_1 is the number of occurrences of σ_1 in ε and similarly for the other variables, and $\delta \subseteq \delta'$ can be obtained in the following way. First, $\delta(R_{1,1}, R_{1,2}, \dots, R_{2,1} \dots)$ is obtained from $\varepsilon(\sigma_1, \sigma_2, \dots)$ by substituting each occurrence of σ_1 for a *distinct* variable from the sequence $R_{1,1}, R_{1,2}, \dots, R_{1,j_1}$ and similarly for the other variables. This first step can be performed essentially in a unique way, modulo renamings. Then we require that $\delta'(R_{1,1}, R_{1,2}, \dots, R_{2,1} \dots)$ is obtained from $\varepsilon'(\sigma_1, \sigma_2, \dots)$ by substituting every occurrence of σ_1 for some variable from the sequence $R_{1,1}, R_{1,2}, \dots, R_{1,j_1}$, in any order and with the possibility of repetitions. Similarly for σ_2 and for all the other variables. This second step can be performed in many nonequivalent ways. For example, for every f as in Condition (2) in Proposition 4.1, the identity obtained in (2)(a) is one of the many possible expansions of the identity from (1) in Proposition 4.1.

Proposition 4.4. *Suppose that $\varepsilon(\sigma_1, \sigma_2, \dots) \subseteq \varepsilon'(\sigma_1, \sigma_2, \dots)$ is an inclusion built using the operations $\cap$, $\circ$ and $^{\smile}$, where σ_1, σ_2, … are intended to be variables for U-admissible relations. Then a variety $\mathcal{V}$ satisfies $\varepsilon(\sigma_1, \sigma_2, \dots) \subseteq \varepsilon'(\sigma_1, \sigma_2, \dots)$ if and only if $\mathcal{V}$ satisfies at least one expansion $\delta(R_{1,1}, R_{1,2}, \dots, R_{2,1}, \dots) \subseteq \delta'(R_{1,1}, R_{1,2}, \dots, R_{2,1}, \dots)$ of $\varepsilon \subseteq \varepsilon'$, where the $R_{i,j}$'s are interpreted as admissible relations.*

Acknowledgement. We thank an anonymous referee for many useful comments which helped improving the paper. We thank the students of Tor Vergata University for stimulating discussions.

The author considers that it is highly inappropriate, and strongly discourages, the use of indicators extracted from the following list (even in aggregate forms in combination with similar lists) in decisions about individuals (job opportunities, career progressions etc.), attributions of funds and selections or evaluations of research projects.

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