AN INFINITE NATURAL PRODUCT

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ABSTRACT. We study an infinite countable iteration of the natural product between ordinals. We present an "effective" way to compute this countable natural product; in the non trivial cases the result depends only on the natural sum of the degrees of the factors, where the degree of a nonzero ordinal is the largest exponent in its Cantor normal form representation. Thus we are able to lift former results about infinitary sums to infinitary products. Finally, we provide an order-theoretical characterization of the infinite natural product; this characterization merges in a nontrivial way a theorem by Carruth describing the natural product of two ordinals and a known description of the ordinal product of a possibly infinite sequence of ordinals.

1. INTRODUCTION

The usual addition $+$ and multiplication $\cdot$ between ordinals can be defined by transfinite recursion and have a clear order-theoretical meaning; for example, $\alpha + \beta$ is the order-type of a copy of α to which a copy of β is added at the top. However, $+$ and $\cdot$ have very poor algebraic properties; though both are associative, they are neither commutative nor cancellative, left distributivity fails, etc. See, e. g., Bachmann [B], Hausdorff [Hau] and Sierpiński [S3] for full details.

In certain cases it is useful to consider the so-called *natural* operations $\#$ and $\otimes$. These operations are defined by expressing the operands in Cantor normal form and, roughly, treating the expressions as if they were polynomials in ω. The natural operations have the advantage of satisfying good algebraic properties; moreover, in a few cases they have found mathematical applications even outside logic. See e. g., Carruth [Ca] and Toulmin [T]. Further references, including some variants and historical remarks, can be found in Altman [A] and Ehrlich [E, pp. 24–25]. It is also interesting to observe that the natural operations

are the restriction to the ordinals of the surreal operations on Conway "Numbers" [Co]. Limited to the case of the ordinal natural sum, the corresponding two-arguments recursive definition appears implicitly as early as in the proof of de Jongh and Parikh [dJP, Theorem 3.4].

An infinitary generalization of the natural sum in which the supremum is taken at the limit stage has been considered in Wang [W] and Väänänen and Wang [VW] with applications to infinitary logics. This infinite natural sum has been studied in detail in [L1], where an order-theoretical characterization has been provided. In the present note we introduce and study the analogously defined infinitary product. The computation of the infinite natural product can be reduced to the computation of some—possibly infinite—natural sum; in particular, we can directly transfer results from sums to products, rather than repeating essentially the same arguments. Curiously, the method applies to the usual infinitary operations, too. See Theorem 2.8 and Corollary 2.9; we are not aware of previous uses of this technique.

Order-theoretical characterizations of the finitary natural operations have been provided in Carruth [Ca]. Some characterizations seem to have been independently rediscovered many times, e. g., Toulmin [T] and de Jongh and Parikh [dJP]. As we showed in [L1], Carruth characterization of the finite natural sum cannot be generalized as it stands to the infinitary natural sum; see, in particular, the comments at the beginning of [L1, Section 4]. A similar situation occurs for the infinitary natural product; see Subsection 3.4 below. In the case of the infinite natural sum the difficulty can be circumvented by imposing a finiteness condition to a Carruth-like description: see [L1, Theorem 4.7]. As we shall show in Section 3, a similar result holds for the infinite natural product, though the situation gets technically more involved.

To explain our construction in some detail, it is well known that the usual finite ordinal product is order-theoretically characterized by taking the reverse lexicographical order on the product of the factors. Less known, the same holds in the infinite case, too, provided that in the product one takes into account only elements with finite support. See Hausdorff [Hau, § 16] and Matsuzaka [Ma]. We show that the infinite natural product can be characterized by merging the representation by Carruth and the just mentioned one; roughly, by using Carruth order for a sufficiently large but finite product of factors and then working as in an ordinary product when we approach infinity. Quite surprisingly, a local version of the result holds. Namely, if $\leqslant'$ is a linear order on the finite-support-product of a sequence $(\alpha_i)_{i<\omega}$ of ordinals and $\leqslant'$ is such that, for every element c, the set of the $\leqslant'$ predecessors of c is built in a way similar to above, then $\leqslant'$ is a well-order of order type less than or

equal to the infinite natural product of the α_i's. Moreover, the infinite natural product is actually the maximum of the ordinals obtained this way, that is, the order-type of the product can always be realized in the above way. See Section 3 and in particular Theorem 3.1 for full details.

1.1. **Preliminaries.** We assume familiarity with the basic theory of ordinal numbers. Unexplained notions and notations are standard and can be found, e. g., in the mentioned books [B, Hau, S3]. Notice that here sums, products and exponentiations are always considered in the ordinal sense. The usual ordinal product of two ordinals α and β is denoted by $\alpha\beta$ or sometimes $\alpha \cdot \beta$ for clarity. The classical infinitary ordinal product is denoted by $\prod_{i<\omega} \alpha_i$. Recall that every nonzero ordinal α can be expressed in *Cantor normal form* in a unique way as follows

$$\alpha = \omega^{\xi_k} r_k + \omega^{\xi_{k-1}} r_{k-1} + \cdots + \omega^{\xi_1} r_1 + \omega^{\xi_0} r_0$$

for some integers $k \geqslant 0$, $r_k, \ldots, r_0 > 0$ and ordinals $\xi_k > \xi_{k-1} > \cdots > \xi_1 > \xi_0$. If β is another ordinal expressed in Cantor normal form, the *natural sum* $\alpha \# \beta$ is obtained by summing the two expressions as if they were polynomials in ω. The *natural product* $\alpha \otimes \beta$ is computed by using the rule $\omega^\xi \otimes \omega^\eta = \omega^{\xi \# \eta}$ and then expanding again by "linearity". Both $\#$ and $\otimes$ are commutative, associative and cancellative (except for multiplication by 0, of course). If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, $\#_{i<\omega} \alpha_i = \sup_{n<\omega}(\alpha_0 \# \alpha_1 \# \cdots \# \alpha_{n-1})$. See [W, VW, L1] for further details about $\#$.

2. An infinite natural product

Definition 2.1. Suppose that $(\alpha_i)_{i<\omega}$ is a sequence of ordinals.

For $i < \omega$, let P_i denote the partial natural product $\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_{i-1}$, with the convention $P_0 = 1$.

Define $\bigotimes_{i<\omega} \alpha_i = \lim_{i<\omega} P_i$.

This means that $\bigotimes_{i<\omega} \alpha_i = 0$ if at least one α_i is 0 and $\bigotimes_{i<\omega} \alpha_i = \sup_{i<\omega} P_i$ if each α_i is different from 0 (cf. Clause (5) in Proposition 2.2 below).

From now on, when not otherwise specified, $(\alpha_i)_{i<\omega}$ is a fixed sequence of ordinals and the partial natural products P_i are always computed as above and with respect to the sequence $(\alpha_i)_{i<\omega}$.

We now state some simple facts about the infinitary operation $\bigotimes$. Only (2), (5) and (6) will be used in what follows.

Proposition 2.2. *Let α_i, β_i be two sequences of ordinals and $n, m < \omega$.*

(1) $\prod_{i<\omega} \alpha_i \leqslant \bigotimes_{i<\omega} \alpha_i$

(2) If $\beta_i \leqslant \alpha_i$, for every $i < \omega$, then $\bigotimes_{i<\omega} \beta_i \leqslant \bigotimes_{i<\omega} \alpha_i$

(3) If π is a permutation of ω, then $\bigotimes_{i<\omega} \alpha_i = \bigotimes_{i<\omega} \alpha_{\pi(i)}$

(4) More generally, suppose that $(F_h)_{h<\omega}$ is a partition of ω into finite subsets, say, $F_h = \{j_1, \ldots, j_{r(h)}\}$, for every $h \in \omega$. Then

$$\bigotimes_{i<\omega} \alpha_i = \bigotimes_{h<\omega} \bigotimes_{j \in F_h} \alpha_j = \bigotimes_{h<\omega} (\alpha_{j_1} \otimes \alpha_{j_2} \otimes \cdots \otimes \alpha_{j_{r(h)}})$$

Suppose in addition that $\alpha_i \neq 0$, for every $i < \omega$.

(5) If $i < j$, then $P_i \leqslant P_j$; equality holds if and only if $\alpha_i = \cdots = \alpha_{j-1} = 1$.

(6) For every $n < \omega$, we have $P_n \leqslant \bigotimes_{i<\omega} \alpha_i$; equality holds if and only if $\alpha_i = 1$, for every $i \geqslant n$.

Proof. Easy, using the properties of the finitary $\otimes$. We just comment on (3). This is trivial if at least one α_i is 0. Otherwise, every partial product on the left is bounded by some sufficiently long partial product on the right (by monotonicity, associativity and commutativity of $\otimes$, and since all factors are nonzero by assumption); the converse holds as well, hence the infinitary products are equal.

Let us mention that (3) and (4) can be also proved by using Theorem 2.6 below and the corresponding properties of $\#$ given in [L1, Proposition 2.4(5)(6)]. □

Lemma 2.3. *If $(\beta_i)_{i<\omega}$ is a sequence of ordinals and $\beta = \#_{i<\omega} \beta_i$, then*

$$\bigotimes_{i<\omega} \omega^{\beta_i} = \omega^{\beta}$$

Proof. Since ω^{β_i} is always different from 0, we have $\bigotimes_{i<\omega} \alpha_i = \sup_{i<\omega} P_i$. Here, of course, P_i is computed with respect to the sequence given by $\alpha_i = \omega^{\beta_i}$, for $i < \omega$.

By a property of the natural product (or the definition, if you like), we have $P_i = \omega^{\beta_0} \otimes \omega^{\beta_1} \otimes \cdots \otimes \omega^{\beta_{i-1}} = \omega^{\beta_0 \# \beta_1 \# \cdots \# \beta_{i-1}}$, for every $i < \omega$. Letting $B_i = \beta_0 \# \beta_1 \# \cdots \# \beta_{i-1}$, we have by definition $\beta = \#_{i<\omega} \beta_i = \sup_{i<\omega} B_i$. But then $\bigotimes_{i<\omega} \alpha_i = \sup_{i<\omega} P_i = \sup_{i<\omega} \omega^{B_i} = \omega^{\beta}$ by continuity of the exponentiation. □

Let α be a nonzero ordinal expressed as $\omega^{\xi_k} r_k + \cdots + \omega^{\xi_0} r_0$ in Cantor normal form. The ordinal $d(\alpha) = \xi_k$ will be called the *degree* or the *largest exponent* of α. The ordinal $m(\alpha) = \omega^{\xi_k} r_k$ will be called the *leading monomial* of α. By convention, we set $m(0) = 0$. The following lemma is trivial, but it will be useful in many situations.

Lemma 2.4. *For every ordinal α,*

$$m(\alpha) \leqslant \alpha \leqslant m(\alpha) + m(\alpha) = m(\alpha) \cdot 2 \leqslant m(\alpha) \otimes 2$$

Lemma 2.5. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals which is not eventually* 1, *then*

$$\bigotimes_{i<\omega} \alpha_i = \bigotimes_{i<\omega} m(\alpha_i)$$

Proof. This is trivial if at least one α_i is 0.

Hence suppose that $\alpha_i \neq 0$, for all $i < \omega$, and that $(\alpha_i)_{i<\omega}$ is not eventually 1. The inequality $\bigotimes_{i<\omega} \alpha_i \geqslant \bigotimes_{i<\omega} m(\alpha_i)$ is trivial by monotonicity (Proposition 2.2(2)), since $\alpha \geqslant m(\alpha)$, for every ordinal α.

To prove the converse, we show that, for every $h < \omega$, there is $k < \omega$ such that $\bigotimes_{i<h} \alpha_i \leqslant \bigotimes_{i<k} m(\alpha_i)$. This is enough since if all the α_i's are nonzero, then the succession of the partial products is nondecreasing (Proposition 2.2(5)). It is a trivial property of the finitary natural product that $m(\bigotimes_{i<h} \alpha_i) = \bigotimes_{i<h} m(\alpha_i)$ (by strict monotonicity of $\#$). Since $(\alpha_i)_{i<\omega}$ is not eventually 1, there is $k > h$ such that $\alpha_{k-1} \geqslant 2$. Then $\bigotimes_{i<k} m(\alpha_i) \geqslant \left(\bigotimes_{i<k-1} m(\alpha_i)\right) \otimes 2 \geqslant \left(\bigotimes_{i<h} m(\alpha_i)\right) \otimes 2 = m\left(\bigotimes_{i<h} \alpha_i\right) \otimes 2 \geqslant \bigotimes_{i<h} \alpha_i$, by Lemma 2.4. $\square$

Theorem 2.6. *Let $(\alpha_i)_{i<\omega}$ be a sequence of ordinals and let $\beta = \#_{i<\omega} d(\alpha_i)$. The infinite natural product $\bigotimes_{i<\omega} \alpha_i$ can be computed according to the following rules.*

(1) $\quad \bigotimes_{i<\omega} \alpha_i = 0 \quad$ *if (and only if) at least one α_i is equal to* 0;

(2) $\quad \bigotimes_{i<\omega} \alpha_i = \alpha_0 \otimes \cdots \otimes \alpha_{n-1} \quad$ *if $\alpha_i = 1$, for every $i \geqslant n$;*

(3) $\quad \bigotimes_{i<\omega} \alpha_i = \omega^{d(\alpha_0)\#\cdots\#d(\alpha_{n-1})+1} = \omega^{\beta+1} \quad$ *if $\alpha_i \neq 0$, for all $i < \omega$,*

$\quad\quad\quad \alpha_i < \omega$, *for all $i \geqslant n$, and the sequence is not eventually* 1;

(4) $\quad \bigotimes_{i<\omega} \alpha_i = \bigotimes_{i<\omega} \omega^{d(\alpha_i)} = \omega^\beta \quad$ *if none of the above cases applies,*

> *that is, no element of the sequence is* 0 *and the members of the sequence are not eventually $< \omega$.*

Before proving Theorem 2.6, we notice that it gives an effective way to compute $\bigotimes_{i<\omega} \alpha_i$, for every sequence $(\alpha_i)_{i<\omega}$ of ordinals. Apply (1) if at least one α_i is equal to 0; if this is not the case, apply (2) if the α_i are eventually 1; if not, then exactly one of (3) or (4) occurs. Notice that conditions (1) and (2) in Theorem 2.6 might overlap, and n in (2) and (3) is not uniquely defined, but the conditions give the same outcome in any overlapping case. Notice also that the expression $d(\alpha_0) \# \cdots \# d(\alpha_{n-1}) + 1$ in (3) causes no ambiguity, since $(d(\alpha_0) \# \cdots \# d(\alpha_{n-1})) + 1 = d(\alpha_0) \# \cdots \# (d(\alpha_{n-1}) + 1)$.

Proof. The result follows trivially from the definitions if some α_i is equal to 0 or when the sequence is eventually 1.

If we are in the case given by (3), then, for every $\ell < \omega$, there is $h > n$ such that there are at least ℓ-many $\alpha_i \geqslant 2$, where the index i varies between n and h. Thus $\bigotimes_{i < h+1} \alpha_i \geqslant \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_{n-1})} \otimes 2^\ell$ since $\alpha \geqslant \omega^{d(\alpha)}$, for every nonzero ordinal α. Since ℓ is arbitrary, we get $\bigotimes_{i < \omega} \alpha_i = \sup_{h < \omega} \bigotimes_{i < h+1} \alpha_i \geqslant \sup_{\ell < \omega} (\omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_{n-1})} \otimes 2^\ell) = \sup_{\ell < \omega} \omega^{d(\alpha_0) \# \cdots \# d(\alpha_{n-1})} 2^\ell = \omega^{d(\alpha_0) \# \cdots \# d(\alpha_{n-1})+1}$, since $\omega^\varepsilon \otimes 2 = \omega^\varepsilon 2$ and $\sup_{p < \omega} \omega^\varepsilon p = \omega^{\varepsilon+1}$, for every ordinal ε.

In the other direction, we have $\bigotimes_{i < \omega} \alpha_i = \bigotimes_{i < \omega} m(\alpha_i)$ from Lemma 2.5, hence it is enough to prove $\bigotimes_{i < \omega} m(\alpha_i) \leqslant \omega^{d(\alpha_0) \# \cdots \# d(\alpha_{n-1})+1}$. If $h < \omega$, and, for every $i < \omega$, letting s_i be the only natural number such that $m(\alpha_i) = \omega^{d(\alpha_i)} s_i$, then by associativity and commutativity of $\otimes$, we get $\bigotimes_{i < h+1} m(\alpha_i) = \omega^{d(\alpha_0)} \otimes s_0 \otimes \cdots \otimes \omega^{d(\alpha_h)} \otimes s_h = \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_h)} \otimes s_0 \otimes \cdots \otimes s_h \leqslant \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_{n-1})} \otimes \omega = \omega^{d(\alpha_0) \# \cdots \# d(\alpha_{n-1})+1}$, since $d(\alpha_i) = 0$, for $i \geqslant n$ and, by construction, $s_i < \omega$, for every $i < \omega$. Hence $\bigotimes_{i < \omega} m(\alpha_i) = \sup_{h < \omega} \bigotimes_{i < h} m(\alpha_i) \leqslant \omega^{d(\alpha_0) \# \cdots \# d(\alpha_{n-1})+1}$.

The last identity in (3) follows from the already mentioned fact that $d(\alpha_i) = 0$, for $i \geqslant n$, hence $\beta = \#_{i < \omega} d(\alpha_i) = \#_{i < n} d(\alpha_i)$.

The case given by (4) is similar and somewhat easier. The inequality $\bigotimes_{i < \omega} \alpha_i \geqslant \bigotimes_{i < \omega} \omega^{d(\alpha_i)}$ is trivial by monotonicity.

For the converse, we use again the identity $\bigotimes_{i < \omega} \alpha_i = \bigotimes_{i < \omega} m(\alpha_i)$ from Lemma 2.5. Arguing as in case (3), we have that, for every $h < \omega$, $\bigotimes_{i \leqslant h} m(\alpha_i) \leqslant \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_h)} \otimes \omega$, but there is some $k > h$ such that $\alpha_k \geqslant \omega$, since the members of the sequence are not eventually $< \omega$. Hence $\bigotimes_{i \leqslant h} m(\alpha_i) \leqslant \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_h)} \otimes \omega \leqslant \omega^{d(\alpha_0)} \otimes \cdots \otimes \omega^{d(\alpha_h)} \otimes \cdots \otimes \omega^{d(\alpha_k)} \leqslant \bigotimes_{i < \omega} \omega^{d(\alpha_i)}$. In conclusion, $\bigotimes_{i < \omega} \alpha_i = \bigotimes_{i < \omega} m(\alpha_i) = \sup_{h < \omega} \bigotimes_{i \leqslant h} m(\alpha_i) \leqslant \bigotimes_{i < \omega} \omega^{d(\alpha_i)}$.

The last identity is from Lemma 2.3. $\qquad\square$

Corollary 5.1 in [L1] can be used to provide a more precise evaluation of β in case (4) in Theorem 2.6.

Corollary 2.7. *Suppose that $(\alpha_i)_{i < \omega}$ is a sequence of ordinals such that no element of the sequence is 0 and the members of the sequence are not eventually $< \omega$ (thus the sequence $(d(\alpha_i))_{i < \omega}$ is not eventually 0). Let ξ be the smallest ordinal such that $\{i < \omega \mid d(\alpha_i) \geqslant \omega^\xi\}$ is finite, and enumerate those α_i's such that $d(\alpha_i) \geqslant \omega^\xi$ as $\alpha_{i_0}, \ldots, \alpha_{i_k}$ (the sequence might be empty). Then $\bigotimes_{i < \omega} \alpha_i = \omega^\beta$, where $\beta = (d(\alpha_{i_0}) \# \cdots \# d(\alpha_{i_k})) + \omega^\xi$.*

We need the results analogous to Theorem 2.6 for the classical ordinal product.

Theorem 2.8.   (1) If $(\beta_i)_{i<\omega}$ is a sequence of ordinals and $\beta = \sum_{i<\omega} \beta_i$, then

$$\prod_{i<\omega} \omega^{\beta_i} = \omega^\beta$$

(2) If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals which is not eventually 1, then

$$\prod_{i<\omega} \alpha_i = \prod_{i<\omega} m(\alpha_i)$$

(3) Suppose that $(\alpha_i)_{i<\omega}$ is a sequence of nonzero ordinals which is not eventually 1 and let $\beta = \sum_{i<\omega} d(\alpha_i)$. Then

$$\prod_{i<\omega} \alpha_i = \omega^{d(\alpha_0)+\cdots+d(\alpha_{n-1})+1} = \omega^{\beta+1} \quad \text{if } \alpha_i < \omega, \text{ for all } i \geqslant n$$

$$\prod_{i<\omega} \alpha_i = \prod_{i<\omega} \omega^{d(\alpha_i)} = \omega^\beta \quad \text{if the sequence is not eventually } < \omega$$

Proof. (1) Like the proof of Lemma 2.3, using the identity $\omega^{\beta_0} \omega^{\beta_1} \dots \omega^{\beta_{i-1}} = \omega^{\beta_0 + \beta_1 + \cdots + \beta_{i-1}}$.

(2) Like the proof of Lemma 2.5. In fact, we do have $m(\prod_{i<h} \alpha_i) = \prod_{i<h} m(\alpha_i)$ and this is enough for the proof.

(3) is proved as Theorem 2.6. $\square$

Theorems 2.6 and 2.8 can be used to "lift" some results from infinitary sums to infinitary products. To show how the method works, we first present a simple example, though only feebly connected with the rest of this note.

Sierpiński [S1] showed that a sum $\sum_{i<\omega} \alpha_i$ of ordinals can assume only finitely many values, by permuting the α_i's. A proof can be found also in [L1]. Then Sierpiński in [S2] showed the analogous result for an infinite product. Using Theorem 2.8 we show that the result about products is immediate from the result about sums.

Corollary 2.9. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, one obtains only a finite number of ordinals by considering all products of the form $\prod_{i\in I} \gamma_i$, where $(\gamma_i)_{i<\omega}$ is a permutation of $(\alpha_i)_{i<\omega}$, that is, there exists a bijection $\pi : \omega \to \omega$ such that $\gamma_i = \alpha_{\pi(i)}$ for every $i < \omega$.*

Proof. This is trivial if some α_i is 0, or if the α_i's are eventually 1, so let us assume that none of the above cases occurs.

If the sequence is eventually $< \omega$, then the first equation in Theorem 2.8(3) shows that we obtain only a finite number of products by taking rearrangements of the factors, since the resulting products are given by $\omega^{\delta+1}$, where δ is a sum of $d(\alpha_0), \ldots, d(\alpha_{n-1})$, taken in some order, but there is only a finite number of rearrangements of this finite set,

hence there are only a finite number of possibilities for δ (notice that if $\alpha_i < \omega$, then $d(\alpha_i) = 0$, hence the degrees of finite ordinals do not contribute to the sum).

In the remaining case, the products obtained by rearrangements have the form ω^δ, with $\delta = \sum_{i<\omega} d(\gamma_i)$, by the second equation in Theorem 2.8(3). The quoted result from [S1] shows that we have only a finite number of possibilities for δ, hence there are only a finite number of possibilities for the values of the rearranged products. $\qquad\square$

We now use Theorems 2.6 and 2.8 in a slightly more involved situation in order to transfer some results from [L1] about infinite natural sums to results about infinite natural products.

Corollary 2.10. *For every sequence $(\alpha_i)_{i<\omega}$ of ordinals there is $m < \omega$ such that, for every $n \geqslant m$,*

$$(5) \qquad \bigotimes_{n \leqslant i < \omega} \alpha_i = \prod_{n \leqslant i < \omega} \alpha_i \quad and$$

$$
(6) \qquad
\begin{aligned}
\bigotimes_{i<\omega} \alpha_i &= (\alpha_0 \otimes \cdots \otimes \alpha_{n-1}) \cdot \bigotimes_{n \leqslant i < \omega} \alpha_i \\
&= (\alpha_0 \otimes \cdots \otimes \alpha_{n-1}) \cdot \prod_{n \leqslant i < \omega} \alpha_i
\end{aligned}
$$

and if, moreover, every α_i is nonzero and the sequence is not eventually 1, then

$$
(7) \qquad
\begin{aligned}
\bigotimes_{i<\omega} \alpha_i &= \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \cdot \bigotimes_{n \leqslant i < \omega} \alpha_i \\
&= \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \cdot \prod_{n \leqslant i < \omega} \alpha_i
\end{aligned}
$$

where $\beta_i = d(\alpha_i)$, for every $i < \omega$.

Proof. The result is trivial if the sequence is eventually 1; moreover, (6) is trivial if some α_i is 0. Furthermore, (5) is trivial if, for every $i < \omega$, there is $j > i$ such that $\alpha_j = 0$. Otherwise, by taking m large enough, we have $\alpha_i > 0$, for $i > m$. Henceforth it is enough to prove the result in the case when the sequence is not eventually 1 and all the α_i's are nonzero.

We shall first prove (5) and (7) and then derive (6). If the sequence is eventually $< \omega$, then (5) is trivial, since in this case, for large enough n, both sides are equal to ω. Then (7) is immediate from equation (3) in Theorem 2.6, since $\omega^{\beta_0 \# \cdots \# \beta_{n-1}} \omega = \omega^{\beta_0 \# \cdots \# \beta_{n-1}+1}$.

Suppose now that the sequence $(\alpha_i)_{i<\omega}$ is not eventually $< \omega$, hence the sequence $(\beta_i)_{i<\omega}$ is not eventually 0. By [L1, Theorem 3.1], there is $m < \omega$ such that, for every $n \geqslant m$, we have $\#_{n \leqslant i < \omega} \beta_i = \sum_{n \leqslant i < \omega} \beta_i$.

Fixing some $n \geq m$ and letting $\beta' = \#_{n \leq i < \omega} \beta_i$, we get $\bigotimes_{n \leq i < \omega} \alpha_i = \omega^{\beta'} = \prod_{n \leq i < \omega} \alpha_i$ from, respectively, equation (4) in Theorem 2.6 and the last equation in Theorem 2.8(3). This proves (5).

Letting $\beta = \#_{i < \omega} \beta_i$, we notice that in [L1, Theorem 3.1] it has also been proved that $\beta = (\beta_0 \# \cdots \# \beta_{n-1}) + \beta'$. Using the above identity and applying equation (4) in Theorem 2.6 twice, we get $\bigotimes_{i < \omega} \alpha_i = \omega^\beta = \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \omega^{\beta'} = \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \bigotimes_{n \leq i < \omega} \alpha_i$, that is, (7) (the second identity in (7) could be proved in the same way, but now it follows immediately from (5)).

Equation (6) remains to be proved. We shall prove that in the non-trivial cases the expressions given by (6) and (7) are equal. One direction is trivial, since $\omega^{\beta_0 \# \cdots \# \beta_{n-1}} = \omega^{\beta_0} \otimes \cdots \otimes \omega^{\beta_{n-1}} \leq \alpha_0 \otimes \cdots \otimes \alpha_{n-1}$. For the other direction, let us observe that, in the nontrivial cases, again by Theorem 2.6, $\bigotimes_{n \leq i < \omega} \alpha_i$ has the form ω^β, for some $\beta \geq 1$. Then $(\alpha_0 \otimes \cdots \otimes \alpha_{n-1}) \cdot \bigotimes_{n \leq i < \omega} \alpha_i = (\alpha_0 \otimes \cdots \otimes \alpha_{n-1}) \cdot \omega^\beta \leq m(\alpha_0 \otimes \cdots \otimes \alpha_{n-1}) \cdot 2 \cdot \omega^\beta = (m(\alpha_0) \otimes \cdots \otimes m(\alpha_{n-1})) \cdot \omega^\beta = (\omega^{d(\alpha_0)} s_0 \otimes \cdots \otimes \omega^{d(\alpha_{n-1})} s_{n-1}) \cdot \omega^\beta = (\omega^{\beta_0} \otimes \cdots \otimes \omega^{\beta_{n-1}} \otimes s_o \ldots s_{n-1}) \cdot \omega^\beta = \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \cdot s_o \ldots s_{n-1} \cdot \omega^\beta = \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \cdot \omega^\beta = \omega^{\beta_0 \# \cdots \# \beta_{n-1}} \cdot \bigotimes_{n \leq i < \omega} \alpha_i$, where we used Lemma 2.4 and the facts that $k\omega^\beta = \omega^\beta$, whenever $k < \omega$ and $\beta \geq 1$, and that $\omega^\xi \otimes k = \omega^\xi \cdot k$, for all ordinals $k < \omega$ and ξ. $\square$

3. AN ORDER-THEORETICAL CHARACTERIZATION

3.1. We refer to, e. g., Harzheim [Har] for a general reference about ordered sets. As usual, when no risk of ambiguity is present, we shall denote a (partially) ordered set $(P, \leq)$ simply as P. However, in many situations, we shall have several different orderings on the same set; in that case we shall explicitly indicate the order. It is sometimes convenient to define $\leq$ in terms of the associated $<$ relation and conversely. As a standard convention, $a \leq b$ is equivalent to "either $a = b$ or $a < b$" (strict disjunction). We shall be quite informal about the distinction and we shall use either $\leq$ or $<$ case by case according to convenience, even when we are dealing with (essentially) the same order.

In order to avoid notational ambiguity, let us denote the cartesian product of a family $(A_i)_{i \in I}$ of sets by $\times_{i \in I} A_i$. If each A_i is an ordered set, with the order denoted by $\leq_i$, then a partial order $\leq_\times$ can be defined on $\times_{i \in I} A_i$ componentwise. Namely, we put $a \leq_\times b$ if and only if $a_i \leq_i b_i$, for every $i \in I$. Apparently, for our purposes, the above definition has little use when dealing with ordinals (more precisely, order-types of well-ordered sets), since ordinals are linearly ordered, but generally the above construction furnishes only a partially ordered set. However, see below for uses of the ordered set $(\times_{i \in I} A_i, \leq_\times)$.

3.2. An order theoretical characterization of the ordinal product can be given using lexicographic products. If $(I, \leqslant_I)$ is reverse-well-ordered and each A_i is an ordered set, then the *anti-lexicographic order* $\mathrm{L}^*_{i \in I} A_i = (\mathrm{L}^*_{i \in I} A_i, \leqslant_*)$ on the set $\times_{i \in I} A_i$ is obtained by putting

> $a \leqslant_* b$ if and only if either $a = b$, or $a_i <_i b_i$, where i is the largest element of I such that $a_i \neq b_i$.

In other words, $\leqslant_*$ orders $\mathrm{L}^*_{i \in I} A_i$ by the *last* difference. The definition makes sense, since I is reverse-well-ordered. It turns out that if each A_i is linearly ordered, then $\mathrm{L}^*_{i \in I} A_i$ is linearly ordered and if in addition I is finite, then $\mathrm{L}^*_{i \in I} A_i$ is well-ordered. Moreover, for two ordinals α_0 and α_1, it happens that $\alpha_0 \alpha_1$ is exactly the order-type of $\mathrm{L}^*_{i < 2} \alpha_i$. This can be obviously generalized to *finite* products.

  A similar characterization can be given for infinite products, but some details should be made precise. Suppose that each A_i is an ordered set with order $\leqslant_i$ and with a specified element $0_i \in A_i$. If $a \in \times_{i \in I} A_i$, the *support supp(a)* of a is the set $\{ i \in I \mid a_i \neq 0_i \}$. Let $\times^0_{i \in I} A_i$ be the subset of $\times_{i \in I} A_i$ consisting of those elements with finite support. Of course, $\times^0_{i \in I} A_i$ inherits a partial order $\leqslant^0_\times$ as a sub-order of $(\times_{i \in I} A_i, \leqslant_\times)$. If I is linearly ordered, we can consider another order $\mathrm{L}^0_{i \in I} A_i = (\mathrm{L}^0_{i \in I} A_i, \leqslant_L)$ on the set $\times^0_{i \in I} A_i$ defined as follows.

> (*) $a \leqslant_L b$ if and only if either $a = b$, or $a_i <_i b_i$, where i is the largest element of $supp(a) \cup supp(b)$ such that $a_i \neq b_i$.

  The definition makes sense, since $supp(a) \cup supp(b)$ is finite and I is linearly ordered. Of course, when I is finite, $\times^0_{i \in I} A_i$ and $\times_{i \in I} A_i$ are the same set and $\mathrm{L}^0_{i \in I}$ and $\mathrm{L}^*_{i \in I}$ are the same order. It is known that, for every sequence $(\alpha_i)_{i < \delta}$ of ordinals, $\mathrm{L}^0_{i < \delta} \alpha_i$ is well-ordered and has order-type $\prod_{i < \delta} \alpha_i$. Here the specified element 0_i is always chosen to be the ordinal 0. See [B, III, § 10 1.2 and § 11 Satz 7], Hausdorff [Hau, § 16] and Matsuzaka [Ma] for details. Of course, we could have seen just by cardinality considerations that $\times_{i < \delta} A_i$ does not work in order to obtain an order theoretical characterization of $\prod_{i < \delta} \alpha_i$ in the case when δ is infinite.

3.3. Dealing now with natural products, a characterization in the finite case has been found by Carruth [Ca]. He proved that $\alpha_0 \otimes \alpha_1$ is the largest ordinal which is the order-type of some linear extension of the componentwise order on $\alpha_0 \times \alpha_1$. Here $\alpha_0 \times \alpha_1$ is ordered as $(\times_{i < 2} \alpha_i, \leqslant_\times)$ in the above notation. Carruth result includes the proof that such an ordinal exists, that is, that each such linear extension is a well-order and that the set of order-types of such linear extensions has a maximum, not just a supremum. From the modern point of view, this

can be seen as a special case of theorems by Wolk [Wo] and de Jongh, Parikh [dJP], since the product of two well quasi-orders (in particular, two well-orders) is still a well quasi order. See, e. g., [Mi]. Carruth result obviously extends to the case of any finite number of factors.

3.4. Some difficulties are encountered when trying to unify the results recalled in 3.2 and 3.3. Let us limit ourselves to the simplest infinite case of an ω-indexed sequence, which is the main theme of the present note. It would be natural to consider the supremum of the order-types of well-ordered linear extensions of the restriction $\leqslant_\times^0$ of $\leqslant_\times$ to $\times_{i<\omega}^0 \alpha_i$. However, just considering $\times_{i<\omega}^0 2$, we see that $(\times_{i<\omega}^0 2, \leqslant_\times^0)$ has linear extensions which are not well-ordered. Indeed, for every $j < \omega$, let $b^j \in \times_{i<\omega}^0 2$ be defined by $b_j^j = 1$ and $b_i^j = 0$ if $i \neq j$. Then the b^j's form a countable set of pairwise $\leqslant_\times^0$-incomparable elements of $\times_{i<\omega}^0 2$, hence every countable linear order is isomorphic to a subset of some linear extension of $\times_{i<\omega}^0 2$ (e. g., by [Har, Theorem 3.3]). Even if we restrict ourselves to well-ordered extensions of $\times_{i<\omega}^0 2$, we get from the above considerations that the supremum of their order-types is ω_1, hence this supremum is not a maximum and anyway it is too large to have the intended meaning, that is, $\omega = \bigotimes_{i<\omega} 2$.

The situation is parallel to [L1] and, as in [L1], an order-theoretical characterization can be found provided we restrict ourselves to linear extensions satisfying some finiteness condition.

3.5. We need a bit more notation in order to state the next theorem. Recall that if $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, then $\times_{i<\omega}^0 \alpha_i$ is the set of the sequences with finite support and the (partial) order $\leqslant_\times^0$ is defined componentwise. If $a, b \in \times_{i<\omega}^0 \alpha_i$ and $a \neq b$, let $diff(a,b)$ be the largest element i of $supp(a) \cup supp(b)$ such that $a_i \neq b_i$. Thus (*) above introduces a linear order $<_L$ on $\times_{i<\omega}^0 \alpha_i$ defined by $a <_L b$ if and only if $a_i < b_i$, for $i = diff(a,b)$ (here $<$ is the standard order on α_i, hence there is no need to explicitly indicate the index). By the results recalled in Subsection 3.2, the above order $<_L$ has type $\prod_{i<\omega} \alpha_i$. On the other hand, in the finite case, by Carruth theorem mentioned in 3.3, $\bigotimes_{i<n} \alpha_i$ is the order-type of the largest linear extension of $\leqslant_\times$ on $\times_{i<n} \alpha_i = \times_{i<n}^0 \alpha_i$. We show that in the infinite case $\bigotimes_{i<\omega} \alpha_i$ can be evaluated by combining the above constructions.

If $a \in \times_{i<\omega} \alpha_i$ is a sequence and $n < \omega$, let $a_{\restriction n}$ be the restriction of a to n, that is, $a_{\restriction n}$ is the element of $\times_{i<n} \alpha_i$ defined as follows. If $a = (a_i)_{i<\omega}$, then $a_{\restriction n} = (a_i)_{i<n}$. Here, as usual, we adopt the convention $n = \{0, 1, \ldots, n-1\}$.

Let us say that a linear order $<'$ on $\times^0_{i<\omega} \alpha_i$ is *finitely Carruth* if $<'$ extends $<^0_\times$ and there are an $n < \omega$ and an order $\leqslant''$ extending $<_\times$ on $\times_{i<n} \alpha_i$ and such that if $a \neq b \in \times^0_{i<\omega} \alpha_i$ and $i = \mathit{diff}(a,b)$, then

   (1) if $i \geqslant n$, then $a <' b$ if and only if $a_i < b_i$, and
   (2) if $i < n$, then $a <' b$ if and only if $a_{\upharpoonright n} <'' a_{\upharpoonright n}$.

A linear order $<'$ on $\times^0_{i<\omega} \alpha_i$ is *locally finitely Carruth* if $<'$ extends $<^0_\times$ and, for every $c \in \times^0_{i<\omega} \alpha_i$, there is $n = n_c < \omega$ such that, for every $a \neq b \in \times^0_{i<\omega} \alpha_i$, if $a, b <' c$ and $i = \mathit{diff}(a,b) \geqslant n$, then $a <' b$ if and only if $a_i < b_i$. In other words, for every c, (1) above holds, restricted to those pairs of elements a, b which are $<' c$, while no version of (2) is assumed.

Theorem 3.1. *If $(\alpha_i)_{i<\omega}$ is a sequence of ordinals, then $\bigotimes_{i<\omega} \alpha_i$ is the order-type of some finitely Carruth linear order on $\times^0_{i<\omega} \alpha_i$.*

Every (locally) finitely Carruth linear order on $\times^0_{i<\omega} \alpha_i$ is a well-order; moreover, $\bigotimes_{i<\omega} \alpha_i$ is the largest order-type of all such orderings.

Proof. By Corollary 2.10, in particular, equation (6), there is some $m < \omega$ such that $\bigotimes_{i<\omega} \alpha_i = (\alpha_0 \otimes \cdots \otimes \alpha_{m-1}) \cdot \prod_{m \leqslant i < \omega} \alpha_i$. By Carruth theorem, there is some linear extension $\leqslant_C$ of $\leqslant_\times$ on $\times_{i<m} \alpha_i$ such that $P_0 = (\times_{i<m} \alpha_i, \leqslant_C)$ has order-type $\bigotimes_{i<m} \alpha_i$. By the results recalled in Subsection 3.2, $\prod_{m \leqslant i < \omega} \alpha_i$ is the order-type of $P_1 = (\times^0_{m \leqslant i < \omega} \alpha_i, \leqslant_L)$. Then $P = \mathrm{L}^*_{h<2} P_h$ has order-type $(\alpha_0 \otimes \cdots \otimes \alpha_{m-1}) \cdot \prod_{m \leqslant i < \omega} \alpha_i = \bigotimes_{i<\omega} \alpha_i$, since, as we mentioned, for two ordinals γ_0 and γ_1 the order-type of $\mathrm{L}^*_{i<2} \gamma_i$ is $\gamma_0 \gamma_1$. Through the canonical bijection between $\times^0_{i<\omega} \alpha_i$ and $\times_{i<m} \alpha_i \times \times^0_{m \leqslant i < \omega} \alpha_i$, the order $<'$ we have constructed on $\mathrm{L}^*_{i<2} P_i$ is clearly finitely Carruth. Indeed, in the definition of finitely Carruth, take $n = m$ and take $\leqslant''$ as $\leqslant_C$. If $i = \mathit{diff}(a,b) < m$, then the ordering between a and b is determined by their $\upharpoonright m$ part, since sequences with the same P_1-component are ordered according to their P_0 component. Hence (2) holds. On the other hand, if $i = \mathit{diff}(a,b) \geqslant m$, then $a_i < b_i$ if and only if $a <' b$, by the definition of $\leqslant_L$, thus (1) holds. Obviously, $<'$ extends $<_\times$, since both $\leqslant_C$ and $\leqslant_L$ extend $\leqslant_\times$ on their respective components. Hence $\bigotimes_{i<\omega} \alpha_i$ can be realized as the order-type of some finitely Carruth order on $\times^0_{i<\omega} \alpha_i$ and the first statement is proved.

Since every finitely Carruth order on $\times^0_{i<\omega} \alpha_i$ is obviously locally finitely Carruth, it is enough to prove that every locally finitely Carruth order on $\times^0_{i<\omega} \alpha_i$ is a well-order of type $\leqslant \bigotimes_{i<\omega} \alpha_i$. So let us assume from now on that $<'$ is locally finitely Carruth.

Claim. *If $c \in \bigtimes^0_{i<\omega} \alpha_i$ and $C = \{a \in \bigtimes^0_{i<\omega} \alpha_i \mid a <' c\}$, then $(C, <'_{\upharpoonright C})$ is a well-ordered set of type $\leqslant \bigotimes_{i<\omega} \alpha_i$.*

Proof. Let $n = n_c$ be given by local Carruth finiteness and, as above, let $P_1 = (\bigtimes^0_{n \leqslant i < \omega} \alpha_i, \leqslant_L)$. Notice that, by Subsection 3.2, P_1 is well-ordered and has type $\prod_{n \leqslant i < \omega} \alpha_i$. If $a \in \bigtimes^0_{i<\omega} \alpha_i$, say, $a = (a_i)_{i<\omega}$, recall that $a_{\upharpoonright n}$ is the element $(a_i)_{i<n}$ of $\bigtimes_{i<n} \alpha_i$. Similarly, let $a_{\geqslant n}$ be the element $(a_i)_{i \geqslant n}$ of $\bigtimes^0_{i \geqslant n} \alpha_i$. Thus the position $a \mapsto (a_{\upharpoonright n}, a_{\geqslant n})$ gives the canonical bijection (mentioned but not described above) from $\bigtimes^0_{i<\omega} \alpha_i$ to $\bigtimes_{i<n} \alpha_i \times \bigtimes^0_{n \leqslant i < \omega} \alpha_i$. If $P = \{d \in \bigtimes^0_{n \leqslant i < \omega} \alpha_i \mid d = a_{\geqslant n}, \text{ for some } a \in \bigtimes^0_{i<\omega} \alpha_i \text{ such that } a <' c\}$, then $P \subseteq P_1$ hence P as a suborder of P_1 inherits a well-order of type $\leqslant \prod_{n \leqslant i < \omega} \alpha_i$. Moreover, by local Carruth finiteness, if $a, b <' c$ and $a_{\geqslant n} <_L b_{\geqslant n}$, then $a <' b$. If $d \in P$, let $Q_d = \{a_{\upharpoonright n} \mid a \in \bigtimes^0_{i<\omega} \alpha_i, a <' c \text{ and } a_{\geqslant n} = d\}$. Then, for every $d \in P$, the order $<'$ induces an order $<_d$ on Q_d by letting $a_{\upharpoonright n} <_d b_{\upharpoonright n}$ if and only if $a <' b$ (notice that, since we are assuming $a, b \in Q_d$, then a and b have the same $\geqslant n$ components). Since, by assumption, $<'$ extends $<_\times$ on $\bigtimes^0_{i<\omega} \alpha_i$, then $<_d$ extends the restriction of $<_\times$ on $\bigtimes_{i<n} \alpha_i$ to Q_d. Hence, by Carruth theorem, for every $d \in P$ we have that $(Q_d, <_d)$ is well-ordered and has type $\leqslant \bigotimes_{i<n} \alpha_i$.

The above considerations show that $(C, <'_{\upharpoonright C})$ is isomorphic to the lexicographical product $L_{d \in P} Q_d$ (recall that if $a, b <' c$ and $a_{\geqslant n} <_L b_{\geqslant n}$, then $a <' b$). Since, as we showed, P is a well-ordered set of type $\leqslant \prod_{n \leqslant i < \omega} \alpha_i$ and each Q_d is a well-ordered set of type $\leqslant \bigotimes_{i<n} \alpha_i$, then $(C, <'_{\upharpoonright C})$ is a well-ordered set of order-type $\leqslant \bigotimes_{i<n} \alpha_i \cdot \prod_{n \leqslant i < \omega} \alpha_i$. If the m given by Corollary 2.10 is $\leqslant n$, then we immediately get from equation (6) that $(C, <'_{\upharpoonright C})$ has order-type $\leqslant \bigotimes_{i<\omega} \alpha_i$. Otherwise, notice that if the condition for local Carruth finiteness is satisfied for c and for some n_c, then the condition is satisfied for any $n' \geqslant n_c$ in place of n_c, hence it is no loss of generality to suppose that the m given by Corollary 2.10 is $\leqslant n$ and we are done as before. $\square_{Claim}$

To complete the proof of the theorem, we have from the Claim that, for every $c \in \bigtimes^0_{i<\omega} \alpha_i$, the set of the $<'$-predecessors of c is well-ordered; this implies that $<'$ is a well-order.

It remains to show that $<'$ has order-type $\leqslant \bigotimes_{i<\omega} \alpha_i$. This is vacuously true if some α_i is 0 and it follows from Carruth theorem if the α_i's are eventually 1. Otherwise, local Carruth finiteness implies that $<'$ has no maximum. Indeed, if $a \in \bigtimes^0_{i<\omega} \alpha_i$, then there is $k < \omega$ such that $a_i = 0$, for $i > k$. Since the sequence of the α_i's is not eventually 1 and no α_i is 0, there is $\bar\imath > k$ such that $\alpha_{\bar\imath} > 1$. If b is equal to a on each component, except that $b_{\bar\imath} = 1$, then $a <_\times b$, hence $a <' b$, since, by

assumption, $<'$ extends $<_\times$. Since a above has been chosen arbitrarily, we get that $<'$ has no maximum. Then the Claim implies that $<'$ has order-type $\leqslant \bigotimes_{i<\omega} \alpha_i$. $\square$

Remark 3.2. For the sake of simplicity, we have limited our study here to sequences of length ω. However, essentially all the results here admit a reformulation for the case of ordinal-indexed transfinite sequences of arbitrary length, modulo the case of the transfinite natural sums studied in [L2]. It should be remarked that there are different ways to extend the natural sums and products to sequences of length $> \omega$. See [L2, Section 5], in particular, Definitions 5.2 and Problems 5.6. We just give here the relevant definitions relative to transfinite products.

Suppose that $(\alpha_\gamma)_{\gamma<\bar\varepsilon}$ is a sequence of ordinals. The *iterated natural product* $\prod^\#_{\gamma<\delta} \alpha_\gamma$ is defined for every $\delta \leqslant \bar\varepsilon$ as follows.

$$\prod_{\gamma<0}^\# \alpha_\gamma = 1; \qquad \prod_{\gamma<\delta+1}^\# \alpha_\gamma = \left(\prod_{\gamma<\delta}^\# \alpha_\gamma\right) \otimes \alpha_\delta$$

$$\prod_{\gamma<\delta}^\# \alpha_\gamma = \lim_{\delta'<\delta} \prod_{\gamma<\delta'}^\# \alpha_\gamma \quad \text{for } \delta \text{ limit}$$

Moreover, we set

$$\bigotimes_{\gamma<\delta}^o \alpha_\gamma = \inf_\pi \prod_{\gamma<\delta}^\# \alpha_{\pi(\gamma)}$$

where π varies among all the permutations of δ. In the above definition we are keeping δ fixed. In the next definition, on the contrary, we let the ordinal δ vary. Suppose that I is *any* set and $(\alpha_i)_{i\in I}$ is a sequence of ordinals. Define

$$\bigotimes_{i\in I}' \alpha_i = \inf_{\delta,f} \prod_{\gamma<\delta}^\# \alpha_{f(\gamma)}$$

where δ varies among all the ordinals having cardinality $|I|$ and f varies among all the bijections from δ to I. Furthermore, let $\lambda = |I|$ and define

$$\bigotimes_{i\in I}^\bullet \alpha_i = \inf_f \prod_{\gamma<\lambda}^\# \alpha_{f(\gamma)}$$

where f varies among all the bijections from λ to I.

and the theory of well-quasi-orders. We thank Harry Altman for stimulating discussions. We thank Arnold W. Miller for letting us know of Toulmin paper [T] in the MR review of [L1].

The literature on the subject of ordinal operations is so vast and sparse that we cannot claim completeness of the following list of references.

It is not intended that each work in the list has given equally significant contributions to the discipline. Henceforth the author disagrees with the use of the list (even in aggregate forms in combination with similar lists) in order to determine rankings or other indicators of, e. g., journals, individuals or institutions. In particular, the author considers that it is highly inappropriate, and strongly discourages, the use (even in partial, preliminary or auxiliary forms) of indicators extracted from the list in decisions about individuals (especially, job opportunities, career progressions etc.), attributions of funds, and selections or evaluations of research projects.

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