Relation identities in 3-distributive varieties

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Abstract. Let α, β, $\gamma, \dots$ Θ, $\Psi, \dots$ R, S, $T, \dots$ be variables for, respectively, congruences, tolerances and reflexive admissible relations. Let juxtaposition denote intersection. We show that if the identity

$$\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta \circ \alpha\beta$$

holds in a variety $\mathcal{V}$, then $\mathcal{V}$ has a majority term, equivalently, $\mathcal{V}$ satisfies $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma$. The result is unexpected, since in the displayed identity we have one more factor on the right and, moreover, if we let Θ be a congruence, we get a condition equivalent to 3-distributivity, which is well-known to be strictly weaker than the existence of a majority term.

The above result is optimal in many senses; for example, we show that slight variations on the displayed identity, such as $R(S \circ \gamma) \subseteq RS \circ R\gamma \circ RS$ or $R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS$ hold in every 3-distributive variety, hence do not imply the existence of a majority term. Similar identities are valid even in varieties with 2 Gumm terms, with no distributivity assumption. We also discuss relation identities in n-permutable varieties and present a few remarks about implication algebras.

1. Introduction

Congruence identities and relation identities. By a classical and surprising result by Nation [26] in the 1970's, there are non-equivalent lattice identities which are equivalent when considered as identities satisfied in all congruence lattices of algebras in some variety. Many results of this kind followed, for example, Freese and Jónsson [4] proved that modularity is equivalent to the Arguesian identity for congruence lattices in varieties. See also Day and Freese [3]. The survey [12] by Jónsson is an excellent introduction to earlier results on the subject. The field is still very active today; more recent results and further references can be found, among others, in Czédli, Horváth and Lipparini [1], Freese and McKenzie [5], Gumm [7], Hobby and McKenzie [9], Kearnes and Kiss [14] and Tschantz [27].

The research in the present note is motivated by two facts related to the study of congruence identities. First, it almost invariably happens that tolerances, and sometimes even reflexive and admissible relations, are irreplaceable tools in proving results about congruences. This is already evident in the proofs

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of the famous and well-known characterizations by Jónsson [11] and Day [2] of, respectively, congruence distributive and congruence modular varieties. The relevance of tolerances and admissible relations in the study of congruence identities appears clearly in [1, 7, 27], just to mention some examples.

The second aspect that plays a role in our motivations is that, almost at the same time of Nation's discovery mentioned at the beginning, non trivial "relation identities" have been discovered. By a *relation identity* we mean an identity satisfied by reflexive and admissible relations on some algebra, where the operations considered are usually intersection and relational composition, possibly also converse and transitive closure. Non trivial is intended in a sense similar to that of Nation's results; namely, we call an implication between two relation identities *non trivial* if it holds when considered for all algebras in a variety, but it does not necessarily hold for single algebras. Werner [28] showed that a variety is congruence permutable if and only if the relation identity $R \circ R = R$ holds in $\mathcal{V}$. The result is also due independently to Hutchinson [10]. It is easy to see that the result is non trivial in the sense specified above; see Remark 4.3(b) below. A similar relational characterization of congruence n-permutable varieties is mentioned in Hagemann an Mitschke [8]. See Proposition 4.2 below for some generalizations. Further examples, details, comments, applications and references about the use of tolerances or reflexive admissible relations, and about relation identities can be found in [1, 6, 7, 14, 17, 19, 20, 21, 22, 27]. Because of the above comments we believe that the study of relation identities is interesting, as a non trivial generalization of the study of congruence identities.

Relation identities in congruence distributive varieties. In [22] we used relation identities in order to approach the problem of the relationships between the numbers of Gumm and of Day terms in a congruence modular variety. While the problem is still largely unsolved, the partial results confirm the usefulness of the approach. Though our main aim has been the study of relation identities satisfied in modular varieties, we encountered delicate issues already in the relatively well-behaved case of 4-distributive varieties [21].

Here we show that the problem of the satisfaction of relation identities is not trivial even for identities related to 3-distributivity. A detailed study of 3-distributive varieties in a different direction appears in Kiss and Valeriote [15]. A variety $\mathcal{V}$ is *n-distributive* if $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\beta \circ \alpha\gamma \dots$ (n factors), where "n factors" means $n-1$ occurrences of $\circ$ on the right-hand side. In general, we say that some identity *holds in a variety* $\mathcal{V}$ if the identity holds in the set of reflexive and admissible relations on every algebra in $\mathcal{V}$. We shall adopt the conventions introduced in the first paragraph of the abstract, in particular, juxtaposition denotes intersection and R, S... are interpreted as reflexive and admissible relations. All the binary relations considered in this note are assumed to be reflexive, hence we shall sometimes simply say *admissible* in place of reflexive and admissible.

The definition of n-distributivity as given above is equivalent to the classical notion of being Δ_n, as introduced in Jónsson [11]. Among other, Jónsson [11] showed that a variety is congruence distributive if and only if it is Δ_n, for some n. As remarked in [18], a recent result by Kazda, Kozik, McKenzie and Moore [13] can be used to strengthen Jónsson's Theorem to the effect that a variety $\mathcal{V}$ is congruence distributive if and only if there is some k such that $\mathcal{V}$ satisfies the relation identity

$$\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha S \dots \quad (k \text{ factors}). \tag{1.1}$$

Hence it is interesting to study relation identities satisfied by congruence distributive varieties; in particular, to evaluate the best possible value of k for which (1.1) holds in any given variety.

Summary of the main results. It is easy to see that if $\mathcal{V}$ has a majority term (and is not a trivial variety) then the best possible value of k for $\mathcal{V}$ in (1.1) above is 2. In [21] we provided examples of 4-distributive varieties for which the best possible value of k is 4. The main results of the present note assert that there is no variety for which the best possible value of k is 3, even if we allow T to be a tolerance; moreover, the best value of k is 4 for every 3-distributive not 2-distributive variety. Combining the present results with [21], we get that the best value of k in (1.1) does not determine the distributivity level of a variety.

We do not know whether there is a variety for which $k = 5$ is the best value. More generally, we know little about the best possible values of k for arbitrary congruence distributive varieties, though some bounds are provided in [18, 21] for, say, varieties with h directed Jónsson terms and for varieties with an n-ary near-unanimity term. Bounds are provided in Section 4 below for congruence distributive n-permutable varieties.

The identities we consider in this note are extremely sensible to minimal variations. For example, if $k = 3$ and we let T be a congruence in (1.1), we get a condition equivalent to 3-distributivity. See identity (3.3) in Theorem 3.1. It is well-known that there are 3-distributive not 2-distributive varieties; in particular, any such variety fails to satisfy the identity

$$\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta \circ \alpha\beta \tag{1.2}$$

displayed in the abstract, but does satisfy

$$\alpha(\Psi \circ \gamma) \subseteq \alpha\Psi \circ \alpha\gamma \circ \alpha\Psi. \tag{1.3}$$

The symmetry between (1.2) and (1.3) is only apparent. In (1.3) the relation assumed to be a congruence is placed in the middle of the right-hand side, while in (1.2) it appears two times on the edges.

Further relation identities satisfied by 3-distributive varieties are presented in Section 3. There we show that, in many cases, we do not even need the equation $j_2(z, y, z) = z$ from the set of Jónsson's equations characterizing 3-distributivity. Namely, the results apply to congruence modular varieties with 2 Gumm terms, where in the numbering we are not counting the trivial projection included in the original presentation by Gumm of his set of terms.

Congruence distributive and congruence modular varieties which are further assumed to be n-permutable are discussed in Section 4. Finally, in Section 5 we show directly by an example that the displayed identity in the abstract fails in the variety of implication algebras. We also show that most properties of implication algebras are preserved if we add a 4-ary near-unanimity term.

2. A tolerance identity implying 2-distributivity

Theorem 2.1. *If $\mathcal{V}$ is a variety and the identity*

$$\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta \circ \alpha\beta \tag{2.1}$$

holds in every algebra $\mathbf{A}$ in $\mathcal{V}$, for all congruences α, β and tolerance Θ on $\mathbf{A}$, then $\mathcal{V}$ has a majority term. In fact, it is enough to assume that (2.1) *holds in $\mathbf{F}_{\mathcal{V}}(3)$, the free algebra in $\mathcal{V}$ generated by* 3 *elements.*

Theorem 2.1 shall be proved after a lemma, whose proof is standard, but not completely usual, since it involves a tolerance, not only congruences. Before stating the lemma we recall the *Jónsson terms for* 3*-distributivity.* These terms will appear only marginally in the proof of the lemma and will assume a prominent role in Section 3. According to [11], a variety $\mathcal{V}$ is Δ_3 if and only if $\mathcal{V}$ has terms j_1 and j_2 satisfying the following set of equations:

$$\begin{array}{lll} (\mathrm{J}_L)\ x = j_1(x,x,z), & (\mathrm{J}_C)\ j_1(x,z,z) = j_2(x,z,z), & (\mathrm{J}_R)\ j_2(x,x,z) = z, \\ (\mathrm{J}_1)\ x = j_1(x,y,x), & & (\mathrm{J}_2)\ j_2(z,y,z) = z. \end{array} \tag{J}$$

It is implicit in [11], and only a minor part of it, that a variety $\mathcal{V}$ is Δ_3 if and only if $\mathcal{V}$ obeys the definition of 3-distributivity that we have given in the introduction.

In the equations (J) we could have done with just two variables, but we shall not need this observation, which nevertheless is important and relevant in different contexts. We believe that here maintaining three variables looks intuitively clearer; the same applies to the equations in (2.2) below.

Remark 2.2. Notice the asymmetry between j_1 and j_2 in the equations (J). We get different identities if we reverse both the order of the terms and of the variables. A probably better way to appreciate the asymmetry goes as follows. Recall that a *majority term* is a term j_1 satisfying the equations (J_L), (J_1), as well as $j_1(x,z,z) = z$. In (J) j_1 is required to satisfy two thirds of the majority condition. On the other hand, a *Maltsev term for congruence permutability* is a term j_2 satisfying (J_R), as well as $x = j_2(x,z,z)$. In (J) j_2 is required to satisfy one third of the majority condition and one half of the permutability condition. The terms j_1 and j_2 are tied by equation (J_C), but (J_C) is not symmetric, either. Had we required $j_1(x,x,z) = j_2(x,x,z)$ in place of (J_C), we would have obtained $x = z$, by (J_L) and (J_R). In spite of the apparent similarity, j_1 and j_2 behave in a quite different way! Let us remark that the above observation is only part of much more general considerations, see Remark 4.2 in [22].

Now we state and prove the lemma we need. In some of the equations below we shall use a semicolon in place of a comma in order to improve readability.

Lemma 2.3. *If* $\mathbf{F}_{\mathcal{V}}(3)$ *satisfies the identity* (2.1)*, then* $\mathcal{V}$ *has a* 5*-ary term* w *such that the following equations hold in* $\mathcal{V}$*:*

$$\begin{array}{llll} \text{(A)} & x = w(x,x,z;x,z) & \text{(B)} & w(x,x,z;z,x) = z \\ \text{(C)} & x = w(x,y,x;y,x) & \text{(D)} & w(x,y,x;x,y) = x. \end{array} \tag{2.2}$$

Proof. Let x, y and z be the generators of $\mathbf{F}_{\mathcal{V}}(3)$ and let α and β be the congruences generated by, respectively, the pairs (x,z) and (x,y). Let Θ be the smallest tolerance containing (y,z), thus $(x,z) \in \alpha(\beta \circ \Theta)$, as witnessed by the element y. By assumption, $(x,z) \in \alpha\beta \circ \alpha\Theta \circ \alpha\beta$, hence $x \ \alpha\beta \ j_1 \ \alpha\Theta \ j_2 \ \alpha\beta \ z$, for certain elements $j_1, j_2 \in F_{\mathcal{V}}(3)$. Notice that the condition $j_1 \ \alpha \ j_2$ is redundant, since it follows from the assumption that α is a congruence and from $j_1 \ \alpha \ x \ \alpha \ z \ \alpha \ j_2$. In fact, we shall not use $j_1 \ \alpha \ j_2$ explicitly.

Since we are working in the free algebra generated by x, y and z, we can think of j_1 and j_2 as ternary terms. Classically, the α-relations imply that j_1 and j_2 satisfy the equations (J_1) and (J_2) from (J), while the β-relations entail (J_L) and (J_R). Were Θ assumed to be a congruence, we had also (J_C), thus getting the Jónsson's condition for 3-distributivity. Since Θ is only assumed to be a tolerance, we have to proceed in a different fashion. It is easy to check that

$$\Theta = \{(w(x,y,z;y,z), w(x,y,z;z,y)) \mid w \text{ a 5-ary term of } \mathcal{V}\}. \tag{2.3}$$

Indeed, the relation defined by the condition on the right in (2.3) is obviously reflexive, admissible, symmetrical and contains (y,z). On the other hand, any reflexive, admissible and symmetrical relation containing (y,z) has to contain all the pairs appearing on the right in (2.3). Hence equality follows.

Since $j_1(x,y,z) \ \Theta \ j_2(x,y,z)$ by construction, then by (2.3) there exists some 5-ary term w such that

$$j_1(x,y,z) = w(x,y,z;y,z) \qquad \text{and} \qquad w(x,y,z;z,y) = j_2(x,y,z). \tag{2.4}$$

Since the equations in (2.4) involve only three variables and hold in $\mathbf{F}_{\mathcal{V}}(3)$, these equations hold throughout $\mathcal{V}$. Notice that in (2.4) we actually need three variables; two variables are not enough. Substituting the equations (2.4) in (J_L), (J_R), (J_1) and (J_2), we get respectively the equations (A), (B), (C) and (D) in (2.2). □

Remark 2.4. Of course, the proof of Lemma 2.3 provides an "if and only if" condition, since we can retrieve j_1 and j_2 from w, using the equations (2.4); then the classical homomorphism argument shows that these three terms witness that the identity (2.1) holds throughout $\mathcal{V}$. However, we do not need this argument, since the conclusion of Theorem 2.1 is much stronger; indeed, the existence of a majority term implies the tolerance identity $\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta$, actually, the relation identity $R(S \circ T) \subseteq RS \circ RT$. In other words, we get the quite surprising result that, globally, that is, within a variety, the locally weaker tolerance identity $\alpha(\beta \circ \Theta) \subseteq \alpha\beta \circ \alpha\Theta \circ \alpha\beta$ implies (and hence is equivalent to) the much stronger relation identity $R(S \circ T) \subseteq RS \circ RT$. Notice that the latter identity is stronger in two senses: first, we have 2 factors instead of 3 factors on the right; moreover, it is more general, since it deals with reflexive and admissible relations, rather than with tolerances or congruences.

But we are getting too far ahead! A proof of Theorem 2.1 is necessary in order to fully justify the above comment.

Proof of Theorem 2.1. Suppose that $\mathcal{V}$ satisfies the identity (2.1). By Lemma 2.3, we have a 5-ary term w satisfying the equations (2.2). Define

$$m(x, y, z) = w(w(x, y, z; y, z), w(x, y, z; z, y), y; z, w(x, z, y; y, z)). \tag{2.5}$$

Using (2.5) and (2.2) repeatedly, we get

$$\begin{aligned} m(x, x, z) &= w(w(x, x, z; x, z), w(x, x, z; z, x), x; z, w(x, z, x; x, z)) \\ &=^{(A)(B)(D)} w(x, z, x; z, x) =^{(C)} x, \\ m(x, y, x) &= w(w(x, y, x; y, x), w(x, y, x; x, y), y; x, w(x, x, y; y, x)) \\ &=^{(C)(D)(B)} w(x, x, y; x, y) =^{(A)} x, \\ m(x, z, z) &= w(w(x, z, z; z, z), w(x, z, z; z, z), z; z, w(x, z, z; z, z)) =^{(B)} z, \end{aligned}$$

where the superscripts indicate the specified equations of (2.2) that we have used, in the respective order. Hence m is a majority term and the theorem is proved. ☐

3. Relation identities valid in 3-distributive varieties

Since there are 3-distributive varieties without a majority term (see, e. g., Section 5 below), we have that 3-distributive varieties do not necessarily satisfy identity (2.1). On the other hand in the present section we show that 3-distributive varieties satisfy identities which are slightly weaker than (2.1), but very similar to it. For example, 3-distributive varieties do satisfy $R(S \circ \gamma) \subseteq RS \circ R\gamma \circ RS$ and $R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS$. Notice that if we take R, S and T to be congruences in the above identities, we get back 3-distributivity, hence the results are optimal.

As mentioned in the introduction, the results form Kazda, Kozik, McKenzie and Moore [13] and the observations in [18] show that, for every n, there is some (possibly quite large) k such that $\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha S \circ \dots$ (k factors) holds in every n-distributive variety. We show that in the case $n = 3$ the best possible value of k is 4, hence not particularly large; moreover, we can equivalently consider a reflexive and admissible relation R in place of the congruence α.

Theorem 3.1. *If $\mathcal{V}$ is a 3-distributive variety, then $\mathcal{V}$ satisfies*

$$R(S \circ T) \subseteq RS \circ RT \circ RT \circ RS, \tag{3.1}$$

$$R(S \circ T) \subseteq RS \circ RT \circ RS \circ RT, \text{ and} \tag{3.2}$$

$$R(S \circ \gamma) \subseteq RS \circ R\gamma \circ RS. \tag{3.3}$$

Recall the equations displayed in (J) at the beginning of the previous section. Before proving Theorem 3.1 and for later use, we shall state in a lemma those parts of the proof which do not need the equation (J_2). We first give a name to some elements which will be frequently used.

Convention 3.2. If a, b and c are elements of some algebra and j_1 and j_2 are ternary terms, we shall denote by e, f, g, $g^\bullet$ the following elements.

$$\begin{aligned}
e &= j_1(j_1(a,b,c),b,c),\\
f &= j_1(j_1(a,c,c),c,c) =^{(J_C)} j_2(j_2(a,c,c),c,c),\\
g &= j_2(j_2(a,c,c),j_2(b,c,c),c),\\
g^\bullet &= j_2(j_2(a,c,c),j_2(a,b,c),c).
\end{aligned}$$

Since the elements a, b and c and the terms j_1 and j_2 shall be kept fixed throughout this section, we do not indicate explicitly the dependence.

Lemma 3.3. *Under the above Convention, if $a\ R\ c$, $a\ S\ b\ T\ c$ and j_1 and j_2 satisfy (J_L), (J_C), (J_R) and (J_1), then the following relations hold.*

$$e\ R\ c, \qquad f\ R\ c, \qquad a\ RS\ e\ RT\ f\ T\ g\ S\ c, \qquad f\ S\ g^\bullet\ T\ c.$$

Proof. The proof is given by the following computations in which, for clarity, we underline the elements which are moved by the appropriate relations and, as usual by now, we indicate as superscripts the identities we are using.

$$\begin{aligned}
e &= j_1(j_1(\underline{a},b,c),b,c)\ R\ j_1(j_1(\underline{c},b,c),b,c) =^{(J_1)} c,\\
f &= j_2(j_2(\underline{a},c,c),c,c)\ R\ j_2(j_2(\underline{c},c,c),c,c) =^{(J_R)} c,\\
a &=^{(J_1)} j_1(j_1(a,b,\underline{a}),b,\underline{a})\ R\ j_1(j_1(a,b,\underline{c}),b,\underline{c}) = e,\\
a &=^{(J_L)} j_1(j_1(a,\underline{a},c),\underline{a},c)\ S\ j_1(j_1(a,\underline{b},c),\underline{b},c) = e,\\
e &=^{(J_R)} j_2(j_2(a,\underline{a},c),c,\underline{e})\ R\ j_2(j_2(a,\underline{c},c),c,\underline{c}) = f.\\
e &= j_1(j_1(a,\underline{b},c),\underline{b},c)\ T\ j_1(j_1(a,\underline{c},c),\underline{c},c) = f.\\
f &=^{(J_R)} j_2(j_2(a,c,c),j_2(b,\underline{b},c),c)\ T\ j_2(j_2(a,c,c),j_2(b,\underline{c},c),c) = g,\\
g &= j_2(j_2(\underline{a},c,c),j_2(b,c,c),c)\ S\ j_2(j_2(\underline{b},c,c),j_2(b,c,c),c) =^{(J_R)} c,\\
f &=^{(J_R)} j_2(j_2(a,c,c),j_2(a,\underline{a},c),c)\ S\ j_2(j_2(a,c,c),j_2(a,\underline{b},c),c) = g^\bullet,\\
g^\bullet &= j_2(j_2(a,c,c),j_2(a,\underline{b},c),c)\ T\ j_2(j_2(a,c,c),j_2(a,\underline{c},c),c) =^{(J_R)} c. \quad \square
\end{aligned}$$

Proof of Theorem 3.1. Suppose that $(a,c) \in R(S \circ T)$, hence $a\ R\ c$ and $a\ S$ $b\ T\ c$, for some b. By 3-distributivity and [11], we have Jónsson's terms j_1 and j_2 satisfying the displayed set of equations (J) at the beginning of the previous section. Because of Lemma 3.3, in order to prove (3.1) and (3.2) it is enough to prove $f \mathrel{R} g \mathrel{R} c$ and $f \mathrel{R} g^\bullet \mathrel{R} c$. We first prove $f \mathrel{R} j_2(a,c,c)$ and we shall also use $f \mathrel{R} c$, as proved in Lemma 3.3. Here are the computations.

$$f =^{(\mathrm{J}_R)} j_2(j_2(a,c,c),c,j_2(a,\underline{a},c)) \mathrel{R} j_2(j_2(a,c,c),c,j_2(a,\underline{c},c)) =^{(\mathrm{J}_2)} j_2(a,c,c),$$
$$f =^{(\mathrm{J}_2)} j_2(\underline{f},j_2(b,c,c),\underline{f}) \mathrel{R} j_2(\underline{j_2(a,c,c)},j_2(b,c,c),\underline{c}) = g,$$
$$g = j_2(j_2(\underline{a},c,c),j_2(b,c,c),c) \mathrel{R} j_2(j_2(\underline{c},c,c),j_2(b,c,c),c) =^{(\mathrm{J}_2)} c.$$
$$f =^{(\mathrm{J}_2)} j_2(\underline{f},j_2(a,b,c),\underline{f}) \mathrel{R} j_2(\underline{j_2(a,c,c)},j_2(a,b,c),\underline{c}) = g^\bullet,$$
$$g^\bullet = j_2(j_2(\underline{a},c,c),j_2(a,b,c),c) \mathrel{R} j_2(j_2(\underline{c},c,c),j_2(a,b,c),c) =^{(\mathrm{J}_2)} c.$$

Were $R\gamma$ a transitive relation, (3.3) would follow from (3.1) with $T = \gamma$. In the general case, since γ is supposed to be a congruence, hence transitive, we have $e \mathrel{\gamma} g$ from Lemma 3.3. In order to prove (3.3) it is then enough to show $e \mathrel{R} g$. We have already proved that $e \mathrel{R} c$, thus

$$e =^{(\mathrm{J}_R)} j_2(a,\underline{a},\underline{e}) \mathrel{R} j_2(a,\underline{c},\underline{c}), \text{ hence}$$
$$e =^{(\mathrm{J}_2)} j_2(\underline{e},j_2(b,c,c),\underline{e}) \mathrel{R} j_2(\underline{j_2(a,c,c)},j_2(b,c,c),\underline{c}) = g.$$

In the special case when R is a congruence, a direct simpler proof of (3.3) is possible. Take $R = \alpha$ and notice that if $(a,c) \in \alpha(S \circ \gamma)$, as witnessed by b, then $a \mathrel{\alpha S} j_1(a,b,c) \mathrel{\alpha\gamma} j_2(a,b,c) \mathrel{\alpha S} c$. For example, $j_1(a,b,c) \mathrel{\gamma} j_1(a,c,c) = j_2(a,c,c) \mathrel{\gamma} j_2(a,b,c)$. The rest is more direct and easier. Cf. also [17, Remark 17]. $\square$

Remark 3.4. As we mentioned, if we take R, S and T congruences in the identities (3.1) and (3.3), we get back 3-distributivity. On the other hand, if we take R, S and T congruences in (3.2), we get only 4-distributivity. This suggests that perhaps, for general n-distributive varieties, one gets cleaner results when considering identities of the form

$$\alpha(S \circ T) \subseteq \alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \alpha S \circ \alpha T \dots,$$

rather than identities of the form (1.1). Compare also identities (4.11) - (4.13) below. More generally, there is the possibility of considering identities like

$$\alpha(R_1 \circ R_2) \subseteq \alpha R_{f(0)} \circ \alpha R_{f(1)} \circ \alpha R_{f(2)} \circ \dots, \tag{3.4}$$

where f is a function with codomain $\{1,2\}$. Some arguments presented in [20, Section 4] suggest that considering identities like (3.4) is a very natural choice.

Two Gumm terms. Let us say that a variety $\mathcal{V}$ has 2 *Gumm terms* if $\mathcal{V}$ has terms j_1 and j_2 satisfying the identities (J_L), (J_C), (J_R) and (J_1) from (J) in Section 2; thus we are leaving out identity (J_2), as we did in Lemma 3.3. Sometimes a variety as above is said to have 3 Gumm terms, since a trivial projection j_0 onto the first coordinate is counted. Notice that some authors give the definition of Gumm terms with the ordering of variables and of terms reversed. This applies to some papers of ours, too, but here it is more convenient to maintain the analogy with the common condition for 3-distributivity. A variety $\mathcal{V}$ has 2 Gumm terms if and only if $\mathcal{V}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha(\gamma \circ \beta)$. Notice that if we take converses and exchange β and γ, we get $\alpha(\beta \circ \gamma) \subseteq \alpha(\gamma \circ \beta) \circ \alpha\gamma$. See, e. g., [7, 13, 18, 22] for further details and information.

If S is a reflexive binary relation on some algebra, we denote by $\overline{S}$ the smallest admissible relation containing S. In the following theorem we shall provide bounds for expressions like, say, $R(S \circ T) \circ RW$, rather than simply $R(S \circ T)$. If 0 denotes the minimal congruence, i.e., the identity relation, then one can always obtain the latter expression from the former, by taking $W = 0$. The reader might always assume to be in the simpler situation when $W = 0$ with just an exception. We need the more involved formula (3.5) below with the actual presence of RW in order to get the relation $R(S \circ T) \circ R(S \circ T) \subseteq RS \circ RT \circ R(S \circ T)$ in the course of the proof of Corollary 3.7 below. Theorem 3.1, too, allows a version with an RW factor added; we have not included this more general version in the statement for simplicity and since we do not need it here. In detail, arguing as in the proof of identities (3.6) and (3.7) below, we get that a 3-distributive variety satisfies $R(S \circ T) \circ RW \subseteq RS \circ RT \circ RT \circ R(\overline{S \cup W})$, as well as $R(S \circ T) \circ RW \subseteq RS \circ RT \circ RS \circ R(\overline{T \cup W})$.

Theorem 3.5. *If $\mathcal{V}$ has 2 Gumm terms, then $\mathcal{V}$ satisfies*

$$R(S \circ T) \circ RW \subseteq RS \circ RT \circ R(\overline{S \cup T \cup W}), \tag{3.5}$$

$$R(S \circ RT) \circ RW \subseteq RS \circ RT \circ RT \circ R(\overline{S \cup W}), \tag{3.6}$$

$$R(RS \circ T) \circ RW \subseteq RS \circ RT \circ RS \circ R(\overline{T \cup W}). \tag{3.7}$$

Proof. Suppose that $(a, c^\bullet) \in R(S \circ T) \circ RW$ with $a \ R \ c$ and $a \ S \ b \ T \ c \ RW \ c^\bullet$. By Lemma 3.3 we have $a \ RS \ e \ RT \ f$, hence in order to prove (3.5) it is enough to compute

$$f =^{(\mathrm{J}_R)} j_2(j_2(\underline{a}, c, c), j_2(b, \underline{b}, c), \underline{c}) \ \overline{S \cup T \cup W} \ j_2(j_2(\underline{b}, c, c), j_2(b, \underline{c}, c), \underline{c}^\bullet) =^{(\mathrm{J}_R)} c^\bullet,$$

$$f = j_2(j_2(\underline{a}, c, c), c, \underline{c}) \ R \ j_2(j_2(\underline{c}, c, c), c, \underline{c}^\bullet) =^{(\mathrm{J}_R)} c^\bullet.$$

To prove (3.6), take $T = RT$ in Lemma 3.3, thus $f \ RT \ g$. We can also assume $b \ R \ c$, hence

$$g = j_2(j_2(\underline{a}, c, c), j_2(\underline{b}, c, c), \underline{c}) \ R \ j_2(j_2(\underline{c}, c, c), j_2(\underline{c}, c, c), c^\bullet) =^{(\mathrm{J}_R)} c^\bullet,$$

$$g = j_2(j_2(\underline{a}, c, c), j_2(b, c, c), \underline{c}) \ \overline{S \cup W} \ j_2(j_2(\underline{b}, c, c), j_2(b, c, c), \underline{c}^\bullet) =^{(\mathrm{J}_R)} c^\bullet.$$

The proof of (3.7) is similar, using the element $g^\bullet$. This time we assume $S = RS$ and $a \ R \ b$. The relations still needing a proof are

$$g^\bullet = j_2(j_2(\underline{a}, c, c), j_2(\underline{a}, b, c), \underline{c}) \ R \ j_2(j_2(\underline{c}, c, c), j_2(\underline{b}, b, c), \underline{c}^\bullet) =^{(\mathrm{J}_R)} c^\bullet,$$

$$g^\bullet = j_2(j_2(a, c, c), j_2(a, \underline{b}, c), c) \ \overline{T \cup W} \ j_2(j_2(a, c, c), j_2(a, \underline{c}, c), \underline{c}^\bullet) =^{(\mathrm{J}_R)} c^\bullet. \quad \square$$

Remark 3.6. Using the methods in the proof of Lemma 2.3 we could have given Maltsev conditions for the identities (3.1) - (3.5) by means of the existence of terms satisfying appropriate equations. Then, following the lines of the proofs of Theorems 3.1 and 3.5, one can construct explicitly these terms in function of j_1 and j_2. In the case at hand, dealing directly with relation identities seems simpler. However, there is an aspect dealing with terms which deserves mention.

The proofs of Theorems 3.1 and 3.5 implicitly use the terms $d_1(x,y,z) = j_1(j_1(x,y,z),y,z)$ and $d_2(x,y,z) = j_2(j_2(x,z,z),j_2(x,y,z),z)$. If j_1 and j_2 are Jónsson terms, then the terms d_1 and d_2 constitute a set of two *directed Jónsson terms* [13, 29], namely, they satisfy the equations

$$\begin{aligned} &x = d_1(x,x,z), \qquad d_1(x,z,z) = d_2(x,x,z), \qquad d_2(x,z,z) = z,\\ &x = d_1(x,y,x), \qquad\qquad\qquad\qquad\qquad\qquad\quad\; d_2(x,y,x) = x. \end{aligned} \tag{3.8}$$

Compare the middle equation in (3.8) with (J_C). Moreover, in comparison with Remark 2.2, notice that here there is full symmetry between d_1 and d_2.

Among other, we have showed that a 3-distributive variety has two directed Jónsson terms. This is the poor man's version of the far more general result proved by Kazda, Kozik, McKenzie and Moore [13] that *every* congruence distributive variety has a (generally longer) set of directed Jónsson terms.

Yet it is interesting to check which parts of Theorem 3.1 follow just from the existence of two directed Jónsson terms. The existence of two directed Jónsson terms implies identity (3.2), limited to the case when R is assumed to be a tolerance, by the case $\ell = 2$ in equation (3.1) in [18, Proposition 3.1]. Notice that the number of terms is counted in a different way in [18], including also two trivial projections; here we adopt the counting convention from [13].

On the other hand, there are varieties having two directed Jónsson terms and which are not 3-distributive: see [18, p. 11] and [21]. Since the identities (3.1) and (3.3) do imply 3-distributivity, we get that the existence of two directed Jónsson terms does not imply (3.1) and (3.3). In particular, our arguments go far beyond the simple proof of the existence of directed Jónsson terms and exploit special particularities of 3-distributive varieties.

Similar remarks apply to Theorem 3.5, with reference to *directed Gumm terms* [13].

If R is a binary relation, we let R^* denote the transitive closure of R.

Corollary 3.7. *(1) A* 3*-distributive variety satisfies*

$$R^*(S \circ T)^* = (RS \circ RT)^* \text{ and, more generally,} \tag{3.9}$$

$$(R \circ V)^*(S \circ T)^* = (RS \circ RT \circ VS \circ VT)^*. \tag{3.10}$$

(2) If $\mathcal{V}$ has 2 *Gumm terms, then $\mathcal{V}$ satisfies*

$$R^*(S \circ T)^* = (RS \circ RT)^* \circ R(\overline{S \cup T}), \tag{3.11}$$

$$R^*(RS \circ T)^* = (RS \circ RT)^*, \text{ and} \tag{3.12}$$

$$R^*S^* = (RS)^*. \tag{3.13}$$

Proof. We first prove identity (3.13) in (2). Of course, (3.13) is the particular case $S = T$ of (3.11), however, it seems convenient to obtain the latter identity as a consequence of the former, hence let us prove (3.13). Let S^n denote the iterated relational composition of S with itself with a total of n factors and recall that 0 denotes the minimal congruence. By taking $W = 0$ and S^n in place of both S and T in (3.5), we get $RS^{2n} = R(S^n \circ S^n) \subseteq RS^n \circ RS^n \circ RS^n$. Since $RS^* = \bigcup_{m\in\omega} RS^m$, we get by induction that $RS^* \subseteq (RS)^*$. By taking R^* in place of R in the above identity and then exchanging the role of R and S we get $R^*S^* \subseteq (R^*S)^* \subseteq ((RS)^*)^* = (RS)^*$. The reverse inclusion in (3.13) is trivial.

Having proved (3.13), then by taking $S \circ T$ in place of S in (3.13), we get $R^*(S \circ T)^* = (R(S \circ T))^*$. By taking $W = S \circ T$ in (3.5), we get $R(S \circ T) \circ R(S \circ T) \subseteq RS \circ RT \circ R(S \circ T)$. Then by induction, always factoring out the first compound factor on the left, we get

$$(R(S \circ T))^* \subseteq (RS \circ RT)^* \circ R(S \circ T). \tag{3.14}$$

Applying again (3.5) with $W = 0$, we get $R(S \circ T) \subseteq RS \circ RT \circ R(\overline{S \cup T})$. Summing everything up, we get the $\subseteq$ inclusion in (3.11). The reverse inclusion is trivial, since $S \circ T \supseteq \overline{S \cup T}$.

Applying (3.7) with $W = 0$, we get $R(RS \circ T) \subseteq RS \circ RT \circ RS \circ RT$. By (3.14) with RS in place of S we get the $\subseteq$ inclusion in (3.12). Again, the reverse inclusion is trivial, We have proved (2).

Now (3.9) is immediate from (3.11) and (3.2). Identity (3.10) follows by repeated applications of (3.9). Of course, identity (3.9) can be also proved directly from (3.2) by using only a minimal part of the above arguments. □

Further comments. Theorems 3.1 and 3.5 can be improved in many ways. For example, notice that the "middle" element $f = j_1(j_1(a, c, c), c, c)$ in the proof of Theorem 3.1 does not depend on b. Hence the proof provides a bound for $R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3)\dots$. Moreover, the element f is the same in the proofs of 3.1 and 3.5. In addition, the arguments in the proof of Theorem 3.5 clearly apply to 3-distributive varieties, as well (here take $c^\bullet = c$ or, which is the same, $W = 0$). Moreover, in the notations from the proof of Theorem 3.1, we have $a =^{(\mathrm{J}_L)} j_1(j_1(a, \underline{a}, c), \underline{a}, c)\ R\ j_1(j_1(a, \underline{c}, c), \underline{c}, c) = f$. Summing everything up, we get items (1) and (2) in the following proposition.

As for (3) below, the definition of two directed Jónsson terms recalled in (3.8) from Remark 3.6 can be extended to any chain of terms; see [13] for full details. Then, merging the above arguments with the proof of [18, Proposition 3.1], we get a proof of (3) below.

Proposition 3.8. (1) *A* 3*-distributive variety satisfies*

$$R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3)\dots \subseteq R(RS_1 \circ RT_1)(RS_2 \circ RT_2)(RS_3 \circ RT_3)\dots \circ$$
$$R(RS_1 \circ RT_1)(RT_1 \circ RS_1)(RS_2 \circ RT_2)(RT_2 \circ RS_2)\dots(\overline{S_1 \cup T_1})(\overline{S_2 \cup T_2})\dots$$

(2) *A variety with* 2 *Gumm terms satisfies*

$$R(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3) \dots \subseteq R(RS_1 \circ RT_1)(RS_2 \circ RT_2)(RS_3 \circ RT_3) \dots \circ R(\overline{S_1 \cup T_1})(\overline{S_2 \cup T_2})(\overline{S_3 \cup T_3}) \dots$$

(3) *A variety with* n *(*$n+2$ *in the terminology from* [18]*) directed Jónsson terms satisfies*

$$\Theta(S_1 \circ T_1)(S_2 \circ T_2)(S_3 \circ T_3) \dots \subseteq (\Theta(\Theta S_1 \circ \Theta T_1)(\Theta S_2 \circ \Theta T_2)(\Theta S_3 \circ \Theta T_3) \dots)^n.$$

By Corollary 3.7 we get an affirmative solution to Problem 2.12 in Gyenizse and Maróti [6] in the special cases of 3-distributive varieties and of varieties with 2 Gumm terms. Recall that a *preorder* is a reflexive and transitive relations.

Corollary 3.9. *If* $\mathbf{A}$ *belongs to a* 3*-distributive variety (a variety with* 2 *Gumm terms) then the lattice of admissible preorders of* $\mathbf{A}$ *is congruence distributive (respectively, congruence modular).*

Proof. Immediate from identities (3.9) and (3.12). □

4. Assuming n-permutabiliy

The length of a chain of iterated relational compositions is bounded in a congruence n-permutable variety. Hence, assuming also congruence distributivity, we get bounds for the value of k in the identity (1.1) from the introduction. This observation shall be dealt with in Theorem 4.5 below. Since we are dealing with admissible relations, not congruences, the bound is not exactly n, in general. See Corollary 4.4 below. Recall that a variety $\mathcal{V}$ is *congruence* n*-permutable* if the congruence identity $\beta \circ \gamma \circ \beta \dots = \gamma \circ \beta \circ \gamma \dots$ (n factors on each side) holds in $\mathcal{V}$.

We shall use the following conventions, in order to make the notation more compact.

Convention 4.1. The expression $S \circ T \circ \overset{m}{\dots}$ denotes $S \circ T \circ S \dots$ with m factors, that is, with $m-1$ occurrences of $\circ$. If, say, m is odd, we sometimes write $S \circ T \circ \overset{m}{\dots} \circ S$ when we want to make clear that S is the last factor in the expression. We shall use the above notation even when R or T are replaced with compound factors, or even when they do not strictly alternate, such as in the expression

$$\alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \alpha S \circ \alpha T \circ \overset{m}{\dots} \tag{4.1}$$

In any case, the number above the dots always indicates the number of $\circ$'s minus one, that is, the number of "factors", with the provision that compound factors are always counted as one factor. For example, if $m = 10$ in (4.1), the expression reads

$$\alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \alpha S \circ \alpha T.$$

In any case, when we write only two factors on the right as in $\alpha S \circ \alpha T \circ \overset{m}{\ldots}$ or in $\alpha S \circ \alpha T \circ \overset{2m-1}{\ldots} \circ \alpha S$, we always mean that these two factors alternate as $\alpha S \circ \alpha T \circ \alpha S \circ \alpha T \ldots$

Moreover, recall that if R is a reflexive binary relation on some algebra, $\overline{R}$ denotes the smallest admissible relation containing R and that R^* denotes the transitive closure of R. Recall also the definition of R^m. In the above notation, R^m is $R \circ R \circ \overset{m}{\ldots}$.

Parts of the following proposition are known. See, e. g., Werner [28], Hagemann and Mitschke [8] and reference [3] quoted there. However it seems that some parts of the proposition are new.

Proposition 4.2. *Suppose that $n \geq 2$ and $\mathcal{V}$ is a variety. Each of the following identities is equivalent to congruence n-permutability of $\mathcal{V}$.*

$$R_1 \circ R_2 \circ \ldots \circ R_n \subseteq \overline{R_1 \cup R_2} \circ \overline{R_2 \cup R_3} \circ \ldots \circ \overline{R_{n-1} \cup R_n}\,, \tag{4.2}$$

$$R^n \subseteq R^{n-1}, \tag{4.3}$$

$$R^* = R^{n-1}, \tag{4.4}$$

$$(S \circ T)^* = S \circ T \circ S \circ T \circ \overset{2n-2}{\ldots} \circ S \circ T, \tag{4.5}$$

$$(S \circ T)^* = S \circ T \circ T \circ S \circ S \circ T \circ \overset{2n-2}{\ldots}, \tag{4.6}$$

$$S \circ T \circ T \circ S \circ S \circ T \circ \overset{2n-2}{\ldots} = T \circ S \circ S \circ T \circ T \circ S \circ \overset{2n-2}{\ldots}, \tag{4.7}$$

$$S \circ T \circ \overset{2n-1}{\ldots} \circ S \subseteq T \circ S \circ \overset{2n-2}{\ldots} \circ S, \tag{4.8}$$

$$S \circ T \circ \overset{2n-1}{\ldots} \circ S = T \circ S \circ \overset{2n-1}{\ldots} \circ T. \tag{4.9}$$

In all the above identities we can equivalently let $R_1, \ldots, R, S, T$ be either reflexive and admissible relations or tolerances.

Moreover, identities (4.2)*,* (4.6) *and* (4.7) *are still equivalent to congruence n-permutability if we let $R_1, \ldots, S, T$ be congruences.*

Proof. We shall give a proof which works in both cases, either when $R_1, \ldots, R$, S, T are reflexive and admissible relations or when they are tolerances. Notice that in the identities (4.4) - (4.6) the left-hand side is always larger than the right-hand side, hence it is enough to prove the reverse inclusion. Let (n-perm) denote congruence n-permutability. We shall prove that (n-perm) $\Rightarrow$ (4.2) for varieties and that (4.2) $\Rightarrow$ (4.3) $\Rightarrow$ (4.4) $\Rightarrow$ (4.6) $\Rightarrow$ (4.7) $\Rightarrow$ (n-perm) and (4.4) $\Rightarrow$ (4.5) $\Rightarrow$ (4.8) $\Rightarrow$ (4.9) $\Rightarrow$ (4.3) hold in every algebra.

If, say, $\mathcal{V}$ is a congruence 3-permutable variety, then, by [8], there are terms t_1 and t_2 such that $x = t_1(x, y, y)$, $t_1(x, x, y) = t_2(x, y, y)$ and $t_2(x, x, y) = y$ are equations valid in $\mathcal{V}$. If $a\ R_1\ b\ R_2\ c\ R_3\ d$, then $a = t_1(\underline{a}, b, \underline{b})\ \overline{R_1 \cup R_2}\ t_1(\underline{b}, b, \underline{c}) = t_2(\underline{b}, c, \underline{c})\ \overline{R_2 \cup R_3}\ t_2(\underline{c}, c, \underline{d}) = d$, hence (4.2) follows in the case $n = 3$. In general, for $n \geq 2$, the same argument shows that every congruence n-permutable variety satisfies (4.2).

Taking $R_1 = R_2 = \cdots = R$ in (4.2), we get (4.3). Then (4.4) follows from a trivial induction.

In order to prove that (4.4) implies (4.6), observe that $S \circ T$ is a reflexive and admissible relation which contains both S and T, thus $S \circ T \supseteq \overline{S \cup T}$. Since $S, T \subseteq \overline{S \cup T}$, hence $S \circ T \subseteq \overline{S \cup T} \circ \overline{S \cup T}$, we get $(S \circ T)^* = (\overline{S \cup T})^*$. By taking the admissible relation $\overline{S \cup T}$ in place of R in (4.4) and using alternatively $\overline{S \cup T} \subseteq S \circ T$ and $\overline{S \cup T} \subseteq T \circ S$, we get (4.6). Notice that if S and T are tolerances, then $R = \overline{S \cup T}$ is a tolerance, hence the proof works also in the case when we are dealing with tolerances. On the other hand, notice that $S \circ T$ is not necessarily a tolerance, even when S and T are.

The implication (4.6) $\Rightarrow$ (4.7) is obvious, since $(S \circ T)^* = (T \circ S)^*$.

By taking S and T congruences in (4.7), we get $\alpha \circ \beta \circ \overset{n}{\ldots} = \beta \circ \alpha \circ \overset{n}{\ldots}$, that is, congruence n-permutability. Notice that, since congruences are transitive, $n - 2$ factors annihilate on each side of (4.7), hence we end up with n actual factors on each side. Notice that we have only used transitivity of tolerances, we do not need symmetry.

The proof that (4.4) implies (4.5) is similar to the proof that (4.4) implies (4.6). Take $R = \overline{S \cup T}$ and always use $\overline{S \cup T} \subseteq S \circ T$. Notice that, were S and T reflexive and admissible relations, we could have proceeded in a simpler way, since $S \circ T$ is reflexive and admissible, too, hence (4.5) is the special case of (4.4) obtained by considering $S \circ T$ in place of R. However, the previous argument works also for tolerances.

The implication (4.5) $\Rightarrow$ (4.8) is trivial, exchanging the role of S and T, since $S \circ T \circ \overset{2n-1}{\ldots} \circ S \subseteq (S \circ T)^*$.

By identity (4.8) we get $S \circ T \circ \overset{2n-1}{\ldots} \circ S \subseteq T \circ S \circ \overset{2n-2}{\ldots} \circ S \subseteq T \circ S \circ \overset{2n-1}{\ldots} \circ S \circ T$, thus by symmetry we get equality of the outer expressions, that is, (4.9).

Recall that 0 denotes the minimal congruence. Taking $S = R$ and $T = 0$ in (4.9) we get back (4.3).

Now we prove the last statement. If some identity holds for reflexive and admissible relations, then it trivially holds for congruences, hence congruence n-permutable varieties satisfy the versions of (4.2), (4.6) and (4.7) when the variables range on such a restricted scope. We already mentioned that (4.7) relative to congruences does imply congruence n-permutability and the argument is the same for (4.6), namely, take S and T congruences. Finally, if $R_1 = R_3 = \cdots = \alpha$ and $R_2 = R_4 = \cdots = \beta$ are congruences, then (4.2) implies $\alpha \circ \beta \circ \overset{n}{\ldots} \subseteq (\overline{\alpha \cup \beta})^{n-1} \subseteq \beta \circ \alpha \circ \alpha \circ \beta \circ \beta \circ \alpha \circ \overset{2n-2}{\ldots} = \beta \circ \alpha \circ \overset{n}{\ldots}$. $\square$

Remarks 4.3. (a) The proof of Proposition 4.2 shows that, for every n, the identities (4.3) - (4.6), (4.8) - (4.9) are equivalent for every algebra. Indeed, identity (4.6) implies (4.4), by taking $T = 0$, all the rest has been mentioned in the proof of 4.2.

(b) We now justify the assertion in the introduction that the equivalence between congruence permutability and the identity $R \circ R = R$ is non trivial, in the sense that it holds for varieties, but not for single algebras.

On one hand, in every algebra, the identity $R \circ R = R$ for relations, or even the identity $\Theta \circ \Theta = \Theta$ for tolerances, do imply congruence permutability.

This follows from the implication (4.3) $\Rightarrow$ (4.6) in Proposition 4.2 in the case $n = 2$. More directly. if α and β are congruences, take $\Theta = \overline{\alpha \cup \beta}$, the smallest admissible relation containing α and β, getting $\alpha \circ \beta \subseteq \Theta \circ \Theta = \Theta = \overline{\alpha \cup \beta} \subseteq \beta \circ \alpha$. Compare the proof of [17, Proposition 13(a)].

On the other hand, let $A = \{0, 1, 2\}$ and $\mathbf{A}$ be the algebra on A with only one unary operation g defined by $g(0) = 1$, $g(1) = g(2) = 2$. The lattice $\mathbf{Con}(\mathbf{A})$ has 3 elements, hence consists of pairwise permutable congruences, the relation $R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2)\}$ is reflexive and admissible, but $R \circ R = R$ fails.

Corollary 4.4. *For every variety $\mathcal{V}$ and every $n \geq 2$, the following conditions are equivalent.*

(1) *$\mathcal{V}$ is congruence n-permutable.*
(2) *$\mathcal{V}$ is admissible-preorder n-permutable.*
(3) *$\mathcal{V}$ is admissible-relation $2n-1$-permutable.*
(4) *$\mathcal{V}$ is tolerance $2n-1$-permutable.*

Proof. The equivalence of (1) and (2) follows from identity (4.7).

The equivalence of (1), (3) and (4) follows from identity (4.9). ☐

In particular, apart from the degenerate case, congruence n-permutability is equivalent to admissible-relation n-permutability if and only if $n = 2$. Notice that the above results do not exactly clarify the meaning of admissible-relation n-permutability for n even. In fact, by identity (4.5) in Proposition 4.2, we get that every congruence n-permutable variety satisfies $S \circ T \circ S \circ T \circ \overset{2n-2}{\dots} = T \circ S \circ T \circ S \circ \overset{2n-2}{\dots}$; however, we do not know the exact consequences of this last identity.

Recall the notational conventions established in the abstract and in 4.1.

Theorem 4.5. *Suppose that $n \geq 2$ and $\mathcal{V}$ is a variety. Each of the following identities is equivalent to the conjunction of congruence distributivity and n-permutability of $\mathcal{V}$.*

$$(\Theta(S \circ T)^*)^* = \Theta S \circ \Theta T \circ \Theta S \circ \Theta T \circ \overset{2n-2}{\dots} \circ \Theta S \circ \Theta T, \tag{4.10}$$

$$(\Theta(S \circ T)^*)^* = \Theta S \circ \Theta T \circ \Theta T \circ \Theta S \circ \Theta S \circ \Theta T \circ \overset{2n-2}{\dots}, \tag{4.11}$$

$$\Theta(S \circ T \circ T \circ S \circ S \circ \overset{2n-2}{\dots}) \subseteq \Theta T \circ \Theta S \circ \Theta S \circ \Theta T \circ \Theta T \circ \Theta S \circ \overset{2n-2}{\dots}, \tag{4.12}$$

$$\alpha(S \circ T \circ T \circ S \circ S \circ \overset{2n-2}{\dots}) = \alpha T \circ \alpha S \circ \alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \overset{2n-2}{\dots}, \tag{4.13}$$

$$\alpha(S \circ T \circ \overset{2n-1}{\dots} \circ S) = \alpha T \circ \alpha S \circ \overset{2n-1}{\dots} \circ \alpha T. \tag{4.14}$$

The variables S and T can be equivalently taken to be tolerances in (4.10) - (4.14) *and congruences in* (4.11) - (4.13).

Proof. The $\supseteq$ inclusions in (4.10), (4.11) and (4.13) are obvious. Notice that in (4.13) we need transitivity of α.

If $\mathcal{V}$ is congruence distributive, then, by Kazda, Kozik, McKenzie and Moore [13] and by [18, Proposition 3.1], $\Theta(S \circ T)^* \subseteq (\Theta S \circ \Theta T)^*$. If in addition $\mathcal{V}$ is congruence n-permutable, we can apply the identities (4.5), (4.6) and (4.9) with ΘS and ΘT in place of, respectively, S and T, obtaining a bounded number of factors on the right. Hence (4.10) - (4.14) follow from the assumptions. To prove the $\supseteq$ inclusion in (4.14) use (4.9) in order to get $\alpha T \circ \alpha S \circ \overset{2n-1}{\dots} \circ \alpha T = \alpha S \circ \alpha T \circ \overset{2n-1}{\dots} \circ \alpha S \subseteq \alpha(S \circ T \circ \overset{2n-1}{\dots} \circ S)$, again since α is transitive.

Conversely, by taking Θ, S and T congruences in each of (4.10) - (4.14), we get identities implying congruence distributivity by [11]. If we take $\Theta = \alpha = 1$, that is, the largest congruence, in (4.10) - (4.14), we get one of (4.5) - (4.7) or (4.9). These identities imply congruence n-permutability by Theorem 4.2. □

Theorem 4.6. *Suppose that $n \geq 2$ and $\mathcal{V}$ is a variety. Each of the following identities is equivalent to the conjunction of congruence modularity and n-permutability of $\mathcal{V}$.*

$$\Theta R^n \subseteq (\Theta R)^{n-1}, \tag{4.15}$$

$$\alpha R^* = (\alpha R)^{n-1}, \tag{4.16}$$

$$\alpha(S \circ \alpha T)^* = \alpha S \circ \alpha T \circ \alpha S \circ \alpha T \circ \overset{2n-2}{\dots} \circ \alpha S \circ \alpha T, \tag{4.17}$$

$$\alpha(S \circ \alpha T)^* = \alpha S \circ \alpha T \circ \alpha T \circ \alpha S \circ \alpha S \circ \alpha T \circ \overset{2n-2}{\dots}, \tag{4.18}$$

$$\alpha(S \circ \alpha T \circ \overset{2n-1}{\dots} \circ \alpha T \circ S) = \alpha T \circ \alpha S \circ \overset{2n-1}{\dots} \circ \alpha T. \tag{4.19}$$

The variables R, S and T can be equivalently taken to be tolerances in (4.15) - (4.19) *and congruences in* (4.18).

Proof. We have from [19, Theorem 1] that a congruence modular variety satisfies $\Theta R^* \subseteq (\Theta R)^*$. Again, this result relies heavily on [13]. By congruence n-permutability and (4.4) we get the nontrivial inclusions in (4.15) and (4.16).

By taking $R = S \circ \alpha T$ in (4.16) we get $\alpha(S \circ \alpha T)^* = (\alpha(S \circ \alpha T))^{n-1}$. But $\alpha(S \circ \alpha T) = \alpha S \circ \alpha T$, hence we get $\alpha(S \circ \alpha T)^* = (\alpha S \circ \alpha T)^*$. Then, as usual by now, by applying Theorem 4.2 with αS in place of S and αT in place of T, we get the nontrivial inclusions in (4.17) - (4.19).

Taking $R = \beta \circ \alpha\gamma$ in (4.16) and using again $\alpha(S \circ \alpha T) = \alpha S \circ \alpha T$ we get congruence modularity. Using [2], we prove in a similar way that (4.15) implies congruence modularity.

The above arguments work when R, S and T are assumed to be reflexive and admissible relations. If we assume that $R = \Theta$, $S = \Psi$ and $T = \Phi$ are tolerances, argue as above using $R = \Theta = \alpha\Phi \circ \Psi \circ \alpha\Phi$ or $R = \Theta = \alpha\gamma \circ \beta \circ \alpha\gamma$ notice that these are tolerances and that $\alpha(\alpha\Phi \circ \Psi \circ \alpha\Phi) = \alpha\Phi \circ \alpha\Psi \circ \alpha\Phi$.

By taking S and T congruences, we have that (4.17) - (4.19), too, imply congruence modularity. As in the proof of Theorem 4.5, the identities (4.15) - (4.19) imply congruence n-permutability, taking $\Theta = \alpha = 1$ and considering the corresponding identities in Theorem 4.2. □

Remarks 4.7. (a) There are obviously many other intermediate equivalent conditions in Proposition 4.2, Theorems 4.5 and 4.6. In order to keep the statements within a reasonable length, we have not explicitly stated such equivalent conditions. For example, for every $m \geq n$, the identity $R^m \subseteq R^{n-1}$ is equivalent to congruence n-permutability, as follows from the proof of 4.2. More generally, we can equivalently augment the number of factors on the left in (4.7) - (4.9), (4.12) - (4.14), provided we replace $=$ with $\subseteq$. Similarly, we can replace transitive closure with a sufficiently long iteration of compositions in each of (4.4) - (4.6), (4.10), (4.11). Moreover, in (4.10) - (4.11) we can replace $(\Theta(S \circ T)^*)^* = \Theta S \circ \Theta T \circ \dots$ with either $(\Theta(S \circ T))^* = \Theta S \circ \Theta T \circ \dots$ or $\Theta(S \circ T)^* \subseteq \Theta S \circ \Theta T \circ \dots$ Similar remarks apply to Theorem 4.6. We leave details to the interested reader.

(b) Congruence identities characterizing the conjunction of n-permutabilty and distributivity or modularity appeared in [16, Propositions 4 and 5].

(c) By equations (3.9) [(3.13) and (3.12)] in Corollary 3.7 we have that if the assumption of congruence distributivity [congruence modularity] is strengthened to 3-distributivity [to the existence of two Gumm terms], then we can allow α and Θ to be reflexive and admissible relations in the identities (4.10) - (4.14) [(4.15) - (4.19)], provided we replace $=$ by $\subseteq$.

5. Implication algebras and a 4-ary near-unanimity term

We denote by $+$, $\cdot$ and $'$ the operations of a Boolean algebra. The variety $\mathcal{I}$ of *implication algebras* is the variety generated by polynomial reducts of Boolean algebras in which $i(x,y) = xy'$ is the only basic operation. Equivalently, $\mathcal{I}$ is the variety of algebras with a binary operation i which satisfies all the equations satisfied by the term xy' in Boolean algebras. A more frequent description of implication algebras uses the term $x' + y$, instead, but Boolean duality implies that (if we reverse the order of variables) we get the same variety. Mitschke [24] showed that $\mathcal{I}$ is 3-distributive, not 2-distributive, congruence 3-permutable and not congruence permutable.

In the above notations, $i(x, i(y,z))$ represents the Boolean term $f(x,y,z) = x(yz')' = x(y'+z)$. Sometimes it is simpler to deal with the corresponding reduct $\mathcal{I}^-$ of Boolean algebras. Namely, $\mathcal{I}^-$ is the variety generated by reducts of Boolean algebras having f as the only basic operation. The varieties $\mathcal{I}$ and $\mathcal{I}^-$ have many properties in common, for example, $\mathcal{I}^-$ is still 3-distributive, congruence 3-permutable and, obviously, not 2-distributive and not congruence permutable. The terms $j_1 = f(x, f(x,y,z), z) = x(x' + yz' + z) = x(y+z)$ and $j_2 = f(z,y,x) = z(y'+x)$ are Jónsson terms witnessing 3-distributivity of $\mathcal{I}^-$. The terms f and j_2 are Hagemann-Mitschke terms [8] for congruence 3-permutability of $\mathcal{I}^-$.

On the other hand, $\mathcal{I}^-$ is much simpler to deal with. For example, free algebras in $\mathcal{I}^-$ are much smaller and more easily described than free algebras in $\mathcal{I}$. Further details about $\mathcal{I}^-$ can be found in the former version [23] of this note. Now [23] is largely subsumed by the present version.

By Theorem 4.5 and the above comments, both $\mathcal{I}$ and $\mathcal{I}^-$ satisfy the identities (4.10) and (4.11) with four factors on the right and moreover we can take an admissible relation R in place of Θ in these identities, by Remark 4.7(c). A direct proof of related identities appeared in [23]. By Theorem 2.1, we get that $\mathcal{I}$ fails to satisfy the identity (2.1). Again, a direct proof appeared in [23]. Some features of the counterexamples presented in [23] are reported below.

Example 5.1. Let $\mathbf{2}$ be the 2-elements Boolean algebra and consider the following elements of 2^5: $x = (1,1,1,0,0)$; $y = (1,0,0,1,0)$ and $z = (0,1,0,1,1)$. Let A be the subset of 2^5 consisting of those elements which are $\leq$ than at least one among x, y or z. Then $\mathbf{A} = (A, i)$ is an implication algebra. Let α be the kernel of the second projection, β be the intersection of the kernels of the first and of the fifth projections, γ be the intersection of the kernels of the third and of the fourth projections.

Let Ψ be the binary relation on A defined as follows: two elements $a, b \in A$ are Ψ-related if and only if at least one of the following conditions holds:

(a) both $a \leq x$ and $b \leq x$, or

(b) (either $a \leq y$ or $a \leq z$, possibly both) and (either $b \leq y$ or $b \leq z$, possibly both).

The relation Ψ is trivially symmetric; Ψ is also reflexive, since, by construction, every element of A is $\leq$ than either x, y or z. We claim that Ψ is admissible in $\mathbf{A}$, thus a tolerance. Indeed, if $a \ \Psi \ c$ is witnessed by (a), then $ab' \leq a \leq x$ and $cd' \leq c \leq x$, for all $b, d \in A$, hence $ab' \ \Psi \ cd'$ (we do not even need the assumption that c and d are Ψ-related). Similarly, if $a \ \Psi \ c$ is given by (b), then ab' is $\leq$ than either y or z and the same holds for cd', hence $ab' \ \Psi \ cd'$. We have proved that Ψ is a tolerance on $\mathbf{A}$,

Let $\Theta = \gamma\Psi$ be the intersection of γ and Ψ . Then $(x, z) \in \alpha(\beta \circ \Theta)$, as witnessed by y. The only (other) element of A which is $\alpha\beta$-related to x is $x_1 = (1,1,0,0,0) = x(y+z)$ and the only (other) element $\alpha\beta$-related to z is $z_1 = (0,1,0,0,1) = z(y'+x)$. No non trivial Θ relation holds among the above elements, besides $x\Theta x_1$ and $z\Theta z_1$ and the converses, hence $(x, z) \notin \alpha\beta \circ \alpha\Theta \circ \alpha\beta$ and this shows that the identity (2.1) fails in $\mathbf{A}$. See [23] for further details, comments and variations related to the present example.

Mitschke [25] showed that $\mathcal{I}$ has no near-unanimity term. It is thus natural to ask whether there exists a congruence 3-permutable, 3-distributive not 2-distributive variety with a near-unanimity term. We show that some expansion of $\mathcal{I}$ has these properties.

Let u be the lattice term defined by $u(x_1, x_2, x_3, x_4) = \prod_{j \neq j}(x_i + x_j)$, where the indices on the product vary on the set $\{1,2,3,4\}$. The term u is clearly a near-unanimity term in every lattice, in particular, in Boolean algebras. Let $\mathcal{I}^{nu}$ denote the variety generated by polynomial reducts of Boolean algebras in which both i and u are taken as basic operations. We denote by $\mathcal{I}^{nu-}$ the variety in which only f and u are considered.

Proposition 5.2. *Both $\mathcal{I}^{nu}$ and $\mathcal{I}^{nu-}$ are 3-distributive and congruence 3-permutable varieties with a 4-ary near-unanimity term. The varieties $\mathcal{I}^{nu}$ and $\mathcal{I}^{nu-}$ are neither congruence permutable, nor 2-distributive.*

Proof. Since any congruence permutable and distributive variety is arithmetical, hence 2-distributive, it is enough to show that $\mathcal{I}^{nu}$ is not 2-distributive. All the rest follows from previous remarks and the mentioned results from [24].

Essentially, we are going to show that the counterexample to 2-distributivity presented in Mitschke [24, p. 185]—and credited in that form to the referee—is closed under u. Notice that we are working with the dual. The subset $B = 2^3 \setminus \{(1,1,1)\}$ of the 8-elements Boolean algebra $\mathbf{2}^3$ is clearly closed under i. It is also closed under u, since if $b_1, b_2, b_3, b_4 \in B$, then at least two b_j's have a 0 in the same position, hence $u(b_1, b_2, b_3, b_4)$ has 0 in the same position. If α, β and γ are, respectively, the kernels of the 2nd, 1st and 3rd projections, then $((1,1,0),(0,1,1)) \in \alpha(\beta \circ \gamma)$, as witnessed by the element $(1,0,1)$. But $((1,1,0),(0,1,1)) \notin \alpha\beta \circ \alpha\gamma$ in B, since the only element which could do the job is $(1,1,1)$ and $(1,1,1) \notin B$. □

Remark 5.3. In [17] we showed that, under a fairly general hypothesis, a variety $\mathcal{V}$ satisfies an identity for congruences if and only if $\mathcal{V}$ satisfies the same identity for tolerances, provided that only tolerances representable as $R \circ R^{\smile}$ are taken into account. By [24], $\mathcal{I}$ is 3-distributive that is, $\mathcal{I}$ satisfies the congruence identity $\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta$. On the other hand, $\mathcal{I}$ fails to satisfy this identity when γ is interpreted as a tolerance, by Example 5.1. Alternatively, use Theorem 2.1 and again [24], where implication algebras are shown not to be 2-distributive. Hence the assumption of representability is necessary in [17], even in the case of 3-distributive congruence 3-permutable varieties. A similar counterexample for 4-distributive varieties has been presented in [21].

Notice that the results from [17] imply that every 3-distributive variety satisfies, for example, $\alpha(\beta \circ R \circ R^{\smile}) \subseteq \alpha\beta \circ \alpha(R \circ R^{\smile}) \circ \alpha\beta$, a result formally incomparable with the identity (3.1).

This is a preliminary version, it might contain inaccuraccies (to be precise, it is more likely to contain inaccuracies than planned subsequent versions).

The author considers that it is highly inappropriate, and strongly discourages, the use of indicators extracted from the list below (even in aggregate forms in combination with similar lists) in decisions about individuals (job opportunities, career progressions etc.), attributions of funds and selections or evaluations of research projects.

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