Large deviations for risk processes with reinsurance

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Abstract

We consider risk processes with reinsurance. A general family of reinsurance contracts is allowed, including proportional and excess-of-loss policies. The claim occurrence is regulated by a classical compound Poisson process or by a Markov modulated compound Poisson process. We provide some large deviation results concerning these two risk processes in the small claim case. Finally we derive the so called Lundberg's estimate for the ruin probabilities, and we present a numerical example.

Keywords: Large deviations, risk process, reinsurance, ruin probability, Lundberg's estimate.

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1 Introduction

The model without reinsurance. We consider the risk process $(X^x(t))$ defined by

$$X^x(t) = x + pt - S(t) \tag{1}$$

where $x > 0$ is the initial capital, $p > 0$ is the (constant) premium rate and the aggregate claims process $(S(t))$ is a compound Poisson process (classical case) or a Markov modulated compound Poisson process (Markov modulated case). More precisely we have $S(t) = \sum_{k=1}^{N(t)} U_k$ and $N(t) = \sum_{k\geq 1} 1_{T_k \leq t}$, where (U_k) is a sequence of positive random variables and $(N(t))$ is a counting process with points (T_k); further details will be given when we present the classical case and the Markov modulated case separately. Roughly speaking, when we deal with the Markov modulated case, claims intensity and claims size distribution depend on the evolution of a finite state space Markov chains; from the actuarial point of view the Markov chain describes the environmental conditions that influence the phenomena, such as weather conditions in car insurance.

We assume that the random variables (U_k) have finite expected values and that $\frac{S(t)}{t}$ converges to some limit value ℓ as $t \to \infty$. The (infinite horizon) ruin probabilities $(\psi(x))_{x>0}$ are defined by

$$\psi(x) = P(\tau^x < \infty), \text{ where } \tau^x = \inf\{t \geq 0 : X^x(t) < 0\}$$

and, in order to avoid the trivial case $\psi(x) = 1$ for all $x > 0$, the so called *net profit condition* is required, i.e. $p = (1 + \kappa)\ell$ for some relative safety loading $\kappa > 0$.

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Reinsurance policies. In our model reinsurance is allowed, i.e. the insurance company may insure part of the risk at another company (the reinsurance company) in return for a part of the premium pt. A reinsurance policy is described by a measurable function $\mathcal{R} : [0,\infty) \times [0,\infty) \to [0,\infty)$, for which we use the notation $\mathcal{R}(t,\alpha) = R_t(\alpha)$; for such a function we require the condition $0 \le R_t(\alpha) \le \alpha$ for all $t, \alpha \ge 0$. This means that $R_t(\alpha)$ is the part of the claim that the company pays when a claim of size α occurs at time t. Since the reinsurance policy is chosen dynamically, the premium rate for the reinsurer is in general not constant in time, as it happen in the classical risk model. We denote with $q_{\mathcal{R}}(t)$ the premium up to time t paid by the insurer to the reinsurer, and we shall see in detail below its determination. We assume that reinsurer uses the expected value principle with relative safety loading $\eta > 0$ for premium calculation. We shall point out below that it is interesting to consider $\eta > \kappa$, i.e. the case in which reinsurance is more expensive than insurance, otherwise the insurer would reinsure the whole portfolio. In conclusion the reserve process $(X^x_{\mathcal{R}}(t))$ under the reinsurance policy $\mathcal{R}$ is defined by

$$\begin{cases} X^x_{\mathcal{R}}(t) = x + p_{\mathcal{R}}(t) - S_{\mathcal{R}}(t), \text{ where} \\ S_{\mathcal{R}}(t) = \sum_{k=1}^{N(t)} R_{T_k}(U_k) \text{ and } p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t). \end{cases} \tag{2}$$

Large deviations and outline of the paper. In this paper we present some large deviation results concerning the risk process under the reinsurance policy $\mathcal{R}$ and we refer to the claim surplus process $(Z_{\mathcal{R}}(t))$ defined by

$$Z_{\mathcal{R}}(t) = x - X^x_{\mathcal{R}}(t) = S_{\mathcal{R}}(t) - p_{\mathcal{R}}(t).$$

In particular we refer to the concept of *large deviation principle* (see e.g. Dembo and Zeitouni [4], pages 4-5, for the definition); from now on we write LDP for short. We present two kinds of LDPs. The first one concerns the classical case (section 2) and it is a sample path large deviation result because it is a LDP on the space of càdlàg functions $D[0,1]$; the second one concerns the Markov modulated case (section 3) and it is a LDP on $\mathbb{R}$. More precisely (we use the standard notation B° for the interior of B and $\overline{B}$ for the closure of B) in the first case we have

$$-\inf_{f \in B^\circ} I_{\mathcal{R}}(f) \le \liminf_{\alpha\to\infty} \frac{1}{\alpha} \log P\Big(\frac{Z_{\mathcal{R}}(\alpha\cdot)}{\alpha} \in B\Big) \le \limsup_{\alpha\to\infty} \frac{1}{\alpha} \log P\Big(\frac{Z_{\mathcal{R}}(\alpha\cdot)}{\alpha} \in B\Big) \le -\inf_{f\in\overline{B}} I_{\mathcal{R}}(f)$$

for all Borel set B in $D[0,1]$, where $I_{\mathcal{R}}$ is the rate function; in the second case we have

$$-\inf_{y \in B^\circ} \Lambda^*_{\mathcal{R}}(y) \le \liminf_{t\to\infty} \frac{1}{t} \log P\Big(\frac{Z_{\mathcal{R}}(t)}{t} \in B\Big) \le \limsup_{t\to\infty} \frac{1}{t} \log P\Big(\frac{Z_{\mathcal{R}}(t)}{t} \in B\Big) \le -\inf_{y\in\overline{B}} \Lambda^*_{\mathcal{R}}(y)$$

for all Borel set B in $\mathbb{R}$, where $\Lambda^*_{\mathcal{R}}$ is the rate function. It is useful to point out that $I_{\mathcal{R}}$ is a good rate function, i.e. the level sets of $I_{\mathcal{R}}$

$$\{f \in D[0,1] : I_{\mathcal{R}}(f) \le c\} \text{ (for all } c > 0)$$

are compact sets.

The Markov modulated case is a generalization of the classical case. The proof of the sample path LDP concerning the classical case is based on Proposition 2.1, which is a known result in the literature. For the Markov modulated case we do not have an analogous sample path large deviation result, so that we can only prove the LDP on $\mathbb{R}$.

In section 4 we present some results for the ruin probabilities $(\psi_{\mathcal{R}}(x))_{x>0}$ under the reinsurance policy $\mathcal{R}$, which are defined by

$$\psi_{\mathcal{R}}(x) = P(\tau^x_{\mathcal{R}} < \infty), \text{ where } \tau^x_{\mathcal{R}} = \inf\{t \ge 0 : X^x_{\mathcal{R}}(t) < 0\}. \tag{3}$$

In subsection 4.1 we prove the so called Lundberg's estimate for the ruin probabilities in (3); this estimate shows that, in a sense related to large deviations, $\psi_{\mathcal{R}}(x)$ decays exponentially as $x \to \infty$. Some comments and a numerical example are presented in subsection 4.2.

The hypothesis (H) for the reinsurance policies. Since we have in mind two prototype examples of reinsurance policies presented below, in all the results in this paper we refer to the following condition:
(H): *Let* $R_\infty : [0,\infty) \to [0,\infty)$ *be a measurable function. Then for all* $\varepsilon > 0$ *there exists* t_ε *such that for all* $t \geq t_\varepsilon$ *we have* $|R_t(\alpha) - R_\infty(\alpha)| \leq \varepsilon \max\{\alpha, 1\}$ *for all* $\alpha \geq 0$.

One could consider the bound $\alpha + 1$ which is simpler than $\max\{\alpha, 1\}$ (but it is larger); in such a case some details presented in the paper have to be accordingly changed.

We remark that, when **(H)** holds, we have the pointwise convergence of R_t to R_∞ as $t \to \infty$:

$$\lim_{t\to\infty} R_t(\alpha) = R_\infty(\alpha) \text{ for all } \alpha \geq 0. \tag{4}$$

Indeed for all $\varepsilon > 0$ and for all $\alpha \geq 0$ let $t_{\varepsilon,\alpha}$ be defined by $t_{\varepsilon,\alpha} := t_{\varepsilon/\max\{\alpha,1\}}$; then, for all $t \geq t_{\varepsilon,\alpha}$, we have $|R_t(\alpha) - R_\infty(\alpha)| \leq \frac{\varepsilon}{\max\{\alpha,1\}} \max\{\alpha, 1\} = \varepsilon$.

Prototype example 1: proportional policies. Set $R_t(\alpha) = b_t\alpha$ for some $b_t \in [0, 1]$ and assume that $\lim_{t\to\infty} b_t = b_\infty \in [0, 1]$. We check **(H)** as follows. Set $R_\infty(\alpha) = b_\infty\alpha$. For all $\varepsilon > 0$ there exists t_ε such that $|b_t - b_\infty| \leq \varepsilon$ for all $t \geq t_\varepsilon$; thus for all $t \geq t_\varepsilon$

$$|R_t(\alpha) - R_\infty(\alpha)| = |b_t - b_\infty|\alpha \leq \varepsilon\alpha \leq \varepsilon \max\{\alpha, 1\} \text{ for all } \alpha \geq 0.$$

Prototype example 2: excess-of-loss policies. Set $R_t(\alpha) = \min\{a_t, \alpha\}$ for some $a_t \in [0,\infty)$ and assume that $\lim_{t\to\infty} a_t = a_\infty \in [0,\infty)$. We check **(H)** as follows. Set $R_\infty(\alpha) = \min\{a_\infty, \alpha\}$. For all $\varepsilon > 0$ there exists t_ε such that $|a_t - a_\infty| \leq \varepsilon$ for all $t \geq t_\varepsilon$; thus for all $t \geq t_\varepsilon$

$$|R_t(\alpha) - R_\infty(\alpha)| = |\min\{a_t, \alpha\} - \min\{a_\infty, \alpha\}| \leq |a_t - a_\infty| \leq \varepsilon \leq \varepsilon \max\{\alpha, 1\} \text{ for all } \alpha \geq 0.$$

2 Classical case

In this section we consider the model (1) where $(S(t))$ is a compound Poisson process. Thus we have $S(t) = \sum_{k=1}^{N(t)} U_k$ and $N(t) = \sum_{k\geq1} 1_{T_k\leq t}$, where: (U_k) and $(N(t))$ are independent; (U_k) are i.i.d.; $(N(t))$ is a Poisson process with intensity λ, i.e. the random variables $(T_k - T_{k-1})$ are i.i.d. exponentially distributed with expected value $\frac{1}{\lambda}$.

We assume the following superexponential condition for the random variables (U_k):
(S1): $\mathbb{E}[e^{\theta U_1}] < \infty$ for all $\theta \in \mathbb{R}$.
As a consequence of **(S1)** the (common) expected value μ of the random variables (U_k) is finite and $\frac{S(t)}{t}$ converges to $\ell = \lambda\mu$ as $t \to \infty$. The reserve process $(X^x_{\mathcal{R}}(t))$ under the reinsurance policy $\mathcal{R}$ is defined by (2) where

$$p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t) = (1 + \kappa)\lambda\mu t - (1 + \eta)\lambda \int_0^t \Big[\underbrace{\mathbb{E}[U_1]}_{=\mu} - \mathbb{E}[R_s(U_1)]\Big] ds;$$

with some easy computations we have

$$p_{\mathcal{R}}(t) = (1 + \eta)\lambda \int_0^t \mathbb{E}[R_s(U_1)]ds - (\eta - \kappa)\lambda\mu t.$$

In all our results we assume that **(H)** holds; thus we have (4), whence we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}[R_s(U_1)]ds = \mathbb{E}[R_\infty(U_1)]; \tag{5}$$

then

$$\lim_{t\to\infty}\frac{p_{\mathcal{R}}(t)}{t}=\overline{p}_{\mathcal{R}},\ \text{where}\ \overline{p}_{\mathcal{R}}=(1+\eta)\lambda\mathbb{E}[R_\infty(U_1)]-(\eta-\kappa)\lambda\mu.$$

The net profit condition for the insurance company under the reinsurance policy $\mathcal{R}$ is $\overline{p}_{\mathcal{R}}>\lambda\mathbb{E}[R_\infty(U_1)]$, i.e.

$$(1+\eta)\lambda\mathbb{E}[R_\infty(U_1)]-(\eta-\kappa)\lambda\mu>\lambda\mathbb{E}[R_\infty(U_1)];$$

thus, after some easy computations, we obtain

$$\mathbb{E}[R_\infty(U_1)]\geq\Big[1-\frac{\kappa}{\eta}\Big]\mu. \tag{6}$$

We point out that (6) always holds when $\kappa\geq\eta$. We also need to consider the process $(\overline{Z}_{\mathcal{R}}(t))$ defined by

$$\overline{Z}_{\mathcal{R}}(t)=\sum_{k=1}^{N(t)}R_\infty(U_k)-\overline{p}_{\mathcal{R}}t.$$

We remark that the process $(\overline{Z}_{\mathcal{R}}(t))$ and the net profit condition presented above are a generalization of the analogous items presented by Hald and Schmidli [6] (section 2) for the proportional reinsurance policies; for instance the process (X_t^b) in eq. (3) in [6] is the analogous of $(\overline{Z}_{\mathcal{R}}(t))$. The net profit is the same if we have $(\overline{Z}_{\mathcal{R}}(t))$ in place of $(Z_{\mathcal{R}}(t))$. Finally we consider the rate function $I_{\mathcal{R}}$ defined by

$$I_{\mathcal{R}}(f)=\left\{\begin{array}{ll}\int_0^1\Lambda_{\mathcal{R}}^*(\dot{f}(t))dt & \text{if } f\in AC_0[0,1]\\ \infty & \text{otherwise}\end{array}\right., \tag{7}$$

where

$$\Lambda_{\mathcal{R}}^*(y)=\sup_{\theta\in\mathbb{R}}[\theta y-\Lambda_{\mathcal{R}}(\theta)]\ \text{and}\ \Lambda_{\mathcal{R}}(\theta)=\lambda(\mathbb{E}[e^{\theta R_\infty(U_1)}]-1)-\overline{p}_{\mathcal{R}}\theta. \tag{8}$$

Our aim is to prove Proposition 2.3, i.e. the LDP of $(\frac{Z_{\mathcal{R}}(\alpha\cdot)}{\alpha})$ with rate function $I_{\mathcal{R}}$ as in (7). In order to do that we shall show that $(\frac{Z_{\mathcal{R}}(\alpha\cdot)}{\alpha})$ is exponentially equivalent to $(\frac{\overline{Z}_{\mathcal{R}}(\alpha\cdot)}{\alpha})$ as $\alpha\to\infty$ (see Definition 4.2.10 in [4]); then Proposition 2.3 will be proved by considering Theorem 4.2.13 in [4] and the next known result of Borovkov [2] (see also de Acosta [3] and the references cited therein).

Proposition 2.1 *Assume* $\mathbb{E}[e^{\theta R_\infty(U_1)}]<\infty$ *for all* $\theta\in\mathbb{R}$, *and* ***(H)***. *Then* $(\frac{\overline{Z}_{\mathcal{R}}(\alpha\cdot)}{\alpha})$ *satisfies the LDP with rate function* $I_{\mathcal{R}}$ *as in (7). Moreover the rate function* $I_{\mathcal{R}}$ *is good.*

Some preliminaries are needed for proving Proposition 2.3. We shall use the symbol $[\cdot]$ to denote the integer part of a real number. Let (A_n) be the sequence defined by

$$A_n=\sum_{k=1}^n\Big|R_{T_k}(U_k)-R_\infty(U_k)\Big|$$

and let us consider the following lemma.

Lemma 2.2 *Assume* ***(S1)*** *and* ***(H)***. *Then* $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}]=0$ *for all* $\theta>0$.

Proof. Let $\theta>0$ be arbitrarily fixed. We have $\mathbb{E}[e^{\theta A_n}]\geq 1$ for all $n\geq 1$ since $\theta>0$ and the random variable A_n is nonnegative; thus we can immediately say that $\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}]\geq 0$. Thus we complete the proof showing that $\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}]\leq 0$.

The following asymptotic estimate (10) is needed. Let $n\geq 1$ and $\rho,r,\varepsilon>0$ be arbitrarily fixed. Then we have

$$P(N(r)\geq[n\varepsilon])\leq e^{-\rho[n\varepsilon]}\mathbb{E}[e^{\rho N(r)}]=e^{-\rho[n\varepsilon]+\lambda r(e^\rho-1)},$$

and therefore $\limsup_{n\to\infty}\frac{1}{n}\log P(N(r))\geq[n\varepsilon])\leq-\rho\varepsilon$; thus, since $\rho>0$ is arbitrary,

$$\lim_{n\to\infty}\frac{1}{n}\log P(N(r)\geq[n\varepsilon])=-\infty.$$

In conclusion, since we have $P(T_{[n\varepsilon]}\leq r)=P(N(r)\geq[n\varepsilon])$, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log P(T_{[n\varepsilon]}\leq r)=-\infty. \tag{9}$$

Furthermore note that

$$A_n=\sum_{k=1}^{n}|R_{T_k}(U_k)-R_\infty(U_k)|\leq\sum_{k=1}^{n}[R_{T_k}(U_k)+R_\infty(U_k)]\leq 2\sum_{k=1}^{n}U_k;$$

then we have

$$\mathbb{E}[e^{\theta A_n}1_{T_{[n\varepsilon]}<r}]\leq\mathbb{E}[e^{2\theta\sum_{k=1}^{n}U_k}1_{T_{[n\varepsilon]}<r}]=(\mathbb{E}[e^{2\theta U_1}])^nP(T_{[n\varepsilon]}<r)$$

since $\theta>0$, $(U_1,\ldots,U_n,T_{[n\varepsilon]})$ are independent and $(U_1,\ldots,U_n)$ are i.i.d.; thus we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}1_{T_{[n\varepsilon]}<r}]=-\infty \tag{10}$$

holds by (9) and **(S1)**. Now let us consider the sum

$$\mathbb{E}[e^{\theta A_n}]=\mathbb{E}[e^{\theta A_n}1_{T_{[n\varepsilon]}<t_\varepsilon}]+\mathbb{E}[e^{\theta A_n}1_{T_{[n\varepsilon]}\geq t_\varepsilon}],$$

where t_ε is the value in **(H)**. As far as the second the addendum is concerned, we obtain

$$\mathbb{E}[e^{\theta A_n}1_{T_{[n\varepsilon]}\geq t_\varepsilon}]\leq\mathbb{E}[e^{2\theta\sum_{k=1}^{[n\varepsilon]-1}U_k}e^{\theta\varepsilon\sum_{k=[n\varepsilon]}^{n}\max\{U_k,1\}}1_{T_{[n\varepsilon]}\geq t_\varepsilon}]\leq$$
$$\leq(\mathbb{E}[e^{2\theta U_1}])^{[n\varepsilon]-1}(\mathbb{E}[e^{\theta\varepsilon\max\{U_1,1\}}])^{n-[n\varepsilon]+1}.$$

Thus, by (10) with $r=t_\varepsilon$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}]\leq\varepsilon\log\mathbb{E}[e^{2\theta U_1}]+(1-\varepsilon)\log\mathbb{E}[e^{\theta\varepsilon\max\{U_1,1\}}].$$

In conclusion we have $\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta A_n}]\leq 0$ since **(S1)** holds and $\varepsilon>0$ can be chosen arbitrarily small. $\square$

Proposition 2.3 *Assume **(S1)** and **(H)**. Then $(\frac{Z_{\mathfrak{R}}(\alpha\cdot)}{\alpha})$ satisfies the LDP with rate function $I_{\mathfrak{R}}$ as in (7).*

Proof. Proposition 2.1 provides the LDP of $(\frac{\overline{Z}_{\mathfrak{R}}(\alpha\cdot)}{\alpha})$ and the goodness of the rate function $I_{\mathfrak{R}}$ in (7); indeed, when **(S1)** holds, we have $\mathbb{E}[e^{\theta R_\infty(U_1)}]<\infty$ for all $\theta\in\mathbb{R}$. Thus, by Theorem 4.2.13 in [4], we only need to show that $(\frac{Z_{\mathfrak{R}}(\alpha\cdot)}{\alpha})$ and $(\frac{\overline{Z}_{\mathfrak{R}}(\alpha\cdot)}{\alpha})$ are exponentially equivalent as $\alpha\to\infty$, i.e.

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(\frac{1}{\alpha}\sup_{t\in[0,1]}|Z_{\mathfrak{R}}(\alpha t)-\overline{Z}_{\mathfrak{R}}(\alpha t)|>\delta\Big)=-\infty\text{ (for all }\delta>0). \tag{11}$$

Let $\delta>0$ be arbitrarily fixed. We have

$$\Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}|Z_{\mathfrak{R}}(\alpha t)-\overline{Z}_{\mathfrak{R}}(\alpha t)|>\delta\Big\}\subset$$

$$\subset \Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|>\frac{\delta}{2}\Big\}\cup\Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}\Big|\sum_{k=1}^{N(\alpha t)}R_{T_k}(U_k)-R_{\infty}(U_k)\Big|>\frac{\delta}{2}\Big\}\subset$$

$$\subset \Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|>\frac{\delta}{2}\Big\}\cup\Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}\sum_{k=1}^{N(\alpha t)}\Big|R_{T_k}(U_k)-R_{\infty}(U_k)\Big|>\frac{\delta}{2}\Big\};$$

then

$$\Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}|Z_{\mathcal{R}}(\alpha t)-\overline{Z}_{\mathcal{R}}(\alpha t)|>\delta\Big\}\subset E_1^\alpha\cup E_2^\alpha$$

where $E_1^\alpha=\Big\{\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|>\frac{\delta}{2}\Big\}$ and $E_2^\alpha=\Big\{\frac{A_{N(\alpha)}}{\alpha}>\frac{\delta}{2}\Big\}$; thus by the union bound we obtain

$$P\Big(\frac{1}{\alpha}\sup_{t\in[0,1]}|Z_{\mathcal{R}}(\alpha t)-\overline{Z}_{\mathcal{R}}(\alpha t)|>\delta\Big)\leq P(E_1^\alpha)+P(E_2^\alpha). \tag{12}$$

In view of what follows it is useful to remark that in general E_1^α is a deterministic event; moreover $E_1^\alpha=\emptyset$ and $\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|\leq\frac{\delta}{2}$ are equivalent conditions. For α large enough we have $E_1^\alpha=\emptyset$ because

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|=0;$$

indeed

$$0\leq\frac{1}{\alpha}\sup_{t\in[0,1]}|p_{\mathcal{R}}(\alpha t)-\overline{p}_{\mathcal{R}}\alpha t|=\frac{(1+\eta)\lambda}{\alpha}\sup_{t\in[0,1]}\Big|\int_0^{\alpha t}\mathbb{E}[R_s(U_1)]ds-\mathbb{E}[R_\infty(U_1)]\alpha t\Big|=$$

$$=\frac{(1+\eta)\lambda}{\alpha}\sup_{t\in[0,1]}\Big|\int_0^{\alpha t}\mathbb{E}[R_s(U_1)]-\mathbb{E}[R_\infty(U_1)]ds\Big|\leq\frac{(1+\eta)\lambda}{\alpha}\sup_{t\in[0,1]}\int_0^{\alpha t}|\mathbb{E}[R_s(U_1)]-\mathbb{E}[R_\infty(U_1)]|ds=$$

$$=\frac{(1+\eta)\lambda}{\alpha}\int_0^{\alpha}|\mathbb{E}[R_s(U_1)]-\mathbb{E}[R_\infty(U_1)]|ds\leq\frac{(1+\eta)\lambda}{\alpha}\int_0^{\alpha}\mathbb{E}[|R_s(U_1)-R_\infty(U_1)|]ds,$$

and $\lim_{\alpha\to\infty}\frac{(1+\eta)\lambda}{\alpha}\int_0^\alpha\mathbb{E}[|R_s(U_1)-R_\infty(U_1)|]ds=0$ by (4). Thus, for α large enough, (12) becomes

$$P\Big(\frac{1}{\alpha}\sup_{t\in[0,1]}|Z_{\mathcal{R}}(\alpha t)-\overline{Z}_{\mathcal{R}}(\alpha t)|>\delta\Big)\leq P(E_2^\alpha);$$

then, since $E_2^\alpha=\Big\{\frac{A_{N(\alpha)}}{\alpha}>\frac{\delta}{2}\Big\}$, the exponential equivalence (11) is proved if we show that

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(A_{N(\alpha)}>\alpha\frac{\delta}{2}\Big)\leq-\infty. \tag{13}$$

Now let $\rho>0$ and a positive integer K be arbitrarily fixed. Then, since T_n is the sum of n exponential random variables with mean $\frac{1}{\lambda}$, we have

$$P(T_{K[\alpha]}<\alpha)\leq e^{\rho\alpha}\mathbb{E}[e^{-\rho T_{K[\alpha]}}]=e^{\rho\alpha}\Big(\frac{\lambda}{\lambda+\rho}\Big)^{K[\alpha]},$$

and therefore

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\log P(T_{K[\alpha]}<\alpha)\leq\rho+K\log\frac{\lambda}{\lambda+\rho}. \tag{14}$$

Furthermore we have $\{T_{K[\alpha]}\geq\alpha\}=\{N(\alpha)\leq K[\alpha]\}$; then, since (A_n) is nondecreasing, we obtain the inequality

$$P\Big(A_{N(\alpha)}>\alpha\frac{\delta}{2},T_{K[\alpha]}\geq\alpha\Big)\leq P\Big(A_{K[\alpha]}>\alpha\frac{\delta}{2}\Big).$$

Thus, for all $\theta > 0$,

$$P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}, T_{K[\alpha]} \geq \alpha\Big) \leq P\Big(A_{K[\alpha]} > \alpha\frac{\delta}{2}\Big) \leq e^{-\theta\alpha\frac{\delta}{2}}\mathbb{E}[e^{\theta A_{K[\alpha]}}]$$

and, by Lemma 2.2, $\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P(A_{N(\alpha)} > \alpha\frac{\delta}{2}, T_{K[\alpha]} \geq \alpha) \leq -\theta\frac{\delta}{2}$. In conclusion

$$\lim_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}, T_{K[\alpha]} \geq \alpha\Big) = -\infty \tag{15}$$

holds since $\theta > 0$ is arbitrarily chosen.

Now we are ready to prove (13). Let $\rho > 0$ and a positive integer K be arbitrarily fixed, as before. By the union bound we get

$$P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}\Big) \leq P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}, T_{K[\alpha]} \geq \alpha\Big) + P(T_{K[\alpha]} < \alpha);$$

hence, by (14) and (15), we have

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}\Big) \leq \rho + K\log\frac{\lambda}{\lambda+\rho}.$$

Moreover, for $K > \lambda$, we can set $\rho = K - \lambda$ and we have

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha}\log P\Big(A_{N(\alpha)} > \alpha\frac{\delta}{2}\Big) \leq K - \lambda - K\log\frac{K}{\lambda};$$

thus (13) holds by taking $K \to \infty$ in the latter right hand side. $\square$

3 Markov modulated case

In this section we consider the Markov modulated risk model in [10] (chapter 12, section 3; see also chapter 12, section 2, subsection 2, example 4 at page 506), i.e. the model (1) where $(S(t))$ is a Markov modulated compound Poisson process. Roughly speaking, let $J = (J(t))$ be an irreducible continuous time Markov chain with finite state space E and, in any finite time interval in which we have $J(t) = i$ for some $i \in E$, $(S(t))$ behaves like a compound Poisson process with claim intensity λ_i and claim size distribution G_i. More precisely we have $S(t) = \sum_{k=1}^{N(t)} U_k$ and $N(t) = \sum_{k\geq 1} 1_{T_k \leq t}$ where: (U_k) and $(N(t))$ are conditionally independent given J; $(N(t))$ is a Markov modulated Poisson process, i.e. a doubly stochastic Poisson process with intensity $(\lambda_{J(t)})$; (U_k) are independent given J and, for all $k \geq 1$, the conditional distribution of U_k given J is $G_{J(T_k)}$.

In general we shall use the notation $\mathbb{E}_i[f(U)]$ to denote the expected value of a random variable $f(U)$, where U is a random variable with distribution G_i.

In view of what follows it is useful to consider the following function $L : \mathbb{R}^E \to \mathbb{R}$; for details on this function see Baldi and Piccioni [1] (section 2). Let $\underline{v} = [v_i]_{i\in E}$ be arbitrarily fixed and let $(p_{ij})_{i,j\in E}$ be the intensity matrix of J; moreover let us consider the matrix $P(\underline{v}) = (p_{ij} + \delta_{ij}v_i)_{i,j\in E}$, where

$$p_{ij} + \delta_{ij}v_i = \begin{cases} p_{ij} + v_i & \text{if } i = j \\ p_{ij} & \text{if } i \neq j \end{cases}.$$

Then Perron Frobenius Theorem guarantees the existence of a simple and positive eigenvalue of the exponential matrix $e^{P(\underline{v})}$, which is equal to the spectral radius of $e^{P(\underline{v})}$; then $L(\underline{v})$ is the logarithm of such eigenvalue. It is important to point out that

$$L(\underline{v}) = \lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\int_0^t v_{J(s)}ds}] \text{ (for all } \underline{v} \in \mathbb{R}^E) \tag{16}$$

whatever is the initial distribution of J. The function $L(\underline{v})$ is convex, nondecreasing with respect to each component v_i of $\underline{v}$ and $\nabla L(\underline{0}) = \pi$, where $\underline{0}$ is the null vector in $\mathbb{R}^E$ and $\pi = (\pi_i)_{i\in E}$ is the stationary distribution of J.

We assume the following condition **(S2)** which is a generalization of **(S1)** presented for the classical case.

(S2): for all $i \in E$ we have $\mathbb{E}_i[e^{\theta U}] < \infty$ for all $\theta \in \mathbb{R}$.

As a consequence of **(S2)** the expected values $(\mathbb{E}_i[U])_{i\in E}$ are finite and $\frac{S(t)}{t}$ converges to $\ell = \sum_{i\in E} \pi_i\lambda_i\mathbb{E}_i[U]$ as $t \to \infty$. The reserve process $(X^x_{\mathcal{R}}(t))$ under the reinsurance policy $\mathcal{R}$ is defined by (2) where

$$p_{\mathcal{R}}(t) = pt - q_{\mathcal{R}}(t) = (1+\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]t - (1+\eta)\sum_{i\in E}\pi_i\lambda_i\int_0^t \Big[\mathbb{E}_i[U] - \mathbb{E}_i[R_s(U)]\Big]ds;$$

with some easy computations we have

$$p_{\mathcal{R}}(t) = (1+\eta)\sum_{i\in E}\pi_i\lambda_i\int_0^t \mathbb{E}_i[R_s(U)]ds - (\eta-\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]t.$$

In all our results we assume that **(H)** holds. Thus some items presented above can be adapted to the Markov modulated case. Let us start with the limit (5) and the definition of $\overline{p}_{\mathcal{R}}$:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}_i[R_s(U)]ds = \mathbb{E}_i[R_\infty(U)] \text{ for all } i \in E;$$

$$\lim_{t\to\infty}\frac{p_{\mathcal{R}}(t)}{t} = \overline{p}_{\mathcal{R}}, \text{ where } \overline{p}_{\mathcal{R}} = (1+\eta)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)] - (\eta-\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]. \tag{17}$$

The net profit condition for the insurance company under the reinsurance policy $\mathcal{R}$ is $\overline{p}_{\mathcal{R}} > \sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)]$, i.e.

$$(1+\eta)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)] - (\eta-\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U] > \sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)];$$

thus, with some easy computations, we obtain

$$\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)] > \Big[1-\frac{\kappa}{\eta}\Big]\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]. \tag{18}$$

We point out that (18) always holds when $\kappa \geq \eta$. Finally let us consider the rate function $\Lambda^*_{\mathcal{R}}$ defined by

$$\Lambda^*_{\mathcal{R}}(y) = \sup_{\theta\in\mathbb{R}}[\theta y - \Lambda_{\mathcal{R}}(\theta)], \tag{19}$$

where

$$\Lambda_{\mathcal{R}}(\theta) = L([\lambda_i(\mathbb{E}_i[e^{\theta R_\infty(U)}]-1)]_{i\in E}) - \overline{p}_{\mathcal{R}}\theta. \tag{20}$$

Our aim is to prove Proposition 3.1, i.e. the LDP of $(\frac{Z_{\mathcal{R}}(t)}{t})$ with rate function $\Lambda^*_{\mathcal{R}}$ as in (19). In order to do that we shall use Gärtner Ellis Theorem (see section 3 of chapter 2 in [4]).

Proposition 3.1 *Assume **(S2)** and **(H)**. Then $(\frac{Z_{\mathcal{R}}(t)}{t})$ satisfies the LDP with rate function $\Lambda^*_{\mathcal{R}}$ as in (19).*

Before proving Proposition 3.1 the following Lemma is needed.

Lemma 3.2 *We have* $\mathbb{E}[e^{\theta\sum_{k=1}^{N(t)}R_{T_k}(U_k)}] = \mathbb{E}\Big[\exp\Big(\int_0^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}]-1)ds\Big)\Big]$.

Proof of Lemma 3.2. This formula can be proved following the lines of the proof of Lemma 2.3 in [9] with $\varphi(s,u) = R_s(u)1_{[0,t]}(s)$; more precisely we have to consider the extension considered in Lemma A.1 in [8]. $\square$

Proof of Proposition 3.1. We want to apply Gärtner Ellis Theorem and, for all $\theta \in \mathbb{R}$, we need to check the limit

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{t\theta \frac{Z_{\mathcal{R}}(t)}{t}}] = \Lambda_{\mathcal{R}}(\theta).$$

First of all note that, by **(S2)**, we have $\mathbb{E}_i[e^{\theta R_\infty(U)}] < \infty$ for all $i \in E$ and for all $\theta \in \mathbb{R}$. Furthermore we have

$$\mathbb{E}[e^{t\theta \frac{Z_{\mathcal{R}}(t)}{t}}] = \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}]e^{-p_{\mathcal{R}}(t)\theta},$$

whence we obtain

$$\frac{1}{t} \log \mathbb{E}[e^{t\theta \frac{Z_{\mathcal{R}}(t)}{t}}] = \frac{1}{t} \log \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] - \frac{p_{\mathcal{R}}(t)}{t}\theta;$$

thus, by (17) and (20), we only have to check the following limit for all $\theta \in \mathbb{R}$:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] = L([\lambda_i(\mathbb{E}_i[e^{\theta R_\infty(U)}] - 1)]_{i\in E}). \tag{21}$$

One can immediately say that (21) holds when $\theta = 0$ since $L([\lambda_i(\mathbb{E}_i[e^{0\cdot R_\infty(U)}]-1)]_{i\in E}) = L(\underline{0}) = 0$.

In order to prove (21) when $\theta \neq 0$ we distinguish the cases $\theta > 0$ and $\theta < 0$; moreover in both the cases we start from Lemma 3.2 with $t \geq t_\varepsilon$, where t_ε is the value in **(H)**:

$$\mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] = \mathbb{E}\Big[\exp\Big(\int_0^{t_\varepsilon} \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1)ds + \int_{t_\varepsilon}^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1)ds\Big)\Big]. \tag{22}$$

Case $\theta > 0$. Set $M(\theta) = \max_{i\in E} \lambda_i(E_i[e^{\theta U}] - 1)$ and, by (22), we have

$$\mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1)ds\Big)\Big] \leq \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] \leq$$

$$\leq e^{M(\theta)t_\varepsilon}\mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1)ds\Big)\Big].$$

Moreover, by **(H)**, we obtain

$$\mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta(R_\infty(U)-\varepsilon\max\{U,1\})}] - 1)ds\Big)\Big] \leq \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] \leq$$

$$\leq e^{M(\theta)t_\varepsilon}\mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^t \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta(R_\infty(U)+\varepsilon\max\{U,1\})}] - 1)ds\Big)\Big].$$

Then, by (16),

$$L([\lambda_i(\mathbb{E}_i[e^{\theta(R_\infty(U)-\varepsilon\max\{U,1\})}] - 1)]_{i\in E}) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] \leq$$

$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] \leq L([\lambda_i(\mathbb{E}_i[e^{\theta(R_\infty(U)+\varepsilon\max\{U,1\})}] - 1)]_{i\in E}).$$

In conclusion, by the continuity of $L(\cdot)$, (21) holds since $\varepsilon > 0$ is arbitrary.

Case $\theta < 0$. We simply adapt the procedure presented for the case $\theta > 0$. Set $m = \min_{i \in E} -\lambda_i$; then, by (22) and **(H)**, we have

$$e^{m \cdot t_\varepsilon} \mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^{t} \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta(R_\infty(U)+\varepsilon \max\{U,1\})}] - 1) ds\Big)\Big] \leq$$

$$\leq e^{m \cdot t_\varepsilon} \mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^{t} \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1) ds\Big)\Big] \leq \mathbb{E}[e^{\theta \sum_{k=1}^{N(t)} R_{T_k}(U_k)}] \leq$$

$$\leq \mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^{t} \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta R_s(U)}] - 1) ds\Big)\Big] \leq \mathbb{E}\Big[\exp\Big(\int_{t_\varepsilon}^{t} \lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta(R_\infty(U)-\varepsilon \max\{U,1\})}] - 1) ds\Big)\Big].$$

In conclusion (21) can be easily checked: we can use (16) in a suitable way, $L(\cdot)$ is continuous and $\varepsilon > 0$ is arbitrary. $\square$

Remark 3.3 *The Markov modulated case is a generalization of the classical case. This can be trivially explained by considering the set E reduced to a single point. A more interesting way consists to consider the following condition **(C)** and some consequences:*
***(C)**: the distributions $(G_i)_{i \in E}$ are all the same G and the values $(\lambda_i)_{i \in E}$ are all the same λ.*
*A first consequence is that $(S(t))$ is a compound Poisson process $S(t) = \sum_{k=1}^{N(t)} U_k$ according to the presentation for the classical case, G is the common distribution the random variables (U_k), and $(S(t))$ and $(J(t))$ are independent. Furthermore **(S2)** coincides with **(S1)**. Finally $p_{\mathcal{R}}(t)$, $\overline{p}_{\mathcal{R}}$ and $\Lambda_{\mathcal{R}}$ coincides with the items denoted by the same symbols for the classical case; for $\Lambda_{\mathcal{R}}$ this fact can be motivated by noting that, if for some $v \in \mathbb{R}$ we have $v_i = v$ for all $i \in E$, then $L([v_i]_{i \in E}) = v$.*

4 Results on ruin probabilities

In this section we focus on the asymptotics of the ruin probability when the initial capital is large. In subsection 4.1 we derive the so called Lundberg's estimate for the ruin probabilities in (3), i.e. the ruin probabilities under the reinsurance policy $\mathcal{R}$. In subsection 4.2 we present some comments and a numerical example.

4.1 Lundberg's estimate

The Lundberg's estimate for the ruin probabilities $(\psi_{\mathcal{R}}(x))_{x>0}$ will be proved in the next Proposition 4.1. The Lundberg's estimate consists of the limit (23) below; roughly speaking this limit shows that $\psi_{\mathcal{R}}(x)$ decays exponentially as $x \to \infty$ in the fashion of large deviations.

By taking into account Remark 3.3 it is not restrictive to consider the Markov modulated case; indeed the classical case can be seen as a particular case.

Proposition 4.1 *Assume **(S2)** and **(H)**. Finally assume $\overline{p}_{\mathcal{R}} > \sum_{i \in E} \pi_i \lambda_i \mathbb{E}_i[R_\infty(U)] > 0$. Then there exists $w_{\mathcal{R}} > 0$ such that $\Lambda_{\mathcal{R}}(w_{\mathcal{R}}) = 0$ and*

$$\lim_{x \to \infty} \frac{1}{x} \log \psi_{\mathcal{R}}(x) = -w_{\mathcal{R}}. \tag{23}$$

Proof. First of all we have $\Lambda'_{\mathcal{R}}(0) < 0$ by (20), $\nabla L(\underline{0}) = \pi$ and the net profit condition $\overline{p}_{\mathcal{R}} > \sum_{i \in E} \pi_i \lambda_i \mathbb{E}_i[R_\infty(U)]$. Moreover

$$\Lambda_{\mathcal{R}}(\theta) \geq \sum_{i \in E} \pi_i \lambda_i (\mathbb{E}_i[e^{R_\infty(U)\theta}] - 1) - \overline{p}_{\mathcal{R}} \theta$$

by the convexity of L and by $\nabla L(\underline{0}) = \pi$, so that the right hand side diverges as $\theta \to \infty$ since $\pi_i, \lambda_i > 0$ for all $i \in E$ and at least one of the functions $(\mathbb{E}_i[e^{R_\infty(U)\theta}] - 1)_{i \in E}$ diverges by the hypothesis $\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[R_\infty(U)]>0$. Thus $\Lambda_{\mathcal{R}}(\theta)$ diverges as $\theta\to\infty$. In conclusion the existence of $w_{\mathcal{R}}$ is guaranteed by the convexity of $\Lambda_{\mathcal{R}}$, $\Lambda'_{\mathcal{R}}(0)<0$, $\lim_{\theta\to\infty}\Lambda_{\mathcal{R}}(\theta)=\infty$ and $\Lambda_{\mathcal{R}}(\theta)<\infty$ for all $\theta\in\mathbb{R}$ (the latter statement holds by **(S2)**).

In order to prove (23) we refer to Corollary 2.3 and Lemma 2.1 of Duffield and O'Connell [5] (taking linear scaling functions); thus we have to check that $\Lambda^*_{\mathcal{R}}$ is continuous at every point of $(0,\infty)$ and the inequality

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{E}[e^{\theta(Z^*_{\mathcal{R}}(n)-Z_{\mathcal{R}}(n))}]\leq 0 \text{ (for all } \theta>0), \tag{24}$$

where $Z^*_{\mathcal{R}}(n)=\sup_{0\leq r<1}Z_{\mathcal{R}}(n+r)$.

First of all the function $\Lambda^*_{\mathcal{R}}$ is convex and finite on the set $\{\Lambda'_{\mathcal{R}}(\theta):\theta\in\mathbb{R}\}=(-\overline{p}_{\mathcal{R}},\infty)$; thus $\Lambda^*_{\mathcal{R}}$ is continuous on this open set and in particular on $(0,\infty)\subset(-\overline{p}_{\mathcal{R}},\infty)$.

As far as (24) is concerned, first of all we have

$$Z^*_{\mathcal{R}}(n)-Z_{\mathcal{R}}(n)=\sup_{0\leq r<1}\Big\{S_{\mathcal{R}}(n+r)-S_{\mathcal{R}}(n)-(p_{\mathcal{R}}(n+r)-p_{\mathcal{R}}(n))\Big\};$$

moreover

$$S_{\mathcal{R}}(n+r)-S_{\mathcal{R}}(n)=\sum_{k=N(n)+1}^{N(n+r)}R_{T_k}(U_k)\leq\sum_{k=N(n)+1}^{N(n+r)}U_k$$

and

$$p_{\mathcal{R}}(n+r)-p_{\mathcal{R}}(n)=\underbrace{(1+\eta)\sum_{i\in E}\pi_i\lambda_i\int_n^{n+r}\mathbb{E}_i[R_s(U)]ds}_{\geq 0}-(\eta-\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]r\geq$$

$$\geq-(\eta-\kappa)\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]r$$

whence we obtain (we use the notation $z^+=\max\{z,0\}$)

$$Z_{\mathcal{R}}(n+r)-Z_{\mathcal{R}}(n)\leq\sum_{k=N(n)+1}^{N(n+1)}U_k+(\eta-\kappa)^+\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U] \text{ (for all } 0\leq r<1).$$

Now let $\theta>0$ be arbitrarily fixed. Then we have

$$\mathbb{E}[e^{\theta(Z^*_{\mathcal{R}}(n)-Z_{\mathcal{R}}(n))}]\leq\mathbb{E}[e^{\theta\sum_{k=N(n)+1}^{N(n+1)}U_k}]e^{\theta(\eta-\kappa)^+\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]}.$$

Moreover, by Lemma 3.2 (slightly changed), we have

$$\mathbb{E}[e^{\theta\sum_{k=N(n)+1}^{N(n+1)}U_k}]=\mathbb{E}\Big[\exp\Big(\int_n^{n+1}\lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta U}]-1)ds\Big)\Big]$$

and, if we set $M(\theta)=\max_{i\in E}\lambda_i(E_i[e^{\theta U}]-1)$ as in the proof of Proposition 3.1, we obtain

$$\mathbb{E}\Big[\exp\Big(\int_n^{n+1}\lambda_{J_s}(\mathbb{E}_{J_s}[e^{\theta U}]-1)ds\Big)\Big]\leq e^{M(\theta)}.$$

In conclusion

$$\mathbb{E}[e^{\theta(Z^*_{\mathcal{R}}(n)-Z_{\mathcal{R}}(n))}]\leq e^{M(\theta)+\theta(\eta-\kappa)^+\sum_{i\in E}\pi_i\lambda_i\mathbb{E}_i[U]}$$

and (24) holds since $M(\theta)<\infty$ by **(S2)**. $\square$

4.2 Comments and a numerical example

In this subsection we present some comments and a numerical example. In particular, by taking into account our prototype examples presented at the end of section 1, we mainly refer to proportional and excess-of-loss policies.

In the classical case, Schmidli [12] provides the Cramér-Lundberg approximation for proportional reinsurance strategy, that is an estimate sharper than the one presented in Proposition 4.1. Moreover, he asserts that other types of reinsurance can be treated similarly. For Markov modulated risk processes, Hald and Schmidli [6] (section 4.2) treat the problem of how to calculate the proportional reinsurance strategy maximizing the adjustment coefficient. As far as we know, there are no asymptotic results for Markov modulated risk processes with excess-of-loss reinsurance.

A question of interest is the choice of a dynamic reinsurance strategy in order to minimize the infinite time ruin probability. In the case when the risk process is approximated by a Brownian motion with drift, Schmidli [11] determines explicitely the optimal proportional reinsurance policy and the corresponding ruin probability function. The optimal retention level turns out to be a constant. In the case of the classical Cramér-Lundberg risk process, Schmidli [11] and Hipp and Vogt [7] analyze proportional reinsurance and excess-of-loss reinsurance respectively. They prove the existence of a smooth solution of the Hamilton-Jacobi-Bellman (HJB) equation as well as a verification theorem, but it seems that no explicit solution of the HJB equation exists. In both papers it is conjectured that for exponentially distributed claim sizes the optimal reinsurance strategy becomes constant for large values of the initial capital. Thus, in general, it is hard to find an explicit solution for the control problem for the Cramér-Lundberg risk process.

A number of papers focus their analysis on giving asymptotic results for the ruin probability. Waters [14] considers constant reinsurance strategies. He finds that in the case of proportional reinsurance there exists a unique constant strategy that maximizes the adjustment coefficient. In the case of excess of loss reinsurance strategies, he argues that the same result holds if the premium is calculated according to the expected value principle. Schmidli studies the asymptotics for risk processes under optimal proportional reinsurance in the small claim case (see [12]) and large claim case (see [13]). In both case, he provides the Cramér-Lundberg approximation as well as the convergence of the optimal strategies. In particular, in the small claim case he proves that the optimal reinsurance strategy converges to the asymptotically optimal strategy as the initial capital increases to infinity. In conclusion, after all this discussion, the existence of the limit of the strategies R_t as $t \to \infty$ (i.e. the strategy R_∞ in **(H)**; see (4)) may be considered realistic.

It is interesting to determine the asymptotically optimal reinsurance strategy, that is, by taking into account Proposition 4.1, the reinsurance strategy $\mathcal{R}$ that maximizes the adjustment coefficient $w_{\mathcal{R}}$. First of all, consider the complete reinsurance case where all the claims are entirely paid by the reinsurer (obviously **(H)** holds in this case; moreover this can be seen as a proportional policy and as an excess-of-loss policy):

$$R_t^{(0)}(\alpha) = 0 \text{ for all } t, \alpha \geq 0. \tag{25}$$

The reserve process (2) becomes

$$X^x_{\mathcal{R}^{(0)}}(t) = x + (\kappa - \eta) \sum_{i \in E} \pi_i \lambda_i \mathbb{E}_i[U] t.$$

Notice that if $\kappa \geq \eta$ then $\psi_{\mathcal{R}^{(0)}}(x) = 0$ for all $x > 0$. Thus, as pointed out in [6] (section 2) for proportional policies concerning the classical case, the inequality $\kappa \geq \eta$ leads to a trivial situation because the reinsurance policy (25) minimizes the ruin probability. In conclusion it is interesting to consider the inequality $\eta > \kappa$, i.e. the case in which reinsurance is more expensive than insurance; on the other hand, when **(H)** holds, we already pointed out that $\kappa \geq \eta$ trivially provides (6) for the classical case and (18) for the Markov modulated case.

In general it is hard to maximize the adjustment coefficient. Here we present a numerical example and we consider proportional and excess-of-loss policies.

Numerical example. *Let J be a two state Markov chain, and then set $E = \{1, 2\}$. Let $\lambda_1 = 1$ and $\lambda_2 = 2$ be the claim intensities and let G_1 and G_2 be the claim size distributions which are assumed to be both exponential with expected values 1 and 2 respectively. Let*

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$$

be the intensity matrix of J; then the corresponding stationary distribution is $(\pi_1, \pi_2) = (\frac{1}{2}, \frac{1}{2})$. Finally let $\eta = 5$ and $\kappa = 4$ be the relative safety loading for the reinsurer and the insurer respectively.

In the Figure 1, we have depicted the adjustment coefficient $w_{\mathfrak{R}}$ as a function of the retention level in the case of proportional reinsurance as well as in the case of excess-of-loss reinsurance. In both cases, the graphs suggest that $w_{\mathfrak{R}}$ is an uni-modal function of the retention level.

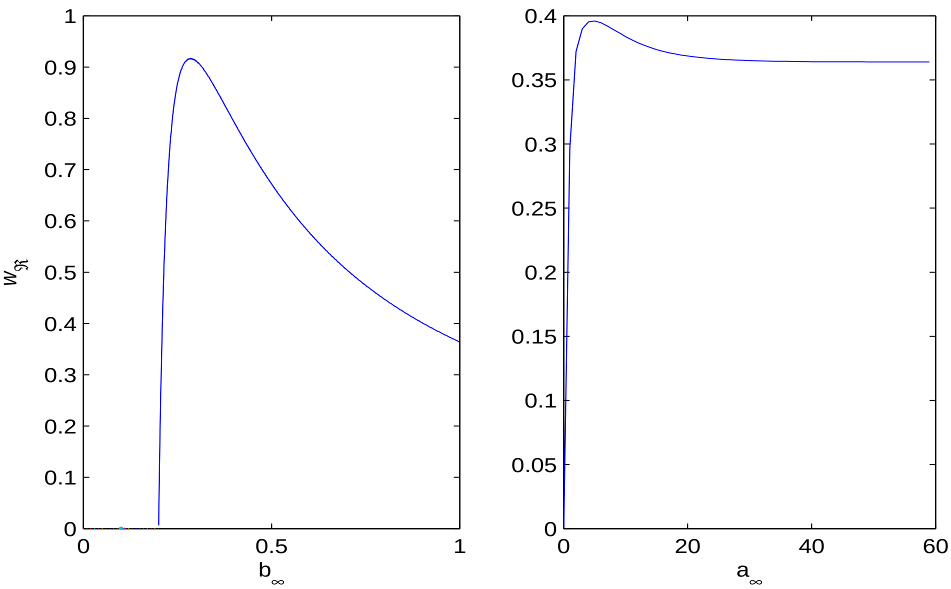

Figure 1: The adjustment coefficient as a function of the retention level in the case of proportional reinsurance and excess of loss reinsurance respectively

We point out that $w_{\mathfrak{R}} = 0$ when (18) fails (in the classical case when (6) fails). In the proportional case (18) is

$$\frac{1}{2}[1 \cdot b_\infty + 2 \cdot 2b_\infty] > \left[1 - \frac{4}{5}\right]\frac{1}{2}[1 \cdot 1 + 2 \cdot 2], \text{ i.e. } b_\infty > 0.2.$$

In the excess-of-loss case (18) is

$$\frac{1}{2}[1 \cdot (1 - e^{-a_\infty}) + 2 \cdot 2(1 - e^{-a_\infty/2})] > \left[1 - \frac{4}{5}\right]\frac{1}{2}[1 \cdot 1 + 2 \cdot 2],$$

and, with some easy computations, we obtain

$$e^{-a_\infty/2} < 2\sqrt{2} - 2, \text{ i.e. } a_\infty > -2\log(2\sqrt{2} - 2) = 0.38.$$

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