Large deviations for fractional Poisson processes*

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Abstract

We prove large deviation principles for two versions of fractional Poisson processes: the *main version* is a renewal process, the *alternative version* is a weighted Poisson process. We also present asymptotic results for the ruin probabilities of an insurance model with a fractional Poisson claim number process.

Keywords: Mittag-Leffler function; renewal process; random time change; ruin probability; weighted Poisson distribution; relative entropy.

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1 Introduction

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale, and we refer to Dembo and Zeitouni (1998) as a reference on this topic. The aim of this paper is to prove some large deviation results for fractional Poisson processes. To the best of our knowledge, these techniques have not been applied so far to the fractional Poisson process.

The study of fractional versions of the usual renewal processes has recently received an increasing interest, starting from the paper by Repin and Saichev (2000). In Jumarie (2001) the so-called fractal Poisson process is introduced (by means of non-standard analysis) and proved to have independent increments. In analogy with the fractional Brownian motion, in Wang and Wen (2003) and in Wang et al. (2006) is proposed a process constructed as a stochastic integral with respect to the Poisson measure. A different approach has been followed by other authors, in the mainstream of the fractional diffusions, in the sense of extending well-known differential equations by introducing fractional-order derivatives with respect to time: the relaxation equation (see e.g. Nonnenmacher (1991)), the heat and wave equations (see e.g. Fujita (1990) and Mainardi (1996a, 1996b)) as well as the telegraph equation (see e.g. Orsingher and Beghin (2004)) and the higher-order heat-type equations (see e.g. Beghin (2008)). In this context the solution of the fractional generalization of the Kolmogorov-Feller equation, together with the distribution of the waiting time for the corresponding process, is derived in Laskin (2003). Many other aspects of this type of fractional Poisson processes have been analyzed: a probabilistic representation of the fractional Poisson process of order ν as a composition of a standard Poisson process with a random time given by a fractional diffusion is given in Beghin and Orsingher (2009) (note that this has some analogy with other results holding for compositions of different processes; see e.g. Beghin et al. (2011) and Orsingher and Beghin (2009)) and, for $\nu = 1/2$, the time argument reduces to the absolute value of a Brownian motion; in Meerschaert et al. (2011) it is proved that we have a renewal process when

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we consider a standard Poisson process time-changed via the inverse of a general subordinator; in Politi et al. (2011) it is given a full characterization of the fractional Poisson process in terms of its multidimensional distributions. Other aspects of the fractional Poisson process are analyzed in Mainardi et al. (2004, 2005) and in Uchaikin et al. (2008).

We also recall other references with different approaches. Applications based on fractional Poisson processes can be found in Uchaikin and Sibatov (2008) (in the field of the transport of charge carriers in semiconductors) and in Laskin (2009) (in the field the fractional quantum mechanics); an inference problem is studied in Cahoy et al. (2010); a version of space-fractional Poisson process where the state probabilities are governed by equations with a fractional difference operator found in time series analysis is presented in Orsingher and Polito (2012).

The outline of the paper is the following. We start with some preliminaries in Section 2. In Section 3 we consider the *main version* which is a slight generalization of the renewal process in Beghin and Orsingher (2009, 2010). We give results for the empirical means of the i.i.d. holding times and for the normalized counting processes; furthermore we study an insurance model with fractional Poisson claim number process. In Section 4 we present large deviation results for an *alternative version*, which is the first version presented in Section 4 in Beghin and Orsingher (2009) with a suitable deterministic time-change. In such a case we have a weighted Poisson process, i.e. all the random variables are weighted Poisson distributed with the same weights (namely the weights do not depend on t). In the literature a weighted Poisson process is defined in Balakrishnan and Kozubowski (2008) and examples of weighted Poisson distributions can be found in del Castillo and Perez-Casany (1998, 2005).

2 Preliminaries

Preliminaries on large deviations. We start by recalling some basic definitions (see Dembo and Zeitouni (1998), pages 4-5). Given a topological space $\mathcal{Z}$, we say that a family of $\mathcal{Z}$-valued random variables $\{Z_t : t > 0\}$ satisfies the large deviation principle (LDP from now on) with rate function I if: the function $I : \mathcal{Z} \to [0, \infty]$ is lower semi-continuous; the upper bound

$$\limsup_{t\to\infty} \frac{1}{t} \log P(Z_t \in C) \leq - \inf_{x\in C} I(x)$$

holds for all closed sets C; the lower bound

$$\liminf_{t\to\infty} \frac{1}{t} \log P(Z_t \in G) \geq - \inf_{x\in G} I(x)$$

holds for all open sets G. The above definition can be given also for a sequence of $\mathcal{Z}$-valued random variables $\{Z_n : n \geq 1\}$ (we mean the discrete parameter denoted by n in place of the continuous parameter t). Moreover a rate function is said to be good if all its level sets $\{\{x \in \mathcal{Z} : I(x) \leq \eta\} : \eta \geq 0\}$ are compact. Throughout this paper we often refer to Cramér's Theorem in $\mathbb{R}$ (see e.g. Theorem 2.2.3 in Dembo and Zeitouni (1998)) which gives the LDP for the empirical means of i.i.d. random variables; furthermore we always set $0 \log 0 = 0$ and $0 \log \frac{0}{0} = 0$.

Preliminaries on (generalized) Mittag-Leffler function. The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(x) := \sum_{r\geq 0} \frac{x^r}{\Gamma(\alpha r + \beta)}$$

(see e.g. Podlubny (1999), page 17). We recall that, if we write $a_t \sim b_t$ to mean that $\frac{a_t}{b_t} \to 1$ as $t \to \infty$, we have

$$E_{\nu,\beta}(z) \sim \frac{1}{\nu} z^{(1-\beta)/\nu} e^{z^{1/\nu}} \text{ as } z \to \infty \tag{1}$$

(see e.g. eq. (1.8.27) in Kilbas et al. (2006)). Finally we recall that the generalized Mittag-Leffler function is defined by

$$E^{\gamma}_{\alpha,\beta}(x) := \sum_{r\geq 0} \frac{(\gamma)_r x^r}{r!\Gamma(\alpha r+\beta)},$$

where $(\gamma)_r$ is the Pochammer symbol defined by

$$(\gamma)_r := \begin{cases} 1 & \text{if } r=0 \\ \gamma(\gamma+1)\cdots(\gamma+r-1) & \text{if } r\in\{1,2,3,\ldots\} \end{cases}$$

(see e.g. eq. (1.9.1) in Kilbas et al. (2006)); note that $E^1_{\alpha,\beta}$ coincides with $E_{\alpha,\beta}$.

3 Results for the main version (renewal process)

Throughout this section we consider a class of fractional Poisson processes defined as renewal processes. More precisely, for $\nu\in(0,1]$ and $h,\lambda>0$, we consider $\{M_{\nu,h,\lambda}(t): t\geq 0\}$ defined by

$$M_{\nu,h,\lambda}(t) := \sum_{n\geq 1} 1_{\{T_1+\cdots+T_n\leq t\}}, \tag{2}$$

where the holding times $\{T_n : n\geq 1\}$ are i.i.d. with *generalized Mittag-Leffler distribution*, i.e. with continuous density $f_{\nu,h,\lambda}$ defined by

$$f_{\nu,h,\lambda}(t) = \lambda^h t^{\nu h-1} E^h_{\nu,\nu h}(-\lambda t^{\nu}) 1_{(0,\infty)}(t).$$

We remark that, if we set $h=1$, we recover the same process in Beghin and Orsingher (2010) (see eq. (2.16)); see also eq. (4.14) in Beghin and Orsingher (2009). Moreover $f_{\nu,k,\lambda}$ coincides with eq. (2.19) in Beghin and Orsingher (2010), where k is integer. Finally we have $f_{1,h,\lambda}(t) = \frac{\lambda^h}{\Gamma(h)} t^{h-1} e^{-\lambda t} 1_{(0,\infty)}(t)$ which is a Gamma density; thus we obtain the classical case with exponentially distributed holding times for $(\nu,h)=(1,1)$.

Now, in view of what follows, we set

$$\kappa_{\nu,h,\lambda}(\theta) := \log \mathbb{E}\left[e^{\theta T_1}\right] = \begin{cases} h\log\frac{\lambda}{\lambda+(-\theta)^{\nu}} & \text{if } \theta\leq 0 \\ \infty & \text{if } \theta>0 \end{cases} \quad \text{for } \nu\in(0,1). \tag{3}$$

Actually this formula can be proved as follows: the case $\theta=0$ is immediate; the case $\theta>0$ is trivial because $\mathbb{E}\left[e^{\theta T_1}\right]=\infty$; finally, if $\theta<0$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{\theta T_1}\right] &= \int_0^{\infty} e^{\theta t}\lambda^h t^{\nu h-1} E^h_{\nu,\nu h}(-\lambda t^{\nu}) dt \\
&= \lambda^h \sum_{r\geq 0} \frac{(h)_r(-\lambda)^r}{r!\Gamma(\nu r+\nu h)} \int_0^{\infty} e^{-(-\theta)t} t^{\nu h+\nu r-1} dt = \lambda^h \sum_{r\geq 0} \frac{(h)_r}{r!} \frac{(-\lambda)^r}{(-\theta)^{\nu h+\nu r}} \\
&= \frac{\lambda^h}{(-\theta)^{\nu h}} \sum_{r\geq 0} \binom{h+r-1}{r} \left(-\frac{\lambda}{(-\theta)^{\nu}}\right)^r = \frac{\lambda^h}{(-\theta)^{\nu h}} \frac{1}{\left(1+\frac{\lambda}{(-\theta)^{\nu}}\right)^h} = \frac{\lambda^h}{(\lambda+(-\theta)^{\nu})^h},
\end{aligned}$$

which is a generalization of eq. (2.18) in Beghin and Orsingher (2010) for all $h>0$. The analogous of (3) for $\nu=1$ is

$$\kappa_{1,h,\lambda}(\theta) := \log \mathbb{E}\left[e^{\theta T_1}\right] = \begin{cases} h\log\frac{\lambda}{\lambda-\theta} & \text{if } \theta<\lambda \\ \infty & \text{if } \theta\geq\lambda; \end{cases}$$

this can be easily checked and we omit the details.

We conclude with the outline of this section. In Subsection 3.1 we present the LDPs for $\{\bar{T}_n : n\geq 1\}$, where $\bar{T}_n := \frac{T_1+\cdots+T_n}{n}$ for all $n\geq 1$, and for $\left\{\frac{M_{\nu,h,\lambda}(t)}{t} : t>0\right\}$. In Subsection 3.2 we present some results for the ruin probabilities concerning an insurance model with a fractional Poisson claim number process.

3.1 The basic LDPs

We start with the LDPs and we conclude with some remarks.

Proposition 3.1. *The sequence* $\{\bar{T}_n : n \geq 1\}$ *satisfies the LDP with rate function* $I^{(T)}_{\nu,h,\lambda}$ *defined by* $I^{(T)}_{\nu,h,\lambda}(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \kappa_{\nu,h,\lambda}(\theta)\}$. *In particular, for* $\nu = 1$, *we have*

$$I^{(T)}_{1,h,\lambda}(x) = \begin{cases} h\left(\frac{\lambda x}{h} - 1 - \log\frac{\lambda x}{h}\right) & if\ x > 0 \\ \infty & if\ x \leq 0, \end{cases}$$

and it is a good rate function. For $\nu \in (0,1)$ *we have:* $I^{(T)}_{\nu,h,\lambda}(x) = \infty$ *for* $x \leq 0$, $I^{(T)}_{\nu,h,\lambda}(x)$ *is decreasing on* $(0,\infty)$, $\lim_{x\downarrow 0} I^{(T)}_{\nu,h,\lambda}(x) = \infty$, $\lim_{x\to\infty} I^{(T)}_{\nu,h,\lambda}(x) = 0$, *the rate function* $I^{(T)}_{\nu,h,\lambda}$ *is not good.*

Proof. It is a straightforward application of Cramér's Theorem in $\mathbb{R}$. □

We remark that, for all $\nu \in (0,1]$ and for all $h > 0$, we have $\kappa_{\nu,h,\lambda}(\theta) = h\kappa_{\nu,1,\lambda}(\theta)$ for all $\theta \in \mathbb{R}$, and therefore $I^{(T)}_{\nu,h,\lambda}(x) = hI^{(T)}_{\nu,1,\lambda}(\frac{x}{h})$ for all $x \in \mathbb{R}$.

Proposition 3.2. *The family* $\left\{\frac{M_{\nu,h,\lambda}(t)}{t} : t > 0\right\}$ *satisfies the LDP with good rate function* $I^{(M)}_{\nu,h,\lambda}$ *defined by*

$$I^{(M)}_{\nu,h,\lambda}(x) := \begin{cases} xI^{(T)}_{\nu,h,\lambda}(1/x) & if\ x > 0 \\ \lambda 1_{\{\nu=1\}} & if\ x = 0 \\ \infty & if\ x < 0. \end{cases}$$

In particular, for $\nu = 1$, *we have*

$$I^{(M)}_{1,h,\lambda}(x) = \begin{cases} hx\log\frac{hx}{\lambda} - hx + \lambda & if\ x \geq 0 \\ \infty & if\ x < 0. \end{cases}$$

For $\nu \in (0,1)$ *we have:* $I^{(M)}_{\nu,h,\lambda}(x) = \infty$ *for* $x < 0$, $I^{(M)}_{\nu,h,\lambda}(x)$ *is increasing on* $[0,\infty)$, $\lim_{x\downarrow 0} I^{(M)}_{\nu,h,\lambda}(x) = I^{(M)}_{\nu,h,\lambda}(0) = 0$, $\lim_{x\to\infty} I^{(M)}_{\nu,h,\lambda}(x) = \infty$.

Proof. The LDP can be proved by combining the LDP in Proposition 3.1 and, for instance, Theorem 1.1(i) in Duffield and Whitt (1998) with u, v, w as the identity function because $I^{(T)}_{\nu,h,\lambda}$ has no peaks with the unique base $x = \infty$ if $\nu \in (0,1)$, and $x = \frac{h}{\lambda}$ if $\nu = 1$. □

As far as the literature is concerned, Duffield and Whitt (1998) provides LDPs for nondecreasing processes and their inverses in a wide generality, allowing non-linear scaling functions; an older reference with linear scaling functions is Glynn and Whitt (1994).

Remarks. If $\nu = \frac{1}{2}$ we can provide explicit formulas for the rate functions presented above. By Proposition 3.1, we have

$$\begin{aligned} I^{(T)}_{\frac{1}{2},h,\lambda}(x) &= \sup_{\theta\leq 0}\left\{\theta x - h\log\left(\frac{\lambda}{\lambda + (-\theta)^{\frac{1}{2}}}\right)\right\} = \left\{\theta x - h\log\left(\frac{\lambda}{\lambda + (-\theta)^{\frac{1}{2}}}\right)\right\}_{\theta = -\left(-\frac{\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2+\frac{2h}{x}}\right)^2} \\ &= -\left(-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 + \frac{2h}{x}}\right)^2 x - h\log\left(\frac{\lambda}{\lambda - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda^2 + \frac{2h}{x}}}\right) \\ &= -x\left(\frac{1}{2}\sqrt{\lambda^2 + \frac{2h}{x}} - \frac{\lambda}{2}\right)^2 + h\log\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2h}{\lambda^2 x}}\right) \quad \text{(for all } x > 0); \end{aligned}$$

thus, by Proposition 3.2, we have

$$I^{(M)}_{\frac{1}{2},h,\lambda}(x) = xI^{(T)}_{\frac{1}{2},h,\lambda}(1/x) = hx\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2hx}{\lambda^2}}\right) - \left(\frac{1}{2}\sqrt{\lambda^2+2hx}-\frac{\lambda}{2}\right)^2 \text{ (for all } x>0).$$

Another minor remark concerns a possible alternative proof of Proposition 3.2 for the case $(\nu, h) = (\frac{1}{2}, 1)$. Actually it is known (see e.g. Remark 2.1 in Beghin and Orsingher (2009)) that, for each fixed $t > 0$, $M_{\frac{1}{2},1,\lambda}(t)$ is distributed as $N_\lambda(|B(2t)|)$ where

$$\begin{cases} \{N_\lambda(t) : t \geq 0\} \text{ and } \{B(t) : t \geq 0\} \text{ are independent,} \\ \{N_\lambda(t) : t \geq 0\} \text{ is a classical Poisson process, i.e. it is distributed as } \{M_{1,1,\lambda}(t) : t \geq 0\}, \\ \text{and } \{B(t) : t \geq 0\} \text{ is a standard Brownian motion} \end{cases}$$

(note that $\{N_\lambda(|B(2t)|) : t \geq 0\}$ does not represent a version of $\{M_{\frac{1}{2},1,\lambda}(t) : t \geq 0\}$ the process $\{N_\lambda(|B(2t)|) : t \geq 0\}$ is nondecreasing with respect to t). Then we can apply Theorem 2.3 in Chaganty (1997) (namely we mean the LDP for marginal distributions) by combining the LDP of $\left\{\frac{|B(2t)|}{t} : t > 0\right\}$ and the LDP of $\left\{\frac{N_\lambda(y_t t)}{t} : t > 0\right\}$ whenever $y_t \to y$ as $t \to \infty$. The rate function obtained in this way is given by a variational formula which reduces to $I^{(M)}_{\frac{1}{2},1,\lambda}$ in Proposition 3.2 with some computations. We omit the details.

3.2 An insurance model with fractional Poisson claim number process

In this subsection we study the ruin probability $\Psi(u) := P(\{\exists t \geq 0 : R(t) < 0\})$ concerning the insurance model

$$R(t) := u + ct - \sum_{k=1}^{M_{\nu,h,\lambda}(t)} U_k,$$

where (we refer to the terminology for eq. (5.1.14) in Rolski et al. (1999)) $\{R(t) : t \geq 0\}$ is the reserve process, $u > 0$ is the initial capital of the company, $c > 0$ is the premium rate and $\{U_k : k \geq 1\}$ are the claim sizes assumed to be i.i.d. positive random variables and independent of the claim number process $\{M_{\nu,h,\lambda}(t) : t \geq 0\}$ defined by (2). Here we consider a slightly different notation for the holding times, which will be denoted by $\left\{T_n^{(\nu)} : n \geq 1\right\}$ instead of $\{T_n : n \geq 1\}$.

We are interested in a fractional claim number process, i.e. we assume that $\nu \in (0, 1)$ and $h > 0$. We recall that, if $(\nu, h) = (1, 1)$, the claim number process is a homogeneous Poisson process and we have the *compound Poisson model* (see e.g. Section 5.3 in Rolski et al. (1999); see also the *Cramér-Lundberg model* in Section 1.1 in Embrechts et al. (1997)).

In our results we need to consider the function $\tilde{\kappa}_\nu$ defined by

$$\tilde{\kappa}_\nu(\theta) := \log\mathbb{E}\left[e^{\theta U_1}\right] + \log\mathbb{E}\left[e^{-c\theta T_1^{(\nu)}}\right] = \log\mathbb{E}\left[e^{\theta U_1}\right] + \kappa_{\nu,h,\lambda}(-c\theta), \tag{4}$$

and the following condition:
(**C**): *there exists* $w_{\nu,h,\lambda} \in (0,\infty) \cap \{\theta \in \mathbb{R} : \tilde{\kappa}_\nu(\theta) < \infty\}^\circ$ *such that* $\tilde{\kappa}_\nu(w_{\nu,h,\lambda}) = 0$.
Condition (**C**) holds if the common distribution of the random variables $\{U_k : k \geq 1\}$ is light tailed, i.e. if $\mathbb{E}\left[e^{\theta U_1}\right] < \infty$ for all $\theta > 0$.

It is well-known that the ruin can occur only at the time epochs of the claims; thus the ruin probability $\Psi(u)$ coincides with a level crossing probability for the random walk $\left\{\sum_{k=1}^n \left(U_k - cT_k^{(\nu)}\right) : n \geq 1\right\}$, i.e.

$$\Psi(u) = P(\tau_u < \infty), \text{ where } \tau_u := \inf\left\{n \geq 1 : \sum_{k=1}^n \left(U_k - cT_k^{(\nu)}\right) > u\right\}$$

(actually the event $\{\tau_u = \infty\}$ occurs if and only if $\sum_{k=1}^{n}\left(U_k - cT_k^{(\nu)}\right) \le u$ for all $n \ge 1$; on the other hand one usually has $\inf \emptyset = \infty$).

Furthermore, if we consider the case $\nu = 1$, the ruin problem is non-trivial (i.e. $\Psi(u) \in (0,1)$) if c is large enough to have $\mathbb{E}\left[U_1 - cT_1^{(1)}\right] < 0$, i.e. if the *net profit condition* $c > \frac{\lambda}{h}\mathbb{E}[U_1]$ holds (note that, for $h = 1$, this meets eq. (5.3.2) in Rolski et al. (1999), or eq. (1.7) in Embrechts et al. (1997)). On the contrary, for the fractional case $\nu \in (0,1)$ considered here, the ruin problem is non trivial for any $c > 0$ because we have $\mathbb{E}\left[U_1 - cT_1^{(\nu)}\right] = -\infty$.

In view of the results presented in this section we need to consider a family of probability measures $\{P_\theta : \tilde{\kappa}_\nu(\theta) < \infty\}$. Let $P_U \otimes P_T$ be the common law of the random variables $\left\{\left(U_n, T_n^{(\nu)}\right) : n \ge 1\right\}$, where P_U and P_T are the marginal laws. Then, for each fixed $n \ge 1$, P_θ is absolutely continuous with respect to $P_U \otimes P_T$ on the σ-field generated by the random variables $\left\{\left(U_k, T_k^{(\nu)}\right) : k \in \{1, \dots, n\}\right\}$ with density

$$\ell_n^\theta = \exp\left(\theta \sum_{k=1}^{n}\left(U_k - cT_k^{(\nu)}\right) - n\tilde{\kappa}_\nu(\theta)\right).$$

Thus, under each P_θ, the random variables $\left\{\left(U_n, T_n^{(\nu)}\right) : n \ge 1\right\}$ are i.i.d. with independent components, the common law P_U^θ of $\{U_n : n \ge 1\}$ has density $\frac{dP_U^\theta}{dP_U}(x) = \exp\left(\theta x - \log \mathbb{E}\left[e^{\theta U_1}\right]\right)$ with respect to P_U and the common law of P_T^θ the random variables $\left\{T_n^{(\nu)} : n \ge 1\right\}$ has density $\frac{dP_T^\theta}{dP_T}(t) = \exp\left(-c\theta t - \log \mathbb{E}\left[e^{-c\theta T_1^{(\nu)}}\right]\right) = e^{-c\theta t - \kappa_{\nu,h,\lambda}(-c\theta)}$ with respect to P_T. In particular we set

$$Q^* = P_{w_{\nu,h,\lambda}}.$$

Remark. Any exponential change of measure P_θ presented above can be considered also for $\nu = 1$. Then we have the two following situations for the corresponding marginal distribution P_T^θ.
• If $\nu = 1$, for $\theta > -\lambda/c$ we have

$$dP_T^\theta(t) = \frac{e^{-c\theta t}\frac{\lambda^h}{\Gamma(h)} t^{h-1} e^{-\lambda t} 1_{(0,\infty)}(t)dt}{\int_0^\infty e^{-c\theta y}\frac{\lambda^h}{\Gamma(h)} y^{h-1} e^{-\lambda y} dy} = \frac{(c\theta + \lambda)^h}{\Gamma(h)} t^{h-1} e^{-(c\theta+\lambda)t} 1_{(0,\infty)}(t)dt;$$

thus $\{P_T^\theta : \theta > -\lambda/c\}$ are all Gamma distributions.
• If $\nu \in (0,1)$, for $\theta \ge 0$ we have

$$\begin{aligned} dP_T^\theta(t) =& \frac{e^{-c\theta t}\lambda^h t^{\nu h-1} E_{\nu,\nu h}^h(-\lambda t^\nu) 1_{(0,\infty)}(t)dt}{\int_0^\infty e^{-c\theta y}\lambda^h y^{\nu h-1} E_{\nu,\nu h}^h(-\lambda y^\nu) dy} \\ =& e^{-c\theta t}(\lambda + (c\theta)^\nu)^h t^{\nu h-1} E_{\nu,\nu h}^h(-\lambda t^\nu) 1_{(0,\infty)}(t)dt; \end{aligned}$$

thus $\{P_T^\theta : \theta > 0\}$ are not generalized Mittag-Leffler distributions as it is P_T^0 because the equality $e^{-c\theta t}E_{\nu,\nu h}^h(-\lambda t^\nu) = E_{\nu,\nu h}^h(-(\lambda + (c\theta)^\nu)t^\nu)$ holds if and only if $\theta = 0$ (on the contrary the equality always holds if $\nu = 1$).

Now we are ready for the first result which provides an asymptotic estimate of $\Psi(u)$ in the fashion of large deviations.

Proposition 3.3. *Assume that* (**C**) *holds. Then we have* $\lim_{u\to\infty} \frac{1}{u}\log \Psi(u) = -w_{\nu,h,\lambda}$.
Remark. *If $\nu_1 < \nu_2$, with $\nu_1, \nu_2 \in (0,1)$, then $w_{\nu_1,h,\lambda} < w_{\nu_2,h,\lambda}$; this can be checked noting that, by* (4) *and the definition of $\kappa_{\nu,h,\lambda}$ in* (3)*, $\tilde{\kappa}_{\nu_1}(\theta) > \tilde{\kappa}_{\nu_2}(\theta)$ for $\theta > 0$. Thus the smaller is the value ν, the more dangerous is the situation (i.e. the more slowly is the decay of $\Psi(u)$ as $u \to \infty$).*

Proof. It is a straightforward application of Theorem 1 in Lehtonen and Nyrhinen (1992) and, for the sake of completeness, we give the main idea. Let $\tilde{\kappa}_\nu^*$ be defined by $\tilde{\kappa}_\nu^*(y) := \sup_{\theta\in\mathbb{R}}\{\theta y - \tilde{\kappa}_\nu(\theta)\}$ and let $x > 0$ be arbitrarily fixed. We have $\{\tau_u < \infty\} \supset \left\{\sum_{k=1}^{[ux]+1}\left(U_k - cT_k^{(\nu)}\right) > u\right\}$, and therefore

$$\Psi(u) \geq P\left(\frac{\sum_{k=1}^{[ux]+1}\left(U_k - cT_k^{(\nu)}\right)}{[ux]+1} > \frac{u}{[ux]+1}\right) \geq P\left(\frac{\sum_{k=1}^{[ux]+1}\left(U_k - cT_k^{(\nu)}\right)}{[ux]+1} > \frac{1}{x}\right);$$

thus $\liminf_{u\to\infty}\frac{1}{u}\log\Psi(u) \geq -x\tilde{\kappa}_\nu^*(1/x)$ by considering the lower bound for open sets for the LDP of $\left\{\frac{\sum_{k=1}^{n}\left(U_k - cT_k^{(\nu)}\right)}{n} : n \geq 1\right\}$ (which is provided by Cramér's Theorem). Then, by taking into account that the equality $w_{\nu,h,\lambda} = \inf\{x\tilde{\kappa}_\nu^*(1/x) : x > 0\}$ (which can be checked with some computations), we get $\liminf_{u\to\infty}\frac{1}{u}\log\Psi(u) \geq -w_{\nu,h,\lambda}$. Furthermore we have $\limsup_{u\to\infty}\frac{1}{u}\log\Psi(u) \leq -w_{\nu,h,\lambda}$ noting that $\Psi(u) = \mathbb{E}_{Q^*}\left[\left(\ell_{\tau_u}^{w_{\nu,h,\lambda}}\right)^{-1}1_{\{\tau_u<\infty\}}\right] \leq e^{-w_{\nu,h,\lambda}u}$. □

Now we recall some preliminaries on importance sampling for the estimation of $\Psi(u)$ when u is large. Let us consider K independent replications of the random walk $\left\{\sum_{k=1}^{n}\left(U_k - cT_k^{(\nu)}\right) : n \geq 1\right\}$ under the original law P; then an unbiased estimator of $\Psi(u)$ is the relative frequency $\widehat{\Psi}(u)$ of the level crossings

$$\widehat{\Psi}(u) := \frac{1}{K}\sum_{i=1}^{K} 1_{\left\{\tau_u^{(i)}<\infty\right\}},$$

where $\tau_u^{(1)}, \ldots, \tau_u^{(K)}$ are the sampled values of τ_u in each replication. Moreover, by Proposition 3.3, this Monte Carlo approach needs K growing exponentially with u to keep a fixed relative precision, indeed the relative precision of $\widehat{\Psi}(u)$ is

$$\frac{1}{\Psi(u)}\sqrt{\frac{\Psi(u)(1-\Psi(u))}{K}}.$$

We overcome this problem by considering K independent replications under another law P°. Firstly P° is such that P is absolutely continuous with respect to P° locally on the event $\{\tau_u < \infty\}$ with positive local density; moreover an unbiased estimator of $\Psi(u)$ is $\widehat{\Psi}(u; P^\circ)$ defined by

$$\widehat{\Psi}(u; P^\circ) := \frac{1}{K}\sum_{i=1}^{K}\ell_{\tau_u^{(i)}}^{P,P^\circ} 1_{\left\{\tau_u^{(i)}<\infty\right\}} = \frac{1}{K}\sum_{i=1}^{K}\left(\ell_{\tau_u^{(i)}}^{P^\circ,P}\right)^{-1} 1_{\left\{\tau_u^{(i)}<\infty\right\}},$$

where in general $\ell_{\tau_u^{(i)}}^{P,P^\circ}$ is the local density of P with respect to P°. Then P° has to be chosen in a class of admissible laws in order to minimize

$$\mathrm{Var}_{P^\circ}\left[\widehat{\Psi}(u; P^\circ)\right] = \frac{\mathrm{Var}_{P^\circ}\left[\left(\ell_{\tau_u}^{P^\circ,P}\right)^{-1}1_{\{\tau_u<\infty\}}\right]}{K} = \frac{\mathbb{E}_{P^\circ}\left[\left(\ell_{\tau_u}^{P^\circ,P}\right)^{-2}1_{\{\tau_u<\infty\}}\right] - \Psi^2(u)}{K}$$

asymptotically as $u \to \infty$, in the fashion of large deviations (actually the minimization for a fixed u is often intractable; anyway, in the applications, one often is interested in large values of u). Roughly speaking P° is said to be *admissible* if $\lim_{n\to\infty}\sum_{k=1}^{n}\left(U_k - cT_k^{(\nu)}\right) = \infty$ almost surely with respect to P°; actually, in such a case, any simulation time under P° is almost surely finite.

We concentrate our attention on the asymptotic behavior of $\frac{1}{u}\log\eta(u; P^\circ)$ as $u \to \infty$, where

$$\eta(u; P^\circ) := \mathbb{E}_{P^\circ}\left[\left(\ell_{\tau_u}^{P^\circ,P}\right)^{-2}1_{\{\tau_u<\infty\}}\right]$$

is the only part of $\mathrm{Var}_{P^\circ}\left[\widehat{\Psi}(u;P^\circ)\right]$ which depends on P°. In this way we can use standard features on large deviations, indeed we have

$$\liminf_{u\to\infty}\frac{1}{u}\log\eta(u;P^\circ)\geq\liminf_{u\to\infty}\frac{1}{u}\log\Psi^2(u)=-2w_{\nu,h,\lambda} \tag{5}$$

by Jensen's inequality and Proposition 3.3. Thus an admissible law P° is said to be an *asymptotically efficient simulation law* for the estimation of $\Psi(u)$ if

$$\lim_{u\to\infty}\frac{1}{u}\log\eta(u;P^\circ)=-2w_{\nu,h,\lambda}; \tag{6}$$

indeed, if K is chosen to guarantee a fixed relative precision

$$\frac{1}{\Psi(u)}\sqrt{\frac{\eta(u;P^\circ)-\Psi^2(u)}{K}}$$

for the estimator $\widehat{\Psi}(u;P^\circ)$, K has chance of growing less than exponentially if and only if (6) holds.

Proposition 3.4. *Assume that* (**C**) *holds. Then Q^* is an asymptotically efficient simulation law for the estimation of $\Psi(u)$.*

Proof. It is a straightforward application of Theorem 2 in Lehtonen and Nyrhinen (1992) and, for the sake of completeness, we give the main idea. The law Q^* is admissible because $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\left(U_k-cT_k^{(\nu)}\right)=\tilde{\kappa}_\nu(w_{\nu,h,\lambda})$ almost surely with respect to Q^*, and $\tilde{\kappa}_\nu(w_{\nu,h,\lambda})>0$. We have $\liminf_{u\to\infty}\frac{1}{u}\log\eta(u;Q^*)\geq -w_{\nu,h,\lambda}$ by (5) with $P^\circ=Q^*$. Furthermore we have $\limsup_{u\to\infty}\frac{1}{u}\log\eta(u;Q^*)\leq -2w_{\nu,h,\lambda}$ noting that $\eta(u;Q^*)=\mathbb{E}_{Q^*}\left[\left(\ell_{\tau_u}^{w_{\nu,h,\lambda}}\right)^{-2}1_{\{\tau_u<\infty\}}\right]\leq e^{-2w_{\nu,h,\lambda}u}$. □

4 Results for the alternative version (weighted Poisson laws)

In this section we consider an alternative version of the fractional Poisson process $\{A_{\nu,\lambda}(t):t\geq 0\}$ which is the first version presented in Section 4 in Beghin and Orsingher (2009) with t^ν in place of t:

$$P(A_{\nu,\lambda}(t)=k)=\frac{(\lambda t^\nu)^k}{\Gamma(\nu k+1)}\frac{1}{E_{\nu,1}(\lambda t^\nu)}\ \text{for all } k\in\mathbb{N}^*:=\{0,1,2,3,\ldots\}.$$

We remark that each random variable $A_{\nu,\lambda}(t)$ has a particular weighted Poisson distribution (we refer to the terminology in Johnson et al. (1992), page 90; see also the references cited therein), and the weights do not depend on t. More precisely, for each fixed t, the discrete density of $A_{\nu,\lambda}(t)$ is

$$q_w(k):=\frac{w(k)q(k)}{\sum_{j\geq 0}w(j)q(j)}\ \text{for all } k\in\mathbb{N}^*,$$

where the density $\{q(k):k\in\mathbb{N}^*\}$ and the weights $\{w(k):k\in\mathbb{N}^*\}$ are defined by $q(k):=\frac{(\lambda t^\nu)^k}{k!}e^{-\lambda t^\nu}$ (the classical Poisson density with mean λt^ν) and $w(k):=\frac{k!}{\Gamma(\nu k+1)}$, respectively.

Our aim is to prove the LDP of $\left\{\frac{A_{\nu,\lambda}(t)}{t}:t>0\right\}$ and to provide a formula (see eq. (8) below) for the rate function $I_{\nu,\lambda}^{(A)}$.

Proposition 4.1. *For $\nu\in(0,1]$, $\left\{\frac{A_{\nu,\lambda}(t)}{t}:t>0\right\}$ satisfies the LDP with good rate function $I_{\nu,\lambda}^{(A)}$ defined by*

$$I_{\nu,\lambda}^{(A)}(x):=\begin{cases}\nu x\log\frac{\nu x}{\lambda^{1/\nu}}-\nu x+\lambda^{1/\nu} & if\ x\geq 0\\ \infty & if\ x<0.\end{cases}$$

Remark. *For each fixed $t\geq 0$, $A_{1,\lambda}(t)$ is distributed as $M_{1,1,\lambda}(t)$ in (2). Thus, if $\nu=1$, we recover Proposition 3.2 with $h=1$ and, actually, one can check that $I_{1,\lambda}^{(A)}$ coincides with $I_{1,1,\lambda}^{(M)}$.*

Proof. We want to apply Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in Dembo and Zeitouni (1998)). Firstly we can immediately check that

$$\mathbb{E}\left[e^{\theta A_{\nu,\lambda}(t)}\right] = \frac{E_{\nu,1}(e^{\theta}\lambda t^{\nu})}{E_{\nu,1}(\lambda t^{\nu})}$$

for all $\theta \in \mathbb{R}$; note that $\mathbb{E}\left[e^{\theta A_{\nu,\lambda}(t)}\right] = m(e^{\theta})$, where $m(\cdot)$ is the probability generating function in eq. (4.4) in Beghin and Orsingher (2009) (with t^{ν} in place of t). Therefore, by using (1), we get

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[e^{\theta A_{\nu,\lambda}(t)}\right] = \lambda^{1/\nu}(e^{\theta/\nu} - 1) =: \Lambda_{\nu,\lambda}(\theta) \text{ for all } \theta \in \mathbb{R}.$$

Then we can apply Gärtner Ellis Theorem because the function $\Lambda_{\nu,\lambda}(\cdot)$ is finite valued and differentiable; thus the LDP holds with good rate function $I^{(A)}_{\nu,\lambda}$ defined by $I^{(A)}_{\nu,\lambda}(x) := \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda_{\nu,\lambda}(\theta)\}$ which coincides with the rate function in the statement (we omit the details). □

It is well-known (see e.g. the discussion in Varadhan (2003)) that the rate functions are often expressed in terms of the relative entropy of a probability measure Q_1 with respect to another one Q_2, which will be denoted by $H(Q_1|Q_2)$ (see the rigorous definition below). Here, for instance, we recall Sanov's Theorem (see e.g. Theorem 6.2.10 in Dembo and Zeitouni (1998)) which concerns the LDP for the empirical laws of i.i.d. random variables with common law μ, and the corresponding rate function is defined by $H(\cdot|\mu)$ on the family of the probability measures.

In what follows we give a formula for the rate function $I^{(A)}_{\nu,\lambda}$ in Proposition 4.1. To this aim we recall the definition and some properties of the relative entropy (see e.g. Section 2.3 in Cover and Thomas (1991)). Given two probability measures Q_1 and Q_2 on the same measurable space $(\Omega, \mathcal{B}(\Omega))$, we write $Q_1 \ll Q_2$ to mean that Q_1 is absolutely continuous with respect to Q_2 and, in such a case, the density will be denoted by $\frac{dQ_1}{dQ_2}$. Then the relative entropy of Q_1 with respect to Q_2 is defined by

$$H(Q_1|Q_2) = \begin{cases} \int_\Omega \log(\frac{dQ_1}{dQ_2}(\omega))Q_1(d\omega) & \text{if } Q_1 \ll Q_2 \\ \infty & \text{otherwise.} \end{cases}$$

It is known that $H(Q_1|Q_2)$ is nonnegative and it is equal to zero if and only if $Q_1 = Q_2$.

If we look at the definition of the process $\{A_{\nu,\lambda}(t) : t \geq 0\}$, it is quite natural to consider a suitable limit of normalized relative entropies (see eq. (7) below); this has some analogy with a recent result for stationary Gaussian processes (see Section 2 in Macci and Petrella (2010)). In view of what follows we denote the law of $A_{\nu,\lambda}(t)$ by $Q_{\nu,\lambda,t}$; here we also allow the case $\lambda = 0$, and $Q_{\nu,0,t}$ is the law of the constant random variable equal to 0. Then, if we consider the following limit of normalized relative entropies

$$\mathcal{H}_\nu(\lambda_1|\lambda_2) := \lim_{t\to\infty} \frac{1}{t} H(Q_{\nu,\lambda_1,t}|Q_{\nu,\lambda_2,t}) \tag{7}$$

(for $\nu \in (0,1]$ and $\lambda_1, \lambda_2 \geq 0$), we have

$$I^{(A)}_{\nu,\lambda}(x) = \mathcal{H}_\nu((\nu x)^\nu|\lambda) \text{ for all } x \geq 0 \tag{8}$$

as an immediate consequence of the following result.

Proposition 4.2. *For $\nu \in (0,1]$ and $\lambda_1, \lambda_2 \geq 0$, we have $\mathcal{H}_\nu(\lambda_1|\lambda_2) = \lambda_1^{1/\nu} \log \frac{\lambda_1^{1/\nu}}{\lambda_2^{1/\nu}} - \lambda_1^{1/\nu} + \lambda_2^{1/\nu}$.*

Proof. We start assuming that $\lambda_1, \lambda_2 > 0$. We have the following chain of equalities where, for the last equality, we take into account eq. (4.6) in Beghin and Orsingher (2009) (with t^{ν} in place of t)

for the expected value $\sum_{k=0}^{\infty} kP(A_{\nu,\lambda_1}(t)=k)$:

$$\begin{aligned}\frac{1}{t}H(Q_{\nu,\lambda_1,t}|Q_{\nu,\lambda_2,t}) =& \frac{1}{t}\sum_{k=0}^{\infty} P(A_{\nu,\lambda_1}(t)=k)\log\left(\frac{P(A_{\nu,\lambda_1}(t)=k)}{P(A_{\nu,\lambda_2}(t)=k)}\right)\\ =& \frac{1}{t}\sum_{k=0}^{\infty} P(A_{\nu,\lambda_1}(t)=k)\log\left(\frac{\lambda_1^k}{\lambda_2^k}\frac{E_{\nu,1}(\lambda_2 t^\nu)}{E_{\nu,1}(\lambda_1 t^\nu)}\right)\\ =& \frac{1}{t}\log\frac{\lambda_1}{\lambda_2}\sum_{k=0}^{\infty} kP(A_{\nu,\lambda_1}(t)=k) + \frac{1}{t}\log\left(\frac{E_{\nu,1}(\lambda_2 t^\nu)}{E_{\nu,1}(\lambda_1 t^\nu)}\right)\\ =& \frac{1}{t}\frac{\lambda_1 t^\nu}{\nu}\frac{E_{\nu,\nu}(\lambda_1 t^\nu)}{E_{\nu,1}(\lambda_1 t^\nu)}\cdot\log\frac{\lambda_1}{\lambda_2} + \frac{1}{t}\log\left(\frac{E_{\nu,1}(\lambda_2 t^\nu)}{E_{\nu,1}(\lambda_1 t^\nu)}\right).\end{aligned}$$

Then, by using (1), the limit in (7) holds with

$$\mathcal{H}_\nu(\lambda_1|\lambda_2) = \frac{\lambda_1^{1/\nu}}{\nu}\log\frac{\lambda_1}{\lambda_2} + \lambda_2^{1/\nu} - \lambda_1^{1/\nu} = \lambda_1^{1/\nu}\log\frac{\lambda_1^{1/\nu}}{\lambda_2^{1/\nu}} - \lambda_1^{1/\nu} + \lambda_2^{1/\nu}.$$

Thus the proof of the proposition is complete when $\lambda_1, \lambda_2 > 0$, and now we give some details for the other cases. If $\lambda_1 = 0$ and $\lambda_2 > 0$, we can consider this procedure, but the above sum reduces to the first addendum (the one with $k = 0$) and we have $\mathcal{H}_\nu(\lambda_1|\lambda_2) = \lambda_2^{1/\nu}$. If $\lambda_2 = 0$, we have

$$\mathcal{H}_\nu(\lambda_1|0) = \begin{cases} 0 & \text{if } \lambda_1 = 0 \\ \infty & \text{if } \lambda_1 > 0 \end{cases}$$

because, for all $t > 0$, we trivially have $H(Q_{\nu,0,t}|Q_{\nu,0,t}) = 0$ and, if $\lambda_1 > 0$, $H(Q_{\nu,\lambda_1,t}|Q_{\nu,0,t}) = \infty$. $\square$

Finally we remark that

$$\begin{aligned}\frac{1}{t}H(Q_{1,\lambda_1,t}|Q_{1,\lambda_2,t}) &= \frac{1}{t}\sum_{k=0}^{\infty} P(A_{1,\lambda_1}(t)=k)\log\left(\frac{\lambda_1^k}{\lambda_2^k}\frac{E_{1,1}(\lambda_2 t)}{E_{1,1}(\lambda_1 t)}\right)\\ &= \frac{1}{t}\log\frac{\lambda_1}{\lambda_2}\sum_{k=0}^{\infty} k\frac{(\lambda_1 t)^k}{k!}e^{-\lambda_1 t} + \frac{1}{t}\log\left(e^{(\lambda_2-\lambda_1)t}\right)\\ &= \lambda_1\log\frac{\lambda_1}{\lambda_2} - \lambda_1 + \lambda_2 = H(Q_{1,\lambda_1,1}|Q_{1,\lambda_2,1})\end{aligned}$$

does not depend on $t > 0$, and therefore coincides with $\mathcal{H}_1(\lambda_1|\lambda_2)$.

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