Money and Collateral*

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Abstract

This paper presents a model in which collateralized monetary loans are essential as trading instruments. Money and private debt collateralized by real assets complement each other as allocative tools, in an environment with informational and commitment limitations. Illiquid public debt may play a socially beneficial role, when collateral is scarce.

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1 Introduction

When agents are unable to commit to fulfill their obligations, lenders may require borrowers to pledge collateral. This alternative to unsecured lending is widespread in modern economies[1] and its macroeconomic consequences have given rise to a vast literature pioneered by Kiyotaki and Moore (1997). This literature, however, is mainly concerned with real credit, while actual credit is mostly monetary. On the other hand, the large literature on the microfoundations of money, pioneered by Kiyotaki

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[1] According to Azariadis, Kaas and Wen (2016), about 45% of liabilities of non-financial US firms is secured by collateral. The percentage is higher for loans to small businesses and consumers.

and Wright (1989), has shown that the use of money as trading instruments helps when it is hard to keep a record of the agents' past trades.[2]

Given these considerations, one would be tempted to assume that, in a world with multiple assets, the combination of limited commitment and record-keeping could immediately give rise to asset-based lending of money, as a natural response to both these frictions. Credit imperfections alone, however, turn out not to be enough to explain why marketable assets are commonly pledged as collateral to borrow cash, rather than, say, being pledged to borrow directly goods or being used as payment instruments, or even replaced from the start with cash holdings. In fact, a common presumption in the literature is that these are all equivalent arrangements, differing only for immaterial details.[3]

In contrast, this paper asks whether there is a rationale for the ample use of collateralized monetary loans, instead of other feasible and seemingly equivalent arrangements. Answering this question requires figuring out which features of the environment may make collateralized monetary loans *essential.* An instrument is essential if it helps achieve outcomes that could not be obtained in other ways.[4] The essentiality question is central if one wants to avoid ad hoc modelling assumptions hiding logical inconsistencies, as argued by Townsend (1988) and Wallace (2001), but it can also have far-reaching consequences for policy, since the types of intervention that are feasible and optimal in a given environment turn out to depend crucially on the imperfections that give rise to the adoption of different trading arrangements.

Modelling monetary loans collateralized by real assets requires setting up an environment in which money, credit and real assets, combined in a specific way, are all useful to allocate resources, and money is lent against the value of the real assets. In order to establish the essentiality of these loans, one needs to show that such a mix outperforms any other feasible alternative, including different combinations of these three ingredients.

[2]Kocherlakota (1998) has shown that money is a substitute for record-keeping in an economy without commitment.

[3]For instance, Lagos (2011) argues that direct payment with an asset, repos and collateralized loans are essentially the same: "Once stripped from the subsidiary contractual complexities, the essence of these transactions is that the asset helps the untrustworthy buyer to obtain what he wants from the seller". Similarly,Venkateswaran and Wright (2013) write: "While these two ways in which assets may facilitate intertemporal exchange - serving as a medium of exchange or as collateral- look different on the surface, they are often equivalent" (p.228).

[4]Wallace (2001) attributes this notion to Frank Hahn.

The literature with multiple trading instruments, including money, credit and real assets,[5] has mostly taken the point of view that the various instruments are competing for the single role of medium to exchange goods. This has led to consider credit mostly in the form of real, rather than monetary loans.[6] As for real assets, no fundamental distinction has been drawn, so far, between their role as collateral or as direct payment instruments,[7] which, as we argued above, have been considered as equivalent ways of achieving the same outcomes. Moreover, although absence of commitment and record-keeping are necessary and, in search economies, sufficient for money to be essential as a medium of exchange, no general joint essentiality result is available for environments with multiple trading instruments.[8] The lesson one can draw from the existing literature with multiple trading instruments is that it is hard to distinguish their transaction functions and have them all simultaneously essential. To overcome the difficulties, this paper explores the idea that the instruments may actually cooperate to solve an allocation problem which is not just confined to the goods but includes the assets themselves.

We build a search environment based on Lagos and Wright (2005), in which the allocative problem, made non-trivial by anonimity, concerns both the goods and the assets, that may be temporarily misallocated relative to best use. Anonimity aside, the features of the environment that give an essential role to collateralized monetary loans are two, namely, the specificity of the real assets which are productive only for some of the agents[9] and the need to allocate all the assets to the same side of the market. In this scenario, it is crucial that the property rights over the real assets are assigned correctly, making them better as collateral than direct means of payment.

[5] The relevant literature includes Kocherlakota and Wallace (1998), Corbae and Ritter (2003) and Jin and Temzelides (2004) in indivisible money models; Berentsen, Camera and Waller (2007), Geromichalos, Licari and Suarez Lledo (2007), Lagos (2010,2011), Li and Li (2013) and He, Wright and Zhu (2014), He, Huang and Wright (2005,2008), Telyukova and Wright (2008) with divisible money.

[6] A recent example is Venkateswaran and Wright (2013). Exceptions include Shi (1996), Berentsen, Camera and Waller (2007), Ferraris and Watanabe (2008), Ferraris (2010).

[7] Ferraris and Watanabe (2008) is an example of the first approach, Lagos and Rocheteau (2008) of the second.

[8] There are both negative and positive results. Whether money and credit can be jointly essential or not, at this stage, seems to hinge on the possibility to use the interest rate involved in credit transactions as a way to reward idle cash. For instance, in Gu, Mattesini and Wright (2016) this is not allowed and money and credit are not simultaneously essential; while in Araujo and Ferraris (2018), where this is allowed, they are both essential.

[9] This is reminiscent of Kiyotaki and Moore (1997, 2002, 2005).

The need to allocate all the assets to the same side of the market creates room for different trading instruments to cooperate rather than compete as media of trade.

We show that the best way to have both goods and assets allocated to the agents who most need them consists in purchasing real assets with cash, then, pledging the assets to borrow extra cash, and, finally, spend the cash to buy the goods: the complementary use of money and credit, through collateralized monetary loans, is the best option among the feasible trading arrangements. We show this by comparing the allocation obtained under this scheme with the allocations obtained under the four alternative arrangements that are feasible in this environment: *i.* without money, in which the agents use only collateralized credit; *ii.* with money and credit as substitute instruments, working independently of each other; *iii.* with money and loans of the asset, that are financed using the asset future value; and *iv.* with real rather than monetary loans.

In terms of policy intervention, the model suggests that public authorities should refrain from expanding the money stock, as the optimal policy is either no-intervention or deflationary. It has become customary, in the recent monetary theory literature, to exclude recourse to lump-sum taxation - which is the classic way to implement a deflation- arguing that the agents' anonymity should apply symmetrically to all dealings, including those with the public authorities.[10] In the current model, however, the agents can use the real assets to commit to make delayed payments to other private agents, who can seize the real assets in case the payments are not received.

Symmetrically, we allow for lump-sum taxation up to the value of the agents' real asset holdings, assuming that the public authorities can seize the assets in case taxes are not paid. This allows the public authorities to implement some deflationary policies. Interestingly, in an economy with collateralized monetary loans, the public authorities may be able to reach policies, including the Friedman rule (Friedman (1969)), that cannot be reached under the other feasible alternative arrangements. This is becuase the economy with collateralized monetary loans, making a better use of the available assets, gives rise to a larger value of the assets that can be seized in case of evasion than other inferior arrangements. Finally, we show that, when the economy is collateral constrained, the public authorities may beneficially augment

[10] Andolfatto (2013) has devised an incentive feasible scheme that reproduces the effects of a deflation when lump-sum taxation is not feasible because of the agents' anonymity. Such a scheme, however, is not immune to group defection.

the collateralizable asset base of the economy by issuing illiquid public bonds. This is reminiscent of Kocherlakota's (2003) rationale for illiquid bonds, except that here the role of illiquid bonds emerges when the economy is collateral constrained.

There is a recent related literature, surveyed in Martin and Ventura (2018),[11] on bubbles with credit market imperfections. In this literature, a non-fundamental component of the value of an asset which happens to be misallocated with respect to best use, may sometimes emerge at equilibrium and help reallocate the underlying asset to its best users. The interest of this literature is in the dynamic properties of such equilibria, especially those that replicate real-world phenomena such as boom-bust cycles in asset values. In our framework there are two assets, real and monetary, whose values may both have a non-fundamental component, that are both misallocated relative to best use. This paper shows that, in some circumstances, the best way to solve this double allocation problem is to use the non-fundamental the value of one asset to reallocate the other and viceversa, ending up with a situation in which money buys the real asset and the real asset helps borrow money. Although, the focus here is on steady state equilibria, the present model can also generate cyclical and sunspot equilibria, resembling boom-bust cycles. These are discussed briefly at the end of the paper.

This paper is also related to the recent literature on indirect asset liquidity,[12] which highlights that assets can be liquid not only because they may serve as media of exchange or collateral, but because agents can sell them in a secondary market for money. In this literature, agents in need of liquidity visit a secondary market to sell bonds or real assets for cash. Also in this model there is a secondary market used to boost liquidity, but here the best arrangement is uniquely identified as a collateralized monetary loan rather than other trading schemes involving, for instance, an outright sale of the asset.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 and 4 present the results. Section 5 discusses them. Section 6 concludes. The derivation of the equilibrium conditions and the proofs are in the Appendix.

[11] The intellectual origins can be traced back to Tirole (1985). Recent contributions include Kocherlakota (2009) and Martin and Ventura (2012).

[12] see Mattesini and Nosal (2016) and Geromichalos and Herrenbrueck (2016).

2 The Model

Fundamentals Time is discrete and continues forever. Each period is divided into two sub-periods, day and night, in which two goods, x and X, are produced, traded and consumed by a continuum of mass one of infinitely-lived, anonymous agents, who cannot commit to future actions either within or across periods. During the day, an agent may turn out to be, with equal probability, a buyer or a seller of x, whose consumption yiels utility $u(\cdot)$, with $u'(\cdot) > 0$ and $u''(\cdot) < 0$, and production costs $c(\cdot)$, with $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. Usual Inada conditions are assumed. During the night, agents can produce, trade and consume good X, which serves as the numeraire of the economy. The buyers of good x during the day are the producers of good X at night, with a technology that uses a durable asset, a, available in fixed supply, A, which yields R units of X per unit of asset during the night only if the asset remains in the hands of the day-time buyer for the entire period. All the agents derive linear utility from the consumption of X, and discount future payoffs at a positive rate $\beta < 1$ across periods. There is no discounting between sub-periods.

Trade All the markets are competitive. The price of x in units of X is p. The asset a can be traded in a primary market at night, at a price ψ in units of X and in a secondary market open during the day, after the resolution of uncertainty, at a price q in units of X. An intrinsically worthless, perfectly divisible and storable asset called money, m, is available initially in amount M_0, equally distributed among the agents. The value of money in units of X is ϕ. Debt contracts, due to the agents' anonymity and inability to commit, need to be collateralized. A debtor agrees to repay the amount borrowed with interest by the end of the same period. Should he fail to repay, the creditors have the right and ability to seize the amount of the asset pledged as collateral. The interest rate on monetary loans is $i \geq 0$.

Government There is a consolidated monetary-fiscal authority, called the government, that can alter the money supply using lump-sum taxes or transfers, τ, denominated in units of X and collected from or distributed to all agents equally at the end of the night after all transactions have occurred. The supply of fiat money, M, changes at a constant gross rate $\gamma \geq \beta$ over time, thus, its evolution is governed by $M_{+1} = \gamma M$.

Efficiency Let x^* be the unique value of x that satisfies $u'(x) = c'(x)$, i.e. first best day-time output. The efficient allocation for the night good involves only the feasibility condition, due to the linearity of the payoff. Efficiency requires the real asset to be assigned entirely to the buyers, being its best users.

3 Collateralized Monetary Loans

Our aim is to show that collateralized monetary loans are the best option, given the imperfections of the environment. We proceed as follows. First, we propose a trading arrangement that uses collateralized monetary loans and show that it can be sustained as an equilibrium. Then, we show that it cannot be outperformed by any feasible alternative arrangement.

3.1 Money and Collateral Equilibrium

Trade The trading arrangement with collateralized monetary loans works as follows. During the day, after the realization of uncertainty, first, the buyers acquire the real asset from the sellers in a competitive and anonymous market, spending the cash they brought from the previous period. Second, the buyers borrow money from the sellers in a competitive and anonymous market place, using the amount of the asset just acquired and the amount brought from the previous period as collateral. Third, the buyers spend money to purchase the day-time consumption good. This gives rise to the following decision problem for the agents. A buyer chooses consumption x^b, asset holdings α^b, debt d^b, to solve

$$V^b(m,a) = Max \quad u(x^b) + W^b\left(\widetilde{m}^b, \widetilde{d}^b, \widetilde{a}^b\right), \tag{1}$$

where $W^b\left(\widetilde{m}^b, \widetilde{d}^b, \widetilde{a}^b\right)$ represents the value of operating in the night market with money, debt and assets holdings, $\widetilde{m}^b$, $\widetilde{d}^b$ and $\widetilde{a}^b$, to be specified below, subject to three constraints, with their non-negative multipliers in square brackets. First, a constraint that reflects the purchase of the asset with cash, limited by its initial amount,

$$q\alpha^b \leq \phi m; \qquad [\mu] \tag{2}$$

second, a constraint that reflects the loan of cash, including the interest payment to be made at night, obtained against the total value of the asset, including the amount just purchased, used as collateral to secure repayment,

$$\phi d^b (1+i) \leq \psi \left(a + \alpha^b\right); \qquad [\lambda] \tag{3}$$

third, a constraint that reflects the purchase of the consumption good with the cash just borrowed plus the amount unspent in the asset transaction,

$$px^b \leq \phi d^b + \phi m - q\alpha^b. \qquad [\delta] \tag{4}$$

Given these transactions, at the beginning of the night a buyer will have the following cash holdings, $\widetilde{m}^b = m + d^b - \frac{q}{\phi}\alpha^b - \frac{p}{\phi}x^b$, debt $\widetilde{d}^b = -(1+i)\, d^b$ and asset holdings $\widetilde{a}^b = a + \alpha^b$. A seller chooses an amount of the good x^s, of the asset α^s and loans d^s, to solve

$$V^s(m,a) = Max \ \ -c(x^s) + W^s\left(\widetilde{m}^s, \widetilde{d}^s, \widetilde{a}^s\right), \tag{5}$$

where $W^s\left(\widetilde{m}^s, \widetilde{d}^s, \widetilde{a}^s\right)$ represents the value of operating in the night market with money, loans and assets holdings, $\widetilde{m}^s$, $\widetilde{d}^s$ and $\widetilde{a}^s$, to be specified below, subject to two constraints. First, a constraint that reflects the sale of the asset, limited by its initial amount, $\alpha^s \leq a$, $[\zeta]$; second, a constraint that reflects the monetary loan extended in the current sub-period, limited by the initial cash holdings plus those acquired in the asset transaction,

$$\phi d^s \leq \phi m + q\alpha^s. \qquad [\theta] \tag{6}$$

The seller at the beginning of the night will have money holdings, $\widetilde{m}^s = m - d^s + \frac{q}{\phi}\alpha^s + \frac{p}{\phi}x^s$, credit $\widetilde{d}^s = (1+i)\, d^s$ and asset holdings $\widetilde{a}^s = a - \alpha^s$. The expected value of entering any given period, before the realization of uncertainty, is $V(m,a) = \frac{1}{2}\sum_j V^j(m,a)$ for $j = b, s$. During the night, the day-time buyers produce the returns of the asset, debts are settled, good X is traded and consumption occurs, and assets are accumulated for the following period. Let $R(j)$ be $R(b) = R$, $R(s) = 0$. An agent j chooses consumption X^j, money and asset holdings for the future, $m_{+1}$and a_{+1}, to solve

$$W^j\left(\widetilde{m}^j, \widetilde{d}^j, \widetilde{a}^j\right) = Max \quad X^j + \beta V\left(m_{+1}, a_{+1}\right), \tag{7}$$

where $V(m_{+1}, a_{+1})$ represents the expected value of operating in the following day market with money holdings m_{+1} and asset holdings a_{+1}, subject to the budget constraint,

$$X^j + \phi m_{+1} + \psi a_{+1} = \phi \widetilde{m}^j + \phi \widetilde{d}^j + [\psi + R(j)] \widetilde{a}^j + \tau, \tag{8}$$

whereby the real value of current asset holdings, night-time production, if any, and government transfers can be used to acquire night-time consumption and assets for the future. We have incorporated the idea, borrowed from Lagos and Wright (2005), that the assets accumulation decisions are the same for all the agents. This is due to the linearity of the night-time payoff, which allows to separate the decisions about future asset holdings from current holdings. Market clearing for the day-time good requires $x^b = x^s \equiv x$, for the asset during the day $\alpha^b = \alpha^s \equiv \alpha$, for debt $d^b = d^s \equiv d$, for the asset at night $a = A$, and for money $m = M$. Since the night market for good X clears whenever the other markets do by Walras Law, we omit its market clearing condition.

Government The government needs to satisfy its budget constraint, $\phi M_{+1} = \phi M + \tau$, hence, $\tau = \phi M(\gamma - 1)$. Given that the agents are anonymous, they cannot be forced to pay taxes. However, in this environment there are physical assets that can be used as commitment devices. We assume that the government can tax the agents at the end of the night up to the value of the real asset held at that point in time, hence, $\tau \geq -\psi a_{+1}$. The idea is that the government can force agents to liquidate their end of period real asset holdings should they refuse to pay taxes.

Summary of Events In sum, during the day, first, the agents trade the asset for cash in an Asset Market (AM), then, borrow cash against the asset in a Money Market (MM), and, finally, purchase the day-time good with cash in a Decentralized Market (DM); subsequently, during the night, the returns of the assets are generated by the former buyers, debts are repaid, and all the agents participate in a Centralized Market (CM), in which the real asset, cash holdings and the night-time good are traded together. Taxation or subsidization by the government occurs at the end of trade. The figure below summarizes the sequence of events within a period.

Day	**Night**
AM $\mapsto$ MM $\mapsto$ DM	R & debt repayment, CM

3.2 Equilibrium Characterization

We solve for a stationary equilibrium with valued money and loans collateralized by the real asset. Since we are interested in the best allocation, we focus on equilibria in which the sellers do not keep any amount of the real asset during the day, hence, $\alpha^s = a$. Substituting from the budget constraint, (8), for X^j into (7) and this, in turn, into (1) and (5), we obtain the maximization problem subject to constraints (2), (3), (4), (6), which gives the optimality conditions for the choice of x and d and the accumulation of m and a.[13] Since the nominal interest rate cannot be negative, otherwise agents would refuse to lend, define $r = \max\{1+i, 1\}$. Since the market for x is competitive, its price equals the marginal cost, $p = c'(x)$. By arbitrage, the day-time price of the asset reflects its discounted night-time value, $q = \frac{\psi}{\gamma}$. The value of money evolves according to $\frac{\phi}{\phi_{+1}} = \gamma$. The Euler equation is

$$1 = \frac{\beta}{2}\left[\frac{u'(x)}{rc'(x)} + \frac{R}{\psi} + \frac{r}{\gamma}\right], \tag{9}$$

which reflects the benefit of holding an extra unit of asset, that can be used to purchase assets and goods during the following day if held by a buyer, or sold out by a seller. Moreover, there are the complementary slackness conditions for the constraints (2), (3), (4) and (6) above, namely,

$$\left[\frac{R}{\psi} + \frac{u'(x)}{rc'(x)} - \frac{u'(x)}{\gamma c'(x)}\right](\gamma\phi M - \psi A) = 0. \tag{10}$$

$$\left[\frac{u'(x)}{rc'(x)} - 1\right](2\psi A - r\phi d) = 0, \tag{11}$$

$$\left[\frac{u'(x)}{c'(x)} - 1\right][\gamma\phi M + \gamma\phi d - \psi A - \gamma c'(x)\, x] = 0, \tag{12}$$

$$(r-1)(\gamma\phi M + \psi A - \gamma\phi d) = 0, \tag{13}$$

Finally, since there is an endogenous lower bound on taxation, $\tau = \phi M(\gamma - 1) \geq -\psi A$, due to the limited enforcement that characterizes this environment, we have to take into account that the ability of the government to shrink the stock of money

[13] The complete derivation is in the Appendix.

is limited by

$$\gamma \geq \frac{\phi M - \psi A}{\phi M}. \tag{14}$$

Next, we define a stationary equilibrium in which money is valuable[14] and the agents, during the day, trade money for the asset, borrow money using the asset as collateral and, then, purchase the good with money.

Definition 1 *A stationary money and collateral equilibrium (SMCE) is a time invariant six-tuple,* (x, d, r, ψ, q, p) *and a positive* ϕ*, satisfying:* $p = c'(x)$*,* $q = \frac{\psi}{\gamma}$*, and (9)-(13), for any* $\gamma \geq \beta$ *satisfying (14).*

The equilibrium system can be considerably simplified, as shown in the Appendix. At an SMCE, if $\gamma \geq 1$, $r = \gamma$ and the only constraint that needs to be checked is (3), whose complementary slackness condition is

$$\left[\frac{u'(x)}{c'(x)}\frac{1}{\gamma} - 1\right]\left[2\psi A - c'(x)\, x\gamma\right] = 0; \tag{15}$$

if $\gamma < 1$, $r = 1$ and the only constraint that needs to be checked is (2), whose complementary slackness condition is

$$\left[\frac{2\gamma - \beta}{\beta} - \frac{u'(x)}{c'(x)}\right]\left[c'(x)\, x - 2\psi A\right] = 0. \tag{16}$$

Hence, the problem of finding a stationary equilibrium reduces to three equations, namely, (9) with $r = \max\{\gamma, 1\}$, and either (15) if $\gamma \geq 1$ or (16) if $\gamma < 1$, in the allocation of the good during the day, x, and the price of the asset at night, ψ. The rest of the equilibrium system determines uniquely the remaining variables, once x and ψ have been pinned down. The first Proposition establishes the existence and uniqueness of this stationary equilibrium. To simplify the notation, define $f(x) \equiv u'(x)\, x$ and $\rho \equiv \frac{\beta R}{1-\beta}$. Assume $f(x)$ monotonic in x.

Proposition 1 *a) If* $\rho A \geq f(x^*)$*, a unique SMCE exists for* $\gamma \geq 1$*; b) if* $f(x^*) > \rho A > \frac{f(x^*)}{2}$*, there exists a* $\widetilde{\gamma} \in (\beta, 1)$*, such that, a unique SMCE exists for* $\gamma \geq \widetilde{\gamma}$*; c) if* $\frac{f(x^*)}{2} \geq \rho A$*, a unique SMCE exists for* $\gamma \geq \beta$*.*

[14] There is also an equilibrium without trade, in which $\phi = 0$ at all times.

Money and the real asset are traded for each other at equilibrium. First, the buyers acquire the asset with cash and, then, borrow cash back, against the value of the asset. Finally, cash is spent on consumption. The only impediment to the smooth working of this scheme may be the scarcity of the asset or of its returns, which may limit the amount of cash the agents can borrow. Next, we show by means of an example how the SMCE works.[15]

An example Suppose $u(x) = \ln x$, $c(x) = x$ and $A = 1$. Define $B \equiv 2\rho(1-\beta)+\beta$. Equation (9) gives $\psi = \frac{R}{\frac{2}{\beta}-\frac{r}{\gamma}-\frac{1}{rx}}$. When $\gamma \geq 1$, $r = \gamma$. Insert ψ into (15) to obtain,

$$(1 - x\gamma)\left[B - (2-\beta)\,x\gamma\right] = 0; \tag{17}$$

when $\gamma < 1$, $r = 1$, inserting ψ into (16), obtain

$$\left[(2\gamma - \beta)\,x - \beta\right]\left[(2\gamma-\beta)\,x - B\gamma\right] = 0. \tag{18}$$

These two equations give the day-time SMCE allocation, x, as a function of parameters in the two cases. When the discounted stream of returns of the asset is sufficiently high, $\rho \geq 1$, the equilibrium is always unconstrained, and the day-time allocation is $x = \frac{1}{\gamma}$, for $\gamma \geq 1$. Since at the SMCE it has to be that $x \leq 1$, this is the only possible equilibrium configuration. When the discounted asset returns are lower, $\rho \in \left[\frac{1}{2}, 1\right)$, the equilibrium is always constrained, with the day-time allocation given by $x = \frac{B}{2\gamma-\beta\gamma}$ for $\gamma \geq 1$ and $x = \frac{\gamma B}{2\gamma-\beta}$ for $\gamma < 1$. When the discounted returns are even lower, $\rho < \frac{1}{2}$, then, the day-time allocation is given by $x = \frac{B}{2\gamma-\beta\gamma}$ for $\gamma \geq 1$, $x = \frac{\gamma B}{2\gamma-\beta}$ for $\gamma \in \left[\frac{\beta}{B}, 1\right)$ and, finally, $x = \frac{\beta}{2\gamma-\beta}$ for $\gamma < \frac{\beta}{B}$. When the economy operates in the unconstrained region, the price of the asset reflects simply its expected discounted returns, $\psi = \frac{\rho}{2}$; when the economy operates in the constrained region, instead, the price of the asset carries a liquidity premium, since the asset plays an active role as collateral in this case.

3.3 Efficiency

Next, we examine the question whether the first best allocation, x^*, can be achieved at the SMCE. The real asset is entirely allocated to the buyers during the day at

[15] The complete characterization is in the Appendix.

any SMCE, hence, efficiency is achieved along this dimension.[16] Define $\omega \equiv \frac{\beta}{4\beta-1-2\beta^2}$. Assume $\beta \geq \frac{1}{2}$.

Proposition 2 *a) If $\rho A \geq f(x^*)$, then, x^* is attained at the SMCE for $\gamma = 1$; b) if $f(x^*) > \rho A > \frac{f(x^*)}{2}$, then, x^* is attained at the SMCE at $\gamma = \widetilde{\gamma}$; c) if $\frac{f(x^*)}{2} \geq \rho A$, then, x^* is attained at the SMCE for $\gamma = \beta$, provided $\rho A \geq \omega f(x^*)(1-\beta)$.*

When the real asset is sufficiently abundant and productive, efficiency is achieved without intervention, which automatically satisfies the taxation constraint (14). When the real asset is less abundant or productive, achieving efficiency may sometimes require implementing the Friedman rule. Since the taxation constraint (14) may be binding, a restriction on the rate of impatience and enough assets are required to guarantee that the government can implement the policy that induces the first best. Going back to the previous example, in this case, the efficient allocation is $x^* = 1$, which at the SMCE is induced by $\gamma = 1$ if $\rho \geq 1$, by $\gamma^* = \frac{\beta}{B-2} > \beta$ if $\rho \in \left(\frac{1}{2}, 1\right)$ and by $\gamma = \beta$ if $\rho \in \left[\omega(1-\beta), \frac{1}{2}\right]$.

4 Essentiality

The trading scheme analyzed in the previous section uses both money and the real asset in the following complementary way: first, money buys the asset; then, the asset borrows money; finally, money buys consumption. Here, we show that this scheme is essential. We conduct the comparison with the alternative arrangements in terms of day-time output, since, as we have seen this is enough to evaluate their social benefits in this environment. We proceed in several steps. First, we exclude some arrangements that are either clearly not feasible or obviously inferior. Second, we consider the relevant feasible arrangements: *i.* with collateralized credit but without money; *ii.* with money and credit as substitute instruments, working independently from each other; *iii.* with loans and money not used to purchase the real asset; *iv.* with real loans instead of monetary. In each case, we find regions of the parameters space in which the arrangement with monetary collateralized loans improves upon the feasible alternatives, inducing a larger day-time output.

[16] Due to transferable utility in the night-time payoff and equal probability of being buyer and seller, ex-ante welfare is $\frac{1}{2}[u(x) - c(x) + RA]$, since the asset is properly allocated to the best user. Since x^* satisfies $u'(x) = c'(x)$, and welfare is strictly concave in x, achieving x^* is necessary and sufficient to maximize welfare.

4.1 Alternative Schemes

Let us eliminate the easy cases first. The anonymity of the agents and the absence of commitment make several arrangements infeasible in this environment. For instance, any scheme that involves complete contracts is not feasible. Since only the day-time buyers have the know-how to use the asset as a productive input, the asset should remain in their hands, hence, using the asset as a direct payment instrument cannot be optimal. However, the agents could still trade the shares of the asset rather than the asset itself. The exchange of equity is similar to the exchange of collateralized debt, in this setting. The crucial difference rests on the allocation of property rights, which remain with the buyer under colletaralized debt but are transfered to outsiders under equity, opening the door to a renegotiation problem, whereby equity holders might try to appropriate part of the returns, threatening to withdraw the asset before it pays off. As usual in settings where agents cannot commit not to renegotiate contracts, the allocation of property rights is a delicate matter and a general result is that such rights should remain with the best user of the asset.[17] In this environment, any arrangement involving debt contracts has to rely on collateral in some form, but it does not necessarily have to use money. Next, we consider a pure credit arrangement.

Credit arrangement Consider the following cash-less arrangement. The buyers borrow the asset from the sellers against its night-time value, i.e. mortgage it, then, pledge their initial asset holdings as collateral to borrow consumption. Assets acquired with a mortgage cannot be used as collateral to obtain a second loan, otherwise the incentives to repay would be jeopardized. This scheme is feasible and leads to the following constraints on the agents trading possibilities: the buyers are subject to two constraints: $q\alpha^b \leq \psi\alpha^b$ and $px^b \leq \psi a$; the sellers are subject only to $\alpha^s \leq a$, which we take to be binding as before. The competitive assumption implies $p = c'(x)$. For the sellers to have an incentive to lend the asset, the day-time price of the asset should not fall short of its night-time value, $q \geq \psi$, hence, by the first constraint for the buyers, we have $q = \psi$. The stationary equilibrium conditions are given by the Euler condition,

$$1 = \frac{\beta}{2}\left[\frac{u'(x)}{c'(x)} + \frac{R}{\psi} + 1\right], \tag{19}$$

[17] as in Hart and Moore (1990).

which reflects the use of the asset to obtain a loan and in production; and the complementary slackness condition for the collateral constraint,

$$\left[\frac{u'(x)}{c'(x)} - 1\right]\left[\psi A - c'(x)\,x\right] = 0, \tag{20}$$

which reflects the fact that only the amount of the asset owned at the beginning of the day can be pledged as collateral.

Definition 2 *A Stationary Credit Equilibrium (SCE) is a time-invariant four-tuple, (x, ψ, q, p), satisfying $p = c'(x)$, $q = \psi$ and (19)-(20).*

An SCE can be shown to exist and be unique with standard arguments. The following Proposition compares the allocation at the SMCE, i.e. the amount of day-time consumption at such an equilibrium, with the allocation at the SCE, i.e. the amount of day-time consumption at such an equilibrium.

Proposition 3 *a) If $\rho A \geq 2f(x^*)$, the SCE allocation is x^*; b) if $2f(x^*) > \rho A$, the SCE allocation is inefficient and there exists a $\widehat{\gamma} > 1$, such that if $\gamma < \widehat{\gamma}$, the SMCE allocation is larger than the SCE allocation.*

Hence, if the asset is sufficiently abundant and productive, the arrangement based only on credit works so well that the presence of money cannot add any extra benefit. Otherwise, having an arrangement with both money and credit helps, provided money is sufficiently valuable. When the asset is not plentiful enough, the use of money is beneficial since it allows to purchase, rather than borrow the real asset. This, in turn, allows the new owner to pledge it as collateral, thus, making a more efficient use of a limited collateral base. Going back to our example, we have that, if $\rho \geq 2$, the SCE is unconstrained and the day-time allocation is efficient; if $\rho < 2$, the SCE is constrained, and the day-time allocation is $\frac{\rho(1-\beta)+\beta}{2-\beta}$. Consider the case in which $\rho \in [1, 2)$. The SMCE allocation is $x = \frac{1}{\gamma}$, hence, if $\gamma \in \left[1, \frac{2-\beta}{\rho(1-\beta)+\beta}\right)$, the allocation is larger at the SMCE than the SCE. When $\rho < 1$, the SMCE allocation is $x = \frac{1}{\gamma}\frac{2\rho(1-\beta)+\beta}{2-\beta}$, hence, if $\gamma \in \left[1, \frac{2\rho(1-\beta)+\beta}{\rho(1-\beta)+\beta}\right)$, the allocation is larger at the SMCE than the SCE. For $\gamma < 1$, the allocation is larger at the SMCE than the SCE.

Monetary arrangement Next, consider an arrangement with cash in which, the real asset is acquired on credit, pledging the available real asset as collateral, and consumption is purchased with cash. This scheme is also feasible and leads to the following constraints on the agents trading possibilities: the buyers are subject to two constraints: $q\alpha^b \leq \psi a$ and $px^b \leq \phi m$; the sellers are subject only to $\alpha^s \leq a$, which we take to be binding as before. The competitive assumption implies $p = c'(x)$. For the sellers to have an incentive to lend the asset, $q \geq \psi$, hence, by the first constraint for the buyers, at equilibrium, we have $q = \psi$. The stationary equilibrium conditions are given by the Euler condition for the asset,

$$1 = \beta \left(\frac{R}{\psi} + 1 \right), \tag{21}$$

which reflects the use of the asset as a way to obtain a loan of goods and in production; the Euler condition for money,

$$1 = \frac{\beta}{2\gamma} \left[\frac{u'(x)}{c'(x)} + 1 \right], \tag{22}$$

which reflects the use of money to acquire consumption; and the complementary slackness condition for the cash constraint,

$$\left[\frac{u'(x)}{c'(x)} - 1 \right] \left[\phi M - c'(x) x \right] = 0. \tag{23}$$

Moreover, the lower bound on taxation, (14), needs to be taken into account. We call this arrangement monetary, since cash works in a completely independent way from credit deals, and serves only to acquire consumption. Any system that uses money and the asset so that substitute for each other, as in Gu, Mattesini and Wright (2016), gives rise to equation (22), hence, will induce the same allocation as this system.

Definition 3 *A Stationary Monetary Equilibrium (SME) is a time-invariant four-tuple,* (x, ψ, q, p) *and a positive* ϕ*, satisfying* $p = c'(x)$*,* $q = \psi$ *and (21)-(23).*

Existence and uniqueness of an SME is established with standard arguments. The following Proposition identifies a region of parameters where the efficient allocation can be achieved by the SMCE for some value of the monetary policy parameter but never by the SME. Define the set $\Sigma \equiv ((1-\beta)\,\omega f(x^*), (1-\beta) f(x^*))$.

Proposition 4 *The SME can achieve x^* if and only if $\gamma = \beta$ is feasible. If $\beta > \frac{1}{2}$ and $\rho A \in \Sigma$, x^* can be achieved at the SMCE but not the SME.*

By equation (22), efficiency would be achieved at the SME for $\gamma = \beta$, i.e. the Friedman rule. Achieving the Friedman rule, however, would require a level of taxation which cannot be enforced since the value of the asset is limited and the agents would rather give it up than having to pay such high taxes. The lower bound on taxation, (14), is endogenous in this economy, and depends on the relative value of the assets, which is more favorable with the SMCE arrangement than the SME. This is due to the fact that the SMCE makes a more economic use of the assets relative to the SME. In terms of the example, when $\rho < 1 - \beta$, the government cannot achieve the Friedman rule at the SME, since the lower bound on taxation is tight, but it can achieve it at the SMCE, for $\rho > (1 - \beta)\,\omega$. Hence, for $\rho \in ((1 - \beta)\,\omega, 1 - \beta)$, only the SMCE can achieve efficiency for any feasible γ. The next Proposition compares the allocation at the SMCE with the one at the SME, i.e. the amount of day-time consumption across these equilibria, for the same value of the policy parameter, γ.

Proposition 5 *a) If $\rho A > \frac{f(x^*)}{2}$, the SMCE allocation is larger than the SME allocation for all $\gamma \geq \widetilde{\gamma}$; b) if $\rho A \leq \frac{f(x^*)}{2}$, the SMCE allocation is at least equal and sometimes larger the SME allocation for all feasible γ.*

The SMCE induces an allocation that is always at least the same and sometimes strictly larger than the SME. In terms of the example we considered above, using (22), gives the allocation $\frac{\beta}{2\gamma - \beta}$ at the SME, which is at best the same as the SMCE allocation, when $\rho \leq \frac{1}{2}$ and $\gamma < \frac{\beta}{B}$, and in all other cases strictly lower than the SMCE allocation.

Loan and money arrangement Next, consider an arrangement in which the asset borrows money and money buys consumption but not the asset, which is borrowed against its own value at night. In such a case, the buyers borrows the asset from the sellers against its night-time value, then, pledge their initial asset holdings as collateral to borrow cash from the sellers, and, finally, spend the cash to acquire consumption. The constraints for the buyers are: first, $q\alpha^b \leq \psi\alpha^b$, which reduces to the requirement that $q \leq \psi$; second, $\phi d^b\,(1 + i) \leq \psi a$; and, third, $px^b \leq \phi d^b + \phi m$. The constraints for the seller are: $\alpha^s \leq a$, which we assume binding as usual, and

$\phi d^s \leq \phi m + q\alpha^s$. Again, as before, $p = c'(x)$ by perfect competition, and $q = \psi$, since the non-negativity of the multiplier of the seller's asset constraint requires $q \geq \psi$. The equilibrium conditions are the Euler condition for money holdings,

$$1 = \frac{\beta}{2\gamma}\left[\frac{u'(x)}{c'(x)} + r\right], \tag{24}$$

the Euler condition for the asset,

$$1 = \frac{\beta}{2}\left[\frac{u'(x)}{rc'(x)} + \frac{R}{\psi} + 1\right], \tag{25}$$

and the complementary slackness conditions for the three constraints left,

$$\left[\frac{u'(x)}{rc'(x)} - 1\right][\psi A - \phi dr] = 0, \tag{26}$$

$$\left[\frac{u'(x)}{c'(x)} - 1\right][\phi M + \phi d - c'(x)x] = 0, \tag{27}$$

$$(r-1)(\phi M - \phi d) = 0. \tag{28}$$

Definition 4 *A Stationary Loan and Money Equilibrium (SLME) is a time-invariant six-tuple,* (x, d, r, ψ, q, p) *and a positive* ϕ*, satisfying:* $p = c'(x)$*,* $q = \psi$*, and (24)-(28).*

An SLME can be shown to exist and be unique with standard arguments. As for the SME, the SLME achieves the efficient allocation only if $\gamma = \beta$, which, in turn, cannot be attained if $\rho A < (1-\beta) f(x^*)$, since the lower bound on taxation prevents the government from running the Friedman rule. Hence, Proposition 4 applies to this system as well. The following Proposition compares the allocation at the SMCE with the allocation at the SLME, i.e. the amount of day-time consumption across these equilibria.

Proposition 6 *a) If* $\rho A > \frac{f(x^*)}{2}$*, the SMCE allocation is larger than the SLME allocation for all* $\gamma \geq \widetilde{\gamma}$*; b) if* $\rho A \leq \frac{f(x^*)}{2}$*, the SMCE is at least equal and sometimes larger than the SLME allocation for all feasible* γ*.*

At the SLME, cash is not used to acquire the asset, unlike at the SMCE. Hence, as can be seen from (24), the benefit of holding cash does not include a share in the returns of the real asset. To compensate the agents for holding money, the interest rate needs to be higher than before, which is a symptom of inefficiency. This shows that in this environment the role of money does not consist simply in allowing to enlarge the available liquidity, but specifically to purchase the asset, instead of borrowing it. In terms of the foregoing example, we have that, if $\rho \geq 1$, then, $x = \frac{\beta}{\gamma}$, which is strictly smaller than day-time SMCE output in the corresponding case; if $\rho < 1$, there exists a cut-off value for γ such that for γ above the cut-off, the allocation is $\frac{\beta}{\gamma}\frac{\rho(1-\beta)+1}{2-\beta}$ and below the cut-off, $\frac{\beta}{2\gamma-\beta}$, which are, respectively, strictly smaller than and equal to the SMCE allocation in the corresponding cases.

Real credit arrangement Finally, consider the case in which money buys the asset, but the asset is used to borrow directly the consumption good rather than money. This is an arrangement with real, rather than monetary, loans. The buyers are subject to two constraint: $q\alpha^b \leq \phi m$ and $px^b \leq \psi\left(a+\alpha^b\right)+\phi m - q\alpha^b$. The sellers face the constraint: $\alpha^s \leq a$, which we assume binding as usual. The equilibrium conditions are the following: the condition that pins down the price of the good, $p = c'(x)$, the arbitrage condition, $q = \frac{\psi}{\gamma}$, the Euler condition,

$$1 = \frac{\beta}{2}\left[\frac{u'(x)}{c'(x)} + \frac{R}{\psi} + \frac{1}{\gamma}\right], \tag{29}$$

and the complementary slackness conditions,

$$\left[\frac{2\gamma-\beta}{\beta} - \frac{u'(x)}{c'(x)}\right](\gamma\phi M - \psi A) = 0, \tag{30}$$

$$\left[\frac{u'(x)}{c'(x)} - 1\right]\left[\gamma\phi M + (2\gamma-1)\psi A - \gamma c'(x)x\right] = 0. \tag{31}$$

Definition 5 *A Stationary Real Credit Equilibrium (SRCE) is a time-invariant four-tuple,* (x, ψ, q, p) *and a positive* ϕ*, satisfying:* $p = c'(x)$*,* $q = \frac{\psi}{\gamma}$*, and (29)-(31).*

As the next Proposition shows, the existence of an SRCE requires some restrictions on parameters. The next Proposition compares also the allocation at the SRCE and SMCE, i.e. the amount of day-time consumption across such equilibria, when the former exists.

Proposition 7 *An SRCE exists only if* $\gamma \leq 1$*. When it exists, the SRCE induces the same allocation and the same price of the asset as the SMCE.*

Hence, this arrangement is an incomplete version of the SMCE, which does not even exists when $\gamma > 1$. The reason is that, at the SRCE arrangement, during the day the real asset is sold for cash but cash balances never earn an interest. However, in an inflationary environment, holding cash balances should be rewarded with interest, otherwise the sellers would rather keep the real asset than give it away for cash. Hence, this system cannot work with inflation. On the other hand, in a deflationary environment, it is indistinguishable from the SMCE.

4.2 The Result

Having exhausted the logical possibilities for the alternative arrangements, we conclude with the main result of the paper. A trading arrangement is undominated (resp. dominant) if it induces, for the same parameters values, an allocation, in terms of day-time consumption, that is never smaller than (resp. strictly larger than) the one induced by the feasible alternative arrangements. We call the arrangement with monetary collateralized loans that induces the SMCE, the SMCE arrangement. We compare the SMCE arrangement with the SCE, SME and SLME arrangements. We ignore the SRCE, since either it does not exist in some region of the parameters space or, when it exists, is indistinguishable from the SMCE.

Proposition 8 *The SMCE arrangement is undominated if* $\rho A < 2f(x^*)$ *and* $\gamma < \widehat{\gamma}$*; it is dominant if* $\rho A \in \left(\frac{f(x^*)}{2}, 2f(x^*)\right)$ *and* $\gamma \in [\widetilde{\gamma}, \widehat{\gamma})$*.*

Hence, monetary loans collateralized by real assets dominate the alternatives in a non-empty region of the parameters space. The restrictions on parameters that determine when the SMCE system dominates, concern the value of the real and monetary assets. The real asset should not be too valuable or abundant nor money too devalued, otherwise the option with only credit would be better than the SMCE. On the other hand, the real asset should not be too devalued or scarce and money not too valuable, otherwise the other systems combining money and credit in different ways would replicate monetary loans. Notice that, as established in Proposition 2, in the region of parameters where the SMCE is dominant, the efficient allocation can be achieved by the SMCE setting $\gamma = \widetilde{\gamma} > \beta$, while the alternative arrangements cannot achieve efficiency. Moreover, also by Proposition 2, in a subset of such a region, when $\rho A \in [f(x^*), 2f(x^*))$, the SMCE achieves the efficient allocation even without monetary intervention, while all the other systems cannot achieve efficiency. Finally, as established in Proposition 4, which applies not only to the SME but also to the SLME, in the non-empty region Σ of values of ρA, the SMCE allows to reach the efficient allocation at the Friedman rule, while the other systems cannot achieve it for any attainable value of the monetary policy instrument, due to a tight lower bound on taxation.

5 Discussion

The crucial elements that make monetary collateralized loans essential are two, namely the assumption that some agents are the natural users of the real asset and the assumption that these are the same that also need cash. Coherently with the monetary search theoretic literature, where typically agents can produce only after having consumed,[18] the day-time buyers were assumed to be the only agents who were able to generate the returns with the asset. This assumption, however, can be relaxed. The main result of the paper continues to hold as long as the return to the seller remains smaller than the one accruing to the buyer. In this section, we discuss what would happen if we were to reverse the assumption and have the day-time sellers as the best users of the asset. Then, we discuss how to relax some of the ancillary assumptions. Finally, we briefly discuss some policy issues and the presence of cyclical and sunspot equilibria.

5.1 Direct Asset Payment

Consider reversing our main assumption and suppose that the day-time sellers are the productive agents at night. Under the best scheme, at the beginning of every period, the buyers sell the real asset held from the previous period to acquire money and purchase the day-time good, while sellers acquire the real asset with money, sell the day-time good and generate the night-time returns of the asset. At a stationary equilibrium, the allocation and the asset price are still pinned down exactly as before. The difference is that, instead of having a combination of monetary payment and collateralized debt, the model generates a combination of monetary payment and

[18] as in Kiyotaki and Wright (1993).

direct payment with the real asset.[19] The model can generate coexistence of different payment instruments in both cases: if the agent who can generate the returns with the asset is the buyer of the day-time good, a combination of money and collateralized debt obtains, if it is the seller, a combination of money and direct asset (or equity) payment, obtains. Hence, the model suggests that the choice of using a real asset as a means of payment rather than as collateral is connected with the different productive characteristics of the asset itself. One could envision an extension in which, instead of having just two types of agents, with the know-how to use the asset perfectly correlated with the day-time activity, there are four types, mixing the different day-time activities with the know-how to use the asset in an idiosincratic way. This model would generate a mix of payment arrangements, some with monetary collateralized loans and some with direct asset payment.[20]

5.2 Relaxing Assumptions

Several assumptions can be considerably relaxed without altering the gist of the paper. In particular, some stark assumptions on the payoffs, the market structure and the type of asset can be relaxed without altering the main results.

Fundamentals For instance, the assumption that the probability of being either a buyer or a seller is the same can be relaxed without affecting the results. The crucial element is the presence of some uncertainty over who will turn out to be the best user of the assets when they are acquired. The linearity of the night-time payoff can be relaxed along the lines of Gu et al. (2016).

Market Structure All markets were assumed competitive. As shown by Rocheteau and Wright (2005) and Amendola and Ferraris (2018), the competitive assumption is compatible with the informational imperfections that characterize monetary trade. However, different market structures can be considered without altering the gist of the paper. The day markets need not be competitive. We could allow for bilateral meetings and bargaining, for instance, or more generally, following Gu et al. (2016), simply postulate that the terms of trade in the market for the day good are determined by a non-linear, increasing function of the allocation, thus, capturing several potential

[19] An equity-like arrangement that is reminiscent of Lagos and Rocheteau (2008).

[20] We thank an anonymous referee for suggesting this point.

deviations from the competitive assumption. The same could be done for the asset market and the credit market. The main idea of the paper, which does not hinge on the specific market structure assumed, would continue to hold. The only result that would not survive would be the possibility to reach the first best allocation at the optimal policy, since the economy would suffer from an inefficiency driven by the imperfections in the market structure, with their ensuing distortions of the pricing away from the marginal cost. However, such a distortion would emerge symmetrically for the trading arrangements alternative to the one with monetary collateralized loans.

Asset The asset does not need to be in fixed supply. The framework can be easily adapted to represent an economy with entrepreneurs and workers, rather than buyers and sellers, exchanging labor and reproducible capital to be used as inputs in the production of a final good that can be both consumed and accumulated for the future. In this context, capital would be used as collateral for monetary loans as well as a productive input.

5.3 Monetary Policy

The model suggests as the optimal policy the adoption of the Friedman rule (Friedman (1969)), namely a zero nominal rate of interest, which corresponds in some cases to no-intervention, rather than a contraction of the stock of money as in other monetary models.[21] Increasing the stock of money, instead, is always a bad idea for day-time output, which is decreasing in γ at an SMCE in all cases. When the economy operates in the credit constrained region, an illiquid instrument such as a public bond that cannot be traded directly for goods but is collateralizable, could help reallocate misallocated liquidity and price it correctly, thus, improving upon the SMCE allocation. The bond would need to be illiquid, in a sense similar to Kocherlakota (2003), since it would otherwise be equivalent to a direct increase in the stock of money, which cannot improve the allocation.[22] To see this point, suppose the equilibrium is collateral constrained and the government issues one period bonds every night, selling them in exchange for money in an open market operation at a price π in units of numeraire, committing to convert them one for one in cash during the following night. Let ζ be

[21] by Proposition 2.

[22] The illiquidity, here relative to Kocherlakota (2003), is only partial since the bonds cannot be used directly to buy goods, but can be used to obtain loans.

the proportion of outstanding bonds to the money stock. The government satisfies its budget constraint whereby the stock of money for the next period equals the current stock net of the amount withdrawn through the open market operation plus the new stock created to convert the oustanding bonds. The law forbids the use of the bonds to purchase any object, but allows to pledge them as collateral to obtain loans.[23] Relative to the previous situation, buyers face a new collateral constraint, namely $\phi d^b (1+i) \leq \psi \left(a + \alpha^b\right) + \phi b$, which replaces constraint (3). The rest of the model remains the same. If ζ is appropriately chosen by the government, at an equilibrium with bonds, the collateral constraint is never binding, liquidity is correctly priced, the allocation is determined by $u'(x) = (1+i)\, c'(x)$, and the price of the real asset equals its fundamental value. The Euler condition for the accumulation of the bond pins down π, which, in turn, allows to compute the equilibrium nominal interest rate, which turns out to be increasing in ζ. Hence, when the real asset is scarce, for a given rate of money growth, the day-time allocation is larger than the constrained one without bonds and open market operations have a sort of liquidity effect, whereby a larger bond issue increases the nominal rate as in Lucas (1990).

5.4 Cycles and Sunspots

For the sake of the comparison with the literature on credit imperfections and bubbles referred to in the Introduction, we briefly comment on some non-stationary equilibria of the model, which resemble boom-bust cycles. Indeed, together with stationary equilibria, the model can generate, in some circumstances, non-trivial dynamic trajectories, including cyclical ones. When the collateral constraint is not binding, consumption is time invariant, while the price of the asset follows a dynamic path governed by a linear difference equation with a unique stationary solution. Hence, in this case, cyclical behavior cannot arise. When the collateral constraint is binding, instead, using standard bifurcation techniques, local deterministic cycles of period two and sunspot equilibria of order two around the SMCE can be shown to exist, when the risk aversion of the utility function is sufficiently large. Intuitively, when the agents expect the asset price to be, say, high in the future, they plan to borrow

[23] Alternatively, the bonds could be illiquid in the goods market but liquid in the money market. In this case, they would be used as payment instruments to acquire cash when needed. The allocation would be identical. What is crucial is that bonds cannot be spent directly on consumption and the Government has the ability to commit to repay the bonds.

more and finance higher consumption, since a higher price of the asset tends to relax their borrowing constraint. However, a high price of the asset induces a lower demand for it, thus putting a downward pressure on the price, which tends to tighten the borrowing constraint, leading to lower consumption. The strenght of this latter effect depends on the price elasticity of demand which is controlled by the curvature of utility. When the curvature is sufficiently large, the effect is sufficiently strong to drive down the price. However, when the agents expect a low asset price, they plan to borrow less, reducing consumption, but they also tend to increase their demand for the asset, and so on. Next to local cycles and sunspots, exploiting the existence of the no-trade equilibrium, global cycles and sunspots can also be shown to exist, for large enough values of the risk aversion of utility.

6 Conclusion

We have presented a model in which collateralized monetary loans are essential for the allocation of resources. The central idea of the paper is that there are two assets, a nominal one, without intrisic value, and a real one, with intrinsic value, both of which are held for precautionary reasons, and both of which may turn out to be misallocated after the realization of uncertainty. In the absence of well functioning markets, due to the agents anonymity, an arrangement whereby the two are traded in a complementary way is the best option.

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7 Appendix

In this Appendix, we derive fully the optimality conditions, the equilibrium conditions and prove some ancillary results and the Propositions in the text.

Optimality Conditions The first order conditions for x^b, α^b, d^b, x^s, α^s, d^s, m_{+1} and a_{+1}, are, respectively,

$$u'(x^b) - p\delta - \frac{p}{\phi} W_m^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right) = 0, \tag{32}$$

$$-q\mu + \psi\lambda - q\delta - \frac{q}{\phi} W_m^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right) + W_a^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right) = 0, \tag{33}$$

$$-\phi(1+i)\lambda + \phi\delta - iW_m^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right) = 0, \tag{34}$$

$$-c'\left(x^s\right) + \frac{p}{\phi} W_m^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right) = 0, \tag{35}$$

$$-\zeta + q\theta + \frac{q}{\phi} W_m^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right) - W_a^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right) = 0, \tag{36}$$

$$-\phi\theta + iW_m^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right) = 0, \tag{37}$$

$$-\phi + \beta V_m'\left(m_{+1}, a_{+1}\right) = 0, \tag{38}$$

$$-\psi + \beta V_a'\left(m_{+1}, a_{+1}\right) = 0. \tag{39}$$

The envelope conditions are, respectively,

$$V_m'\left(m, a\right) = \frac{1}{2}\left[\phi\mu + \phi\delta + W_m^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right)\right] + \frac{1}{2}\left[\phi\theta + W_m^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right)\right], \tag{40}$$

$$V_a'\left(m, a\right) = \frac{1}{2}\left[\psi\lambda + W_a^{b\prime}\left(\widetilde{m}^b, \widetilde{a}^b\right)\right] + \frac{1}{2}\left[\psi\zeta + W_a^{s\prime}\left(\widetilde{m}^s, \widetilde{a}^s\right)\right], \tag{41}$$

$$W_m^{j\prime}\left(\widetilde{m}^j, \widetilde{a}^j\right) = \phi, \tag{42}$$

$$W_a^{j\prime}\left(\widetilde{m}^j, \widetilde{a}^j\right) = \psi + R\left(j\right). \tag{43}$$

Use (42) and (43) into (40) and (41), delay them one period and insert them into (38) and (39), to obtain, respectively,

$$\phi = \beta\phi_{+1}\left[\frac{1}{2}\left(\mu_{+1} + \delta_{+1} + 1\right) + \frac{1}{2}\left(\theta_{+1} + 1\right)\right], \tag{44}$$

$$\psi = \beta\psi_{+1}\left[\frac{1}{2}\frac{R}{\psi_{+1}} + \frac{1}{2}\left(\lambda_{+1} + 1\right) + \frac{1}{2}\left(\frac{\zeta_{+1}}{\psi_{+1}} + 1\right)\right]. \tag{45}$$

Next, we derive the multipliers of the constraints. Combine (37) and (42), to obtain $\theta = r - 1$. Combine (32), (42) and $p = c'\left(x^s\right)$, to obtain $\delta = \frac{u'\left(x^b\right)}{c'\left(x^s\right)} - 1$. Combine (34), (42), $\delta = \frac{u'\left(x^b\right)}{c'\left(x^s\right)} - 1$ and $p = c'\left(x^s\right)$, to obtain $\lambda = \frac{u'\left(x^b\right)}{rc'\left(x^s\right)} - 1$. Combine (36), $\theta = r - 1$ and (42)-(43), to obtain $\zeta = rq - \psi$. Combine (33), (42)-(43), $p = c'\left(x^s\right)$, $\delta = \frac{u'\left(x^b\right)}{c'\left(x^s\right)} - 1$, $\lambda = \frac{u'\left(x^b\right)}{rc'\left(x^s\right)} - 1$, to obtain $\mu = \frac{R}{q} + \frac{u'\left(x^b\right)}{c'\left(x^s\right)}\left[\frac{\psi}{qr} - 1\right]$.

Equilibrium conditions We now turn to the stationary money and collateral equilibrium, imposing market clearing and stationarity. Inserting the multipliers into (44) and (45), we obtain

$$1 = \frac{\beta}{2\gamma}\frac{\psi}{q}\left[\frac{R}{\psi} + \frac{u'(x)}{rc'(x)} + \frac{rq}{\psi}\right], \tag{46}$$

$$1 = \frac{\beta}{2}\left[\frac{R}{\psi} + \frac{u'(x)}{rc'(x)} + \frac{rq}{\psi}\right]. \tag{47}$$

These equations together imply $\frac{\psi}{q} = \gamma$. Therefore, we obtain

$$1 = \frac{\beta}{2}\left[\frac{R}{\psi} + \frac{u'(x)}{rc'(x)} + \frac{r}{\gamma}\right]. \tag{48}$$

Since the buyers are the best users of the real asset, we restrict attention to a situation in which $\alpha^s = a$. The multiplier of this constraint is $\zeta = q(r - \gamma) \geq 0$. The complementary slackness conditions for the remaining constraints are

$$(r-1)(\gamma\phi M + \psi A - \gamma\phi d) = 0, \tag{49}$$

$$\left[\frac{u'(x)}{c'(x)} - 1\right][\gamma\phi M + \gamma\phi d - \psi A - \gamma c'(x)x] = 0, \tag{50}$$

$$\left[\frac{u'(x)}{rc'(x)} - 1\right](2\psi A - r\phi d) = 0, \tag{51}$$

$$\left[\frac{R}{\psi} + \frac{u'(x)}{rc'(x)} - \frac{u'(x)}{\gamma c'(x)}\right](\gamma\phi M - \psi A) = 0. \tag{52}$$

The next Lemma begins the equilibrium characterization. We set up some notation. Define $g(x) \equiv c'(x)x$, $C(x,\gamma) \equiv 2(1-\beta)\rho A - (2-\beta)\gamma g(x) + \beta f(x)$, $D(x,\gamma) \equiv 2\gamma(1-\beta)\rho A - (2\gamma - \beta)g(x) + \beta\gamma f(x)$, $\lambda(x,\gamma) \equiv \frac{f(x)}{\gamma g(x)} - 1$ and $\mu(x,\gamma) \equiv \frac{2\gamma-\beta}{\beta} - \frac{f(x)}{g(x)}$.

Lemma 1 *At equilibrium, $x \leq x^*$, $r = \max\{\gamma, 1\}$ and: a) if $\gamma \geq 1$, the allocation is determined by $\lambda(x,\gamma)C(x,\gamma) = 0$; b) if $\gamma < 1$ the allocation is either x^* or determined by $\mu(x,\gamma)D(x,\gamma) = 0$.*

Proof. Since $\frac{u'(x)}{c'(x)} \geq 1$ must hold, and $\frac{d}{dx}\frac{u'(x)}{c'(x)} = \frac{u''(x)}{c'(x)} - \frac{u'(x)c''(x)}{c'(x)^2} < 0$, then $x \leq x^*$. There are three cases: *i.* the first term in (49) is positive, and the second term is zero; *ii.* both terms are zero; *iii.* the first term in (49) is zero, the second term is positive. *i.* Suppose $r > 1$ and $\gamma\phi M + \psi A = \gamma\phi d$. By the second term in (52), $\gamma\phi d \geq 2\psi A$.

By the second term in (51), $2\psi A \geq r\phi d$. Hence, $\gamma \geq r$. Since $\zeta = q(r-\gamma) \geq 0$, then, $r = \gamma$. Hence, in this case $\gamma > 1$. Condition (48) is rewritten as

$$\frac{R}{\psi} + \frac{u'(x)}{\gamma c'(x)} = \frac{2-\beta}{\beta}. \tag{53}$$

Since $r = \gamma$, the first term in (52) is strictly positive, hence, $\gamma\phi M = \psi A$. Inserting this into (50), we obtain $\left[\frac{u'(x)}{c'(x)} - 1\right][\phi d - c'(x)x] = 0$. Since $\gamma > 1$, $\frac{u'(x)}{c'(x)} > \frac{u'(x)}{\gamma c'(x)} \geq 1$, hence, $\phi d = c'(x)x$, which we can substitute, together with (53), into (51), obtaining,

$$\left[\frac{u'(x)}{\gamma c'(x)} - 1\right]\left[2RA + u'(x)x - \frac{2-\beta}{\beta}\gamma c'(x)x\right] = 0. \tag{54}$$

By definition, (54) $\Leftrightarrow \lambda(x,\gamma)C(x,\gamma) = 0$. *ii.* Suppose $r = 1$ and $\gamma\phi M + \psi A = \gamma\phi d$. All the derivations of case i. still apply. Since $\gamma = 1$, either $\frac{u'(x)}{c'(x)} = 1$, and, thus, $x = x^*$, or $2RA + u'(x)x - \frac{2-\beta}{\beta}c'(x)x = 0$. Hence, $\lambda(x,\gamma)C(x,\gamma) = 0$ applies to this case as well. *iii.* Suppose $r = 1$ and $\gamma\phi M + \psi A > \gamma\phi d$. Since, $\zeta = q(r-\gamma) \geq 0$, in this case, $\gamma \leq 1$. Rewrite (48) as

$$\frac{R}{\psi} + \frac{u'(x)}{c'(x)} = \frac{2\gamma-\beta}{\gamma\beta}. \tag{55}$$

If $\frac{u'(x)}{c'(x)} > 1$, by (50), we have $\gamma\phi M + \gamma\phi d = \psi A + \gamma c'(x)x$, and by (51), $2\psi A = \phi d$. These imply $\gamma\phi M = \gamma c'(x)x - (2\gamma-1)\psi A$. Inserting this and (54) into (52), we obtain a single equation in x,

$$\left[\frac{2\gamma-\beta}{\beta} - \frac{u'(x)}{c'(x)}\right]\left[\frac{2\gamma-\beta}{\gamma\beta}c'(x)x - u'(x)x - 2RA\right] = 0. \tag{56}$$

By definition, (56) $\Leftrightarrow \mu(x,\gamma)D(x,\gamma) = 0$. If $\frac{u'(x)}{c'(x)} = 1$, the allocation is x^*. With $\gamma = 1$, $\mu(x,\gamma) > 0$ and $D(x,\gamma) = C(x,\gamma) = 0$, hence, we are back to case *ii.* We conclude that $r = \max\{\gamma, 1\}$ and $\lambda(x,\gamma)C(x,\gamma) = 0$ holds for $\gamma \geq 1$ and $\mu(x,\gamma)D(x,\gamma) = 0$ for $\gamma < 1$. ■

Therefore, proving the existence of an SMCE reduces to finding a solution in x of equations $\lambda(x,\gamma)C(x,\gamma) = 0$ if $\gamma \geq 1$, $\mu(x,\gamma)D(x,\gamma) = 0$ if $\gamma < 1$, and, then, use (48) to find the corresponding unique equilibrium value of ψ. The rest of the system determines the remaining equilibrium variables uniquely. Next we prove Proposition 1 in the text. Define $\omega(\gamma) \equiv \frac{\beta(1-\gamma)}{2\gamma(1-\beta)}$ and $\sigma(\gamma) \equiv \frac{(2-\beta)\gamma-\beta}{2\gamma(1-\beta)}$.

Proof of Proposition 1. By Lemma 1, $\lambda(x,\gamma) C(x,\gamma) = 0$ and $\mu(x,\gamma) D(x,\gamma) = 0$ do not apply simultenaously. Hence, the two conditions can be analysed separately. When $\gamma \geq 1$, two cases are possible: i. $\lambda(x,\gamma) = 0$ and $C(x,\gamma) \geq 0$, iff $f(x) \leq \rho A$; ii. $\lambda(x,\gamma) > 0$ and $C(x,\gamma) = 0$, iff $f(x) > \rho A$. Case i. The function $\lambda(x,\gamma)$ is continuous in x. By the Inada conditions, $\lambda(0,\gamma) = \infty$, and $\lambda(x^*,\gamma) = \frac{1}{\gamma} - 1 \leq 0$. By the Intermediate Value Theorem, for any $\gamma \geq 1$, a value $\widetilde{x} \leq x^*$ exists such that $\lambda(\widetilde{x},\gamma) = 0$. Moreover, $\frac{\partial \lambda(x,\gamma)}{\partial x} < 0$, hence, $\widetilde{x}$ is unique for every γ. This case is the only possibility for $f(x^*) \leq \rho A$. Case ii. The function $C(x,\gamma)$ is continuous in x. By the Inada conditions, $C(0,\gamma) > 0$, and $C(x^*,\gamma) < 0$, since $f(x^*) > \rho A$, hence, by the Intermediate Value Theorem, for any $\gamma \geq 1$, a value $\widetilde{x}' < x^*$ exists such that $C(\widetilde{x}',\gamma) = 0$. Moreover, $\frac{\partial C(x,\gamma)}{\partial x} < 0$, hence, $\widetilde{x}'$ is unique for every γ. When $\gamma < 1$, only two cases are possible: i. $\mu(x,\gamma) > 0$ and $D(x,\gamma) = 0$, iff $\omega(\gamma) f(x) \leq \rho A$; ii. $\mu(x,\gamma) = 0$ and $D(x,\gamma) \geq 0$, iff $\omega(\gamma) f(x) > \rho A$. Case i. The function $D(x,\gamma)$ is continuous in x. By the Inada conditions, $D(0,\gamma) < 0$, and $D(x^*,\gamma) \geq 0$ if $\sigma(\gamma) f(x^*) \geq \rho A$, hence, by the Intermediate Value Theorem, a value $\widetilde{x}'' \leq x^*$ exists such that $C(\widetilde{x}'',\gamma) = 0$, for any $\gamma \geq \widetilde{\gamma} \in (\beta, 1)$, where $\widetilde{\gamma}$ is s.t. $D(x^*,\gamma) = 0 \Leftrightarrow \sigma(\gamma) f(x^*) = \rho A$, which gives $\widetilde{\gamma} = \frac{\beta f(x^*)}{(2-\beta) f(x^*) - 2\rho A(1-\beta)} \in (\beta, 1)$, when $\rho A > \frac{f(x^*)}{2}$. Moreover, $\frac{\partial D(x,\gamma)}{\partial x} > 0$, hence, $\widetilde{x}''$ is unique for every γ. Case ii. This case cannot arise if $\rho A > \frac{f(x^*)}{2}$. The function $\mu(x,\gamma)$ is continuous in x. By the Inada conditions, $\mu(0,\gamma) = -\infty$, and $\mu(x^*,\gamma) = 2\left(\frac{\gamma}{\beta} - 1\right) \geq 0$ with equality for $\gamma = \beta$, hence, by the Intermediate Value Theorem, a value $\widetilde{x}''' \leq x^*$ exists such that $\mu(\widetilde{x}''',\gamma) = 0$. Moreover, $\frac{\partial \mu(x,\gamma)}{\partial x} > 0$, hence, $\widetilde{x}'''$ is unique for every γ. We conclude that, for $\rho A \geq f(x^*)$, an SMCE exists and is unique for $\gamma \geq 1$; for $f(x^*) > \rho A > \frac{f(x^*)}{2}$, there exists a $\widetilde{\gamma} \in (\beta, 1)$, such that, an SMCE exists and is unique when $\gamma \geq \widetilde{\gamma}$; for $\frac{f(x^*)}{2} \geq \rho A$, an SMCE exists and is unique when $\gamma \geq \beta$.■

The following three Lemmas provide the complete characterization of the SMCE. The SMCE is unconstrained if the relevant constraints for each case are slack, constrained if they are binding.

Lemma 2 *Suppose $f'(x) > 0$. a) If $\rho A \geq f(x^*)$, then, the SMCE is unconstrained; b) if $f(x^*) > \rho A > \frac{f(x^*)}{2}$, then: bi) if $f(0) < \rho A$, there exists a $\overline{\gamma_1} > 1$ such that, for $\gamma > \overline{\gamma_1}$ the SMCE is unconstrained, for $\gamma \leq \overline{\gamma_1}$ constrained; bii) if $f(0) \geq \rho A$, the SMCE is always constrained; c) If $\frac{f(x^*)}{2} \geq \rho A$, then, there exists a $\underline{\gamma_1} \in (\beta, 1)$ such that: ci) if $f(0) < \rho A$, for $\gamma > \overline{\gamma_1}$ the SMCE is unconstrained, for $\underline{\gamma_1} < \gamma \leq \overline{\gamma_1}$ constrained and for $\gamma \leq \underline{\gamma_1}$ unconstrained; cii) if $f(0) \geq \rho A$, the SMCE is constrained* *for $\gamma > \underline{\gamma_1}$ and unconstrained for $\gamma \leq \underline{\gamma_1}$;*

Proof. *a)* $f(x^*) \leq \rho A$ and $f(0) \leq \rho A$, guarantee $f(x) \leq \rho A$ for all values of $x \leq x^*$. Hence, $\lambda(x,\gamma) = 0$ for all values of $x \leq x^*$. *bi)* $f(x^*) > \rho A$ and $f(0) < \rho A$, imply that there exists a unique $\overline{x} \leq x^*$ such that $f(x) \leq \rho A$ for $x \geq \overline{x}$ and $f(x) > \rho A$ for $x < \overline{x}$ s.t. $\lambda(x,\gamma) = 0$ for $x \geq \overline{x}$ and $\lambda(x,\gamma) > 0$ for $x < \overline{x}$; $C(x,\gamma) > 0$ for $x > \overline{x}$ and $C(x,\gamma) = 0$ for $x \leq \overline{x}$. The equation $C(\overline{x},\gamma) = 0$ can be solved to find the unique cutoff $\overline{\gamma_1} \geq 1$. *bii)* $f(x^*) > \rho A$ and $f(0) \geq \rho A$, imply that $\lambda(x,\gamma) > 0$ for all values of positive $x \leq x^*$ when $\gamma \geq 1$ and $\rho A \geq \frac{f(x^*)}{2}$ implies $\mu(x,\gamma) > 0$ for all values of positive $x \leq x^*$ when $\gamma < 1$. *ci)* $\frac{f(x^*)}{2} > \rho A$ and $f(0) < \rho A$, imply the existence of two ordered cutoffs, $\overline{\gamma_1} \geq 1 > \underline{\gamma_1}$, such that above the first, $\lambda(x,\gamma) = 0$, and below the second $\mu(x,\gamma) = 0$, in between both $\lambda(x,\gamma) > 0$ and $\mu(x,\gamma) > 0$. *cii)* $\frac{f(x^*)}{2} > \rho A$ and $f(0) < \rho A$, imply that the economy is always constrained for $\gamma > \underline{\gamma_1}$, and $\mu(x,\gamma) = 0$ for $\gamma \leq \underline{\gamma_1}$. ■

The proofs of the next two Lemmas are analogous to the proof of the previous Lemma.

Lemma 3 *Suppose $f'(x) < 0$. a) If $\rho A \geq f(0)$, then, the SMCE is unconstrained; b) if $f(0) > \rho A \geq f(x^*)$, there exists a $\overline{\gamma_2} \geq 1$ such that, for $\gamma > \overline{\gamma_2}$ the SMCE is constrained, for $\gamma \leq \overline{\gamma_2}$ unconstrained; c) if $f(x^*) > \rho A > \frac{f(x^*)}{2}$, then, the SMCE is always constrained; d) if $\frac{f(x^*)}{2} \geq \rho A$, then, there exists a $\underline{\gamma_2} \in (\beta, 1)$ such that, for $\gamma > \underline{\gamma_2}$ the SMCE is constrained and for $\gamma \leq \underline{\gamma_2}$ unconstrained.*

Lemma 4 *Suppose $f(x) = k$. a) If $\rho A \geq k$, then, the SMCE is unconstrained; b) if $k > \rho A > \frac{k}{2}$, then, the SMCE is constrained; c) if $\rho A \leq \frac{k}{2}$, then, there exists a $\underline{\gamma}_3 \in (\beta, 1)$ such that the SMCE is constrained for $\gamma > \underline{\gamma}_3$ and unconstrained for $\gamma \leq \underline{\gamma}_3$.*

Next, we prove the remaining Propositions in the text.

Proof of Proposition 2. a) By Proposition 1, if $f(x^*) \leq \rho A$, $\lambda(x^*, 1) = 0$ holds at an SMCE. Inequality (14) is always satisfied. b) By Proposition 1, if $f(x^*) > \rho A > \frac{f(x^*)}{2}$, $D(x^*, \widetilde{\gamma}) = 0$ holds at an SMCE. Inequality (14) requires $\widetilde{\gamma} \geq \frac{1}{2}$, which is guaranteed if $\rho A \geq \frac{2-3\beta}{1-\beta}\frac{f(x^*)}{2}$, with $\frac{2-3\beta}{1-\beta} \leq 1$ iff $\beta \geq \frac{1}{2}$. c) By Proposition 1, if $\rho A \leq \frac{f(x^*)}{2}$, $\mu(x^*, \beta) = 0$ holds at an SMCE. Inequality (14) is satisfied at $\gamma = \beta$, if $\rho A \geq \omega f(x^*)(1-\beta)$, with $\omega(1-\beta) \leq \frac{1}{2}$, iff $\beta \geq \frac{1}{2}$.■

Proof of Proposition 3. a) The unconstrained SCE is first best efficient, since $u'(x) = c'(x)$. b) The SCE is constrained iff $2f(x^*) > \rho A$. At a constrained SCE, $x < x^*$ is determined by $C(x, 1) - \rho A(1-\beta) = 0$. At an SMCE, if $\rho A \geq f(x^*)$, $u'(x) = \gamma c'(x)$ for all $\gamma \geq 1$. By Proposition 2, for $\gamma = 1$, at an SMCE, $\widetilde{x} = x^*$. By continuity, we can find an upper-bound on γ larger than 1, such that SMCE output is strictly larger than SCE output for values of γ strictly smaller than the upper-bound. If $\rho A < f(x^*)$, the SMCE has $C(x, 1) = 0$, which gives a strictly higher x than the SCE, since $C(x, 1)$ is strictly decreasing in x. By continuity, we can find an upper-bound on γ larger than 1, such that SMCE output is strictly larger than SCE output for values of γ strictly smaller than the upper-bound. When $\gamma < 1$, a fortiori the SMCE is higher. Take as $\widehat{\gamma}$ the smallest of the two upper-bounds.■

Proof of Proposition 4. Equation (21) gives $\psi = \rho$. Therefore, equation (14) at a monetary equilibrium writes as $\gamma \geq \frac{g(x) - \rho A}{g(x)}$. Equation (22) pins down day-time output at the SME, thus, the efficient allocation, x^*, is reached iff $\gamma = \beta$. At $x = x^*$, $g(x^*) = f(x^*)$. Hence, at the SME, (14) is violated for $\gamma = \beta$, and, thus, x^* cannot be reached, if $\rho A < f(x^*)(1-\beta)$. By Proposition 2, the SMCE induces x^* if $\rho A > \omega(1-\beta) f(x^*)$ at $\gamma = \beta$. Notice that $\omega < 1 \Leftrightarrow \beta > \frac{1}{2}$.■

Proof of Proposition 5. Equation (22) pins down day-time output at the SME. By equations (17) and (18), output is always strictly higher at the SMCE, except in the cases in which (18) applies and the equilibrium is unconstrained. In those cases SMCE and SME output are the same.■

Proof of Proposition 6. There are two cases. For $r \geq 1$, the system reduces to

$$\left[\frac{u'(x)}{\gamma c'(x)} - \frac{1}{\beta}\right]\left[RA + u'(x)x - \left(\frac{2-\beta}{\beta}\right)c'(x)x\gamma\right] = 0. \tag{57}$$

Comparing (57) with (54), one sees immediately that the SMCE allocation is larger both in the unconstrained and constrained case. Consider $r = 1$. Then, by (24), $\frac{u'(x)}{c'(x)} = \frac{2\gamma - \beta}{\beta}$. The SMCE allocation is larger when $\rho A > \frac{f(x^*)}{2}$, and at least as large when $\rho A \leq \frac{f(x^*)}{2}$, for all feasible γ in each case.■

Proof of Proposition 7. At an SRCE, the multiplier of the constraint $\alpha^s \leq a$, is $\zeta = 1 - \frac{\psi}{q} \geq 0$. The arbitrage condition, $q = \frac{\psi}{\gamma}$, is obtained as before for the SMCE. These two necessary conditions together imply that the SRCE does not exists if $\gamma > 1$. Using (29)-(31), we obtain (56). Hence, when (56) holds at the SMCE, i.e. for $\gamma \leq 1$, the two systems attain the same allocation. The Euler conditions are the same, hence, the equilibrium price of the asset is the same.■

Proof of Proposition 8. By Proposition 3, 6, 7 , whenever the SMCE induces a larger allocation than the SCE is undominated; when it also gives a strictly larger allocation than the SME and SLME, it is dominant.■