A HOMOGENIZATION RESULT FOR INTERACTING ELASTIC AND BRITTLE MEDIA

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Abstract. We consider energies modelling the interaction of two media parameterized by the same reference set, such as those used to study interactions of a thin film with a stiff substrate, hybrid laminates, or skeletal muscles. Analytically, these energies consist of a (possibly non-convex) functional of hyperelastic type and a second functional of the same type such as those used in variational theories of brittle fracture, paired by an interaction term governing the strength of the interaction depending on a small parameter. The overall behaviour is described by letting this parameter tend to zero and exhibiting a limit effective energy using the terminology of Gamma-convergence. Such energy depends on a single state variable and is of hyperelastic type. The form of its energy function highlights an optimization between microfracture and microscopic oscillations of the strain, mixing homogenization and high-contrast effects.

1. Introduction

The subject of this paper is the analysis of a model of interacting media governed by coupled energies in the context of the theory of homogenization for hyperelastic energies. In the simplest "classical" setting homogenization theory studies the effective behaviour of energies for a single medium, that can be written as an integral

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla v\Big)dx.$$

defined on a reference configuration Ω in $\mathbb{R}^n$ ($n = 2$ or 3 in the physical cases) and depending on a function v taking values in some $\mathbb{R}^m$. The assumed periodicity of the function f in the first variable describes the microstructure of the medium. The parameter ε is the scale of the microstructure and is assumed to be small with respect to the size of the sample. The overall behaviour of these energies can be then approximately described by letting ε tend to 0 and computing the Γ-limit of the energies, which is a homogeneous integral functional whose energy function takes the form

$$f_{\rm hom}(z) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{(-\frac{T}{2},\frac{T}{2})^n} f(y, Dv)dy : v(x) = zx \ \text{ on } \ \partial\Big(-\frac{T}{2},\frac{T}{2}\Big)^n \Big\}$$

where the inf is taken over all v in a Sobolev space depending on the growth conditions of f. This formula provides the description of the overall behaviour of the energies F_ε for small ε. First, it highlights that for a given macroscopic "strain" z

the microscopic behaviour depends only on z and is obtained by optimization of oscillations at scale of order ε. Second, that the relevant (microscopic) period of these oscillations is at the same scale ε but may be much larger that the minimal period of the microsctructure. This is a characteristic behaviour of non-convex energies: oscillations with the same period as that of the microstructure are optimal only for convex energies. Third, the existence of the limit shows that oscillations stabilize, so that the behaviour is not greatly influenced by ε as long as ε is small (in mathematical terms, the asymptotic behaviour of F_ε does not depend on subsequences of ε). We note that the problem of the computation of the overall behaviour of the energy makes sense also when there is no microstructure; i.e., when the function f does not depend on the first variable. In that case we refer to the problem as that of *relaxation* of a single functional, and the effective energy is still determined by oscillations, which nevertheless are not constrained to a precise period. We refer to the monograph [1] for an introduction to homogenization and relaxation.

We will examine coupled hyperelastic media both in a homogenization and a relaxation context. The energies that we consider depend on two functions u and v defined on the common reference configuration Ω. While interpreting such energies in the continuum may seem confined to special modelling issues, from an atomistic standpoint they are quite natural. Indeed, we may think of a lattice model mixing strong and weak molecular interactions. Sublattices of molecules linked by strong bonds can be separately approximated by continuum elastic energies (see e.g. [2]). The weak interactions are instead approximated by an integral term coupling the energies, which depends on the characteristic intermolecular distance ε. Such coupled systems are typical of high-constrast systems described by "double-porosity" energies (see e.g. [3]; for a different geometric setting see [4]). In our model we may consider more general microscopic energies than just elastic ones, letting the energy depending on the variable u allow for fracture. More precisely, we interpret u as the deformation of a (possibly, brittle) hyperelastic material governed by Griffith fracture energies, such as those used in recent analyses of crack propagation [5], of the form

$$\int_\Omega f(\nabla u)dx + k\,\mathcal{H}^{n-1}(S(u)),$$

where $S(u)$ is the fracture site of u and $\mathcal{H}^{n-1}$ denotes the $n-1$-dimensional Hausdorff surface measure. The parameter k describes the fracture toughness of the material. The relevant case is when k is small, since in that case it models the possibility of "diffuse" micro-cracks, which may be homogenized. In the context of a passage from discrete to continuum theories for systems of Lennard-Jones interactions, k is proportional to the characteristic intermolecular distance ε (see [6, 7]). With this molecular interpretation in mind we can include more general Griffith-type fracture energies than just the Hausdorff measure of the crack set, of the form $\varepsilon \int_{S(u)} \varphi\big(\frac{u^+-u^-}{\varepsilon}\big) d\mathcal{H}^{n-1}$, with $u^\pm$ the traces of u on both sides of the crack. In the same reference configuration Ω, we consider a second function v that we may interpret as the deformation of a hyperelastic material, with energy function g. The two materials are coupled by an integral penalizing u and v far apart, governed by a second small parameter δ. The complete form of the energy we are going to consider is then

$$F_{\delta,\varepsilon}(u,v) = \int_\Omega \Big(f(\nabla u) + g(\nabla v) + h\big(\frac{u-v}{\delta}\big)\Big)\,dx + \varepsilon \int_{S(u)} \varphi\big(\frac{u^+-u^-}{\varepsilon}\big) d\mathcal{H}^{n-1}, \tag{1}$$

for some function h. In this expression the function v belongs to an appropriate Sobolev space, while the correct space for the variable u is the space of special functions with bounded variation $SBV(\Omega;\mathbb{R}^m)$, commonly used in theories of fracture mechanics [5, 8].

In a one-dimensional setting, energies of the form (1) have been used for the description of different mechanical problems. For instance, Baldelli et al. ([9], see also previous work by Marigo and Truskinovsky [10]) have investigated fracture and debonding processes of a thin film on a stiff substrate with an elastic-brittle interface. In such a model, an additional dissipative energetic term is considered, representing the brittle fracture energy (delamination) of the interface. Instead, the pseudo-ductile response of thin-ply hybrid laminates has been captured by Alessi et al. [11, 12] by considering cohesive interface laws, possibly with a softening part, and an elastic-brittle behavior for both layers, with their corresponding additional energetic terms. We refer to those papers for more physical insight and graphical representation of the energies we will consider (see e.g. Fig. 1 and 3 in [9]). In all theses cases, the relevant scaling for the energies $F_{\delta,\varepsilon}$ is indeed $\delta=\varepsilon$, and we will then consider only that case; i.e. energies

$$F_\varepsilon(u,v)=\int_\Omega\Big(f(\nabla u)+g(\nabla v)+h\big(\frac{u-v}{\varepsilon}\big)\Big)\,dx+\varepsilon\int_{S(u)}\varphi\big(\frac{u^+-u^-}{\varepsilon}\big)d\mathcal{H}^{n-1} \tag{2}$$

(see Section 3, (e)). More in general, we may consider inhomogeneous energies with also an oscillating spatial dependence, but it is interesting to note that the interaction of the two media requires a homogenization process also with no spatial inhomogeneity. For such energies we will describe the asymptotic behaviour as $\varepsilon\to 0$. For an interpretation of δ as a characteristic intermolecular length scale, as mentioned above, we refer to the discrete models in[13, 14], from which the relevance of the scaling $\delta=\varepsilon$ can also be directly derived.

Note that as a particular case we may consider the one where both media are elastic, in which case we consider, with a slight abuse of notation, energies as

$$F_\delta(u,v)=\int_\Omega\Big(f(\nabla u)+g(\nabla v)+h\big(\frac{u-v}{\delta}\big)\Big)\,dx, \tag{3}$$

thus ruling out the possibility of fracture for the medium described by u. These energies are defined on pairs of Sobolev spaces. If h blows up at infinity we expect the interaction term to force $u=v$ in the limit as δ tends to zero. However, for homogeneous energies (3) the effect of h is restricted to the fact that the effective energy may be described by some type of elastic energy governed by a single parameter v, and the resulting effective energy can be simply described by relaxation arguments (see Section 3, (c)). This is due to the fact that oscillations necessary for relaxation can be performed at an arbitrary scale.

For the general energies F_ε in (2), it must be noted that superlinear growth conditions on g immediately imply that sequences $\{v_\varepsilon\}$ such that $F_\varepsilon(u_\varepsilon,v_\varepsilon)$ is equibounded are weakly precompact in some Sobolev space (up to the addition of constants), so that in that case we may assume that v_ε weakly converge to v in some $W^{1,p}$. Moreover, growth conditions on h give that $(u_\varepsilon-v_\varepsilon)/\varepsilon$ must be bounded in some Lebesgue space, so that actually also u_ε converge to the same v (e.g., in L^1). Note that in general u_ε may not weakly converge in any Sobolev space, and that the $\mathcal{H}^{n-1}$-measure of the sets $S(u_\varepsilon)$ may diverge; nevertheless, the limit of u_ε is a Sobolev function. This remark justifies the description of the limit by using only the variable v, integrating out the effects of u also in the case of coupling with a brittle medium.

Our first result is a general *homogenization theorem* that states that the Γ-limit of the energies above is a local functional that can be written as a usual hyperelastic energy

$$F_{\rm hom}(v) = \int_\Omega f_{\rm hom}(\nabla v)\,dx. \tag{4}$$

The energy density is characterized by the *asymptotic homogenization formula*

$$\begin{aligned} f_{\rm hom}(z) \quad &= \lim_{T\to+\infty}\frac{1}{T^n}\inf\Big\{\int_{(-\frac{T}{2},\frac{T}{2})^n}\Big(f(\nabla u)+g(\nabla v)+h(u-v)\Big)\,dx \\ &+\int_{S(u)\cap(-\frac{T}{2},\frac{T}{2})^n}\varphi(u^+-u^-)d\mathcal{H}^{n-1} : u=v=zx \ \text{ on } \ \partial\Big(-\frac{T}{2},\frac{T}{2}\Big)^n\Big\}, \end{aligned} \tag{5}$$

where the inf is taken over all u in $SBV^p(\Omega;\mathbb{R}^m)$ and $v\in W^{1,p}(\Omega;\mathbb{R}^m)$. This formula highlights that minimizing behaviours are obtained by optimizing among microgeometries with interacting oscillations *and discontinuities*. Note that this formula mixes the asymptotic analysis typical of nonlinear periodic media with the interaction between terms depending on the gradient and 'lower-order terms' typical of double-porosity phenomena. Indeed, we may view the scaling of the surface part as playing the same role as that of singularly perturbed gradient terms in theories of high-contrast media.

An interesting remark is that formula (5) is optimal, in the sense that homogenization arguments *must* be used even though no periodicity is present in the original functional. Optimal configurations with average gradient z tend to be periodic with a precise period even in absence of an underlying microgeometry, and the period *depends on z itself*. An example with an explicit computation is described in Remark 14, with the corresponding period $T(z)=1/S(z)$ depicted also in Figure 2. Note that the optimal value for the infima above in general is achieved only as $T\to+\infty$ since affine Dirichlet boundary conditions may be incompatible with oscillations at an optimal scale. In the scalar case $m=1$ (anti-plane case) and isotropic energies we show that optimal patterns are locally one-dimensional; i.e., optimal sequences have discontinuities and oscillations that arrange in the direction of the gradient, and we may restrict to considering the homogenization formula for one-dimensional problems. Note that in this case the direction of optimal cracks or oscillations is locally determined by the orientation of the limit ∇v.

The analysis of a prototypical one-dimensional energy allows to highlight more in detail the effect of fractures and oscillations at the microscopic level. We concentrate on the effect of fracture by choosing f,g and h even and convex, and φ constant with value $k>0$. Then $f_{\rm hom}(z)$ is obtained by minimizing

$$\min\Big\{\frac{1}{S}\int_0^S\big(f(u')+g(v')+h(u-v)\big)\,dx+\frac{k}{S}\ :\ v(0)=0,\ v(S)=Sz\Big\} \tag{6}$$

for $S>0$. The minimum in (6) is performed on u and v which are regular on $(0,S)$, and represents the average energy of periodic optimal arrangements with u having consecutive discontinuities at distance S in the unscaled variables. The case $S=\infty$ corresponds to no fracture, for which the minimal u and v are equal and affine and the minimum is $f(z)+g(z)+\min h$. In the case of quadratic f, g and h we can compute $f_{\rm hom}(z)$ explicitly and highlight that

• $f_{\rm hom}(z)$ is strictly convex;
• there exists z_* such that $f_{\rm hom}(z) = f(z) + g(z)$ if $|z| \le z_*$ (no fracture);
• for large values of z, $f_{\rm hom}(z) - g(z)$ scales as $z^{2/3}$ and the optimal spacing of microfracture $S(z)$ scales as $z^{-2/3}$.

This example already shows interesting issues as the onset of microscopic fracture at a specific positive threshold z^* and the optimal arrangements of cracks following a scaling which is reminiscent of that appearing in the study of periodic minimizing sequences of singularly perturbed non-convex energies (see e.g. [15, 16]).

The plan of the paper is a follows. In Section 2 we prove the general homogenization Theorem 2, where we also include a possible highly oscillating periodic dependence in the energy densities. This generalization allows to include cases when the limit process is non trivial also when no possibility of fracture is taken into account. In Section 4 we consider isotropic energies, for which we show that optimal sequences have a one-dimensional structure. Section 5 contains the analysis of one-dimensional functionals in the case of Griffith fracture, and the explicit computation for quadratic energy densities hinted at above.

2. A general convergence result via homogenization

Before stating our convergence result, we briefly recall some notation. The letter capital C will denote a positive constant depending on the fixed parameters of the problem under consideration, which we will mention explicitly when relevant, and whose value may vary at each its appearance. The cardinality of a set A is denoted by $\#A$. We use standard notation for Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^m)$ of $\mathbb{R}^m$-valued maps defined on an open subset Ω of $\mathbb{R}^n$. We will also use the space of *special functions of bounded variation* $SBV^p(\Omega;\mathbb{R}^m)$ whose approximate gradient is p-integrable. For such a function u we denote by $S(u)$ the jump set of u, on which a measure-theoretical normal ν_u is defined $\mathcal{H}^{n-1}$-almost everywhere, where $\mathcal{H}^{n-1}$ denotes the $n-1$-dimensional Hausdorff surface measure, as well as the traces $u^\pm$ on both sides of $S(u)$. For a precise definition of all these quantities we refer to [8], and for an interpretation within the Griffith theory of brittle fracture to [17, 5].

Let $f, g\colon \mathbb{R}^n \times \mathbb{R}^{n\times m} \to [0,+\infty)$, $h\colon \mathbb{R}^n\times\mathbb{R}^m \to [0,+\infty)$ and $\varphi\colon \mathbb{R}\times\mathbb{R}\times S^{n-1} \to [0,+\infty]$ be such that

(H1) $\frac{1}{c}(|z|^p - 1) \le f(y,z),\ g(y,z),\ h(y,z) \le c(|z|^p+1)$;
(H2) $\varphi(y,tw,\nu) \le c\,\varphi(y,w,\nu)$ for $|t|\le 1$;
(H3) f, g, h, φ are Carathéodory functions, 1-periodic with respect to their first variable and continuous with respect to the other ones.

Given a bounded open set $\Omega\subset\mathbb{R}^n$ with Lipschitz boundary, for $u \in SBV^p(\Omega;\mathbb{R}^m)$, $v\in W^{1,p}(\Omega;\mathbb{R}^m)$, and $\varepsilon > 0$ we define:

$$\begin{aligned} F_\varepsilon(u,v;\Omega) \quad &= \quad \int_\Omega \Big(f\big(\frac{x}{\varepsilon}, \nabla u\big) + g\big(\frac{x}{\varepsilon}, \nabla v\big) + h\big(\frac{x}{\varepsilon}, \frac{u-v}{\varepsilon}\big)\Big)\, dx \\ &\qquad\qquad + \varepsilon \int_{S(u)\cap\Omega} \varphi\big(\frac{x}{\varepsilon}, \frac{u^+-u^-}{\varepsilon}, \nu_u\big)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{7}$$

Note that in (7) we have supposed that the scale of the possible inhomogeneities is the same as that of the fracture toughness and the interaction distance ε. This is coherent with the interpretation of ε as an intermolecular distance, and with the derivation of the energies F_ε from atomistic models. Note that considering homogenous energies (no dependence on x/ε) does not bring any simplification to the proofs. Conversely, the treatment of inhomogeneities at other scales can be performed by a multi-scale analysis (see (d), Section 3).

Remark 1 (Compactness)**.** Let $u_\varepsilon, v_\varepsilon$ be such that $F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \leq C < +\infty$. Then in particular ∇v_ε is bounded in $L^p(\Omega; \mathbb{R}^{m\times n})$ so v_ε weakly converges to some v in $W^{1,p}(\Omega;\mathbb{R}^m)$ (up to subsequences and addition of constants). The growth conditions on h ensure that also u_ε converges to the same v in $L^p(\Omega;\mathbb{R}^m)$. This remark justifies the choice of the convergence $u_\varepsilon, v_\varepsilon \to v$ in L^p in the following result.

Theorem 2 (Homogenization)**.** *The functionals F_ε Γ-converge with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^p(\Omega;\mathbb{R}^m)$ to the functional F defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ by*

$$F_{\rm hom}(v) = \int_\Omega f_{\rm hom}(\nabla v)\, dx, \tag{8}$$

where

$$f_{\rm hom}(z) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{F_1(u,v;Q_T) : u = v = zx \ \ in \ \ \partial Q_T\Big\}, \tag{9}$$

where $Q_T = (-\frac{T}{2}, \frac{T}{2})^n$. In (9) *$F_1$ denotes F_ε with $\varepsilon = 1$ and the* inf *is taken over all (u,v) in the domain of F_1 satisfying the boundary conditions in the sense of inner traces.*

Remark 3 (convergence of minimum problems)**.** As an application of Theorem 2 we obtain, for example, that minima and minimizers of problems of the form ($f_i \in L^p(\Omega;\mathbb{R}^m)$, $\phi \in W^{1,p}(\Omega;\mathbb{R}^m)$)

$$\min\Big\{F_\varepsilon(u,v) + \int_\Omega (\langle f_1, u\rangle + \langle f_2, v\rangle)\, dx : \ v = \phi \text{ on } \partial\Omega\Big\}$$

(or, equivalently, $v = u = \phi$ on $\partial\Omega$) converge to the corresponding minumum and minimizers of

$$\min\Big\{F_{\rm hom}(v) + \int_\Omega \langle f_1 + f_2, v\rangle\, dx : \ v = \phi \text{ on } \partial\Omega\Big\}.$$

This is immediately obtained by Remark 1, the continuity of the second integral, and the compatibility of Γ-convergence with the addition of boundary conditions. The latter follows from a cut-off argument close to the boundary for unconstrained recovery sequences, and is performed explicitly in the first part of the proof of Theorem 2 when Ω is a cube.

In the following, we will use the notation

$$f^T_{\rm hom}(z) = \frac{1}{T^n} \inf\Big\{F_1(u,v;Q_T) : u = v = zx \ \text{ in } \ \partial Q_T\Big\}.$$

Unless otherwise indicated, the infima and minima in the sequel are taken over all (u,v) in the domain of the corresponding functionals.

Remark 4. The existence of the limit in (9) can be proved by an usual argument of subadditivity (see [1, Prop. 14.4]). Following the same type of arguments, if f, g, h and φ do not depend on the spatial variable y, we can prove that

$$f_{\rm hom}(z) = \inf_{T>0} f^T_{\rm hom}(z). \tag{10}$$

Indeed, we fix $\delta > 0$; for any $T > 0$ let u_T, v_T such that $u_T = v_T = zx$ in ∂Q_T and

$$\frac{1}{T^n} F_1(u_T, v_T; Q_T) < f^T_{\mathrm{hom}}(z) + \delta.$$

For $S > T$ we consider the set of indices $\mathcal{I}_S = \{i \in \mathbb{Z}^n : Q_T + Ti \subset Q_S\}$ and define u and v in Q_S by setting $u(x) = u_T(x - Ti) + zTi$, $v(x) = v_T(x - Ti) + zTi$ if $x \in Q_T + Ti,\ i \in \mathcal{I}_S$, and $u(x) = v(x) = zx$ in $Q_S \setminus \bigcup_{i\in\mathcal{I}_S}(Q_T + Ti)$. Hence

$$\frac{1}{S^n} F_1(u, v; Q_S) \quad \leq \quad \frac{1}{S^n}\Big\lfloor\frac{S}{T}\Big\rfloor^n F_1(u_T, v_T; Q_T) + \frac{CT^nS^{n-1}}{S^n} \leq f^T_{\mathrm{hom}}(z) + \delta + \frac{CT^n}{S}.$$

Taking the limit for $S \to +\infty$ we get $f_{\mathrm{hom}}(z) \leq f^T_{\mathrm{hom}}(z) + \delta$; this proves (10) since $\delta > 0$ is arbitrary.

Proof of Theorem 2. *Lower bound.* We prove the lower inequality by using the blow-up technique introduced by Fonseca and Müller (see [18, 19]).

Let $u_\varepsilon, v_\varepsilon$ be such that $F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \leq C$ and $u_\varepsilon, v_\varepsilon \to v$ in L^p, and let $u_j = u_{\varepsilon_j}$ and $v_j = v_{\varepsilon_j}$ be subsequences such that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) = \lim_{j\to+\infty} F_{\varepsilon_j}(u_j, v_j; \Omega).$$

We define the measures μ_j by setting $\mu_j(A) = F_{\varepsilon_j}(u_j, v_j; A)$; since the family $\{\mu_j\}$ is equibounded, we can assume that $\mu_j \overset{*}{\rightharpoonup} \mu$ up to subsequences. The lower bound follows if we show that

$$\frac{d\mu}{d\mathcal{L}^n}(x_0) \geq f_{\mathrm{hom}}(\nabla v(x_0)) \text{ for almost all } x_0 \in \Omega$$

where $\frac{d\mu}{d\mathcal{L}^n}$ denotes the Radon-Nikodym derivative of the measure μ with respect to the Lebesgue measure (see for instance [8, Section 1.1]).

We first remark that, for almost every $x_0 \in \Omega$, we have that

1) x_0 is a Lebesgue point for μ with respect to the Lebesgue measure, so that $\displaystyle\frac{d\mu}{d\mathcal{L}^n}(x_0) = \lim_{\varrho\to 0}\frac{\mu(Q_\varrho(x_0))}{\varrho^n}$, where $Q_\varrho(x_0) = (x_0 - \frac{\varrho}{2}, x_0 + \frac{\varrho}{2})^n$;

2) x_0 is such that $\displaystyle\Big(\frac{1}{\varrho^n}\int_{Q_\varrho(x_0)} |v(x) - v(x_0) - \nabla v(x_0)(x - x_0)|^p\, dx\Big)^{\frac{1}{p}} = o(\varrho)_{\varrho\to 0}$ since $v \in W^{1,p}(\Omega; \mathbb{R}^m)$ (see for instance [20, Th. 6.2]).

It is not restrictive to fix $x_0 = 0$ and $v(x_0) = 0$. For every ϱ except for a countable set we have $\mu(Q_\varrho) = \lim_{j\to+\infty} F_{\varepsilon_j}(u_j, v_j; Q_\varrho)$, where $Q_\varrho = Q_\varrho(0)$. Note that in particular $\lim_j \frac{1}{\varrho^n} F_{\varepsilon_j}(u_j, v_j; Q_\varrho)$ is equibounded with respect to ϱ.

As a first step, following a classical method introduced by De Giorgi for matching boundary values (see [21] and [22, Sec. 4.2.1]), we show that, by modifying v_j and u_j near the boundary of Q_ϱ, it is not restrictive to assume that their boundary value is exactly $\nabla v(0)x$. Indeed, fixed $\delta \in (0, 1)$ and $N \in \mathbb{N}$, for any $i = 0, \dots N$ we define $Q^i = Q_{\varrho-\delta\varrho+i\frac{\delta\varrho}{N}}$. For $i \geq 1$, let ψ_i be a non-negative Lipschitz function such that $\psi_i(x) = 0$ in $Q_\varrho \setminus Q^i$, $\psi_i(x) = 1$ in Q^{i-1} and $|\nabla\psi_i| \leq \frac{2N}{\delta\varrho}$. Setting $u^i_j = \psi_i u_j + (1-\psi_i)\nabla v(0)x$ and $v^i_j = \psi_i v_j + (1-\psi_i)\nabla v(0)x$, the growth hypotheses on f give

$$\begin{aligned}\int_{Q_\varrho} f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx &= \int_{Q^{i-1}} f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx + \int_{Q_\varrho\setminus Q^i} f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx\\ &\quad + \int_{Q^i\setminus Q^{i-1}} f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx\\ &\le \int_{Q_\varrho} f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx + C(|\nabla v(0)|^p+1)\delta\varrho^n\\ &\quad + \frac{CN^p}{\delta^p\varrho^p}\int_{Q^i\setminus Q^{i-1}} |u_j-\nabla v(0)x|^p\,dx\\ &\quad + C\int_{Q^i\setminus Q^{i-1}} f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx,\end{aligned}$$

where C denotes a positive constant depending only on p,n and c. Hence

$$\begin{aligned}\sum_{i=1}^N\int_{Q_\varrho} f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx &\le CN(|\nabla v(0)|^p+1)\delta\varrho^n + (N+C)\int_{Q_\varrho} f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx\\ &\quad + \frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}|u_j-v|^p\,dx + \frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}|v-\nabla v(0)x|^p\,dx\end{aligned}$$

and correspondingly for $\sum_{i=1}^N\int_{Q_\varrho} g\big(\frac{x}{\varepsilon_j},\nabla v_j^i\big)\,dx$ thanks to the growth hypotheses on g. Moreover, the assumptions on h and φ give

$$\begin{aligned}&\sum_{i=1}^N\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j^i-v_j^i}{\varepsilon_j}\Big)\,dx = \sum_{i=1}^N\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\psi_i\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx\\ &\qquad \le CNh(0)\delta\varrho^n + N\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx + C\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx,\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^N\int_{Q_\varrho\cap S(u_j^i)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{(u_j^i)^+-(u_j^i)^-}{\varepsilon_j},\nu_{u_j^i}\Big)\,dx\\ &= \sum_{i=1}^N\int_{Q_\varrho\cap S(u_j^i)} \varphi\Big(\frac{x}{\varepsilon},\psi_i\frac{(u_j)^+-(u_j)^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx\\ &\le N\int_{Q_\varrho\cap S(u_j)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{u_j^+-u_j^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx + C\int_{Q_\varrho\cap S(u_j)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{u_j^+-u_j^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac1N\sum_{i=1}^N F_{\varepsilon_j}(u_j^i,v_j^i;Q_\varrho) &\le F_{\varepsilon_j}(u_j,v_j;Q_\varrho) + \frac{C}{N}F_{\varepsilon_j}(u_j,v_j;Q_\varrho) + C\delta\varrho^n\\ &\quad + \frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}\Big(|u_j-u|^p+|v_j-u|^p+2|u-\nabla u(0)x|^p\Big)\,dx,\end{aligned}$$

where now C stands for a positive constant depending also on $\nabla v(0)$ and $h(0)$. We choose $\underline{i}$ such that $F_{\varepsilon_j}(u_j^{\underline{i}},v_j^{\underline{i}};Q_\varrho)\le\frac1N\sum_{i=1}^N F_{\varepsilon_j}(u_j^i,v_j^i;Q_\varrho)$ and we set $\hat u_j=u_j^{\underline{i}}$ and $\hat u_j=v_j^{\underline{i}}$.

Since $\frac{1}{\varrho^n}F_{\varepsilon_j}(u_j,v_j;Q_\varrho)$ is equibounded, the convergence of $u_j,v_j\to u$ in L^p and the properties of the point $x_0=0$ ensure that for any $\delta\in(0,1)$ and $N\in\mathbb{N}$

$$\lim_{\varrho\to 0}\frac{1}{\varrho^n}\lim_{j\to+\infty}F_{\varepsilon_j}(\hat u_j,\hat v_j;Q_\varrho) \quad\leq\quad \lim_{\varrho\to 0}\frac{1}{\varrho^n}\lim_{j\to+\infty}F_{\varepsilon_j}(u_j,v_j;Q_\varrho)+\frac{\tilde C}{N}+C\delta.$$

This inequality allows us to assume that $u_j=v_j=\nabla v(0)x$ on ∂Q_ϱ.

Now, setting $\overline{u}_j(x)=\frac{1}{\varepsilon_j}u_j(\varepsilon_j x)$ and $\overline{v}_j(x)=\frac{1}{\varepsilon_j}v_j(\varepsilon_j x)$, we get

$$\begin{aligned}\frac{1}{\varrho^n}F_{\varepsilon_j}(u_j,v_j;Q_\varrho) \quad &= \quad \frac{\varepsilon^n}{\varrho^n}\Big(\int_{Q_{\frac{\varrho}{\varepsilon}}} f(y,\nabla\overline{u}_j)+g(y,\nabla\overline{v}_j)+h(y,\overline{u}_j-\overline{v}_j)\,dy\\ &\qquad +\int_{Q_{\frac{\varrho}{\varepsilon}}\cap S(\overline{u}_j)}\varphi(y,\overline{v}_j^+-\overline{v}_j^-)\,d\mathcal{H}^{n-1}\Big)\geq f_{\rm hom}^{\varrho/\varepsilon}(\nabla v(0)).\end{aligned}$$

The result then follows by taking the limit as $\varepsilon\to 0$.

Upper bound. We first prove that the function $f_{\rm hom}$ is continuous. We fix $z\in\mathbb{R}^{m\times n}$. For any $T>0$ and $\delta\in(0,1)$ let $u_{T,\delta},v_{T,\delta}$ be such that $\dfrac{1}{T^n}F_1(u_{T,\delta},v_{T,\delta};Q_T)\leq f_{\rm hom}^T(z)+\delta$, and $u_{T,\delta}=v_{T,\delta}=zx$ in ∂Q_T. For any z' we extend $u_{T,\delta}$ and $v_{T,\delta}$ to $Q_{(1+\delta)T}$ as $\tilde u_{T,\delta}(x)=\tilde v_{T,\delta}(x)=\psi_{T,\delta}(x)z'x+(1-\psi_{T,\delta}(x))zx$ in $Q_{(1+\delta)T}\setminus Q_T$, where $\psi_{T,\delta}$ is a non-negative Lipschitz function such that $\psi_{T,\delta}=0$ in Q_T, $\psi_{T,\delta}=1$ in $\mathbb{R}^n\setminus Q_{(1+\delta)T}$ and $|\nabla\psi_{T,\delta}|\leq\frac{2}{T\delta}$. Hence, $\tilde u_{T,\delta},\tilde v_{T,\delta}$ are test functions for the minimum problem for F_1 in $Q_{(1+\delta)T}$; note that $S(u_{T,\delta})\cap Q_T=S(\tilde u_{T,\delta})\cap Q_{(1+\delta)T}$. The growth conditions on f and g give

$$\begin{aligned}f_{\rm hom}^{(1+\delta)T}(z') \quad&\leq\quad \frac{1}{(1+\delta)^nT^n}F_1(\tilde u_{T,\delta},\tilde v_{T,\delta};[0,(1+\delta)T]^n)\\ &\leq\quad \frac{1}{T^n}F_1(u_{T,\delta},v_{T,\delta};[0,T]^n)\\ &\qquad +C\frac{\delta^{1-p}T^n}{(1+\delta)^nT^n}|z-z'|^p+C\frac{\delta T^n}{(1+\delta)^nT^n}\Big(|z-z'|^p+|z|^p+h(0)\Big)\\ &\leq\quad f_{\rm hom}^T(z)+\delta+C\frac{\delta^{1-p}}{(1+\delta)^n}|z-z'|^p+C\frac{\delta}{(1+\delta)^n}(|z|^p+h(0)),\end{aligned}$$

where the positive constant C depends only on n and on the growth of f e g. If $|z-z'|<\delta$ we get

$$\begin{aligned}f_{\rm hom}(z') \quad&=\quad \lim_{T\to+\infty}f_{\rm hom}^{(1+\delta)T}(z') \;\leq\; \lim_{T\to+\infty}f_{\rm hom}^T(z)+C\delta(1+|z|^p+h(0))\\ &\leq\quad f_{\rm hom}(z)+C\delta(1+|z|^p+h(0))\end{aligned}$$

By exchanging the roles of z and z' the argument above gives the inequality

$$f_{\rm hom}(z)\leq f_{\rm hom}(z')+C\delta(1+|z'|^p+h(0)),$$

and hence the continuity of $f_{\rm hom}$.

Now, let $v\in W^{1,p}(\Omega;\mathbb{R}^m)$ and w_ε be piecewise-affine continuous functions such that $w_\varepsilon\to v$ as $\varepsilon\to 0$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. The continuity of $f_{\rm hom}$ and the dominated convergence give $\lim_{\varepsilon\to 0}F_{\rm hom}(w_\varepsilon)=F_{\rm hom}(v)$. Hence, it is sufficient to construct a recovery sequence for piecewise-affine continuous v. We start by considering the function $v(x)=zx$ on a n-dimensional open simplex S. We fix $\delta>0$. Let $T_\varepsilon\in\mathbb{N}$

such that $T_\varepsilon \to +\infty$ and $\varepsilon T_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $\widetilde{u}_\varepsilon, \widetilde{v}_\varepsilon$ be defined on Q_{T_ε} and such that $\dfrac{1}{T_\varepsilon^n} F_1(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; Q_{T_\varepsilon}) \leq f^{T_\varepsilon}_{\mathrm{hom}}(z) + \delta$.

For $i \in \varepsilon T_\varepsilon \mathbb{Z}^n$ we set $Q^i_\varepsilon = i + Q_{\varepsilon T_\varepsilon}$. Let $\mathcal{I}_\varepsilon$ be the set of the indices $i \in \varepsilon T_\varepsilon \mathbb{Z}^n$ such that $\overline{Q}^i_\varepsilon \subset S$ and define the recovery sequences by setting

$$u_\varepsilon(x) = \begin{cases} \varepsilon \tilde{u}_\varepsilon\big(\frac{x-i}{\varepsilon}\big) + zi & \text{if } x \in Q^i_\varepsilon,\ i \in \mathcal{I}_\varepsilon \\ zx & \text{if } x \in S \setminus \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon \end{cases}$$

and correspondingly v_ε. Note that $u_\varepsilon, v_\varepsilon \to v$ in $L^p(S;\mathbb{R}^m)$ and $S(u_\varepsilon) \cap S = S(u_\varepsilon) \cap \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon$. Since $|S \setminus \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon| \to 0$ as $\varepsilon \to 0$, recalling that f, g, h and φ are 1-periodic with respect to the first variable, and that $T_\varepsilon \in \mathbb{N}$, we get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; S) &\leq \limsup_{\varepsilon\to 0} \Big(\big|S \setminus \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon\big| (f(z) + g(z)) + F_\varepsilon(u_\varepsilon, v_\varepsilon; \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon) \Big) \\
&\leq \limsup_{\varepsilon\to 0} \#\mathcal{I}_\varepsilon \varepsilon^n F_1(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; Q_{T_\varepsilon}) \\
&\leq \limsup_{\varepsilon\to 0} \big| \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon \big| \frac{1}{T_\varepsilon^n} F_1(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; Q_{T_\varepsilon}) \\
&\leq |S| \lim_{\varepsilon\to 0} f^{T_\varepsilon}_{\mathrm{hom}}(z) + \delta|S| = |S| f_{\mathrm{hom}}(z) + \delta|S| \\
&= F_{\mathrm{hom}}(v) + \delta|S|.
\end{aligned}$$

Thanks to the arbitrariness of $\delta > 0$, $(u_\varepsilon, v_\varepsilon)$ is a recovery sequence for $v = zx$. Since $u_\varepsilon(x) = v_\varepsilon(x) = zx$ in a neighbourhood of ∂S and since the functionals are defined up to translations, this inequality ensures that the upper estimate holds for a piecewise-affine and continuous v by repeating the construction in each simplex. $\square$

3. Discussion on the convergence result

(a) As a particular case, we can take $\varphi(y, w, \nu) = +\infty$ if $w \neq 0$ (and equal to 0 if $w = 0$ for completeness. Note that the value $w = 0$ is never taken into account). In this case, F_ε is finite only if $\mathcal{H}^{n-1}(S(u) \cap \Omega) = 0$ or, equivalently, $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and

$$F_\varepsilon(u, v; \Omega) = \int_\Omega \Big(f\big(\frac{x}{\varepsilon}, \nabla u\big) + g\big(\frac{x}{\varepsilon}, \nabla v\big) + h\big(\frac{x}{\varepsilon}, \frac{u - v}{\varepsilon}\big) \Big)\, dx.$$

(b) Given an integer $M \geq 1$ we may consider more in general energies defined for $u_i \in SBV^p(\Omega;\mathbb{R}^m)$, $i \in \{1, \dots, M\}$ and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$\begin{aligned}
F_\varepsilon(u, v; \Omega) \;=\; & \sum_{i=1}^M \int_\Omega \Big(f_i\big(\frac{x}{\varepsilon}, \nabla u\big) + h_i\big(\frac{x}{\varepsilon}, \frac{u_i - v}{\varepsilon}\big) \Big)\, dx + \int_\Omega g\big(\frac{x}{\varepsilon}, \nabla v\big)\, dx \\
& + \varepsilon \sum_{i=1}^M \int_{S(u_i)\cap\Omega} \varphi_i\big(\frac{x}{\varepsilon}, \frac{u_i^+ - u_i^-}{\varepsilon}, \nu_{u_i}\big)\, d\mathcal{H}^{n-1}, \qquad (11)
\end{aligned}$$

where $u = (u_1, \dots, u_M)$, and f_i, h_i and φ_i satisfy the same assumptions as f, h and φ, respectively. Theorem 2 can then be proved without major changes in the proof. Note that even more in general, we may also add a term of the form

$$\int_\Omega \sum_{i\neq j} h_{ij}\big(\frac{x}{\varepsilon}, \frac{u_i - u_j}{\varepsilon}\big)\, dx.$$

We will not discuss the various hypotheses that one can impose to h_i and h_{ij} so that the compactness arguments in Remark 1 still hold.

(c) If we take into account homogeneous energies as in (a), i.e. of the form

$$F_\varepsilon(u,v;\Omega)=\int_\Omega\Big(f(\nabla u)+g(\nabla v)+h\big(\frac{u-v}{\varepsilon}\big)\Big)\,dx,$$

then the homogenized energy density is simply given by $f_{\rm hom}(z)=\mathcal{Q}f(z)+\mathcal{Q}g(z)+\min h$, where $\mathcal{Q}$ denotes the quasiconvexification operator (see e.g. [1]). Indeed, the energy with density the right-hand side is clearly a lower bound (after taking into account Remark 1). To check that this is also an upper bound, by the integral representation in Theorem 2 it suffices to consider the case of a linear target function $v(x)=zx$. This can be seen by taking sequences $u_\varepsilon, v_\varepsilon$ converging to v such that

$$\int_\Omega f(\nabla u_\varepsilon)\,dx\to|\Omega|\;\mathcal{Q}f(z)\qquad\text{and}\qquad\int_\Omega g(\nabla v_\varepsilon)\,dx\to|\Omega|\;\mathcal{Q}g(z)$$

up to an arbitrarily small error, and with $u_\varepsilon-v_\varepsilon=\zeta_0+o(\varepsilon)$, where $h(\zeta_0)=\min h$. This can be done remarking that we may take recovery sequences converging to v in L^∞ by a truncation argument in the target space (see e.g. [23]).

(d) We may consider $\eta=\eta(\varepsilon)$ and generalize the energies F_ε in (7) to

$$\int_\Omega\Big(f\big(\frac{x}{\eta},\nabla u\big)+g\big(\frac{x}{\eta},\nabla v\big)+h\big(\frac{x}{\eta},\frac{u-v}{\varepsilon}\big)\Big)\,dx+\varepsilon\int_{S(u)\cap\Omega}\varphi\big(\frac{x}{\eta},\frac{u^+-u^-}{\varepsilon},\nu_u\big)\,d\mathcal{H}^{n-1}.$$

If η is at the same scale as ε then we may reduce to the case ε/η constant and then apply Theorem 2. Otherwise, we may apply a separation-of-scale argument using Theorem 2 and classical homogenization results in the proper order. As a result, if $\eta<<\varepsilon$ then the Γ-limit is that formally obtained first using homogenization results keeping ε fixed, and then applying Theorem 2 to the functional with the resulting homogeneous energy densities, while if $\varepsilon<<\eta$ the Γ-limit is obtained by first applying Theorem 2 keeping x/η as a parameter, and then applying homogenization results to the resulting integral. These processes are rather technical and will not be dealt with here. We refer to [24] for a similar argument mixing homogenization and the theory of phase transitions.

(e) As remarked in the Introduction, more in general we may consider energies of the form (1) also depending on a parameter δ. We do not treat this general case since it would involve additional multi-scale arguments which are not central in our analysis (we refer to e.g. [25, 26] for similar multi-scale problems in different contexts). However, in the homogeneous case this analysis simplifies and we note the following.

1) if $\delta<<\varepsilon$ then the interaction term forces $u-v=O(\delta)<<\varepsilon$. This makes the introduction of jump points energetically non-favorable, so that the analysis of $F_{\delta,\varepsilon}$ simply reduces to that of

$$\int_\Omega\big(f(\nabla u)+g(\nabla v)+\min h\big)\,dx,\tag{12}$$

with $u=v+\delta\zeta_0+o(\delta)$, where $h(\zeta_0)=\min h$. This energy can be analyzed by relaxation methods as in (c) above;

2) if $\varepsilon<<\delta$ then the condition $u-v=O(\delta)>>\varepsilon$ allows the minimization of the interaction term without influencing the rest of the energy, and the analysis of

$F_{\delta,\varepsilon}$ simply reduces to that of the decoupled energies

$$\int_\Omega f(\nabla u)\,dx+\varepsilon\int_{S(u)}\varphi\Big(\frac{u^+-u^-}{\varepsilon}\Big)d\mathcal{H}^{n-1}+\int_\Omega\Big(g(\nabla v)+\min h\Big)\,dx. \tag{13}$$

In this case, the part of the energy depending on u trivializes since we may approximate all u with piecewise-affine discontinuous functions jumping on a scale much larger than ε, and we reduce the analysis to that of

$$\int_\Omega\big(g(\nabla v)+\min h+\min f\big)\,dx. \tag{14}$$

4. One-dimensional behaviour of homogeneous isotropic energies

In this section we consider a particular case of the Γ-convergence result of Theorem 2, with additional hypotheses of isotropy on f,g,h,φ ensuring that the limit is essentially locally one-dimensional. In particular, within this class fall energies that can be reduced to the examples contained in the next section.

Let $f,g,h\colon[0,+\infty)\to[0,+\infty)$ and $\varphi\colon[0,+\infty)\to[0,+\infty)$ be such that

(H1′) $\frac{1}{c}(|z|^p-1)\le f(z),\ g(z),\ h(z)\le c(|z|^p+1)$;
(H2′) $\varphi(tw)\le c\,\varphi(w)$ for $0\le t\le 1$;
(H3′) f,g are monotone not decreasing,

and consider the functionals

$$G_\varepsilon(u,v;\Omega)=\int_\Omega\Big(f(|\nabla u|)+g(|\nabla v|)+h\Big(\frac{|u-v|}{\varepsilon}\Big)\Big)\,dx+\varepsilon\int_{S(u)\cap\Omega}\varphi\Big(\frac{|u^+-u^-|}{\varepsilon}\Big)\,d\mathcal{H}^{n-1} \tag{15}$$

defined for $u\in SBV^p(\Omega)$ and $v\in W^{1,p}(\Omega)$. The hypotheses (H1′) and (H2′) on f,g,h,φ ensure that the functions $\tilde f(y,z)=f(|z|),\tilde g(y,z)=g(|z|),\tilde h(y,z)=h(|z|)$ and $\tilde\varphi(y,z,\nu)=\varphi(|z|)$ satisfy the hypotheses of Theorem 2, hence the Γ-limit of G_ε with respect to the convergence $u,v\to v$ in $L^p(\Omega)$ is given by

$$G_{\rm hom}(v)=\int_\Omega g_{\rm hom}(\nabla v)\,dx,$$

where

$$g_{\rm hom}(z)=\lim_{T\to+\infty}\frac{1}{T^n}\inf\{G_1(u,v;Q_T):u=v=zx\text{ in }\partial Q_T\}.$$

In the one-dimensional case; i.e., when $n=1$, in order to highlight the dependence on the dimension, we denote $g_{\rm hom}$ as

$$g_{1,\rm hom}(z)=\lim_{T\to+\infty}\frac{1}{T}\inf\Big\{\int_0^T\big(f(|u'|)+g(|v'|)+h(|u-v|)\big)\,dx+\sum_{S(u)\cap\Omega}\varphi(|u^+-u^-|)\\ :u(0)=v(0)=0,u(T)=v(T)=Tz\Big\}. \tag{16}$$

Note that $g_{1,\rm hom}(z)=g_{1,\rm hom}(|z|)$. The following result holds.

Theorem 5. *If f,g,h,φ satisfy* (H1′)-(H3′)*, then*

$$G_{\rm hom}(v)=\int_\Omega g_{1,\rm hom}(|\nabla v|)\,dx.$$

Remark 6. The fact that the energy function of $G_{\rm hom}$ depends only on the norm of the gradient and is expressed through $g_{1,\rm hom}$ highlights that optimal sequences for a given strain are given by "one-dimensional oscillations" (oriented in the direction of the gradient).

Proof. Lower bound. We prove the lower bound by using a slicing argument (see [8, Sec. 4.1], [22, Sec. 3.4]). For each $\xi\in S^{n-1}$ we consider the orthogonal hyperplane passing through 0; that is, $\Pi_\xi=\{x\in\mathbb{R}^n : x\cdot\xi=0\}$. Given a bounded subset A of Ω, for each $y\in\Pi_\xi$ we define the one-dimensional set $A_{\xi,y}=\{t\in\mathbb{R} : y+t\xi\in A\}$, and for w defined on Ω we denote by $w^{\xi,y}$ the one-dimensional function $w^{\xi,y}(t)=w(y+t\xi)$, defined on $\Omega_{\xi,y}$. Note that if $v\in W^{1,p}(A)$ and $u\in SBV^p(A)$, then for any ξ the function $v^{\xi,y}$ belongs to $W^{1,p}(A_{\xi,y})$, the function $u^{\xi,y}$ belongs to $SBV^p(A_{\xi,y})$ for almost all $y\in\Pi_\xi$, and that $S(u_{\xi,y})=\{t\in\mathbb{R} : y+t\xi\in S(u)\}$ (see for instance [8, Th. 4.1]).

Let I be a bounded open subset of $\mathbb{R}$; for $u\in SBV^p(I)$ and $v\in W^{1,p}(I)$ we define

$$G^{\xi,y}_\varepsilon(u,v;I)=\int_I\Big(f(|u'|)+g(|v'|)+h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dt+\varepsilon\sum_{t\in S(u)\cap I}\varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big).$$

By Theorem 2 applied in dimension one, the Γ-limit of $G^{\xi,y}_\varepsilon$ is given by

$$G^{\xi,y}(v;I)=\int_I g_{1,\rm hom}(v')\,dt=\int_I g_{1,\rm hom}(|v'|)\,dt.$$

Now, setting for $v\in W^{1,p}(A)$ and $u\in SBV^p(A)$

$$G^{\xi}_\varepsilon(u,v;A)=\int_{\Pi_\xi}G^{\xi,y}_\varepsilon(u^{\xi,y},v^{\xi,y};A_{\xi,y})\,d\mathcal{H}^{n-1}(y)$$

by an application of Fubini's Theorem we get

$$\begin{aligned}
G^{\xi}_\varepsilon(u,v;A) &= \int_A\Big(f(|\nabla u\cdot\xi|)+g(|\nabla v\cdot\xi|)+h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dx\\
&\quad+\varepsilon\int_{S(u)\cap A}\varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big)|\nu_u\cdot\xi|\,d\mathcal{H}^{n-1}\\
&\le \int_A\Big(f(|\nabla u|)+g(|\nabla v|)+h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dx\\
&\quad+\varepsilon\int_{S(u)\cap A}\varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big)\,d\mathcal{H}^{n-1}\\
&= G_\varepsilon(u,v;A).
\end{aligned}$$

thanks to the monotonicity of f and g. Now, let $u_\varepsilon,v_\varepsilon\to v$ as $\varepsilon\to0$ in $L^p(\Omega)$. Note that, again by Fubini's Theorem, $u^{\xi,y}_\varepsilon,v^{\xi,y}_\varepsilon\to v^{\xi,y}$ in $L^p(\Omega_{\xi,y})$ for almost all

$y \in \Pi_\xi$. Applying Fatou's Lemma we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, v_\varepsilon; A) &\geq \liminf_{\varepsilon\to 0} G^\xi_\varepsilon(u_\varepsilon, v_\varepsilon; A)\\
&= \liminf_{\varepsilon\to 0} \int_{\Pi_\xi} G^{\xi,y}_\varepsilon(u^{\xi,y}_\varepsilon, v^{\xi,y}_\varepsilon; A_{\xi,y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi} \liminf_{\varepsilon\to 0} G^{\xi,y}_\varepsilon(u^{\xi,y}_\varepsilon, v^{\xi,y}_\varepsilon; A_{\xi,y})\, d\mathcal{H}^{n-1}(y)\\
&\geq \int_{\Pi_\xi} \int_{A_{\xi,y}} g_{1,\mathrm{hom}}(|v'_{\xi,y}|)\, dt\ d\mathcal{H}^{n-1}(y)\\
&= \int_A g_{1,\mathrm{hom}}(|\nabla v\cdot\xi|)\, dx.
\end{aligned}$$

The functionals G_ε are local, hence the set function $\mu(A) = \Gamma\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(v; A)$ is super-additive on open sets with disjoint compact closure. Let $\{\xi_i\}$ be a countable dense subset of S^{n-1}. Since $\mu(A) \geq \int_A \psi_i\, d\lambda$ for all i, where $\lambda = \mathcal{L}^n$ and $\psi_i(x) = g_{1,\mathrm{hom}}(|\nabla v(x)\cdot\xi_i|)$, it follows that $\mu(A) \geq \int_A \sup_i g_{1,\mathrm{hom}}(|\nabla v\cdot\xi_i|)\, dx$ (see [22, Lemma 3.1]). The continuity of $g_{1,\mathrm{hom}}$ gives

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(v; A) \geq \int_A g_{1,\mathrm{hom}}(|\nabla v|)\, dx.$$

Upper bound. Since Theorem 2 holds, it is sufficient to construct a recovery sequence for an affine function $w = \alpha\xi\cdot x$ in a n-dimensional open simplex S, with $\alpha \in \mathbb{R}$ and $\xi \in S^{n-1}$.

We fix $\delta > 0$. Let $T > 0$, $u \in SBV^p(0,T)$, and $v \in W^{1,p}(0,T)$ be such that $u(0) = v(0) = 0$, $u(T) = v(T) = |\alpha|T$, and

$$\frac{1}{T} G_1(u, v; (0,T)) < g_{1,\mathrm{hom}}(|\alpha|) + \delta.$$

Given a bounded interval $[a,b]$ we set $M_\varepsilon = \lfloor \frac{b-a}{\varepsilon T} \rfloor$ and consider the intervals $a + \varepsilon T[m, m+1)$ for any $m = 0, \dots, M_\varepsilon - 1$. We introduce the one-dimensional function $\tilde u_\varepsilon \colon [a,b] \to \mathbb{R}$ given by

$$\tilde u_\varepsilon(t) = \varepsilon u\Big(\frac{t - \varepsilon T m - a}{\varepsilon}\Big) + (a + \varepsilon T m)|\alpha| \quad \text{for } t \in a + \varepsilon T[m, m+1)$$

and $\tilde u_\varepsilon(t)$ affine in $[a + \varepsilon T M_\varepsilon, b]$ with $\tilde u_\varepsilon(a + \varepsilon T M_\varepsilon) = \varepsilon T M_\varepsilon + a|\alpha|$ and $\tilde u_\varepsilon(b) = b|\alpha|$. We define $\tilde v_\varepsilon$ correspondingly. We get

$$G_\varepsilon(\tilde u_\varepsilon, \tilde v_\varepsilon; (a,b)) \leq \varepsilon \sum_{m=0}^{M_\varepsilon - 1} G_1(u, v; (0,T)) + C\varepsilon \leq (b-a) g_{1,\mathrm{hom}}(|\alpha|) + \delta + C\varepsilon, \tag{17}$$

where C depends only on α, a, b and $h(0)$. For a given $y \in \Pi_\xi$, if $\{t\xi + y : t \in \mathbb{R}\} \cap S \neq \emptyset$ then there exist $a(y)$, $b(y)$ with $a(y) < b(y)$ and such that $\{t\xi + y : t \in \mathbb{R}\} \cap S = \{t\xi + y : a(y) < t < b(y)\}$. Hence, setting

$$u_\varepsilon(x) = \tilde u_\varepsilon((x-y)\cdot\xi) \quad \text{and} \quad v_\varepsilon(x) = \tilde v_\varepsilon((x-y)\cdot\xi) \tag{18}$$

it follows that $u_\varepsilon, v_\varepsilon \to w$ in $L^p(S)$ and, thanks to (17)

$$\begin{aligned} G_\varepsilon(u_\varepsilon, v_\varepsilon; S) &= \int_{S^\xi}\int_{a(y)}^{b(y)} f(|\xi\tilde{u}_\varepsilon'(t)|) + g(|\xi\tilde{u}_\varepsilon'(t)|) + h\Big(\frac{|\tilde{u}_\varepsilon - \tilde{v}_\varepsilon|}{\varepsilon}\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{S^\xi} \varepsilon \sum_{t\in S(\tilde{u}_\varepsilon)\cap(a(y),b(y))} \varphi\Big(\frac{|\tilde{u}_\varepsilon^+ - \tilde{u}_\varepsilon^-|}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(y) \\ &= \int_{S^\xi} G_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; (a(y), b(y)))\, d\mathcal{H}^{n-1}(y) \\ &\leq \int_{S^\xi} \Big((b(y)-a(y))g_{1,\mathrm{hom}}(|\alpha|) + \delta + C\varepsilon\Big)\, d\mathcal{H}^{n-1}(y) \\ &= \int_S g_{1,\mathrm{hom}}(|\nabla w|)\, dx + |S|\delta + |S|C\varepsilon, \end{aligned}$$

where S^ξ denotes the set of $y \in \Pi_\xi$ such that $\{t\xi + y : t \in \mathbb{R}\} \cap S \neq \emptyset$. Thanks to the arbitrariness of $\delta > 0$, it follows that the sequence $(u_\varepsilon, v_\varepsilon)$ defined in (18) is a recovery sequence for the Γ-limit. □

Example 7 (Bi-stable energy density)**.** As a particular case of a non-convex energy without a jump part, we may consider $f(z) = (|z|-1)^2$, $g(z) = h(z) = z^2$ and $\varphi = +\infty$. By Section 3(c) and Theorem 5 we have

$$g_{1,\mathrm{hom}}(z) = \begin{cases} z^2 & \text{if } |z| \leq 1 \\ 2z^2 - 2|z| + 1 & \text{if } |z| > 1. \end{cases}$$

Note the difference with the corresponding result in [13], where similar energies are considered in the discrete setting where fast oscillations are not allowed, and optimal sequences such as those in (c), Section 3 cannot be constructed.

5. A prototypical one-dimensional example

We analyze fracture-type non convex energies, for which the effective behaviour is determined by an optimal periodic arrangement of discontinuities.

We consider a particular case of the sequence of functional defined in (15) in the one-dimensional frame, with f, g, h not depending on the space variable, h even and strictly convex, f and g strictly convex and φ constant; that is,

$$G_\varepsilon(u, v; I) = \int_I \Big(f(u') + g(v') + h\big(\frac{u-v}{\varepsilon}\big)\Big)\, dx + k\varepsilon\#S(u). \tag{19}$$

Applying Theorem 2 we get that the Γ-limit of G_ε with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^p(I)$ is given by

$$G_{\mathrm{hom}}(v) = \int_I g_{1,\mathrm{hom}}(v')\, dx,$$

where $g_{1,\mathrm{hom}}$ is defined in (16). In the sequel, we denote by $\widetilde{G}_\varepsilon(u, v; I)$ the functional

$$\widetilde{G}_\varepsilon(u, v; I) = \int_I \Big(f(u') + g(v') + h\big(\frac{u-v}{\varepsilon}\big)\Big)\, dx \tag{20}$$

again defined for $v \in W^{1,p}(I)$ and $u \in SBV^p(I)$. The following proposition holds.

Proposition 8. *If f, g, h are even and strictly convex and φ is the positive constant k, then*

$$g_{1,\mathrm{hom}}(z) = \inf_{S>0} \Big\{ \psi(S,z) + \frac{k}{S} \Big\}, \tag{21}$$

where

$$\psi(S,z) = \frac{1}{S} \min \Big\{ \widetilde{G}_1(u,v;(0,S)) : u,v \in W^{1,p}(0,S),\ v(0)=0,\ v(S)=Sz \Big\}. \tag{22}$$

Remark 9. A special case is when the minimum points of f and g coincide. Then, denoting by z^* the common minimum point, we have $\psi(S,z^*) = f(z^*)+g(z^*)+h(0)$, independent of S, so that $g_{1,\mathrm{hom}}(z) = f(z^*) + g(z^*) + h(0)$, and the infimum (21) is achieved for $S \to +\infty$.

Before giving the proof of Proposition 8 we state some preliminary properties.

Remark 10 (Symmetry properties of minimizers of $\psi(S,z)$)**.** Note that the minimum in (22) is achieved by the application of the direct method of the calculus of variations. The strict convexity of f, g and h gives the uniqueness of the minimum point. We observe that, by the strict convexity of f, g and h, if (u,v) realizes the minimum in (22) then

$$u\Big(\frac{S}{2}\Big) = v\Big(\frac{S}{2}\Big) = \frac{Sz}{2}. \tag{23}$$

Indeed, otherwise assume by contradiction that (u,v) solve the problem (22) and do not satisfy (23). We then set

$$\tilde u(t) = \frac{1}{2}\big(u(t) - u(S-t) + Sz\big), \quad \tilde v(t) = \frac{1}{2}\big(v(t) - v(S-t) + Sz\big)$$

so that $(\tilde u, \tilde v)$ is admissible for (22) and satisfies (23). Note that $\tilde u(S) - \tilde u(0) = u(S) - u(0)$. Then, the convexity of f, g and h and the symmetry of h ensure that

$$\begin{aligned}
\widetilde{G}_1(\tilde u, \tilde v; (0,S)) &= \int_0^S \big(f(\tilde u') + g(\tilde v') + h(\tilde u - \tilde v)\big)\, dt \\
&= \int_0^S f\Big(\frac{1}{2}(u'(t) + u'(S-t))\Big)\, dt + \int_0^S g\Big(\frac{1}{2}(v'(t) + v'(S-t))\Big)\, dt \\
&\quad + \int_0^S h\Big(\frac{1}{2}(u(t) - v(t)) + \frac{1}{2}(v(S-t) - u(S-t))\Big)\, dt \\
&\le \int_0^S \big(f(u') + g(v') + h(u-v)\big)\, dt.
\end{aligned}$$

The uniqueness of the minimum point implies that $\tilde u = u$ and $\tilde v = v$, giving the contradiction.

Moreover, consider z not equal to the (possible) common minimum point of f and g, and (u,v) solving (22). Then $u(t) = v(t) = zt$ if and only if $t = \frac{S}{2}$. Indeed, if there exists $\sigma < \frac{S}{2}$ such that $u(\sigma) = v(\sigma) = \sigma z$, then by the minimality we deduce $u = v = zt$ in $[\sigma, \frac{S}{2}]$. Denoting by $[a,b]$ the larger interval containing $[\sigma, \frac{S}{2}]$ such that $u = v = zt$ in $[a,b]$, if $a > 0$ we construct a new pair $(\tilde u, \tilde v)$ by setting

$$\tilde u(t) = \begin{cases} zt & \text{if } 0 \le t \le b-a \\ u(t-(b-a)) + z(b-a) & \text{if } b-a \le t \le b \\ u(t) & \text{if } b \le t \le S \end{cases}$$

and in the same way $\tilde{v}$. Since $\widetilde{G}_1(\tilde{u},\tilde{v};(0,S)) = \tilde{G}_1(u,v;(0,S))$, by uniqueness we deduce that $\tilde{u} = u$ and $\tilde{v} = v$. Hence, $u(t) = v(t) = zt$ in $[0,b]$ which gives a contradiction since $a > 0$ and $[a,b]$ is maximal. If $a = 0$ and $b < S$, a completely similar construction allows to prove that $u(t) = v(t) = zt$ in the whole interval $(0,S)$, which gives a contradiction since the affine and equal functions do not solve the minimum problem (22), where the boundary values are imposed only on v, and z does not coincide with the (possible) common minimum point of f and g.

We now prove a property of the minimum problem for $\widetilde{G}_1(u,v;(0,T))$ with fixed boundary values for v and a fixed number of jump points for u.

Lemma 11. *For any $T > 0$, $M \in \mathbb{N}$ and $z \in \mathbb{R}$ different from the (possible) common minimum point of f and g the following equality holds:*

$$\begin{aligned}\min\{\widetilde{G}_1(u,v;(0,T)) : v(0) = 0, v(T) = Tz, \#S(u) = M\} \\ = \min\{\widetilde{G}_1(u,v;(0,T)) : (u,v) \in \mathcal{E}_M(T,z)\},\end{aligned} \tag{24}$$

where $\mathcal{E}_M(T,z)$ is the set of pairs $(u,v) \in SBV^p(0,T) \times W^{1,p}(0,T)$ satisfying

$$S(u) = \Big\{\frac{iT}{M+1} : 1 \le i \le M\Big\},\ v\Big(\frac{iT}{M+1}\Big) = \frac{iTz}{M+1} \text{ for } 0 \le i \le M+1. \tag{25}$$

Proof. We consider the case $M \ge 1$ (noting that for $M = 0$ the thesis is obvious). We start by proving the existence of the first minimum in (24). Indeed, by semicontinuity and since the constraint is closed, there exists the minimum of $\widetilde{G}_1(u,v;(0,T))$ in

$$\{(u,v) :\ v(0) = 0,\ v(T) = Tz,\ \#S(u) \le M\},$$

where $v \in W^{1,p}(0,T)$ and $u \in SBV^p(0,T)$ as above (see e.g. [8]). Let $(\overline{u},\overline{v})$ realize the minimum.

We prove the thesis by showing that $\#S(\overline{u}) = M$. Set $K = \#S(\overline{u})$ and suppose by contradiction that $K < M$. Since z does not coincide with the (possible) common minimum point z^* of f and g, there exists at least one interval $(a,b) \subset (0,T)$ such that $a,b \in S(\overline{u}) \cup \{0,T\}$, $\overline{u} \in W^{1,p}(a,b)$ and $\overline{v}(b) - \overline{v}(a) = \overline{z}(b-a) \ne z^*(b-a)$. Recalling Remark 10, $\overline{u}(t) = \overline{v}(t) = \overline{v}(a) + \overline{z}(t-a)$ holds in (a,b) if and only if $t = \frac{a+b}{2}$. Denoting by $\tilde{u},\tilde{v}$ the unique solution of the minimum problem defining $\psi(\frac{b-a}{2},\overline{z})$, we set

$$u(t) = \begin{cases} \overline{u}(t) & t \in (0,T)\setminus(a,b) \\ \overline{u}(a) + \tilde{u}(t-a) & t \in (a,\frac{a+b}{2}) \\ \overline{u}(\frac{a+b}{2}) + \tilde{u}(t-\frac{a+b}{2}) & t \in (\frac{a+b}{2},b) \end{cases}$$

and correspondingly we define v. Since $\overline{z}$ does not coincide with z^*, then $(u,v) \ne (\overline{u},\overline{v})$ in (a,b) and $\#S(u) = K+1$. The pair (u,v) is admissible as test function since $\#S(u) = K+1 \le M$, hence by the uniqueness of the solution of $\psi(\frac{b-a}{2},\overline{z})$ we get

$$\widetilde{G}_1(u,v;(0,T)) < \widetilde{G}_1(\overline{u},\overline{v};(0,T))$$

which gives a contradiction.

Now we show equality (24) for $M = 1$. Let (u,v) be such that $S(u) = \{\tau\}$, $v(0) = 0$ and $v(T) = Tz$. Setting $T_1 = \tau$, $T_2 = T - \tau$, $T_1 z_1 = v(\tau)$ and $T_2 z_2 =$

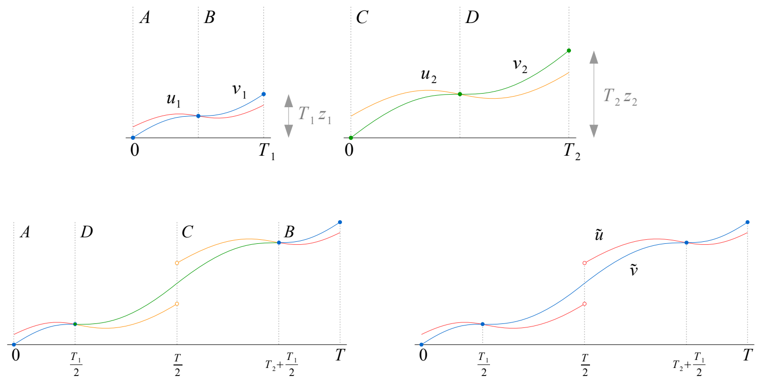

FIGURE 1. Construction of $\tilde{u}, \tilde{v}$.

$Tz - T_1 z_1$, for $i = 1, 2$ we choose (u_i, v_i) solving the minimum problem for $\psi(T_i, z_i)$ and define

$$\tilde{u}(t) = \begin{cases} u_1(t) & \text{if } \ t < \frac{T_1}{2} \\ u_2\big(t + \frac{T_2 - T_1}{2}\big) + \frac{T_1 z_1 - T_2 z_2}{2} & \text{if } \ \frac{T_1}{2} \leq t < \frac{T}{2} \\ u_2\big(t - \frac{T}{2}\big) + \frac{Tz}{2} & \text{if } \ \frac{T}{2} \leq t < \frac{T}{2} + \frac{T_2}{2} \\ u_1(t - T_2) + T_2 z_2 & \text{if } \ \frac{T}{2} + \frac{T_2}{2} \leq t \leq T \end{cases}$$

In the same way we define $\tilde{v}$ (see Fig. 1).

Since (23) holds in $(0, T_i)$, we deduce that $S(\tilde{u}) = \{\frac{T}{2}\}$ and $\tilde{v}\big(\frac{T}{2}\big) = \frac{Tz}{2}$. By construction, $\widetilde{G}_1(\tilde{u}, \tilde{v}; (0,T)) = \widetilde{G}_1(u, v; (0,T))$ concluding the proof of (25) for $M = 1$.

Note that if $\tau \neq \frac{T}{2}$ then $(\tilde{u}, \tilde{v})$ does not solve the minimum problem for $\psi(\frac{T}{2}, \frac{z}{2})$. Indeed, if it were not so, then, recalling Remark 10, both $\tilde{u}$ and $\tilde{v}$ would coincide with the affine function zt in $\frac{T}{4}$ and $\frac{T_1}{2}$. As noticed in Remark 10, the strict convexity of f, g, h would give a contradiction. The same holds if $\tau = \frac{T}{2}$ and $v(\tau) \neq \frac{Tz}{2}$, since the previous construction gives a function $\tilde{v}$ such that $\tilde{v}(\frac{T}{4}) \neq \frac{Tz}{4}$; hence also in this case $(\tilde{u}, \tilde{v})$ does not solve the minimum problem for $\psi(\frac{T}{2}, \frac{z}{2})$.

Then, if (u, v) does not belong to $\mathcal{E}_1(T, z)$, we can always find $\overline{u} \in SBV^p(0,T)$ with $S(u) = \{\frac{T}{2}\}$ and $\overline{v} \in W^{1,p}(0,T)$ with $\overline{v}(0) = 0$, $\overline{v}(\frac{T}{2}) = \frac{Tz}{2}$ and $\overline{v}(T) = Tz$ such that $\widetilde{G}_1(\overline{u}, \overline{v}; (0,T)) < \widetilde{G}_1(\tilde{u}, \tilde{v}; (0,T))$.

If $M > 1$, let (u, v) be such that $v(0) = 0$ and $v(T) = Tz$ and $S(u) = \{\tau_i\}_{i=1}^M$ with $0 = \tau_0 < \tau_i < \tau_{i+1} < \tau_{M+1} = T$ for any $i = 1, \ldots, M-1$. We set $T_i = \tau_{i+1} - \tau_i$. If (u, v) does not satisfy (25) there exists j such that $T_j \neq T_{j-1}$ or $T_j = T_{j-1}$ and $v(\frac{\tau_{i-1} + \tau_{i+1}}{2}) \neq \frac{v(\tau_{i+1}) + v(\tau_{i-1})}{2}$. Since the functionals are invariant by translations, we can suppose $j = 1$. The same argument of the case $M = 1$ allows to find $\overline{u} \in SBV^p(0, \tau_2)$ with $S(u) = \{\frac{\tau_2}{2}\}$ and $\overline{v} \in W^{1,p}(0, \tau_2)$ with $\overline{v}(0) = 0$, $\overline{v}(\frac{\tau_2}{2}) = \frac{v(\tau_2)}{2}$ and $\overline{v}(\tau_2) = v(\tau_2)$ such that $\widetilde{G}_1(\overline{u}, \overline{v}; (0, \tau_2)) < \widetilde{G}_1(u, v; (0, \tau_2))$. By defining $\tilde{u} = \overline{u}$ in $(0, \tau_2)$ and $\tilde{u}(t) = u(t)$ in (τ_2, T), and correspondingly $\tilde{v}$, we deduce $\widetilde{G}_1(\tilde{u}, \tilde{v}; (0,T)) < \widetilde{G}_1(u, v; (0,T))$, concluding the proof. $\square$

Proof of Proposition 8. For any fixed $N \in \mathbb{N}$ we consider the minimum problem

$$(26) \quad \frac{1}{T} \inf\{\widetilde{G}_1(u, v; (0,T)) : u(0) = v(0) = 0, \ u(T) = v(T) = Tz, \ \#S(u) = N\}.$$

Note that for $N = 0$ the minimum in (26) is attained in $u(t) = v(t) = zt$ since f, g, h are even and convex, and

$$\begin{aligned}\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : u(0) = v(0) = 0,\ u(T) = v(T) = Tz,\ \#S(u) = 0\}\\ = f(z) + g(z) + h(0).\end{aligned}$$

If $N \geq 1$, we have

(27)
$$\begin{aligned}&\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : u(0) = v(0) = 0,\ u(T) = v(T) = Tz,\ \#S(u) = N\}\\ &\quad\geq \frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : v(0) = 0,\ v(T) = Tz,\ \#S(u) = N-1\}.\end{aligned}$$

Indeed, let u, v be admissible functions for the minimum problem (26), and let $\tau \in (0,T)$ be such that $\tau \in S(u)$ and $u \in H^1(0,\tau)$. We consider a periodic extension of the problem and define $\tilde{u}, \tilde{v}$ in $[0,T]$ by setting

$$\tilde{u}(x) = \begin{cases} u(x+\tau) - v(\tau) & \text{if } 0 \leq x < T-\tau \\ u(x+\tau-T) - v(\tau) + Tz & \text{if } T-\tau \leq x \leq T \end{cases}$$

and $\tilde{v}$ correspondingly. Hence $\tilde{v}(0) = 0$, $\tilde{v}(T) = Tz$ and $\#S(\tilde{u}) = N-1$, and we get $\widetilde{G}_1(u,v;(0,T)) = G_1(\tilde{u},\tilde{v};(0,T))$, which proves (27).

Recalling that Lemma 11 ensures that for any $N \geq 1$

$$\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : v(0) = 0,\ v(T) = Tz,\ \#S(u) = N-1\} \geq \psi\Big(\frac{T}{N}, z\Big),$$

it follows that

$$\begin{aligned} g_{1,\mathrm{hom}}(z) &= \inf_{T>0}\frac{1}{T}\{\widetilde{G}_1(u,v(0,T)) + k\#S(u) : u(0) = v(0) = 0,\ u(T) = v(T) = Tz\}\\ &\geq \min\Big\{f(z) + g(z) + h(0), \inf_{T>0}\inf_{N\geq 1}\Big\{\psi(\frac{T}{N}, z) + k\frac{N}{T}\Big\}\Big\}\\ &\geq \min\Big\{f(z) + g(z) + h(0), \inf_{S>0}\Big\{\psi(S,z) + \frac{k}{S}\Big\}\Big\}\\ &\geq \inf_{S>0}\Big\{\psi(S,z) + \frac{k}{S}\Big\}\end{aligned}$$

since $f(z) + g(z) + h(0) \geq \inf_{S>0}\{\psi(S,z) + \frac{k}{S}\}$.

Now we have to show the opposite inequality. We fix $\delta > 0$; let S_δ be such that

$$\psi(S_\delta, z) + \frac{k}{S_\delta} < \inf_{S>0}\Big\{\psi(S,z) + \frac{k}{S}\Big\} + \delta$$

and $\overline{u}, \overline{v} \in W^{1,p}(0, S_\delta)$ solving the minimum problem (22). We set

$$\tilde{u}(x) = \begin{cases} u(x + \frac{S_\delta}{2}) - v(\frac{S_\delta}{2}) & \text{if } 0 \leq x < \frac{S_\delta}{2} \\ u(x - \frac{S_\delta}{2}) - v(\frac{S_\delta}{2}) + S_\delta z & \text{if } \frac{S_\delta}{2} \leq x \leq S_\delta \end{cases}$$

and correspondingly $\tilde{v}$. Since (23) holds, then $\tilde{u}(0) = \tilde{v}(0) = 0$, $\tilde{u}(S_\delta) = \tilde{v}(S_\delta) = S_\delta z$, and $\#S(\tilde{u}) \leq 1$. We get

$$\begin{aligned}\psi(S_\delta, z) + \frac{k}{S_\delta} &= \frac{1}{S_\delta}\widetilde{G}_1(\tilde{u},\tilde{v};(0,S_\delta)) + \frac{k}{S_\delta}\\ &\geq \frac{1}{S_\delta}\inf\{G_1(u,v;(0,S_\delta)) : u(0) = v(0) = 0, u(S_\delta) = v(S_\delta) = S_\delta z\}\\ &\geq g_{\mathrm{hom}}(z).\end{aligned}$$

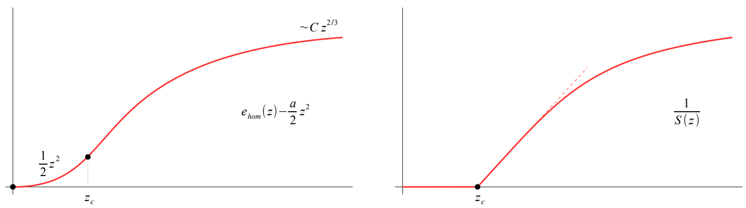

FIGURE 2. The energy $e_{\rm hom}(z) - \frac{a}{2}z^2$ and the density of fracture $\frac{1}{S(z)}$.

Hence $\inf\limits_{S>0}\Big\{\psi(S,z) + \dfrac{k}{S}\Big\} + \delta \geq g_{1,\rm hom}(z)$. Since $\delta > 0$ is arbitrary, this proves (21). □

Example 12. We now compute the homogenized energy density for an explicit example, which has also been analyzed in [9]. This example will allow us to highlight the behaviour of $g_{\rm hom}$ close to 0 and at infinity. A similar discrete energy has been studied in [13]; for a possible future comparison with that paper, we use the same notation therein: we take $f(z) = \frac{1}{2}z^2$, $g(z) = \frac{a}{2}z^2$, $h(z) = \frac{b}{2}z^2$, and $k = \frac{\eta}{2}$. With this choice

$$E_\varepsilon(u,v;I) = \int_I \Big(\frac{1}{2}(u')^2 + \frac{a}{2}(v')^2 + \frac{b}{2}\big(\frac{u-v}{\varepsilon}\big)^2\Big)\,dt + \frac{\eta}{2}\varepsilon\#S(u), \tag{28}$$

where I is a bounded interval, $v \in H^1(I)$ and $u \in SBV(I)$. We can apply Proposition 8, obtaining that the Γ-limit of E_ε with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^2(I)$ is given by

$$E_{\rm hom}(w) = \int_I e_{\rm hom}(v')\,dt$$

with $e_{\rm hom}$ given by Proposition 8.

Proposition 13 (Explicit representation of $e_{\rm hom}$). *The following equality holds*

$$e_{\rm hom}(z) = \inf_{S>0}\Big\{\frac{1}{2}\frac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2 + \frac{\eta}{2S}\Big\}, \tag{29}$$

where $\omega^2 = \frac{(a+1)b}{a}$. Moreover, the function $e_{\rm hom}(z)$ is strictly convex, and

1. *$e_{\rm hom}(z) = \dfrac{a+1}{2}z^2$ in $[0, z_c]$, where $z_c = \sqrt{\frac{\eta\omega a}{2(a+1)}}$;*
2. *for $z \to +\infty$ $e_{\rm hom}(z) \sim \dfrac{a}{2}z^2 + Cz^{2/3}$, where $C > 0$ depends only on a, b, η.*

Remark 14. In the proof of the proposition above, we show that for $z \leq z_c$ the inf in (29) is given by the limit for $S \to +\infty$, and the minimum is attained in a unique $S(z) > 0$ otherwise. Hence, we can introduce the function

$$S(z) = \begin{cases} +\infty & \text{if } \ z \leq z_c \\ \text{arg min}\Big\{\dfrac{1}{2}\dfrac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2 + \dfrac{\eta}{2S} : S > 0\Big\} & \text{if } \ z > z_c \end{cases}$$

representing the optimal distance between fracture points at a given strain. In Fig. 2 we picture the graph of the "effective" energy $e_{\rm hom}(z) - \frac{a}{2}z^2$ (given by the difference between the homogenized density energy and the energy of the elastic substrate) and the density of fracture $\frac{1}{S(z)}$ (see the corresponding pictures in [9, Fig. 5]).

Proof of Proposition 13. By solving the Euler-Lagrange equations for

$$\widetilde{E}_1(u,v;(0,S)) = \frac{1}{2}\int_0^S ((u')^2 + a(v')^2 + b(u-v)^2)\,dt$$

and by minimizing on the boundary values of u we get the explicit expression for ψ in (22) as

$$\psi(S,z) = \frac{1}{2}\frac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2,$$

where $\omega^2 = \frac{(a+1)b}{a}$, proving (29). In order to complete the proof of the proposition, we simplify the expression of $e_{\rm hom}(z)$ by writing

$$e_{\rm hom}(z) = \frac{\eta\omega}{4}\inf_{x>0}\Big\{\frac{2(a+1)}{\eta\omega}z^2\frac{ax}{ax+\tanh(x)} + \frac{1}{x}\Big\} = \frac{\eta\omega}{4}\inf_{x>0} H\Big(x, z\sqrt{\frac{2(a+1)}{\eta\omega}}\Big),$$

where $H(x,y) = \frac{ax}{ax+\tanh(x)}y^2 + \frac{1}{x}$. Since for $y \le \sqrt{a}$

$$\frac{x^2}{y^2}\frac{\partial H}{\partial x}(x,y) = \frac{a\tanh(x) - ax(1-\tanh^2(x))}{(ax+\tanh(x))^2}x^2 - \frac{1}{y^2} < 0,$$

the infimum coincides with the limit for $x \to +\infty$, and we get that

$$e_{\rm hom}(z) = \frac{a+1}{2}z^2 \quad \text{for} \quad z \le z_c = \sqrt{\frac{\eta\omega a}{2(a+1)}}. \tag{30}$$

Now we consider the case $z > z_c$, corresponding to $y > \sqrt{a}$. In this case $\inf_{x>0} H(x,y) = H(x(y),y)$, where $x = x(y)$ is implicitly defined by

$$\frac{a\tanh x - ax(1-\tanh^2 x)}{(ax+\tanh x)^2}y^2 = \frac{1}{x^2}. \tag{31}$$

We deduce that $x(y)$ is strictly decreasing, and tends to 0 as $y \to +\infty$. Moreover, again by (31) we get that $x(y) \sim \sqrt[3]{\frac{3(a+1)^2}{2a}}y^{-\frac{2}{3}}$ for $y \to +\infty$. Hence,

$$\inf_{x>0} H(x,y) = H(x(y),y) \sim \frac{a}{a+1}y^2 + \sqrt[3]{\frac{2a}{3(a+1)^2}}y^{2/3} \quad \text{for} \quad y \to +\infty$$

and consequently $e_{\rm hom}(z) \sim \frac{a}{2}z^2 + \sqrt[3]{\frac{4a}{3(a+1)\eta\omega}}z^{2/3}$ for $z \to +\infty$.

As for the strict convexity of $e_{\rm hom}(z)$, we prove that $(\inf_{x>0} H(x,y))'$ is strictly increasing. By (31) we get

$$\Big(\inf_{x>0} H(x,y)\Big)' = \frac{d}{dy}H(x(y),y) = 2y\frac{ax(y)}{ax(y)+\tanh x(y)}$$

which is strictly positive, and it is strictly increasing if and only if $\big((\inf_{x>0} H(x,y))'\big)^2$ is strictly increasing. Again by (31) we deduce

$$\Big(2y\frac{x(y)}{ax(y)+\tanh x(y)}\Big)^2 = \frac{4}{a\tanh x(y) - ax(y)(1-\tanh^2 x(y))},$$

and $\left(\dfrac{4}{a\tanh x(y) - ax(y)(1-\tanh^2 x(y))}\right)' > 0$ since $x'(y) < 0$, thus concluding the proof. $\square$

6. Conclusions

We have examined the overall behaviour of two interacting hyperelastic media, one of which possibly subject to brittle fracture. The corresponding energies depend on two placement fields and on a small parameter ε governing the microscopic interaction between the two media. The effective description is obtained by computing a limit energy as $\varepsilon \to 0$. This energy is a hyperelastic energy depending on a single placement field, and is described by an asymptotic homogenization formula highlighting the optimization of strain micro-oscillations and microfracture combined, as $\varepsilon \to 0$. We have provided some alternate formulas in one dimension, and examined in detail a prototypical example, showing that for a given macro-stress z micro-patterns tend to be periodic of a period depending on z itself. This then shows that the homogenization process is triggered by a competition between bulk energies, fracture energies and the interaction terms, and not by an underlying material heterogeneity. It is interesting to note that the behaviour of these homogenized energies is in many aspects different from that of discrete models of a seemingly very similar form, where the discreteness translates in constraints on the fracture sites.

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