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Weak q-concavity conditions for *CR* manifolds

Mauro Nacinovich[1] · **Egmont Porten**[2,3]

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Abstract We introduce various notions of q-pseudo-concavity for abstract *CR* manifolds, and we apply these notions to the study of hypoellipticity, maximum modulus principle and Cauchy problems for *CR* functions.

Keywords *CR*-manifolds · *q*-concavity conditions · *CR*-hypoelliptic · *CR* functions · Cauchy problem

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[1] The Witt index of a Hermitian form of signature (p, q) is $\min\{p, q\}$.

✉ Mauro Nacinovich
nacinovi@mat.uniroma2.it

Egmont Porten
Egmont.Porten@miun.se

1 Dipartimento di Matematica, II Università di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy

2 Department of Mathematics, Mid Sweden University, 85170 Sundsvall, Sweden

3 Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Kielce, Poland

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1 Introduction

The definition of q-pseudo-concavity for abstract *CR* manifolds of arbitrary *CR*-dimension and *CR*-codimension, given in [20], required that all scalar Levi forms corresponding to non-characteristic codirections have Witt index[1] larger or equal to q. Important classes of homogeneous examples (see, for example, [1,3–5,33,35]) show that these conditions are in fact too restrictive and that weaker notions of q-pseudo-concavity are needed. For example, the results on the non-validity of the Poincaré lemma for the tangential Cauchy–Riemann complex in [10,23] only involve scalar Levi forms of maximal rank. In [21], the classical notion of 1-pseudo-concavity was extended by a trace condition that was further improved in [2,18,22]. These notions are relevant to the behavior of *CR* functions, being related to hypoellipticity, weak and strong unique continuation, hypoanaliticity (see [38]) and the maximum modulus principle.

In this paper, we continue these investigations. A key point of this approach is the simple observation that the Hermitian-symmetric vector-valued Levi form $\mathcal{L}$ of a *CR* manifold M defines a linear form on $T^{1,1}M = T^{1,0}M \otimes_M T^{0,1}M$. Our notion of pseudo-concavity is the request that its kernel contains elements τ which are positive semidefinite. To such a τ, we can associate an invariantly defined degenerate elliptic real partial differential operators P_τ, which turns out to be related to the dd^c operator of [32]. By consistently keeping this perspective, we prove in this paper some results on $\mathscr{C}^\infty$ hypoellipticity, the maximum modulus principle, and undertake the study of boundary value problems for *CR* functions on open domains of abstract *CR* manifolds, testing the effectiveness of a new notion of weak two-pseudo-concavity by its application to the Cauchy problem for *CR* functions.

The general plan of the paper is the following. In the next section, we define the notion of $\mathcal{Z}$-structure that generalizes *CR* structures insofar that all formal integrability and rank conditions can be dropped, while our focus is *CR* functions, only considered as solutions of a homogeneous overdetermined system of first-order p.d.e.'s, and set the basic notation that will be used throughout the paper. In particular, we introduce the kernel $[\ker \mathcal{L}]$ of the Levi form as a subsheaf of the sheaf of germs of semipositive tensors of type $(1, 1)$.

In Sect. 3, we show how the maximum modulus principle relates to $\mathscr{C}^\infty$-regularity and weak and strong unique continuation of *CR* functions. We also make some comments on generic points of non-embeddable *CR* manifolds, where, by using our results of [38], we can prove, in Proposition 3.5, a result of strong unique continuation and partial hypoanaliticity (cf. [47]).

In Sect. 4, we show that to each semipositive tensor τ in the kernel of the Levi form we can associate a real degenerate elliptic scalar p.d.o. of the second-order P_τ. Real parts of *CR* functions are P_τ-harmonic, and the modulus of a *CR* function is P_τ-subharmonic at points

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where it is different from zero. Then, by using some techniques originally developed for the generalized Kolmogorov equation (cf. [25,26,29]), we are able to enlarge, in comparison with [2], the set of vector fields *enthralled by* $\mathcal{Z}$. Thus, we can improve, by Theorem 4.2, some hypoellipticity result of [2], and, by Theorem 4.7, a propagation result of [22], for the case in which this hypoellipticity fails.

In Sect. 5, we prove the *CR* analogue of Malgrange's theorem on the vanishing of the top degree cohomology under some subellipticity condition. Our result slightly generalizes previous results of [9,30,31], also yielding a Hartogs-type theorem on abstract *CR* manifolds, to recover a *CR* function on a relatively compact domain from boundary values satisfying some momentum condition (Proposition 5.3).

In Sect. 6, we use the dd^c-operator of [32] to show that the operators P_τ are invariantly defined in terms of sections of $[\ker\mathcal{L}]$ (Corollary 6.8). The Hopf Lemma for P_τ is used to deduce pseudo-convexity properties of the boundary of a domain where a *CR* functions has a peak point (Proposition 6.15). This leads to a notion of convexity/concavity for points of the boundary of a domain (Definition 6.4). Most of these notions can be formulated in terms of the scalar Levi forms associated with the covectors of a half-space of the characteristic bundle.

Thus, in Sect. 7, we have found it convenient to consider properties of convex cones of Hermitian-symmetric forms satisfying conditions on their indices of inertia, which are preliminary to the definitions of the next section.

In Sect. 8, we propose various notions of weak-q-pseudo-concavity, give some examples, and show in Proposition 8.7 that on an essentially 2-pseudo-concave manifold strong-1-convexity/concavity at the boundary becomes an *open* condition, i.e., stable under small perturbations. This is used in the last two sections to discuss existence and uniqueness for the Cauchy problem for *CR* functions, with initial data on a hypersurface.

In Sect. 9, after discussing uniqueness in the case of a locally embeddable *CR* manifold, we turn to the case of an abstract *CR* manifold, proving, via Carleman-type estimates, that the uniqueness results of [13,21,22] can be extended by using some convexity condition (see Proposition 9.9). In Sect. 10, an existence theorem for the Cauchy problem is proved for locally embeddable *CR* manifolds, under some convexity conditions.

2 *CR*-and Z-manifolds: preliminaries and notation

Let M be a real smooth manifold of dimension m.

Definition 2.1 A *$\mathcal{Z}$-structure* on M is the datum of a $\mathscr{C}^\infty_{\mathrm{M}}$-submodule $\mathcal{Z}$ of the sheaf $\mathfrak{X}^{\mathbb{C}}_M$ of germs of smooth complex vector fields on M. It is called

- *formally integrable* if $[\mathcal{Z}, \mathcal{Z}] \subset \mathcal{Z}$;
- *of CR* type if $\mathcal{Z} \cap \overline{\mathcal{Z}} = \underline{0}$ (the 0-sheaf);
- *almost-CR* if $\mathcal{Z}$ is of *CR* type and locally free of constant rank;
- *quasi-CR* if it is of *CR* type and formally integrable;
- *CR* if $\mathcal{Z}$ is of *CR* type, formally integrable and locally free of constant rank.

A *Z-manifold* is a real smooth manifold M endowed with a $\mathcal{Z}$-structure. Since $\mathscr{C}^\infty_{\mathrm{M}}$ is a fine sheaf, $\mathcal{Z}$ can be equivalently described by the datum of the space $\mathcal{Z}(M)$ of its global sections.

When M is a smooth real submanifold of a complex manifold X, then

$$\mathcal{Z}(M) = \{Z \in \mathfrak{X}^{\mathbb{C}}(M) \mid Z_p \in T^{0,1}_p\mathrm{X},\ \forall p \in M\}$$

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is formally integrable. Hence, $\mathcal{Z}(M)$ defines a quasi-*CR* structure on M, which is *CR* if the dimension of $T^{0,1}_p\mathrm{X}\cap\mathbb{C}T_pM$ is constant for $p\in M$. This is always the case when M is a real hypersurface in X.

A *complex embedding (immersion)* $\phi : M \hookrightarrow \mathrm{X}$ of a quasi-*CR* manifold M into a complex manifold X is a smooth embedding (immersion) for which the $\mathcal{Z}$-structure on M is the pullback of the complex structure of X:

$$\mathcal{Z}(M)=\{Z\in\mathfrak{X}^{\mathbb{C}}(M)\mid d\phi(Z_p)\in T^{0,1}_{\phi(p)}\mathrm{X},\ \forall p\in M\}.$$

Example 2.1 Let $M=\{w=z_1\bar z_1+i\,z_2\bar z_2\}\subset\mathbb{C}^3_{w,z_1,z_2}=\mathrm{X}$. We can take the real and imaginary parts of z_1,z_2 as coordinates on M, which therefore, as a smooth manifold, is diffeomorphic to $\mathbb{C}^2_{z_1,z_2}$. The embedding $M\hookrightarrow\mathbb{C}^3$ yields the quasi-*CR* structure

$$\mathcal{Z}(M)=\mathscr{C}^\infty(M)\left[z_2\frac{\partial}{\partial\bar z_1}+i z_1\frac{\partial}{\partial\bar z_2}\right]$$

on M. Then, $M\setminus\{0\}$ is a *CR* manifold of *CR*-dimension 1 and *CR*-codimension 2, while all elements of $\mathcal{Z}(M)$ vanish at $0\in M$.

A *Z*-manifold M of *CR* type contains an open dense subset $\mathring{M}$ whose connected components are almost-*CR* for the restriction of $\mathcal{Z}$. Likewise, any quasi-*CR* manifold M contains an open dense subset $\mathring{M}$ whose connected components are *CR* manifolds.

We shall use Ω and $\mathscr{A}$ for the sheaves of germs of *complex-valued* and *real-valued* alternate forms on M (subscripts indicate degree of homogeneity). Starting with the case of an almost-*CR* manifold M, we introduce the notation:

$$\begin{aligned}
&T^{0,1}M=\bigcup\nolimits_{p\in M}\big(T^{0,1}_pM=\{Z_p\mid Z\in\mathcal{Z}(M)\}\big)\subset\mathbb{C}TM,\quad T^{1,0}M=\overline{T^{0,1}M},\\
&HM=\bigcup\nolimits_{p\in M}\big(H_pM=\{\mathrm{Re}\,Z_p\mid Z_p\in T^{0,1}_pM\}\big)\subset TM,\\
&J_M:H_pM\to H_pM,\quad X_p+iJ_MX_p\in T^{0,1}_pM,\quad\forall X_p\in H_pM,\\
&\qquad\qquad\text{(partial complex structure)},\\
&\mathcal{H}=\{\mathrm{Re}\,Z\mid Z\in\mathcal{Z}\},\\
&\pi_M:TM\to TM/HM\quad\text{(projection onto the quotient)},\\
&\mathscr{I}(M)=\{\alpha\in\bigoplus\nolimits_{h=1}^{\nu}\Omega^h(M,\mathbb{C})\mid\alpha|T^{0,1}M=0\},\quad(\mathscr{I}\text{ is the } \textit{ideal sheaf}),\\
&H^0M=\bigcup\nolimits_{p\in M}\big(H^0_pM=\{\xi\in T^*_pM\mid\xi(H_pM)=\{0\}\}\big)\subset T^*M,\\
&H^{1,1}M=\bigcup\nolimits_{p\in M}\big(H^{1,1}_pM=\text{convex hull of }\{(Z_p\otimes\bar Z_p)\mid Z\in\mathcal{Z}(M)\}\big),\\
&H^{1,1,(r)}M=\bigcup\nolimits_{p\in M}\big(H^{1,1,(r)}_pM=\{\tau\in H^{1,1}_pM\mid\mathrm{rank}\ \tau=r\}\big).
\end{aligned}$$

Note that $T^{0,1}M,\ T^{1,0}M,\ HM,\ TM/HM,\ H^0M,\ H^{1,1}M,\ H^{1,1,(r)}M$ define smooth vector bundles because we assumed that the rank n of $\mathcal{Z}$ is constant. This n is called the *CR-dimension* and the difference $k=m-2n$ the *CR-codimension* of M.

For a general $\mathcal{Z}$-manifold, we use the same symbols

$$T^{0,1}M,\quad T^{1,0}M,\quad HM,\quad TM/HM,\quad H^{1,1}M,\quad H^{1,1,(r)}M$$

for the closures of

$$T^{0,1}\mathring{M},\quad T^{1,0}\mathring{M},\quad H\mathring{M},\quad T\mathring{M}/H\mathring{M},\quad H^{1,1}\mathring{M},\quad H^{1,1,(r)}\mathring{M}$$

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in $T^{\mathbb{C}}M$, $T^{\mathbb{C}}M$, TM, TM/HM, $T^{\mathbb{C}}M \otimes_M T^{\mathbb{C}}M$, $T^{\mathbb{C}}M \otimes_M T^{\mathbb{C}}M$, respectively.

Example 2.2 For the M in Example 2.1, the fiber $T^{0,1}_p M$ has dimension 1 at all points p of $\mathring{M} = M \setminus \{0\}$, while $T^{0,1}_0 M = \mathbb{C}[\partial/\bar{z}_1, \partial/\partial\bar{z}_2]$ has dimension 2. By contrast, as we already observed, all elements of $\mathcal{Z}(M)$ vanish at 0.

If $\mathcal{F}$ is a subsheaf of the sheaf of germs of (complex-valued) distributions on M, an element f of $\mathcal{F}$ is said to be *CR* if it satisfies the equations $Zf = 0$ for all $Z \in \mathcal{Z}(M)$. The *CR* germs of $\mathcal{F}$ are the elements of a sheaf that we denote by $\mathcal{F}\mathcal{O}_M$. We will simply write $\mathcal{O}_M$ for $\mathcal{C}^\infty\mathcal{O}_M$.

We will assume in the rest of this section that M is an almost-*CR* manifold.

The fibers of $H^{1,1}M$ are closed convex cones, consisting of the *positive semidefinite* Hermitian-symmetric tensors in $T^{0,1}M \otimes_M T^{1,0}M$. The *characteristic bundle* H^0M is the dual of the quotient TM/HM.

Let us describe more carefully the bundle structure of $H^{1,1,(r)}M$. Set $V = T^{0,1}_p M$ and consider the non-compact Stiefel space $\mathcal{S}t_r(V)$ of r-tuples of linearly independent vectors of V. Two different r-tuples $v_1, \dots, v_r$ and $w_1, \dots, w_r$ in $\mathcal{S}t_r(V)$ define the same τ_p, i.e., satisfy

$$\tau_p = v_1 \otimes \bar{v}_1 + \cdots + v_r \otimes \bar{v}_r = w_1 \otimes \bar{w}_1 + \cdots + w_r \otimes \bar{w}_r,$$

if and only if there is a matrix $a = (a^i_j) \in \mathbf{U}(r)$ (the unitary group of order r) such that $w_j = \sum_j a^i_j v_i$. In fact, the span of $v_1, \dots, v_r$ is determined by the tensor τ_p, so that $w_j = \sum_j a^i_j v_i$ for some $a = (a^i_j) \in \mathbf{GL}_r(\mathbb{C})$ and

$$\sum_{i=1}^r w_i \otimes \bar{w}_i = \sum_{j=1}^r \sum_{i,h=1}^r a^i_j \bar{a}^h_j v_i \otimes \bar{v}_h = \sum_{i,h=1}^r \left(\sum_{j=1}^r a^i_j \bar{a}^h_j \right) v_i \otimes \bar{v}_h$$

shows that $a \in \mathbf{U}(r)$. Hence, $H^{1,1,(r)}M$ is the quotient bundle of the non-compact complex Stiefel bundle of r-frames in $T^{0,1}M$ by the action of the unitary group $\mathbf{U}(r)$. By using the Cartan decomposition

$$\mathbf{U}(r) \times \mathfrak{p}(r) \ni (x, X) \longrightarrow x \cdot \exp(X) \in \mathbf{GL}_r(\mathbb{C}),$$

where $\mathfrak{p}(r)$ is the vector space of Hermitian-symmetric $r \times r$ matrices, we see that $H^{1,1,(r)}M$ can be viewed as a rank r^2 real vector bundle on the Grassmannian $\mathcal{G}r^r(M)$ of r-planes of $T^{0,1}M$. Thus, it is a smooth vector bundle when M is almost-*CR*.

2.1 Scalar and vector-valued Levi forms

The map

$$Z_p \otimes \bar{Z}_p \longrightarrow -\pi_M(i[Z, \bar{Z}]_p), \quad \forall p \in M, \quad \forall Z \in \mathcal{Z}(M), \tag{2.1}$$

extends to a linear map

$$\mathcal{L} : H^{1,1}M \to TM/HM, \tag{2.2}$$

that we call the *vector-valued Levi form*. To each characteristic codirection $\xi \in H^0_p M$, we associate the Hermitian quadratic form

$$\mathcal{L}_\xi(Z_p, \bar{Z}_p) = \mathcal{L}(Z_p \otimes \bar{Z}_p) = -\langle \xi | i[Z, \bar{Z}]_p \rangle, \quad \forall Z \in \mathcal{Z}(M).$$

It extends to a convex function on $H^{1,1}_p M$, which is the evaluation by the covector ξ of the vector-valued Levi form. Thus, the *scalar Levi forms* are

$$\mathcal{L}_\xi(\tau) = \xi(\mathcal{L}(\tau)), \quad \text{for } p \in M, \quad \xi \in H^0_p M, \quad \tau \in H^{1,1}_p M. \tag{2.3}$$

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The range $\Gamma_p M$ of the vector-valued Levi form is a convex cone of T_pM/H_pM, whose dual cone is

$$\Gamma_p^0 M = \{\xi \in H_p^0 M \mid \mathcal{L}_\xi \geq 0\}.$$

Thus, we obtain

Lemma 2.3 *An element $v \in T_pM/H_pM$ belongs to the closure or the range of the vector-valued Levi form if and only if*

$$\langle v|\xi\rangle \geq 0, \quad \forall \xi \in H_p^0 M \quad \text{such that} \quad \mathcal{L}_\xi \geq 0. \tag{2.4}$$

□

Remark 2.4 Note that $\Gamma_p M$ need not be closed. An example is provided by the quadric $M = \{\mathrm{Re}\, z_3 = z_1\bar{z}_1,\ \mathrm{Re}\, z_4 = \mathrm{Re}(z_1\bar{z}_2)\} \subset \mathbb{C}^4$: the cone $\Gamma_0 M$ is the union of the origin and of an open half-plane.

It is convenient to introduce the notation:

$$[\ker\mathcal{L}]^{(q)} = H^{1,1,(q)}M \cap \ker\mathcal{L},\ \ \overline{[\ker\mathcal{L}]} = \bigoplus\nolimits_{q\geq 0}[\ker\mathcal{L}]^{(q)},\ \ [\ker\mathcal{L}] = \bigoplus\nolimits_{q>0}[\ker\mathcal{L}]^{(q)}.$$

Definition 2.2 We call $[\ker\mathcal{L}]$ the *kernel of the Levi form*.

We note that this definition is at variance with a notion that appears in the literature (see, for example, [12]), where the kernel of the Levi form consists of the (1, 0)-vectors which are isotropic for all scalar Levi forms. These vectors are related to $[\ker\mathcal{L}]^{(1)}$, which is trivial in several examples of *CR* manifolds which are not of hypersurface type and have a non-trivial $[\ker\mathcal{L}]$.

Let $\mathcal{Y}$ be a generalized distribution of *real* vector fields on M and $p \in U^{\text{open}} \subset M$. The Sussmann leaf of $\mathcal{Y}$ through p in U is the set $\ell(p;\mathcal{Y},U)$ of points p' which are ends of piecewise $\mathscr{C}^\infty$ integral curves of $\mathcal{Y}$ starting from p and lying in U. We know that $\ell(p;\mathcal{Y},U)$ is always a smooth submanifold of U (see [17]).

Let $\mathcal{H} = \{\mathrm{Re}\, Z \mid Z \in \mathcal{Z}\}$. A $\mathcal{Z}$-manifold M is called *minimal* at p if $\ell(p;\mathcal{H},U)$ is an open neighborhood of p for all $U^{\text{open}} \subset M$ and $p \in U$. (This notion was introduced in [46] for embedded *CR* manifolds.) In the following, by a *Sussmann leaf of* $\mathcal{Z}$ we will mean a Sussmann leaf of $\mathcal{H}$.

A smooth real submanifold N of M (of arbitrary codimension ℓ) is said to be *non-characteristic*, or *generic*, at $p_0 \in N$, when

$$T_{p_0}N + H_{p_0}M = T_{p_0}M. \tag{2.5}$$

If this holds for all $p \in N$, then N is a *generic CR* submanifold of M, of type $(n-\ell, k+\ell)$, as $T_p^{0,1}N = T_p^{\mathbb{C}}N \cap T_p^{0,1}N$ and $H_p^0 N = H_p^0 M \oplus J_M^*(T_pN)^0$ for all $p \in N$. To distinguish from the Levi form $\mathcal{L}$ of M, we write $\mathcal{L}^N$ for the Levi form of N.

A Sussmann leaf for $\mathcal{Z}$ which is not open is *characteristic* at all points.

More generally, when $\Xi(M)$ is any distribution of complex-valued smooth vector fields on M, we say that N is Ξ*-non-characteristic at* $p_0 \in N$ if

$$T_{p_0}N + \{\mathrm{Re}\, Z_{p_0} \mid Z \in \Xi(M)\} = T_{p_0}M. \tag{2.6}$$

In this terminology, *non-characteristic* is equivalent to $\mathcal{Z}$-non-characteristic.

We note that the Ξ-non-characteristic points make an open subset of N.

Revised Proof

3 Hypoellipticity and the maximum modulus principle

In [38], we proved that, for locally embedded *CR* manifolds, the hypoellipticity of its tangential Cauchy–Riemann system is equivalent to the holomorphic extendability of its *CR* functions. Thus, hypoellipticity may be regarded as a weak form of pseudo-concavity. The regularity of *CR* distributions implies a strong maximum modulus principle for *CR* functions (see [21, Theorem 6.2]).

Proposition 3.1 *Let M be a Z-manifold. Assume that all germs of CR distributions on M that are locally L^2 are smooth. Then, for every open connected subset Ω of M, we have*

$$|f(p)| < \sup_{\Omega}|f|, \quad \forall p \in \Omega, \quad \textit{for all non-constant}\, f \in \mathscr{O}_M(\Omega). \tag{3.1}$$

Proof We prove that an $f \in \mathscr{O}_M(\Omega)$ for which $|f|$ attains a maximum value at some inner point p_0 of Ω is constant. Assume that $p_0 \in \Omega$ and $|f(p_0)| = \sup_{\Omega}|f|$. If $f(p_0) = 0$, then f is constant and equal to zero on Ω.

Assume that $f(p_0) \neq 0$. After rescaling, we can make $f(p_0) = |f(p_0)| = 1$.

Let E be the space $\mathscr{O}_M(\Omega)$ endowed with the L^2_{loc} topology. By the hypoellipticity assumption, E is Fréchet. Then, by Banach open mapping theorem, the identity map $E \to \mathscr{O}_M(\Omega)$ is an isomorphism of topological vector spaces. In particular, for all compact neighborhoods K of p_0 in Ω, there is a constant $C_K > 0$ such that

$$|u(p_0)|^2 \leq C_K \int_K |u|^2 d\lambda, \quad \forall u \in \mathscr{O}_M(\Omega).$$

Applying this inequality to f^{ν}, we obtain that

$$1 \leq \int_K |f|^{2\nu} d\lambda \leq \int_K d\lambda.$$

The sequence $\{f^{\nu}\}$ is compact in $\mathscr{O}_M(\Omega)$, because, by the hypoellipticity assumption and the Ascoli-Arzelà theorem, restriction to a relatively compact subset of *CR* functions is a compact map. Hence, we can extract from $\{f^{\nu}\}$ a sequence that converges to a *CR* function ϕ, which is nonzero because it has a positive square-integral on every compact neighborhood of p_0. We note now that $|\phi|$ is continuous and takes only the values 1, at points where $|f| = 1$, and 0 at points where $|f| < 1$. Since $\phi \neq 0$, we have $|\phi| \equiv 1$ on Ω and hence $|f| \equiv 1$ on Ω. By applying the preceding argument to $p \to \frac{1}{2}(1 + f(p))$, we obtain that $|1 + f(p)| = 2$ on Ω. Hence, Re $f \equiv 1$, which yields $f \equiv 1$, on Ω. □

Under the assumptions of Proposition 3.1, a *CR* function $f \in \mathscr{O}_M(\Omega)$ is constant on a neighborhood of any point where $|f|$ attains a local maximum.

Then, we have

Proposition 3.2 *Assume that*

(i) *all germs of CR distribution on M are smooth;*
(ii) *the weak unique continuation principle for CR functions is valid on M.*

Then, any CR function f, defined on a connected open subset Ω of M, for which $|f|$ attains a local maximum at some point of Ω, is constant.

We recall that the weak unique continuation principle (ii) means that a *CR* function $f \in \mathscr{O}_M(\Omega)$ which is zero on an open subset U of Ω is zero on the connected component of U in Ω.

Revised Proof

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Definition 3.1 We say that M has property (H) if (i) holds, and property (WUC) if (ii) holds. We say that (H) (or (WUC)) holds at p if it holds when M is substituted by a sufficiently small open neighborhood of p in M.

For a locally *CR*-embeddable *CR* manifold M, the implication $(H) \Rightarrow (WUC)$ is a consequence of [38]. In fact, (H) implies *minimality*, which implies (WUC) when M is locally *CR*-embeddable (see [46,48]). In fact, in this case (H) implies the *strong unique continuation principle* for *CR* functions.

Proposition 3.3 *Assume that M is a CR submanifold of a complex manifold* X *and that M has property (H). Then, a CR function, defined on a connected open subset Ω of M and vanishing to infinite order at a point p_0 of Ω, is identically zero in Ω.*

Proof Let $f \in \mathscr{O}_M(\Omega)$. It is sufficient to prove that the set of points where f vanishes to infinite order is open in Ω. This reduces the proof to a local statement, allowing us to assume that the embedding $M \hookrightarrow$ X is generic. By Nacinovich and Porten [38], any *CR* function f extends to a holomorphic function $\tilde{f}$, defined on a connected open neighborhood U of p in X. By the assumption that $M \hookrightarrow$ X is generic, $\tilde{f}$ is uniquely determined by the Taylor series of f at p in any coordinate chart and thus vanishes to infinite order at a point $p' \in U \cap \Omega$ if and only if f does. Hence, f vanishes to infinite order at p if and only if $\tilde{f}$ vanishes on U, and this is equivalent to the fact that f vanishes identically on $U \cap \Omega$. The proof is complete. □

When M is *not* locally embeddable, there should be smaller local rings of *CR* functions, so that in fact properties of regularity and unique continuation should even be more likely true. Let us shortly discuss this issue. Set

$$T_p^{*1,0}M = \{\zeta \in \mathbb{C}T_p^*M \mid \zeta(Z) = 0, \ \forall Z \in T_p^{0,1}M\}.$$

Note that, with

$$T_p^{*0,1}M = \overline{T_p^{*1,0}M} = \{\zeta \in \mathbb{C}T_p^*M \mid \zeta(\bar{Z}) = 0, \ \forall Z \in T_p^{0,1}M\},$$

the intersection

$$T_p^{*1,0}M \cap T_p^{*0,1}M = \mathbb{C}H_p^0M$$

is the complexification of the fiber of the characteristic bundle and therefore different from zero, unless $\mathcal{Z}$ is an almost complex structure. Differentials of smooth *CR* functions are sections of the bundle $T^{*1,0}M$. Thus, for a fixed p, we can consider the map

$$\mathscr{O}_{M,p} \ni f \longrightarrow \mathrm{d}f(p) \in T_p^{*1,0}M. \tag{3.2}$$

Clearly, we have

Lemma 3.4 *A necessary and sufficient condition for M to be locally CR-embeddable at p is that* (3.2) *is surjective.* □

We can associate with the map (3.2) a pair (ν_p, k_p) of nonnegative integers, with

$$k_p = \dim_{\mathbb{C}}\{\mathrm{d}f(p) \mid f \in \mathscr{O}_{M,p}\} \cap \mathbb{C}H_p^0M, \quad \text{and} \quad \nu_p + k_p = \dim_{\mathbb{C}}\{\mathrm{d}f(p) \mid f \in \mathscr{O}_{M,p}\}.$$

The numbers ν_p and $\nu_p + k_p$ are upper semicontinuous functions of p and hence locally constant on a dense open subset $\mathring{M}$ of M. Thus, we can introduce

Definition 3.2 We call *generic* the points of the open dense subset $\mathring{M}$ of M, where ν_p and $\nu_p + k_p$ are locally constant.

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Proposition 3.5 *Assume that M has property (H). Then, the strong unique continuation principle is valid at generic points p_0 of M. This means that $f \in \mathscr{O}_{M,p_0}$ is the zero germ if and only if it vanishes to infinite order at p_0.*

Moreover, there are finitely many germs $f_1, \ldots, f_\mu \in \mathscr{O}_{M,p_0}$, vanishing at p_0, such that, for every $f \in \mathscr{O}_{M,p_0}$, we can find $F \in \mathscr{O}_{\mathbb{C}^\mu,0}$ such that $f = F(f_1, \ldots, f_\mu)$.

Proof By the assumption that p_0 is generic, we can fix a connected open neighborhood U of p_0 in M and functions $f_1, \ldots, f_\mu \in \mathscr{O}_M(U)$, vanishing at p_0, such that $df_1(p) \wedge \cdots \wedge df_\mu(p) \neq 0$ for all $p \in U$ and $df_1(p), \ldots, df_\mu(p)$ generate the image of (3.2) for all $p \in U$. Then, by shrinking U, if needed, we can assume that

$$\phi : U \ni p \longrightarrow (f_1(p), \ldots, f_\mu(p)) \in N \subset \mathbb{C}^\mu$$

is a smooth real vector bundle on a generic *CR* submanifold N of $\mathbb{C}^\mu$, of *CR*-dimension ν_{p_0} and *CR*-codimension k_{p_0}.

In fact, we can assume that $\operatorname{Re} df_1, \ldots, \operatorname{Re} df_\mu, \operatorname{Im}(df_1), \ldots, \operatorname{Im}(df_\nu)$ are linearly independent on U. We can fix local coordinates $x_1, \ldots, x_m$ centered at p_0 with $x_1, \ldots, x_{\mu+\nu}$ equal to $\operatorname{Re} f_1, \ldots, \operatorname{Re} f_\mu, \operatorname{Im} f_1, \ldots, \operatorname{Im} f_\nu$. By the assumption, in these local coordinates $\operatorname{Im} f_{\nu+1}, \ldots, \operatorname{Im} f_\mu$ are smooth functions of $x_1, \ldots, x_{\mu+\nu}$ and this yields a parametric representation of N as a graph of $\mathbb{C}^\nu \times \mathbb{R}^{\mu-\nu}$ in $\mathbb{C}^\mu$, which is therefore locally a generic *CR*-submanifold of type $(\nu, \mu - \nu)$ of $\mathbb{C}^\mu$. The map $\phi : U \to N$ is *CR*, and therefore, the pullback of germs of continuous *CR* function on N defines germs of continuous *CR* function on M. If M has property (H), then the $\mathscr{C}^\infty$ regularity of their pullbacks implies the $\mathscr{C}^\infty$ regularity of the germs on N. Thus, N also has property (H), and, since it is embedded in $\mathbb{C}^\mu$, by [38], all *CR* functions on an open neighborhood ω_0 of 0 in N are the restriction of homomorphic functions on a full open neighborhood $\tilde{\omega}_0$ of 0 in $\mathbb{C}^\mu$, with $\omega_0 = \tilde{\omega}_0 \cap N$. Since $f_i = \phi^*(z_i)$ for the holomorphic coordinates $z_1, \ldots, z_\mu$ of $\mathbb{C}^\mu$, we obtain that all germs of *CR* functions at $p_0 \in M$ are germs of holomorphic functions of $f_1, \ldots, f_\mu$. This clearly implies the validity at p_0 of the strong unique continuation principle. The proof is complete. □

4 The kernel of the Levi form and the *(H)* property

To a finite set $Z_1, \ldots, Z_r$ of vector fields in $\mathcal{Z}(M)$, we associate the real-valued vector field

$$Y_0 = \frac{1}{2i} \sum_{j=1}^{r} [Z_j, \bar{Z}_j]. \tag{4.1}$$

Any *CR* function u on M satisfies the degenerate-Schrödinger-type equation

$$S_u = 0, \qquad \text{with} \tag{4.2}$$

$$S = -iY_0 + \tfrac{1}{2} \sum_{j=1}^{r} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) = -iY_0 + \sum_{j=1}^{2r} X_j^2, \tag{4.3}$$

where $X_j = \operatorname{Re} Z_j$, $X_{j+r} = \operatorname{Im} Z_j$ for $1 \leq j \leq r$. In fact, by (4.1), we have

$$S = \tfrac{1}{2} \sum_{i=1}^{r} \bar{Z}_j Z_j,$$

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and thus the operator S belongs to the left ideal, in the ring of scalar linear partial differential operators with complex smooth coefficients, generated by $\mathcal{Z}(M)$.

We note that S is of the second order, with a real principal part which is uniquely determined by $\tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes \bar{Z}_r \in \Gamma(H^{1,1}M)$, while a different choice of the Z_j's would yield a new Y_0', differing form Y_0 by the addition of a linear combination of $X_1, \ldots, X_{2r}$.

If we assume that $\tau \in \ker(\mathcal{L})$, then

$$\sum_{i=1}^{r} [Z_j, \bar{Z}_j] = \bar{L}_0 - L_0 \tag{4.4}$$

for some $L \in \mathcal{Z}(M)$, which is uniquely determined by τ modulo a linear combination with $\mathscr{C}^\infty$ coefficients of $Z_1, \ldots, Z_r$. Thus, the distributions of *real* vector fields

$$\begin{cases} \mathcal{Q}_1(\tau) = \langle \operatorname{Re} Z_1, \ldots, \operatorname{Re} Z_r, \operatorname{Im} Z_1, \ldots, \operatorname{Im} Z_r \rangle, \\ \mathcal{V}_1(\tau) = \mathfrak{L}(\mathcal{Q}_1(\tau)), \\ \mathcal{V}_2(\tau) = \mathfrak{L}(\mathcal{Q}_1(\tau) + \operatorname{Re} L_0), \end{cases} \tag{4.5}$$

are uniquely determined by τ and $\mathcal{Z}$. By $\mathfrak{L}(\ldots)$, we indicate the formally integrable distribution of real vector fields, which is generated by the elements of the set inside the parentheses and their iterated commutators. Note that $\mathcal{V}_1(\tau) \subseteq \mathcal{V}_2(\tau)$, and while $Y_0 = \operatorname{Im} L_0 \in \mathcal{V}_1(\tau)$, the vector field $X_0 = \operatorname{Re} L_0$ may not belong to $\mathcal{V}_1(\tau)$. We also introduce, for further reference, the distributions of *complex* vector fields

$$\begin{cases} \Theta(\tau) = \langle Z_1, \ldots, Z_r \rangle \ \text{ and } \ \Theta = \bigcup_{\tau \in [\ker \mathcal{L}]} \Theta(\tau), \\ \tilde{\Theta}(\tau) = \Theta(\tau) + \langle L_0 \rangle \ \text{ and } \ \tilde{\Theta} = \bigcup_{\tau \in [\ker \mathcal{L}]} \tilde{\Theta}(\tau) \end{cases} \tag{4.6}$$

When there is a $\tau \in [\ker \mathcal{L}](\Omega^{\text{open}})$, we utilize (4.4) to show that the real and imaginary parts of *CR* functions or distributions on $\Omega \subset M$ are solutions of a *real degenerate elliptic scalar second-order differential equation*. Indeed, if f is a *CR* function, or distribution, in Ω, then

$$L_0 f = 0, \quad Z_j f = 0 \Longrightarrow (\bar{L}_0 + L_0) f = \sum_{i=1}^{r} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) f.$$

This is a consequence of the algebraic identities

$$\tfrac{1}{2} \left\{ \sum_{i=1}^{r} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) - (\bar{L}_0 + L_0) \right\} = \sum_{i=1}^{r} \bar{Z}_j Z_j - L_0 = \sum_{i=1}^{r} Z_j \bar{Z}_j - \bar{L}_0. \tag{4.7}$$

It terms of the *real* vector fields $X_0 = \operatorname{Re} L_0$ and $X_j = \operatorname{Re} Z_j$, $X_{r+j} = \operatorname{Im} Z_j$, for $1 \leq j \leq r$, the linear partial differential operator of (4.7) is

$$P_\tau = -X_0 + \sum_{i=1}^{2r} X_j^2, \tag{4.8}$$

which has real-valued coefficients and is degenerate elliptic according to [8]. Thus, *the real and imaginary parts of a CR function, or distribution, both satisfy the homogeneous equation* $P_\tau \phi = 0$.

Actually, P_τ is independent of the choice of $Z_1, \ldots, Z_r$ in the representation of τ, as we will later show in Proposition 6.6, by representing P_τ in terms of the dd^c operator on M. We also have (see [22]):

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Lemma 4.1 *If $u \in \mathscr{O}_M(\Omega)$, then*

$$P_\tau|u| \geq 0, \quad on \ \ \Omega \cap \{u \neq 0\}. \tag{4.9}$$

Proof On a neighborhood of a point where $u \neq 0$, we can consistently define a branch of $\log u$. This still is a *CR* function, and from the previous observation, it follows that $P_\tau(\log|u|) = P_\tau(\operatorname{Re}\log u) = 0$ on $\Omega \cap \{u \neq 0\}$. Hence,

$$\begin{aligned} P_\tau|u| &= P_\tau \exp(\log|u|) = |u| \left(P_\tau(\log|u|) + \sum_{i=1}^{r} |Z_j(\log|u|)|^2 \right) \\ &= |u| \sum_{i=1}^{r} |Z_j(\log|u|)|^2 \geq 0 \end{aligned}$$

there. □

We can use the treatment of the generalized Kolmogorov equation in [25, §22.2] to slightly improve the regularity result of [2, Corollary 1.15]. Let us set

$$\mathcal{V}_2 = \mathfrak{L}\left(\bigcup\nolimits_{\tau\in[\ker\mathcal{L}]} \mathcal{V}_2(\tau)\right), \qquad \mathcal{Y} = \mathfrak{L}(\mathcal{V}_2; \mathcal{H}), \tag{4.10}$$

where we use $\mathfrak{L}(\mathcal{V}_2; \mathcal{H})$ for the $\mathcal{V}_2$-Lie module generated by $\mathcal{H}$, which consists of the linear combinations, with smooth real coefficients, of the elements of $\mathcal{H}$ and their iterated commutators with elements of $\mathcal{V}_2$:

$$\mathfrak{L}(\mathcal{V}_2; \mathcal{H}) = \mathcal{H} + [\mathcal{V}_2, \mathcal{H}] + [\mathcal{V}_2, [\mathcal{V}_2, \mathcal{H}]] + [\mathcal{V}_2, [\mathcal{V}_2, [\mathcal{V}_2, \mathcal{H}]]] + \cdots \tag{4.11}$$

Note that $\mathcal{V}_2 \subset \mathfrak{L}(\mathcal{V}_2; \mathcal{H})$ and that both $\mathcal{V}_2$ and $\mathcal{Y}$ are fine sheaves.

Theorem 4.2 *M has property (H) at all points p where $\{Y_p \mid Y \in \mathcal{Y}(M)\} = T_pM$.*

Before proving the theorem, let us introduce some notation. For $\epsilon > 0$, we denote by $\mathcal{S}_\epsilon(M)$ the set of real vector fields $Y \in \mathfrak{X}(M)$ such that for every $p \in M$, there is a neighborhood $U^{\text{open}} \Subset M$ of p, a constant $C \geq 0$, $\tau_1, \ldots, \tau_h \in [\ker\mathcal{L}](M)$ and complex vector fields $Z_1, \ldots, Z_\ell \in \mathcal{Z}(M)$ such that

$$\|Yf\|_{\epsilon-1} \leq C \left(\sum_{j=1}^{h} \|P_{\tau_j} f\|_0 + \sum_{i=1}^{\ell} \|Z_j f\|_0 + \|f\|_0 \right), \quad \forall f \in \mathscr{C}_0^\infty(U). \tag{4.12}$$

The Sobolev norms of real order (and integrability two) in (4.12) are of course computed after fixing a Riemannian metric on M. Different choices of the metric yield equivalent norms (see, for example, [2,16] for technical details). Beware that the Z_j in the right-hand side of (4.12) are not required to be related to those entering the definition of the P_{τ_j}'s. Set

$$\mathcal{S}(M) = \bigcup\nolimits_{\epsilon>0} \mathcal{S}_\epsilon(M). \tag{4.13}$$

Theorem 4.2 will follow from the inclusion $\mathcal{Y}(M) \subset \mathcal{S}(M)$.

The following Lemmas 4.3 and 4.4 were proved in [2,21].

Lemma 4.3 *If $\tau \in [\ker\mathcal{L}](M)$ and $P_\tau = -X_0 + \sum_{i=1}^{2r} X_i^2$, then $X_1, \ldots, X_{2r} \in \mathcal{S}_1(M)$, and for every $U^{\text{open}} \Subset M$, there is a constant $C > 0$ and $Z_1, \ldots, Z_\ell \in \mathcal{Z}(M)$ such that*

$$\sum_{i=1}^{2r} \|X_i f\|_0 \leq C \left(\|f\|_0 + \sum_{j=1}^{\ell} \|Z_j f\|_0 \right), \quad \forall f \in \mathscr{C}_0^\infty(U). \tag{4.14}$$

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Set $\mathcal{V}_1 = \mathfrak{L}\left(\bigcup_{\tau\in[\ker\mathcal{L}]}\mathcal{V}_1(\tau)\right)$ and

$$\mathfrak{L}(\mathcal{V}_1;\mathcal{H}) = \mathcal{H} + [\mathcal{V}_1,\mathcal{H}] + [\mathcal{V}_1,[\mathcal{V}_1,\mathcal{H}]] + [\mathcal{V}_1,[\mathcal{V}_1,[\mathcal{V}_1,\mathcal{H}]]] + \cdots .$$

Lemma 4.4 *We have the inclusion* $\mathfrak{L}(\mathcal{V}_1;\mathcal{H}) \subset \mathcal{S}$.

To prove Theorem 4.2, we add the following lemma.

Lemma 4.5 *Let* $\tau \in [\ker\mathcal{L}](M)$, *with* $P_\tau = -X_0 + \sum_{i=1}^{2r} X_i^2$. *Then,*

$$[X_0, \mathcal{S}_\epsilon(M)] \subset \mathcal{S}_{\epsilon/4}(M). \tag{4.15}$$

Proof Let $Q_\tau = \mathrm{P}_\tau + c$, for a suitable nonnegative real constant c, to be precised later. We decompose Q_τ into the sum $Q_\tau = Q'_\tau + iQ''_\tau$, where $Q'_\tau = \frac{1}{2}(Q_\tau + Q^*_\tau)$ and $Q''_\tau = \frac{1}{2i}(Q_\tau - Q^*_\tau)$ are self-adjoint. In particular, $Q^*_\tau = Q'_\tau - iQ''_\tau$. We can rewrite Q'_τ as a sum $Q'_\tau = -\sum_{j=1}^{2r} X_j^* X_j + iT + c$, for a p.d.o. T of order ≤ 1, whose principal part of order 1 is a linear combination with $\mathscr{C}^\infty$ coefficients of $X_1, \ldots, X_{2r}$. Moreover, we note that $\mathrm{P}_\tau - \mathrm{P}_\tau{}^* = Q_\tau - Q^*_\tau$. The advantage in dealing with Q_τ instead of P_τ is that, for c positive and sufficiently large,

$$(Q_\tau f|f)_0 = (Q'_\tau f|f)_0 \geq 0, \quad \forall f \in \mathscr{C}_0^\infty(U). \tag{$\star$}$$

This is the single requirement for our choice of c.

In [2], it was shown that $[X_i, \mathcal{S}_\epsilon] \subset \mathcal{S}_{\frac{\epsilon}{2}}$ for $i = 1, \ldots, 2r$ and all $\epsilon > 0$. Then, (4.15) is equivalent to the inclusion $[Q''_\tau, \mathcal{S}_\epsilon] \subset \mathcal{S}_{\frac{\epsilon}{4}}$.

Let $Y \in \mathcal{S}_\epsilon(M)$ and $U^{\text{open}} \Subset M$. We need to estimate $\|[Q''_\tau, Y]f\|_{\frac{\epsilon}{4}-1}$ for $f \in \mathscr{C}_0^\infty(U)$. Let A be any properly supported pseudo-differential operator of order $\frac{\epsilon}{2} - 1$. We have

$$\begin{aligned} i([Q''_\tau, Y]f|Af) &= ((Q'_\tau - Q^*_\tau)Yf|Af)_0 + ((Q_\tau - Q'_\tau)f|Y^*Af)_0 \\ &= (Q'_\tau Yf|Af)_0 - (Yf|Q_\tau Af)_0 + (Q_\tau f|Y^*Af)_0 - (Q'_\tau f|Y^*Af)_0. \end{aligned}$$

While estimating the summands in the last expression, we shall indicate by $C_1, C_2, \ldots$ positive constants independent of the choice of f in $\mathscr{C}_0^\infty(U)$.

Let us first consider the second and third summands. We have

$$\begin{aligned} |(Yf|Q_\tau Af)_0| &\leq \|Yf\|_{\epsilon-1}\|Q_\tau Af\|_{1-\epsilon} \leq \|Yf\|_{\epsilon-1}\left(\|AQ_\tau f\|_{1-\epsilon} + \|[A, Q_\tau]f\|_{1-\epsilon}\right) \\ &\leq C_1\|Yf\|_{\epsilon-1}\left(\|Q_\tau f\|_{-\frac{\epsilon}{2}} + \left\|\left[A, \sum_{j=1}^{2r} X_j^2\right] f\right\|_{1-\epsilon} + \|f\|_{-\frac{\epsilon}{2}}\right). \end{aligned}$$

We have

$$\left[A, \sum\nolimits_{j=1}^{2r} X_j^2\right] = -\sum\nolimits_{j=1}^{2r}\left(2[X_j, A]X_j + [X_j,[X_j, A]]\right).$$

Since $[X_j, A]$ and $[X_j,[X_j, A]]$ have order $\frac{\epsilon}{2} - 1$, and P_τ and Q_τ differ by a constant, we obtain

$$|(Yf|Q_\tau Af)_0| \leq C_2\,\|Yf\|_{\frac{\epsilon}{2}-1}\left(\|\mathrm{P}_\tau f\|_{-\frac{\epsilon}{2}} + \|f\|_{-\frac{\epsilon}{2}} + \sum\nolimits_{j=1}^{2r}\|X_j f\|_{-\frac{\epsilon}{2}}\right).$$

Analogously, for the third summand we have, since $(Y + Y^*)$ has order zero,

$$\begin{aligned} |(Q_\tau f|Y^*Af)_0| &\leq \|Q_\tau f\|_0\left(\|AY^*f\|_0 + \|[Y^*, A]f\|_0\right) \\ &\leq C_2\left(\|\mathrm{P}_\tau f\|_0 + \|f\|_0\right)\left(\|Yf\|_{\frac{\epsilon}{2}-1} + \|f\|_{\frac{\epsilon}{2}-1}\right). \end{aligned}$$

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Next we consider

$$|(Q'_\tau Yf|Af)_0| = |(Yf|Q'_\tau Af)| \le |(Yf|AQ'_\tau f)_0| + |(Yf|[Q'_\tau, A]f)_0|.$$

Let us first estimate the second summand in the last expression.

We have $Q'_\tau = \sum_{i=1}^{2r} X_i^2 + R'_0$ for a first-order p.d.o. R'_0 whose principal part is a linear combination of $X_1, \ldots, X_{2r}$. Hence,

$$[Q'_\tau, A] = [R'_0, A] + \sum \left(2[X_i, A]X_i + [X_i, [X_i, A]]\right),$$

with pseudo-differential operators $[R'_0, A]$, $[X_i, A]$, $[X_i, [X_i, A]]$ of order $\le (\frac{\epsilon}{4} - 1)$. Thus, we obtain

$$|(Yf|[Q'_\tau, A]f)_0| \le C_3 \|Yf\|_{\epsilon-1} \left(\|f\|_{-\frac{\epsilon}{4}} + \sum_{j=1}^{2r} \|X_j f\|_{-\frac{\epsilon}{4}} \right).$$

Because of $(*)$, we have the Cauchy inequality

$$|(Q'_\tau f_1|f_2)| \le \sqrt{(Q'_\tau f_1|f_1)\,(Q'_\tau f_2|f_2)}, \quad \text{for } f_1, f_2 \in \mathscr{C}_0^\infty(U).$$

Hence,

$$|(Yf|AQ'_\tau f)_0|^2 = |(Q'_\tau f|A^*Yf)_0|^2 \le (Q'_\tau A^*Yf|A^*Yf)_0 (Q'_\tau f|f)_0,$$
$$|(Q'_\tau f|Y^*Af)_0| \le (Q'_\tau Y^*Af|Y^*Af)_0 (Q'_\tau f|f)_0.$$

We have, for the second factor on the right-hand sides,

$$(Q'_\tau f|f)_0 = (Q_\tau f|f)_0 \le \|Q_\tau f\|_0 \|f\|_0 \le (\|\mathrm{P}_\tau f\|_0 + |c| \|f\|_0) \|f\|_0.$$

Let us estimate the first factors. We get

$$\begin{aligned}
(Q'_\tau A^*Yf|A^*Yf)_0 &= (Q_\tau A^*Yf|A^*Yf) \le \|Q_\tau A^*Yf\|_{-\frac{\epsilon}{2}} \|A^*Yf\|_{\frac{\epsilon}{2}} \\
&\le \|A^*Yf\|_{\frac{\epsilon}{2}} \left(\|A^*YQ_\tau f\|_{-\frac{\epsilon}{2}} + \|\,[A^*Y, Q_\tau]f\|_{-\frac{\epsilon}{2}} \right) \\
&\le C_3 \|Yf\|_{\epsilon-1} \left(\|Q_\tau f\|_0 + \|\,[A^*Y, Q_\tau]f\|_{-\frac{\epsilon}{2}} \right).
\end{aligned}$$

We need to estimate the second summand inside the parentheses in the last expression. We note that

$$[A^*Y, Q_\tau] = [A^*Y, \mathrm{P}_\tau] = -[A^*Y, X_0] + \sum_{j=1}^{2r} \left(2[A^*Y, X_j]X_j + [X_j, [A^*Y, X_j]] \right).$$

Since the operators $[A^*Y, X_0]$, $[A^*Y, X_j]$, $[X_j, [A^*Y, X_j]]$ have order $\frac{\epsilon}{2}$, we obtain

$$\|\,[A^*Y, Q_\tau]f\|_{-\frac{\epsilon}{2}} \le C_4 \left(\|f\|_0 + \sum_{j=1}^{2r} \|X_j f\|_0 \right).$$

Finally,

$$\begin{aligned}
(Q'_\tau Y^*Af|Y^*Af)_0 &= (Q_\tau Y^*Af|Y^*Af) \\
&\le \|Y^*Af\|_{\frac{\epsilon}{2}} \left(\|Y^*AQ_\tau f\|_{-\frac{\epsilon}{2}} + \|\,[Q_\tau, Y^*A]f\|_{-\frac{\epsilon}{2}} \right) \\
&\le C_5 \|Y^*Af\|_{\frac{\epsilon}{2}} \left(\|Q_\tau f\|_0 + \|\,[Q_\tau, Y^*A]f\|_{-\frac{\epsilon}{2}} \right).
\end{aligned}$$

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Since

$$[Y^*A, Q_\tau] = [Y^*A, \mathrm{P}_\tau] = -[Y^*A, X_0] + \sum_{j=1}^{2r} \left(2[Y^*A, X_j]X_j + [X_j, [Y^*A, X_j]]\right)$$

and the operators $[Y^*A, X_0]$, $[Y^*A, X_j]$, $[X_j, [Y^*A, X_j]]$ have order $\frac{\epsilon}{2}$, we obtain that

$$\| [Q_\tau, Y^*A]f\|_{-\frac{\epsilon}{2}} \leq C_6 \left(\|f\|_0 + \sum_{j=1}^{2r} \|X_j f\|_0 \right).$$

Moreover,

$$Y^*A = -AY + (Y + Y^*)A + [A, Y],$$

with $\{(Y + Y^*)A+[A, Y]\}$ of order $\leq (\frac{\epsilon}{2} - 1)$, because $Y + Y^*$ has order 0. Hence,

$$\|Y^*Af\|_{\frac{\epsilon}{2}} \leq C_7 \left(\|Yf\|_{\epsilon-1} + \|f\|_0\right).$$

Putting all these inequalities together, we conclude that

$$|([X_0, Y]f|Af)_0| \leq C_8 \left(\|f\|_0^2 + \|Yf\|_{\epsilon-1}^2 + \|\mathrm{P}_\tau f\|_0^2 + \sum_{j=1}^{2r} \|X_j f\|_0^2 \right), \quad \forall f \in \mathscr{C}_0^\infty(U).$$

By taking $A = \Lambda_{\frac{\epsilon}{2}-1}[X_0, Y]$ for an elliptic properly supported pseudo-differential operator $\Lambda_{\frac{\epsilon}{2}-1}$ of order $\frac{\epsilon}{2} - 1$, we deduce that

$$\|[X_0, Y]f\|_{\frac{\epsilon}{4}-1} \leq C_9 \left(\|f\|_0 + \|Yf\|_{\epsilon-1} + \|\mathrm{P}_\tau f\|_0 + \sum_{i=1}^{2r} \|X_i f\|_0 \right)$$

and therefore, since $X_1, \ldots, X_{2r} \in \mathcal{S}_1(M)$ and $Y \in \mathcal{S}_\epsilon(M)$, that $[X_0, Y] \in \mathcal{S}_{\frac{\epsilon}{4}}$. □

Corollary 4.6 *We have*

$$\mathfrak{L}(\mathcal{V}_2; \mathcal{S}) \subset \mathcal{S}. \tag{4.16}$$

□

Proof of Theorem 4.2 By the assumption, $\{Y_q \mid Y \in \mathcal{S}(M)\} = T_qM$ for all q in an open neighborhood of p in M. Thus, there are $p \in U^{\text{open}} \Subset M$, $\tau_1, \ldots, \tau_h \in [\ker\mathcal{L}](M)$, $Z_1, \ldots, Z_\ell \in \mathcal{Z}(M)$ and $C > 0$ such that

$$\|f\|_\epsilon \leq C \left(\|f\|_0 + \sum_{j=1}^{h} \|P_{\tau_j} f\|_0 + \sum_{i=1}^{\ell} \|Z_i f\|_0 \right), \quad \forall f \in \mathscr{C}_0^\infty(U). \tag{4.17}$$

Let $\mathrm{P}_{\tau_j} = -X_{0,j} + \sum_{s=1}^{2r_j} X_{s,j}^2$, with $Z_{s,j} = X_{s,j} + iX_{s+r_j,j} \in \mathcal{Z}(M)$ for $1 \leq s \leq r_j$, and let $Z_{0,j}$ be the vector field in $\mathcal{Z}(M)$ with $\mathrm{Re}\, Z_{0,j} = X_{0,j}$. If A is a properly supported pseudo-differential operator, then

$$[\mathrm{P}_{\tau_j}, A] = -[X_{0,j}, A] + \sum_{s=1}^{2r_j} \left(2X_{s,j}, [X_{s,j}, A] + [[X_{s,j}, A], X_{s,j}]\right).$$

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If A has order δ and is zero outside a compact subset K of U, and χ is a smooth function with compact support which equals one neighborhood of K, then we obtain

$$\begin{aligned}
\| \mathrm{P}_{\tau_j} A(\chi f)\|_0 &\leq \|A(\chi\, \mathrm{P}_{\tau_j} f)\|_0 + \| [\mathrm{P}_{\tau_j}, A](\chi f)\|_0 \\
&\leq C' \left(\|\chi\, \mathrm{P}_{\tau_j} f\|_\delta + \|\chi f\|_\delta + \sum_{s=1}^{2r_j} \|X_s[X_s, A](\chi f)\|_0 \right) \\
&\leq C'' \left(\|\chi\, \mathrm{P}_{\tau_j} f\|_\delta + \|\chi f\|_\delta + \sum_{s=0}^{r_j} \|Z_{s,j}[X_s, A](\chi f)\|_0 \right) \\
&\leq C''' \left(\|\chi\, \mathrm{P}_{\tau_j} f\|_\delta + \|\chi f\|_\delta + \sum_{s=0}^{r_j} \|\chi Z_{s,j} f\|_\delta \right), \quad \forall f \in \mathscr{C}^\infty(U),
\end{aligned}$$

for suitable positive constants C', C'', C''', uniform with respect to f. By using similar argument to estimate $\|Z_i A f\|_0$, we obtain that

$$\|A(\chi f)\|_\epsilon \leq \text{const} \left(\|\chi f\|_\delta + \sum_{j=1}^{h} \|\chi P_{\tau_j} f\|_\delta + \sum_{i=1}^{\ell} \|\chi Z_i f\|_0 \right), \quad \forall f \in \mathscr{C}^\infty(U).$$

This shows that for any pair of functions $\chi_1, \chi_2 \in \mathscr{C}_0^\infty(U)$ with $\operatorname{supp}(\chi_1) \subset \{\chi_2 > 0\}$, we obtain the estimate

$$\|\chi_1 f\|_{\epsilon+\delta} \leq \text{const} \left(\|\chi_2 f\|_\delta + \sum_{j=1}^{h} \|\chi_2 P_{\tau_j} f\|_\delta + \sum_{i=1}^{\ell} \|\chi_2 Z_i f\|_0 \right), \quad \forall f \in \mathscr{C}^\infty(U),$$

for some constant const $=$ const$(\chi_1, \chi_2) \geq 0$. By [15], this inequality is valid for all $f \in W^{\delta,2}_{\text{loc}}(U)$ with $\mathrm{P}_{\tau_j} f, Z_i f \in W^{\delta,2}_{\text{loc}}(U)$, where $W^{\delta,2}_{\text{loc}}(U)$ is the space of distributions ϕ in U such that, for all $\chi \in \mathscr{C}_0^\infty(U)$, the product $\chi \cdot \phi$ belongs to the Sobolev space of order δ and integrability two. This implies in particular that any *CR* distribution which is in $W^{\delta,2}_{\text{loc}}(U)$ belongs in fact to $W^{\delta+\epsilon,2}_{\text{loc}}(U)$, and this implies property (H). □

Let us consider the case where $\mathfrak{L}(\mathcal{V}_2; \mathcal{H})$ does not contain all smooth real vector fields. In this case, we have a propagation phenomenon along the leaves of $\mathcal{V}_2$. Let $\tau \in [\ker \mathcal{L}](M)$, and $X_0, Y_0, X_1, \ldots, X_{2r}$ the vector fields introduced above for a given representation of $\tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes \bar{Z}_r$. As we already noticed, while $Y_0 = \operatorname{Im} \sum [Z_i, \bar{Z}_i]$ belongs to the Lie subalgebra of $\mathfrak{X}(M)$ generated by $X_1, \ldots, X_{2r}$, the real part X_0 of $L_0 = X_0 + iY_0 \in \mathcal{Z}(M)$ may not belong to $\mathcal{V}_1(\tau)$. Thus, the following result improves [22, Theorem 5.2], where only the smaller distribution $\mathcal{V}_1(\tau)$ was involved.

Theorem 4.7 *Let $\Omega^{\text{open}} \subset M$ and assume that $\mathcal{V}_2$ has constant rank in Ω. If $f \in \mathscr{O}_M(\Omega)$ and $|f|$ attains a maximum at a point p_0 of Ω, then f is constant on the leaf through p_0 of $\mathcal{V}_2$ in Ω.*

Proof On the integral manifold N of $\mathcal{V}_2$ through p_0 in Ω, we can consider the $\mathcal{Z}'$-structure defined by the span of the restrictions to N of the elements of $\hat{\Theta}$. Indeed, the *CR* functions on Ω restrict to *CR* functions for $\mathcal{Z}'$ on the leaf N. By Corollary 4.6 and Theorem 4.2, the $\mathcal{Z}'$-manifold N has property (H), and therefore, the statement is a consequence of Proposition 3.1. □

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5 Malgrange's theorem and some applications

In this section, we state the obvious generalization of Malgrange's vanishing theorem and its corollary on the extension of *CR* functions under momentum conditions, slightly generalizing results of [9,30,31] to the case where M has property (SH). In this section, we require that M is a *CR* manifold.

We recall that the tangential Cauchy–Riemann complex can be defined as the quotient of the de Rham complex on the powers of the ideal sheaf (for this presentation, we refer to [19]): since $d\mathscr{I} \subset \mathscr{I}$, we have $d\mathscr{I}^a \subset \mathscr{I}^a$ for all nonnegative integers a and the tangential *CR*-complex $(\mathscr{Q}^{a,*}, \bar{\partial}_M)$ on a-forms is defined by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}^{a+1} & \longrightarrow & \mathscr{I}^a & \longrightarrow & \mathscr{Q}^a & \longrightarrow & 0 \\ & & d\big\downarrow & & d\big\downarrow & & \bar{\partial}_M\big\downarrow & & \\ 0 & \longrightarrow & \mathscr{I}^{a+1} & \longrightarrow & \mathscr{I}^a & \longrightarrow & \mathscr{Q}^a & \longrightarrow & 0, \end{array} \tag{5.1}$$

where $\mathscr{Q}^a$ is the quotient $\mathscr{I}^a/\mathscr{I}^{a+1}$. In turn, $\bar{\partial}_M$ is a degree 1 derivation for a $\mathbb{Z}$-grading $\mathscr{Q}^a = \bigoplus_{q=0}^n \mathscr{Q}^{a,q}$, where the elements of $\mathscr{Q}^{a,q}$ are equivalence classes of forms having representatives in $\mathscr{I}^a \cap \mathscr{A}^{\mathbb{C}}_{a+q}$.

We denote by $\mathscr{E}$ the sheaf of germs of smooth complex-valued functions on M. The $\mathscr{Q}^{a,q}$ are all locally free sheaves of $\mathscr{E}$-modules, and therefore, we can form the corresponding sheaves and cosheaves of functions and distributions. We will consider the tangential Cauchy–Riemann complexes $(\mathscr{D}^{a,*}, \bar{\partial}_M)$ on smooth forms with compact support, $(\mathscr{E}^{a,*}, \bar{\partial}_M) = (\mathscr{Q}^{a,*}, \bar{\partial}_M)$ on smooth forms with closed support, $(\mathscr{D}'^{a,*}, \bar{\partial}_M)$ on form distributions, $(\mathscr{E}'^{a,*}, \bar{\partial}_M)$ on form distributions with compact support. We use the notation $H^q(\mathscr{F}^{a,*}(\Omega), \bar{\partial}_M)$ for the cohomology group in degree q on $\Omega^{\text{open}} \subset M$, for $\mathscr{F}$ equal to either one of $\mathscr{E}, \mathscr{D}, \mathscr{D}', \mathscr{E}'$.

Proposition 5.1 *If M has property (SH), and either M is compact or has property (WUC), then $\bar{\partial}_M : \mathscr{E}'^{a,0}(M) \longrightarrow \mathscr{E}'^{a,1}(M)$ and $\bar{\partial}_M : \mathscr{D}^{a,0}(M) \longrightarrow \mathscr{D}^{a,1}(M)$ have closed range for all integers $a = 0, \ldots, m$.*

Proof We can assume that M is connected. It is convenient to fix a Riemannian metric on M, and smooth Hermitian products on the complex linear bundles $Q^{a,q}M$ corresponding to the sheaves $\mathscr{Q}^{a,q}$, to define L^2 and Sobolev norms, by using the associated smooth regular Borel measure.

By property (SH), we have a subelliptic estimate: for every $K \Subset M$, we can find constants $C_K \geq 0,\ c_K > 0,\ \epsilon_K > 0$ such that

$$\|\bar{\partial}_M u\|_0^2 + C_K\|u\|_0^2 \geq c_K\|u\|_{\epsilon_K}^2, \quad \forall u \in \mathscr{D}^{a,0}(K). \tag{5.2}$$

In a standard way, we deduce from (5.2) that

$$u \in \mathscr{D}'^{a,0}(M),\ \ \bar{\partial}_M u \in [\mathrm{W}^r_{\text{loc}}]^{a,1}(M) \Longrightarrow u|_{\mathring{K}} \in [\mathrm{W}^{r+\epsilon_K}_{\text{loc}}]^{a,1}(\mathring{K}), \ \ \forall K \Subset M, \tag{5.3}$$

and that for all $K \Subset M$ and real r, there are constants $C_{r,K} \geq 0$, $c_{r.K} > 0$ such that

$$\begin{aligned} &\|\bar{\partial}_M u\|_r^2 + C_{r,K}\|u\|_r^2 \geq c_{r,K}\|u\|_{r+\epsilon_K}^2, \\ &\qquad \forall u \in \{u \in \mathscr{E}'^{a,0}(M) \mid \bar{\partial}_M u \in [\mathrm{W}^r]^{a,1}(M),\ \operatorname{supp}(u) \subset K\}. \end{aligned} \tag{5.4}$$

This suffices to obtain the thesis when M is compact.

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Let us consider the case where M is connected and non-compact. Let $\{u_\nu\}$ be a sequence in $\mathscr{E}'^{a,0}(M)$ such that all $\bar\partial_M u_\nu$ have support in a fixed compact subset K of M and there is $r \in \mathbb{R}$ such that $\{\bar\partial_M u_\nu\} \subset [\mathrm{W}^r](M)$, $\mathrm{supp}(\bar\partial_M u_\nu) \subset K$ for all ν and $\bar\partial_M u_\nu \to f$ in $[\mathrm{W}^r]^{a,1}(M)$. We can assume that $M\backslash K$ has no compact connected component. Then, since M has property (WUC), it follows that $\mathrm{supp}(u_\nu) \subset K$ for all ν, because the $u_\nu|_{M\backslash K}$ define elements of $\mathscr{O}_M(M\backslash K)$ which vanish on a non-empty open subset of each connected component of $M\backslash K$, and thus on $M\backslash K$. Moreover, this also implies that (5.4) holds with $C_{r,K} = 0$. Then, $\{u_\nu\}$ is uniformly bounded in $[\mathrm{W}^{r+\epsilon}]^{a,0}(M)$ and hence contains a subsequence which weakly converges to a solution $u \in [\mathrm{W}^{r+\epsilon}]^{a,0}(M)$ of $\bar\partial_M u = f$.

The closedness of the image of $\bar\partial_M$ in $\mathscr{D}^{a,1}(M)$ follows from the already proved result for $\mathscr{E}'^{a,1}(M)$ and the hypoellipticity of $\bar\partial_M$ on $(a, 0)$-forms. □

We remind that if M is embedded and has property (H), or is (abstract and) essentially pseudo-concave, then it has property (WUC).

As in [9], one obtains

Proposition 5.2 *Assume that M is a connected non-compact CR manifold of CR-dimension n which has properties (SH) and (WUC). Then, $H^n(\mathscr{E}^{a,*}(M), \bar\partial_M)$ and $H^n(\mathscr{D}'^{a,*}(M), \bar\partial_M)$ are 0 for all $a = 0, \ldots, m$.*

Proof By Proposition 5.1, the sequences

$$\begin{array}{l} 0 \longrightarrow \mathscr{D}^{a,0}(M) \xrightarrow{\bar\partial_M} \mathscr{D}^{a,1}(M), \\ 0 \longrightarrow \mathscr{E}'^{a,0}(M) \xrightarrow{\bar\partial_M} \mathscr{E}'^{a,1}(M) \end{array}$$

are exact and all maps have closed range.

Assume that M is oriented. Then, we can define duality pairings between $\mathscr{D}^{a,q}(M)$ and $\mathscr{D}'^{n+k-a,n-q}(M)$ and between $\mathscr{E}'^{a,q}(M)$ and $\mathscr{E}^{n+k-a,n-q}(M)$, extending

$$\langle [\alpha],\ [\beta]\rangle = \int_M \alpha \wedge \beta,$$

where $\alpha \in \mathscr{A}_{a+q}(M) \cap \mathscr{I}^a(M)$ has compact support and is a representative of $[\alpha] \in \mathscr{D}^{a,q}(M)$ and $\beta \in \mathscr{A}_{m-a-q}(M) \cap \mathscr{I}^{n+k-a}(M)$ a representative of $[\beta] \in \mathscr{E}^{n+k-a,n-q}(M)$. Then, by duality (see, for example, [43]) we obtain exact sequences

$$\begin{array}{l} 0 \longleftarrow \mathscr{D}'^{n+k-a,n}(M) \xleftarrow{\bar\partial_M} \mathscr{D}'^{n+k-a,n-1}(M), \\ 0 \longleftarrow \mathscr{E}^{n+k-a,n}(M) \xleftarrow{\bar\partial_M} \mathscr{E}^{n+k-a,n-1}(M), \end{array}$$

proving the statement in the case where M is orientable.

If M is not orientable, then we can take its oriented double covering $\pi : \tilde M \to M$, which is a *CR*-bundle with the total space $\tilde M$ being a *CR* manifold of the same *CR* dimension and codimension. From the exact sequences

$$\begin{array}{l} 0 \longleftarrow \mathscr{D}'^{n+k-a,n}(\tilde M) \xleftarrow{\bar\partial_{\tilde M}} \mathscr{D}'^{n+k-a,n-1}(\tilde M), \\ 0 \longleftarrow \mathscr{E}^{n+k-a,n}(\tilde M) \xleftarrow{\bar\partial_{\tilde M}} \mathscr{E}^{n+k-a,n-1}(\tilde M), \end{array}$$

we deduce that statement for the non-orientable M by averaging on the fibers. □

We also obtain the analogue of the Hartogs-type theorem in [30].

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Proposition 5.3 *Let $\Omega^{\text{open}} \Subset M$ be relatively compact, orientable, and with a piecewise smooth boundary $\partial\Omega$. If u_0 is the restriction to $\partial\Omega$ of an $(a,0)$-form $\tilde{u}_0$ of class $\mathscr{C}^2$ on M, with $\bar\partial\tilde{u}_0$ vanishing to the second order on $\partial\Omega$, and*

$$\int_{\partial\Omega} u_0 \wedge \phi = 0, \quad \forall \phi \in \ker(\bar\partial_M : \mathscr{E}^{n+k-a,n-1}(M') \to \mathscr{E}^{n+k-a,n}(M')),$$

then there is $u \in \mathscr{Q}^{a,0}(\Omega) \cap \mathscr{C}^1(\bar\Omega)$ with $\bar\partial_M u = 0$ on Ω and $u = u_0$ on $\partial\Omega$.

Proof We restrain for simplicity to the case $a = 0$. The general case can be discussed in an analogous way. If M is not orientable, then the inverse image of Ω in the double covering $\pi : \tilde{M} \to M$ consists of two disjoint open subsets, both *CR*-diffeomorphic to Ω. Thus, we can and will assume that M is orientable.

Let E be a discrete set that intersects each relatively compact connected component of $M \setminus \bar\Omega$ in a single point and $M' = M \setminus E$. Note that M' has been chosen in such a way that no connected component of $M' \setminus \Omega$ is compact.

Extending $\bar\partial_M \tilde{u}_0$ by 0 outside of Ω, we define a $\bar\partial_M$-closed element f of $\mathscr{E}'^{0,1}(M')$, with support contained in $\bar\Omega$. The map $\bar\partial_M : \mathscr{E}'^{0,0}(M') \to \mathscr{E}'^{0,1}(M')$ has a closed image by Proposition 5.1. Hence, to get existence of a solution $v \in \mathscr{E}'^{0,0}(M')$ to $\bar\partial_M v = f$, it suffices to prove that f is orthogonal to the kernel of $\bar\partial_M : \mathscr{E}^{n+k,n-1}(M') \to \mathscr{E}^{n+k,n}(M')$. This is the case because

$$\int_{M'} f \wedge \phi = \int_{\Omega} (\bar\partial_M \tilde{u}_0) \wedge \phi = \int_{\Omega} (du_0) \wedge \phi = \int_{\partial\Omega} u_0 \phi - \int_{\Omega} u_0 d\phi$$

for all $\phi \in \mathscr{E}^{n+k,n-1}(M') = \mathscr{A}^{\mathbb{C}}_{m-1}(M') \cap \mathscr{I}^{n+k}(M')$, and the last summand in the last term vanishes when $d\phi = \bar\partial_M \phi = 0$. A $v \in \mathscr{E}'^{0,0}(M')$ satisfying $\bar\partial_M v = f$ defines a *CR* function on $M' \setminus \bar\Omega$ that vanishes on some open subset of each connected component of $M' \setminus \bar\Omega$. Thus, for (*WUC*) and the regularity (5.3), which are consequences of (*SH*), the solution v is $\mathscr{C}^1$ and has support in $\bar\Omega$. In particular, it vanishes on $\partial\Omega$ and therefore $u = \tilde{u}_0 - v$ satisfies the thesis. □

Remark 5.4 An analogue of this *momentum* theorem for functions on one complex variable states that a function u_0, defined and continuous on the boundary of a rectifiable Jordan curve $\mathbf{c}$, is the boundary value of a holomorphic function on its enclosed domain if and only if $\int_{\mathbf{c}} u_0(z) p(z) \mathrm{d}z = 0$ for all holomorphic polynomials $p(z) \in \mathbb{C}[z]$.

6 Hopf lemma and some consequences

In complex analysis, properties of domains are often expressed in terms of the indices of inertia of the complex Hessian of its exhausting function. Trying to mimic this approach in the case of an (abstract) *CR* manifold M, we are confronted with the fact that pluri-harmonicity and pluri-subharmonicity are well defined only for sections of a suitable vector bundle $\mathcal{T}$ (see [6,32,42]), which can be characterized in terms of 1-jets when M is embedded. We will avoid here this complication, by defining the complex Hessian $\mathrm{dd}^c\rho$ as an affine subspace of Hermitian-symmetric forms on $T^{1,0}M$. As we did for the Levi form, we shall consider its extension to $H^{1,1}M$, and note that it is an invariantly defined function on $[\ker \mathcal{L}]$. Since a *CR* function canonically determines a section of $\mathcal{T}$, we will succeed in making a very implicit use of the sheaf $\mathcal{T}$ of *transversal* 1*-jets* of [32].

In this section, we shall consider the P_τ of Sect. 4, exhibit their relationship to the complex Hessian, and, by using the fact that they are degenerate elliptic operators, draw, from

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their boundary behavior at non-characteristic points, consequences on the properties of *CR* functions on M.

6.1 Hopf lemma

The classical Hopf Lemma also holds for degenerate elliptic operators. We have, from [14, Lemma 4.3]:

Proposition 6.1 *Let Ω be a domain in M and $u \in \mathscr{C}^1(\bar{\Omega}, \mathbb{R})$ satisfy $P_\tau u \geq 0$ on Ω, for the operator $P_\tau = -X_0 + \sum_{i=1}^{2r} X_j^2$ of (4.8). Assume that $p_0 \in \partial\Omega$ is a $\mathscr{C}^2$ non-characteristic point of $\partial\Omega$ for P_τ and that there is an open neighborhood U of p_0 in M such that*

$$u(p) < u(p_0), \quad \forall p \in \Omega \cap U. \tag{6.1}$$

Then,

$$\mathrm{d}u(p_0) \neq 0. \tag{6.2}$$

The condition that $\partial\Omega$ is non-characteristic at p_0 for P_τ means that, if Ω is represented by $\rho < 0$ near p_0, with $\rho \in \mathscr{C}^2$ and $d\rho(p_0) \neq 0$, then $\sum_{i=1}^{2r} |X_j \rho(p_0)|^2 > 0$.

Remark 6.2 If M has property (H), then (6.1) is automatically satisfied if $u = |f|$, for $f \in \mathscr{O}_M(\Omega) \cap \mathscr{C}^0(\bar{\Omega})$, when $u(p_0)$ is a local maximum and f is not constant on a half-neighborhood of p_0 in Ω.

Corollary 6.3 *Let Ω be an open subset of M and $f \in \mathscr{O}_M(\Omega) \cap \mathscr{C}^2(\bar{\Omega})$, $p_0 \in \partial\Omega$ with*

$$|f(p)| < |f(p_0)|, \quad \forall p \in \Omega. \tag{6.3}$$

If $\partial\Omega$ is smooth and Θ-non-characteristic at p_0, then $d|f|(p_0) \neq 0$.

Proof By the assumption that $\partial\Omega$ is Θ-non-characteristic at p_0, the function $u = |f|$ is, for some open neighborhood U of p_0 in M, a solution of $P_\tau u \geq 0$ on $\Omega \cap U$, for an operator P_τ of the form (4.8), obtained from a section τ of $[\ker \mathcal{L}](U)$, and for which $\partial\Omega$ is non-characteristic at p_0.

6.2 The complex Hessian and the operators dd^c, P_τ

Denote by $\mathscr{A}_1$ the sheaf of germs of smooth real-valued 1-forms on M, by $\mathscr{J}_1$ its subsheaf of germs of sections of $H^0 M$ and by $\mathscr{I}_1$ the degree 1-homogeneous elements of the ideal sheaf of M. The elements of $\mathscr{I}_1$ are the germs of smooth complex-valued 1-forms vanishing on $T^{0,1}M$.

Let Ω be an open subset of M.

Lemma 6.4 *If $\alpha \in \mathscr{A}_1(\Omega)$, then we can find $\xi \in \mathscr{A}_1(\Omega)$ such that $\alpha + i\xi \in \mathscr{I}_1(\Omega)$.*

Proof The sequence

$$0 \longrightarrow \mathscr{J}_1 \xrightarrow{\ i\cdot\ } \mathscr{I}_1 \xrightarrow{\ \mathrm{Re}\ } \mathscr{A}_1 \longrightarrow 0$$

of fine sheaves is exact and thus splits on every open subset Ω of M. □

If ρ si a smooth, real-valued function on $\Omega^{\text{open}} \subset M$, by Lemma 6.4, we can find $\xi \in \mathscr{A}_1(\Omega)$ such that $d\rho + i\xi \in \mathscr{I}_1(\Omega)$. If $Z \in \mathcal{Z}(M)$, then $d\rho(Z) = -i\xi(Z)$, $d\rho(\bar{Z}) = i\xi(\bar{Z})$, and we obtain

$$Z\bar{Z}\rho = Z(d\rho(\bar{Z})) = iZ[\xi(\bar{Z})], \quad \bar{Z}Z\rho = \bar{Z}(d\rho(Z)) = -i\bar{Z}[\xi(Z)].$$

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Hence,

$$[Z\bar{Z} + \bar{Z}Z]\rho = i(Z[\xi(\bar{Z})] - \bar{Z}[\xi(Z)]) = i d\xi(Z, \bar{Z}) + i\xi([Z, \bar{Z}]).$$

We note that ξ is only defined modulo the addition of a smooth section $\eta \in \mathscr{J}_1(\Omega)$ of the characteristic bundle H^0M, for which

$$i\mathrm{d}\eta(Z, \bar{Z}) = -i\eta([Z, \bar{Z}]) = \mathcal{L}_\eta(Z, \bar{Z}), \quad \forall Z \in \mathcal{Z}(M).$$

Definition 6.1 The *complex Hessian of* ρ *at* p_0 is the affine subspace

$$\mathrm{Hess}^{1,1}_{p_0}(\rho) = \{i\mathrm{d}\xi_{p_0} \mid \xi \in \mathscr{A}_1(\Omega), \quad \mathrm{d}\rho + i\xi \in \mathscr{I}_1(\Omega)\}. \tag{6.4}$$

Fix a point p_0 where $\mathrm{d}\rho(p_0) \notin H^0_{p_0}M$, i.e., $\bar{\partial}_M\rho(p_0) \neq 0$, and consider the level set $N = \{p \in U \mid \rho(p) = \rho(p_0)\}$, in a neighborhood U of p_0 in Ω where $\bar{\partial}_M\rho(p)$ is never 0. Then N is a smooth real hypersurface and a *CR*-submanifold, of type $(n-1, k+1)$.

Lemma 6.5 *For every* $p \in N$, *we have*

$$\{\xi|_N \mid \xi \in T^*_pM \mid \mathrm{d}\rho(p) + i\xi \in T^{*1,0}_pM\} \subset H^0_pN. \tag{6.5}$$

The left-hand side of (6.5) *is an affine hypersurface in* H^0_pN, *with associated vector space* H^0_pM.

Proof When $Z \in \mathcal{Z}(U)$ is tangent to N, we obtain $0 = \mathrm{d}\rho(Z_p) = -i\xi(Z_p)$ and hence $\xi(\mathrm{Re}\, Z_p) = \xi(\mathrm{Im}\, Z_p) = 0$ because ξ is real. This gives $\xi|_N \in H^0_pN$. The last statement is a consequence of the previous discussion of the complex Hessian. □

Definition 6.2 If ρ is a smooth real-valued function defined on a neighborhood Ω of a point $p_0 \in N$ and $\xi \in \mathscr{A}_1(\Omega)$ is such that $dr + i\xi \in \mathscr{I}_1(\Omega)$, then we set

$$\mathrm{dd}^c\rho_{p_0}(\tau) := \tfrac{i}{2}d\xi(\tau), \quad \forall\tau \in [\ker\mathcal{L}]_{p_0}. \tag{6.6}$$

Let $\tau = Z_1 \otimes \bar{Z}_1 + \cdots + Z_r \otimes Z_r \in [\ker\mathcal{L}](\Omega)$, with $\bar{L}_0 - L_0 = \sum_{i=1}^r [Z_j, \bar{Z}_j]$ and $L_0, Z_1, \ldots, Z_r \in \mathcal{Z}(\Omega)$. Let $\xi \in \mathscr{A}_1(\Omega)$ be such that $d\rho + i\xi \in \mathscr{I}_1(\Omega)$. Then,

$$\begin{aligned}
\mathrm{d}\rho(Z_j) + i\xi(Z_j) = 0 &\Longrightarrow \mathrm{d}\rho(\bar{Z}_j) - i\xi(\bar{Z}_j) = 0\\
\Rightarrow i\mathrm{d}\xi(Z_j, \bar{Z}_j) &= i\left(Z_j\xi(\bar{Z}_j) - \bar{Z}_j\xi(Z_j) - \xi([Z_j, \bar{Z}_j])\right)\\
&= Z_j d\rho(\bar{Z}_j) + \bar{Z}_j \mathrm{d}\rho(Z_j) - i\xi([Z_j, \bar{Z}_j])\\
&= (Z_j\bar{Z}_j + \bar{Z}_jZ_j)\rho - i\xi([Z_j, \bar{Z}_j]).
\end{aligned}$$

We recall that $\sum_{i=1}^r [Z_j, \bar{Z}_j] = \bar{L}_0 - L_0 = 2i\,\mathrm{Im}\, L_0$, with $L_0 \in \mathcal{Z}(\Omega)$. We have

$$(d\rho + i\xi)(L_0) = 0 \Longrightarrow d\rho(\mathrm{Re}\, L_0) = \xi(\mathrm{Im}\, L_0),\ d\rho(\mathrm{Im}\, L_0) = -\xi(\mathrm{Re}\, L_0)$$

and therefore

$$\begin{aligned}
2\mathrm{dd}^c\rho(\tau) &= \sum_{i=1}^r i\mathrm{d}\xi(Z_j, \bar{Z}_j) = \sum_{i=1}^r (Z_j\bar{Z}_j + \bar{Z}_jZ_j)\rho - i\xi\left(\sum_{i=1}^r [Z_j, \bar{Z}_j]\right)\\
&= \sum_{i=1}^r (Z_j\bar{Z}_j + \bar{Z}_jZ_j)\rho + 2\xi(\mathrm{Im}\, L_0)\\
&= \sum_{i=1}^r (Z_j\bar{Z}_j + \bar{Z}_jZ_j)\rho - 2\mathrm{d}\rho(\mathrm{Re}\, L_0) = 2P_\tau\rho.
\end{aligned}$$

As a consequence, we obtain:

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Proposition 6.6 *If* ρ *is a real-valued smooth function on the open set* Ω *of* M *and* $\tau \in [\ker\mathcal{L}](\Omega)$*, then*

$$\mathrm{dd}^c\rho(\tau) = P_\tau\rho \quad on\ \ \Omega. \tag{6.7}$$

Corollary 6.7 *The operator* P_τ *only depends on the section* τ *of* $[\ker\mathcal{L}]$ *and is independent of the choice of the vector fields* $Z_1, \ldots, Z_r \in \mathcal{Z}$ *in* (4.7).

Corollary 6.8 *Let* $\Omega^{\text{open}} \subset M$*. If* $\rho_1, \rho_2 \in \mathscr{C}^\infty(\Omega)$ *are real-valued functions which agree to the second order at* $p_0 \in \Omega$*, then*

$$\mathrm{dd}^c\rho_1(\tau_0) = \mathrm{dd}^c\rho_2(\tau_0), \quad \forall \tau_0 \in [\ker\mathcal{L}]_{p_0}. \tag{6.8}$$

In particular, $\mathrm{dd}^c\rho$ is well defined and continuous on the fibers of $[\ker\mathcal{L}]$ for functions ρ which are of class $\mathscr{C}^2$.

Remark 6.9 There is a subtle distinction between $\mathrm{dd}^c\rho$, which is the (1, 1)-part of an alternate form of degree two, and $\mathrm{Hess}^{1,1}(\rho)$, which is the (1, 1)-part of a symmetric bilinear form. In fact, we multiplied by $(i/2)$ the differential in (6.6) and identified the two concepts, as multiplication by i interchanges skew-Hermitian and Hermitian-symmetric matrices.

We have:

Lemma 6.10 *Let* ρ *be a smooth real-valued function defined on a neighborhood of* $p_0 \in M$*, with* $d\rho(p_0) \neq 0$ *and* $N = \{p \mid \rho(p) = \rho(p_0)\}$*. The following statements:*

(i) *every* $h \in \mathrm{Hess}^{1,1}_{p_0}(\rho)$ *has a nonzero positive index of inertia;*
(ii) *there exists* $\tau \in [\ker\mathcal{L}]_{p_0} \cap H^{1,1}_{p_0}N$ *such that* $\mathrm{dd}^c\rho_{p_0}(\tau) > 0$*;*
(iii) *the restriction of every* $h \in \mathrm{Hess}^{1,1}_{p_0}(\rho)$ *to* $T^{0,1}_{p_0}N$ *has a nonzero positive index of inertia;*

are related by

$$(ii) \iff (iii) \implies (i).$$

Set $U^- = \{p \in U \mid \rho(p) < \rho(p_0)\}$.

Definition 6.3 We set

$$H^0_{M,p_0}(U^-) = \bigcup\nolimits_{\lambda>0}\{\xi|_N \mid \xi \in T^*_{p_0}\partial U^- \mid \lambda d\rho(p_0) + i\xi \in T^{*1,0}_{p}M\}. \tag{6.9}$$

This is an open half-space in H^0_pN. Note that $H^0_{M,p_0}(U^-)$ does not depend on the choice of the defining function ρ.

6.3 Real parts of *CR* functions

In this subsection, we try to better explain the meaning of dd^c by defining a differential operator d^c_λ which associates with a real smooth function a real one form. Its definition depends on the choice of a *CR-gauge* λ on M, but $[\mathrm{d}^c_\lambda]$'s corresponding to different choices of λ differ by a differential operator with values in $\mathscr{J}$, so that all the $d\mathrm{d}^c_\lambda$ agree with our dd^c on $[\ker\mathcal{L}]$.

A *CR* function (or distribution) f is a solution to the equation $du \in \mathscr{I}_1$. In this subsection, we study the characterization of the real parts of *CR* functions.

Lemma 6.11 *Let* Ω *be open in* M*. If* M *is minimal, then a real-valued* $f \in \mathscr{O}_M(\Omega)$ *is locally constant.*

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Proof A real-valued $f \in \mathscr{O}_M(\Omega)$ satisfies $Xf = 0$ for all $X \in \Gamma(M, HM)$ and therefore is constant on the Sussmann leaves of $\Gamma(M, HM)$. □

We have an exact sequence of fine sheaves (the superscript $\mathbb{C}$ means forms with complex-valued coefficients)

$$0 \longrightarrow \mathscr{J}_1^{\mathbb{C}} \xrightarrow{\alpha\to(\alpha,-\alpha)} \mathscr{I}_1 \oplus \bar{\mathscr{I}}_1 \xrightarrow{(\alpha,\beta)\to\alpha+\beta} \mathscr{A}_1^{\mathbb{C}} \longrightarrow 0. \tag{6.10}$$

In [32, §2A], the notion of a *balanced real CR-gauge* was introduced. It was shown that it is possible to define a smooth morphism

$$\lambda : \mathbb{C}TM \longrightarrow T^{*1,0}M \tag{6.11}$$

of $\mathbb{C}$-linear bundles which defines a special splitting of (6.10): with

$$\bar{\lambda} : \mathbb{C}TM \ni \alpha \longrightarrow \overline{\lambda(\bar{\alpha})} \in T^{*0,1}M, \tag{6.12}$$

we have

$$\alpha = \lambda(\alpha) + \bar{\lambda}(\alpha), \quad \forall \alpha \in \mathscr{A}_1^{\mathbb{C}}, \tag{6.13}$$

$$\lambda(\alpha) = \bar{\lambda}(\alpha) = \tfrac{1}{2}\alpha, \ \ \forall \alpha \in \mathscr{J}_1^{\mathbb{C}}. \tag{6.14}$$

Note that

$$\bar{\lambda}(\mathscr{I}_1) \subset \mathscr{J}_1, \ \ \lambda(\bar{\mathscr{I}}_1) \subset \mathscr{J}_1, \ \lambda \circ \bar{\lambda} = \bar{\lambda} \circ \lambda.$$

Explicitly, the splitting of (6.10) is provided by

$$0 \longrightarrow \mathscr{A}_1^{\mathbb{C}} \xrightarrow{\alpha\to(\lambda(\alpha),\,\bar{\lambda}(\alpha))} \mathscr{I}_1 \oplus \bar{\mathscr{I}}_1 \xrightarrow{(\alpha,\beta)\to\,\bar{\lambda}(\alpha)-\lambda(\beta)} \mathscr{J}_1^{\mathbb{C}} \longrightarrow 0.$$

Furthermore, we get

$$\mathscr{A}_1^{\mathbb{C}} = \ker \bar{\lambda} \oplus \mathscr{J}_1^{\mathbb{C}} \oplus \ker \lambda, \quad \mathscr{I}_1 = \ker \bar{\lambda} \oplus \mathscr{J}_1^{\mathbb{C}}, \quad \bar{\mathscr{I}}_1 = \mathscr{J}_1^{\mathbb{C}} \oplus \ker \lambda,$$
$$\lambda(\alpha) = \alpha, \ \forall \alpha \in \ker \bar{\lambda}, \quad \bar{\lambda}(\alpha) = \alpha, \ \forall \alpha \in \ker \lambda, \quad \lambda(\alpha) = \bar{\lambda}(\alpha) = \tfrac{1}{2}\alpha, \ \forall \alpha \in \mathscr{J}_1^{\mathbb{C}}.$$

Let us introduce the first-order linear partial differential operator

$$\mathrm{d}^c_\lambda f = \tfrac{1}{i}(\lambda(\mathrm{d}f) - \bar{\lambda}(\mathrm{d}f)), \quad \forall f \in \mathscr{C}^\infty(M). \tag{6.15}$$

We note that d^c_λ is *real*: this means that $\mathrm{d}^c_\lambda u$ is a real-valued form when u is a real-valued function. Indeed, for a real-valued $u \in \mathscr{C}^\infty(M)$, we have

$$\mathrm{d}^c_\lambda u = 2\,\mathrm{Im}\,\lambda(\mathrm{d}u) = -2\,\mathrm{Im}(\bar{\lambda}(\mathrm{d}u)).$$

Lemma 6.12 *We have* $\mathrm{d}\mathrm{d}^c_\lambda u \in \mathscr{J}_2$ *for every* $u \in \mathscr{A}_0$.

Proof For any germ of real-valued smooth function u, the differential $\mathrm{d}\mathrm{d}^c_\lambda u$ is real and we have

$$i\,\mathrm{d}\mathrm{d}^c_\lambda u = d(\lambda(\mathrm{d}u) - \bar{\lambda}(\mathrm{d}u)) = d(2\lambda(\mathrm{d}u) - \mathrm{d}u) = 2d\,\lambda(\mathrm{d}u) \in \mathscr{I}_2,$$
$$= d(\mathrm{d}u - 2\,\bar{\lambda}(\mathrm{d}u) = -2d\,\bar{\lambda}(\mathrm{d}u) \in \bar{\mathscr{I}}_2,$$

so that $\mathrm{d}\mathrm{d}^c_\lambda u \in \mathscr{I}_2 \cap \bar{\mathscr{I}}_2 \cap \mathscr{A}_2 = \mathscr{J}_2$. □

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Proposition 6.13 *Let Ω be a simply connected open set in M. A necessary and sufficient condition for a real-valued $u \in \mathscr{C}^\infty(\Omega)$ to be the real part of an $f \in \mathscr{O}_M(\Omega)$ is that there exists a section $\xi \in \mathscr{J}_1(\Omega)$ such that*

$$d[\mathrm{d}^c_\lambda u + \xi] = 0 \quad \text{on } \Omega. \tag{6.16}$$

If M is minimal, then ξ is uniquely determined.

Proof Assume that (6.16) is satisfied by some $\xi \in \mathscr{J}_1(\Omega)$. Then, $\mathrm{d}^c_\lambda u + \xi = dv$ for some real-valued $v \in \mathscr{C}^\infty(\Omega)$, and with $f = u + iv$, we obtain

$$\begin{aligned}\lambda(\mathrm{d}u) - \bar\lambda(\mathrm{d}u) = i[\lambda(\mathrm{d}v) + \bar\lambda(\mathrm{d}v) - \xi] \Longrightarrow \bar\lambda(\mathrm{d}f) = \lambda(\mathrm{d}u - i\mathrm{d}v) - i\xi \in \mathscr{J}_1^{\mathbb{C}}(\Omega)\\ \Longrightarrow \mathrm{d}f \in \mathscr{I}_1(\Omega) \Longleftrightarrow f \in \mathscr{O}_M(\Omega).\end{aligned}$$

Assume vice versa that $f = u + iv \in \mathscr{O}_M(\Omega)$, with u and v real-valued smooth functions. Write $\mathrm{d}f = \mathrm{d}u + i\mathrm{d}v = \alpha + \zeta$, with $\alpha \in \mathscr{I}_1(\Omega)$, $\zeta \in \mathscr{J}_1^{\mathbb{C}}(\Omega)$, and $\bar\lambda(\alpha) = 0$. From

$$\bar\lambda(\mathrm{d}u) + i\,\bar\lambda(\mathrm{d}v) = \tfrac{1}{2}\zeta \Longrightarrow \lambda(\mathrm{d}u) - i\lambda(\mathrm{d}v) = \tfrac{1}{2}\bar\zeta,$$

we obtain

$$i\mathrm{d}^c_\lambda u = \lambda(\mathrm{d}u) - \bar\lambda(\mathrm{d}u) = i\lambda(\mathrm{d}v) + \tfrac{1}{2}\bar\zeta + i\bar\lambda(\mathrm{d}v) = i\,\mathrm{d}v - \tfrac{1}{2}(\zeta - \bar\zeta)$$

This is (6.16) with $\xi = (i/2)(\zeta - \bar\zeta)$.

To complete the proof, we note that if $\xi \in \mathscr{J}_1(\Omega)$ and $\mathrm{d}\xi = 0$, then $\xi = \mathrm{d}\phi$ for some real-valued function $\phi \in \mathscr{C}^\infty(\Omega)$. If $\xi_{p_0} \neq 0$ for some $p_0 \in \Omega$, then $\{\phi(p) = \phi(p_0)\}$ defines a germ of smooth hypersurface through p_0 which is tangent at each point to the distribution HM, contradicting the minimality assumption. □

The *Aeppli complex* for *pluri-harmonic* functions on the *CR* manifold M is

$$\begin{aligned}0 \longrightarrow \mathscr{A}_0 \oplus \mathscr{J}_1 \xrightarrow{(u,\xi)\to \mathrm{d}\mathrm{d}^c_\lambda u + d\xi} \mathscr{J}_2 \xrightarrow{d} \mathscr{J}_3 \xrightarrow{d}\\ \cdots \xrightarrow{d} \mathscr{J}_{m-1} \xrightarrow{d} \mathscr{J}_m \longrightarrow 0.\end{aligned}$$

We note that $\mathscr{J}_1 = 0$ if M is a complex manifold (we reduce to the classical case) and $\mathscr{J}_q = \mathscr{A}_q$ for $q > 0$ if M is totally real. In general, the terms of degree $\geq k+2$ make a subcomplex of the de Rham complex.

6.4 Peak points of *CR* functions and pseudo-convexity at the boundary

A non-characteristic point of the boundary of a domain, where the modulus a *CR* function attains a local maximum, is *pseudo-convex*, in a sense that will be explained below.

Lemma 6.14 *Let $\Omega^{\text{open}} \subset M$ and assume there is $f \in \mathscr{O}_M(\Omega) \cap \mathscr{C}^2(\bar\Omega)$ such that $|f|$ attains a local isolated maximum value at $p_0 \in \partial\Omega$. If $\partial\Omega$ is smooth, non-characteristic at p_0, and moreover, $d|f(p_0)| \neq 0$, then there is a nonzero $\xi \in H^0_{M,p_0}(\Omega)$ with $\mathcal{L}^{\partial\Omega}_\xi \geq 0$.*

Proof Let U be an open neighborhood of p_0 in M, and $\rho \in \mathscr{C}^\infty(U, \mathbb{R})$ a defining function for Ω near p_0, with $U^- = \Omega \cap U = \{p \in U \mid \rho(p) < 0\}$, and $d\rho(p) \neq 0$ for all $p \in U$.

We can assume that $f(p_0) = |f(p_0)| > 0$ and exploit the fact that the restriction of $u = \operatorname{Re} f$ to $\partial\Omega$ takes a maximum value at p_0. Since $d_{\partial\Omega}u(p_0) = 0$, the real Hessian of u on $\partial\Omega$ is well defined at p_0, with

$$\operatorname{hess}(u)(X_{p_0}, Y_{p_0}) = (XYu)(p_0), \quad \forall X, Y \in \mathfrak{X}(\partial\Omega),$$

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and $\operatorname{hess}(u)(p_0) \leq 0$ by the assumption that the restriction of u to $\partial\Omega$ has a local maximum at p_0. In particular, it follows that

$$(Z\bar{Z}u)(p_0) = (\bar{Z}Zu)(p_0) \leq 0, \quad \forall Z \in \mathcal{Z}(\partial\Omega).$$

Let $v = \operatorname{Im} f$. Then, $\mathrm{d}f = \mathrm{d}u + i\mathrm{d}v$, and the condition that $d_{\partial\Omega}u(p_0) = 0$ implies that $(Zv)(p_0) = 0$ for all $Z \in \mathcal{Z}(\partial\Omega)$ and thus $\xi = \mathrm{d}v(p_0) \in H^0\partial\Omega$. Moreover,

$$(Zu)(p) = -i(Zv)(p), \quad (\bar{Z}u)(p) = i(\bar{Z}v)(p), \quad \forall Z \in \mathcal{Z}(\partial\Omega), \quad \forall p \in \partial\Omega. \tag{6.17}$$

Hence,

$$2Z\bar{Z}u(p_0) = (Z\bar{Z} + \bar{Z}Z)u(p_0) = i(Z\bar{Z} - \bar{Z}Z)v(p_0) = i\xi(p_0)([Z, \bar{Z}])$$

and thus the condition on the *real* Hessian of u implies that $\mathcal{L}_\xi^{\partial\Omega} \geq 0$. We note that $\mathrm{d}u(p_0)$ is different from 0 and proportional to $d\rho(p_0)$. Indeed, near p_0 we have

$$|f| = u\sqrt{1 + (v^2/u^2)} \simeq u\left(1 + \tfrac{1}{2}(v^2/u^2)\right) = u + 0(2),$$

since $v(p_0) = 0$. Thus, $d|f|(p_0) = du(p_0) \neq 0$.

By the assumption that $\partial\Omega$ is non-characteristic at p_0, we have that $\mathrm{d}u(p_0)$ is nonzero and equal to $\lambda d\rho(p_0)$ for some $\lambda > 0$: therefore, $\xi = \mathrm{d}v(p_0) \in H^0_{M,p_0}(\Omega)$, and this proves our claim. □

Proposition 6.15 *Let Ω be an open subset of M, and assume that there is a CR function $f \in \mathcal{O}_M(\Omega) \cap \mathcal{C}^2(\bar{\Omega})$ and a point $p_0 \in \partial\Omega$ such that:*

$$|f(p_0)| > |f(p)|, \quad \forall p \in \Omega, \qquad (a)$$
$$\partial\Omega \ \textit{is}\ \Theta\textit{-non-characteristic at}\ p_0. \qquad (b)$$

Then, we can find $0 \neq \xi \in H^0_{M,p_0}(\Omega)$ with $\mathcal{L}_\xi^{\partial\Omega} \geq 0$.

[For the meaning of *non-characteristic*, see (2.6).]

Proof To apply Lemma 6.14, we need to check that $d|f|(p_0) \neq 0$. By the assumption that $\partial\Omega$ is Θ-non-characteristic at p_0, there is an open neighborhood U of p_0 in M and $\tau \in [\ker \mathcal{L}](U)$ such that $\partial\Omega$ is non-characteristic for P_τ at p_0. Since $P_\tau|f| \geq 0$, by the Hopf lemma, $d|f|(p_0)| \neq 0$, and therefore, $\mathrm{d}u(p_0)$ is a positive multiple of $\mathrm{d}\rho(p_0)$. Then, $\xi = d\operatorname{Im} f(p_0) \in H^0_{M,p_0}(\Omega)$ and we obtain the statement. □

For characteristic peak points in the boundary of Ω, we have:

Lemma 6.16 *Let Ω be an open subset of M, and assume that there is a CR function $f = u + iv \in \mathcal{O}_M(\Omega) \cap \mathcal{C}^2(\bar{\Omega})$, with u and v real valued, and $p_0 \in \partial\Omega$ such that:*

(a) $v(p_0) = 0, \ \ \mathrm{d}u(p_0) \in H^0_{p_0}N, \ \ u(p_0) > u(p), \ \ \forall p \in \Omega,$
(b) $0 \neq \xi = \mathrm{d}v(p_0).$

Then, $\xi \in H^0_{p_0}M$ and $\mathcal{L}_\xi \geq 0$.

Proof Set $\eta = \mathrm{d}u(p_0)$. Then, $\xi = \mathrm{d}v(p_0) \in H^0_{p_0}M$, because $df(p_0) = \eta + i\xi$ is zero on $\mathcal{Z}(M)$, and hence ξ, vanishing on $\mathcal{Z}(M)$ and being real, belongs to $H^0_{p_0}M$. The conclusion follows by the argument of Lemma 6.14, taking into account that this time all vectors in $T^{0,1}_{p_0}M$ are tangent to $\partial\Omega$ and that (6.17) is valid for $Z \in \mathcal{Z}(M)$ at all points where f is defined and $\mathcal{C}^1$. □

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Proposition 6.15 suggest to introduce some notions of convexity/concavity for boundary points of a domain in M. Let Ω be a domain in M, $p_0 \in \partial\Omega$ a smooth point of $\partial\Omega$, and ρ a defining function for Ω near p_0.

Definition 6.4 We say that Ω is at p_0

- strongly 1-concave if there is $\tau \in [\ker \mathcal{L}] \cap H^{1,1}_{p_0}\partial\Omega$ such that $dd^c\rho_{p_0}(\tau) < 0$;
- strongly 1-convex if there is $\tau \in [\ker \mathcal{L}] \cap H^{1,1}_{p_0}\partial\Omega$ such that $dd^c\rho_{p_0}(\tau) > 0$.

Points where the boundary is strictly 1-concave cannot be peak points for the modulus of *CR* functions.

Proposition 6.17 *Assume that M has property (H). Let Ω be a relatively compact open domain in M and $N \subset \partial\Omega$ a smooth part of $\partial\Omega$ consisting of points where $\partial\Omega$ is smooth, Θ-non-characteristic and strongly 1-concave. Then,*

$$|u(p)| < \sup_{q \in \partial\Omega \setminus N} |u(q)|, \quad \forall p \in \Omega \cup N, \tag{6.18}$$

for every non-constant $u \in \mathscr{O}_M(\Omega) \cap \mathscr{C}^2(\bar{\Omega})$.

Proof Since M has property (H), by Proposition 3.1 we have $|f(p)| < \max_{\partial\Omega}|f|$, for all $p \in \Omega$ and all non-constant $f \in \mathscr{O}_M(\Omega)$. The statement then follows from Proposition 6.15, because $|f|$ cannot have a maximum on N. □

6.5 1-convexity/concavity at the boundary and the vector-valued Levi form

Let $\Omega^{\text{open}} \subset M$ have piecewise smooth boundary and denote by N the *CR* submanifold of type $(n-1, k+1)$ of M consisting of the smooth non-characteristic points of $\partial\Omega$. The quotient $(TN \cap HM)/HN \subset TN/HN$ is a real line bundle on N.

The partial complex structure $J_M : HM \to HM$ restricts to the partial complex structure on HN, and the tangent vectors v in $(HM \cap TN) \setminus HN$ are characterized by the fact that $J_M(v) \notin TN$. Fix a point $p_0 \in N$ and a defining function ρ of Ω on a neighborhood U of p_0 in N, so that $0 \neq d\rho(p_0)$ is an outer conormal to Ω at p_0. The elements $\xi_0 \in H^0_{M,p_0}\Omega$ are defined, modulo multiplication by a positive scalar, by the condition that $d\rho(p_0) + i\xi_0 \in T^{*1,0}_{p_0}M$. Since $v + iJ_Mv \in T^{0,1}_{p_0}M$, we have

$$\begin{aligned} 0 = \langle (d\rho(p_0) + i\xi_0), (v + iJ_Mv)\rangle = i\langle d\rho(p_0), J_Mv\rangle + i\langle \xi_0, v\rangle - \langle \xi_0, J_Mv\rangle \\ \Longrightarrow \langle \xi_0, J_Mv\rangle = 0, \quad \langle \xi_0, v\rangle = -\langle d\rho(p_0), J_Mv\rangle. \end{aligned}$$

The restriction $\xi_0|_N$ is an element of $H^0_{p_0}N$, with $\langle \xi_0, v\rangle \neq 0$ if p_0 is non-characteristic. Therefore, we have shown:

Lemma 6.18 *Let $v = J_M w_{p_0}$ for an outer normal vector in $p_0 \in N \subset \partial\Omega$ to Ω, with $v \in H_{p_0}M$. If $[v]$ belongs to the range of the vector-valued Levi form $\mathcal{L}^N$, then Ω is strongly 1-convex at p_0.*

Vice versa, if Ω is strongly 1-convex at p_0, then $[v]$ belongs to the range of the vector-valued Levi form.

As usual, we used $[v]$ to denote the image of v in the quotient TN/HN.

A similar statement holds for strong-1-concavity.

Revised Proof

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7 Convex cones of Hermitian forms

In a *CR* manifold of arbitrary *CR*-codimension, the *scalar* Levi forms associate with each point a linear space of Hermitian-symmetric quadratic forms. Different notions of pseudo-concavity in [2,21,22] originate from the observation that the polar of a subspace of forms with positive Witt index contains positive definite tensors. As shown in Sect. 6, the analogue on a *CR* manifold *M* of the complex Hessian of a smooth real function yields an *affine* subspace of Hermitian-symmetric forms. Therefore, it was natural to associate with a non-characteristic point of the boundary of a domain in *M* an open half-space of Hermitian-symmetric forms. In this section, we describe some properties of duals of convex cones of Hermitian-symmetric forms, to better understand the notions of pseudo-concavity that are relevant to discuss the extensions of some facts of analysis in several complex variables to the case of *CR* manifolds.

7.1 Convexity in Euclidean spaces

(cf. [28,39]) Let us recall some notions of convex analysis. Let V be an n-dimensional Euclidean real vector space. A non-empty subset C of V is a convex cone (with vertex 0) if

$$v_1, v_2 \in C, \quad t_1 > 0, \quad t_2 \geq 0 \Longrightarrow t_1 v_1 + t_2 v_2 \in C.$$

The *dual cone* of C is

$$C^* = \{\xi \in V \mid (v|\xi) \geq 0, \ \forall v \in C\}.$$

By the Hahn-Banach theorem, one easily obtains:

Lemma 7.1 *For any non-empty convex cone C in V, we have $C^{**} = \bar{C}$.*

Proof If $w \notin \bar{C}$, then, by the Hahn–Banach separation theorem we can find $\xi \in V$ such that $\inf_{v \in C}(v|\xi) > (w|\xi)$. Since C is a cone, this implies that $(v|\xi) \geq 0$ for all $v \in C$, i.e., $\xi \in C^*$, and then $(w|\xi) < 0$ shows that $w \notin C^{**}$. This proves that $C^{**} \subset \bar{C}$. The opposite inclusion trivially follows from the definition. □

We call *salient* a convex cone which does not contain any real line: this means that if $0 \neq v \in C$, then $-v \notin C$. By Lemma 7.1, we have

Lemma 7.2 *A non-empty closed convex cone C is salient if and only if C^* has a non-empty interior.*

Proof If C contains a vector subspace W, then C^* is contained in the orthogonal $W^* = W^{\perp}$, which is a proper linear subspace of V and therefore C^* has an empty interior. Vice versa, if C^* has an empty interior, then its linear span U is a proper linear subspace of V and $W = U^* = U^{\perp}$ is a linear subspace of V of positive dimension contained in $\bar{C} = C$. □

Lemma 7.3 *Let C be a salient closed convex cone and W a linear subspace of V with $W \cap C = \{0\}$. Then, we can find a hyperplane W' with $W \subset W'$ and $W' \cap C = \{0\}$.*

Proof For each $v \in V$, we write $v = v' + v''$ for its decomposition into the sum of its component $v' \in W$ and its component $v'' \in W^{\perp}$. We claim that the orthogonal projection C'' of C into $W^{\perp}$ is still a closed salient cone. Closedness follows by the fact that $\|v'\| \leq C\|v''\|$ for some $C > 0$ for all $v \in C$. To prove that C'' is salient, we argue by contradiction. Assume that C'' contains two opposite nonzero vectors $\pm w''$. Then, there are $w'_+, w'_- \in W$ such that

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$w'_+ + w''$, $w'_- - w'' \in C$. The sum of these two nonzero vectors is nonzero by the assumption that C is salient, but

$$0 \neq (w'_+ + w'') + (w'_- - w'') = (w'_+ + w'_-) \in C \cap W$$

yields a contradiction.

By Lemma 7.2, the interior of the dual cone of C'' in $W^\perp$ is non-empty. This means that there is a $\xi \in W^\perp$ with $(v''|\xi) > 0$ for all $v'' \in C''$ and hence $(\xi|v) > 0$ for all $v \in C$, since $C \subset C'' + W$. □

A closed convex cone C with $\mathring{C}^* = \emptyset$ contains a linear subspace E_C of V and is called a *wedge* with *edge* E_C. Lemma 7.3 generalizes to the case of closed wedges.

Lemma 7.4 *If C is a closed wedge with edge E_C and W a linear subspace of V with $W \cap C \subset E_C$, then there is a hyperplane W' with $W \subset W'$ and $W' \cap C = E_C$.*

Proof C contains all affine subspaces $v + E_C$, for $v \in C$. If $\pi : V \to V/E_C$ is the projection into the quotient, then $\pi(C)$ is a pointed cone and $\pi(W) \cap \pi(C) = \{0\}$. By Lemma 7.3, there is a hyperplane H in V/W with $\pi(W) \subset H$ and $H \cap \pi(C) = \{0\}$. Then, $W' = \pi^{-1}(H)$ is a hyperplane in V which contains W and has $C \cap W' = E_C$. □

7.2 Convex cones in the space of Hermitian-symmetric forms

Let us denote by $\mathcal{P}_n$ the n^2-dimensional real vector space of $n \times n$ Hermitian-symmetric forms on $\mathbb{C}^n$. It is a Euclidean space with the scalar product $(h_1|h_2) = \sum_{i,j=1}^n h_1(e_i, e_j) h_2(e_j, e_i)$, where $e_1, \ldots, e_n$ is any basis of $\mathbb{C}^n$. It will be convenient, however, to avoid fixing any specific scalar product on $\mathcal{P}_n$ and formulate our statements in a more invariant way, involving the dual $\mathcal{P}'_n$ of $\mathcal{P}_n$. It consists of the Hermitian-symmetric covariant tensors that we write as sums $\pm v_1 \otimes \bar{v}_1 \pm \cdots \pm v_r \otimes \bar{v}_r$, for $v_1, \ldots, v_r \in \mathbb{C}^n$. The identification of $\mathcal{P}_n$ with $\mathcal{P}'_n$ provided by the choice of a scalar product on $\mathcal{P}_n$ allows us to apply the previous results of convex analysis in this slightly different formulation.

A matrix corresponding to a Hermitian-symmetric form h has real eigenvalues. The number of positive (resp. negative) eigenvalues is called its *positive* (resp. *negative*) *index of inertia*, the smallest of the two its *Witt index*, the sum of the two its *rank*.

Set $\bar{\mathcal{P}}_n^+ = \{h \geq 0\}$ and $\mathcal{P}_n^+ = \bar{\mathcal{P}}_n^+ \setminus \{0\}$, $\mathring{\mathcal{P}}_n^+ = \{h > 0\}$, and, likewise, $\bar{\mathcal{P}}_n^- = \{h \leq 0\}$ and $\mathcal{P}_n^- = \bar{\mathcal{P}}_n^- \setminus \{0\}$, $\mathring{\mathcal{P}}_n^- = \{h < 0\}$. We shall use the simple

Lemma 7.5

$$[\bar{\mathcal{P}}_n^+]^* = [\mathring{\mathcal{P}}_n^+]^* = \bigcup\nolimits_r \{v_1 \otimes \bar{v}_1 + \cdots + v_r \otimes \bar{v}_r \mid v_1, \ldots, v_r \in \mathbb{C}^n\},$$

$$\{\psi \in \mathcal{P}'_n \mid \psi(h) > 0,\ \forall h \in \mathcal{P}_n^+\} = \{v_1 \otimes \bar{v}_1 + \cdots + v_n \otimes \bar{v}_n \mid \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n\},$$

$$\{\psi \in \mathcal{P}'_n \mid \psi(h) > 0,\ \forall h \in \mathring{\mathcal{P}}_n^+\} = \{v_1 \otimes \bar{v}_1 + \cdots + v_r \otimes \bar{v}_r \mid r > 0,\ \langle v_1, \ldots, v_n \rangle = \mathbb{C}^n\}.$$

Proposition 7.6 *Let $\mathcal{W}$ be a convex closed cone, with vertex in 0, in $\mathcal{P}_n$. Assume that every nonzero element of $\mathcal{W}$ has a nonzero positive index of inertia. Then, there is a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that*

$$\sum_{i=1}^n h(e_i, e_i) \geq 0, \quad \forall h \in \mathcal{W}. \tag{7.1}$$

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Proof Both $\mathcal{W}$ and $\mathcal{W}^+ = \{h_1 + h_2 \mid h_1 \in \mathcal{W},\ h_2 \geq 0\}$ are proper closed convex cones in $\mathcal{P}_n$. Since $\mathcal{W}^+$ does not contain any negative semidefinite nonzero form, its edge has empty intersection with $\mathcal{P}_n^+ = \{h \geq 0, \quad h \neq 0\}$. By Lemma 7.4, we can find a $\psi \in \mathcal{P}_n'$ such that

$$\psi(h) \geq 0,\ \forall h \in \mathcal{W}^+ \quad \text{and} \quad \mathcal{W}^+ \cap \{\psi = 0\} = E_{\mathcal{W}^+}.$$

In particular, $\psi(h) > 0$ for $h \in \mathcal{P}_n^+$ and hence, by Lemma 7.5, ψ is of the form $\psi(h) = \sum_{i=1}^{n} h(e_i, e_i)$ for a basis $e_1, \ldots, e_n$ of V. □

We obtain, as a corollary, the result of [21, Lemma 2.4], which motivated the definition of *essential pseudo-concavity*.

Corollary 7.7 *If $\mathcal{W}$ is a linear subspace of $\mathcal{P}_n$ such that each nonzero element of $\mathcal{W}$ has a positive Witt index, then there exists a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that*

$$\sum\nolimits_{i=1}^{n} h(e_i, e_i) = 0, \forall h \in \mathcal{W}.$$

Proposition 7.8 *Let $\mathcal{W}$ be a relatively open convex cone with vertex at 0 of $\mathcal{P}_n$, and such that every element h of $\mathcal{W}$ has a nonzero positive index of inertia. Then, the elements of $\bar{\mathcal{P}}_n^-$ which are contained in $\overline{\mathcal{W}}$ are all degenerate.*

All the elements of maximal rank in $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$ have the same kernel, which has a positive dimension r and a basis $e_1, \ldots, e_r$ such that

$$\sum_{i=1}^{r} h(e_i, e_i) > 0, \quad \forall h \in \mathcal{W}. \tag{7.2}$$

Proof Let $\mathring{\mathcal{P}}_n^- = \{h \in \mathcal{P}_n \mid h < 0\}$. Then, $\mathcal{W}$ and $\mathring{\mathcal{P}}_n^-$ are disjoint relatively open convex cones of $\mathcal{P}_n$ with vertex in 0 and therefore (see, for example, [44, Thorem 2.7]) are separated by a hyperplane, defined by a linear functional ψ, which is positive on $\mathcal{W}$ and negative on $\mathring{\mathcal{P}}_n^-$. Being negative on $\mathring{\mathcal{P}}_n^-$, by Lemma 7.5, ψ has the form (7.2). This implies that all elements of $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$ are degenerate. Since $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$ is a cone, all its elements of maximal rank belong to its relative interior and have the same kernel, say $U \subset \mathbb{C}^n$, whose positive dimension we denote by r. In fact, for a pair of negative semidefinite forms h_1, h_2, we have $\ker(h_1 + h_2) = \ker h_1 \cap \ker h_2$. The statement follows by applying Proposition 7.6 to $\overline{\mathcal{W}}|_U = \{h|_U \mid h \in \overline{\mathcal{W}}\}$, which is a closed cone in $\mathcal{P}_r$ in which all nonzero elements have a nonzero positive index of inertia. In fact, if there is a nonzero $h \in \overline{\mathcal{W}}$ whose restriction to U is seminegative, and h_0 is an element of maximal rank in the cone $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$, then, for $C > 0$ and large, $h + Ch_0$ would be a negative definite element in $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$. □

Proposition 7.9 *Let $\mathcal{W}$ be a cone in $\mathcal{P}_n$, with the property that all its elements of maximal rank have a nonzero positive index of inertia. Then, all forms in $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$ are degenerate; those of maximal rank have all the same kernel, of dimension $r > 0$, which contains a basis $e_1, \ldots, e_r$ such that*

$$\sum_{i=1}^{r} h(e_i, e_i) \geq 0, \quad \forall h \in \mathcal{W}. \tag{7.3}$$

Proof Let $\mathring{\mathcal{P}}_n^+ = \{h \in \mathcal{P}_n \mid h > 0\}$. Then, $\mathcal{W} + \mathring{\mathcal{P}}_n^+$ is an open cone in $\mathcal{P}_n$ such that all its elements have a nonzero positive index of inertia.

Since $\overline{\mathcal{W} + \mathring{\mathcal{P}}_n^+} \cap \bar{\mathcal{P}}_n^- = (\overline{\mathcal{W}} + \bar{\mathcal{P}}_n^+) \cap \bar{\mathcal{P}}_n^- = \overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$, we know from Proposition 7.8 that all elements of maximal rank in $\overline{\mathcal{W}} \cap \bar{\mathcal{P}}_n^-$ have the same kernel U, which is a subspace of $\mathbb{C}^n$

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of positive dimension r and contains a basis $e_1, \dots, e_r$ for which

$$\sum_{i=1}^{r} h(v_i, v_i) > 0, \quad \forall h \in \mathcal{W} + \mathring{\mathcal{P}}_n^+.$$

This implies (7.3). □

Analogous results can be given to characterize cones of Hermitian forms having some given amount of positive (or negative) eigenvalues. In this case, we need to consider the behavior of the restriction of forms to subspaces of $\mathbb{C}^n$. We use the notation $\mathcal{G}r_h(\mathbb{C}^n)$ for the Grassmannian of complex linear h-planes of $\mathbb{C}^n$.

Proposition 7.10 *Let $\mathcal{W}$ be a proper closed convex cone in $\mathcal{P}_n$, with vertex in 0 and q an integer with $0 < q \le n$. Assume that every nonzero form in $\mathcal{W}$ has a positive index of inertia $\ge q$. Then, for every $V \in \mathcal{G}r_{n-q+1}(\mathbb{C}^n)$, we can find a basis $v_1, \dots, v_{n-q+1}$ of V such that*

$$\sum_{i=1}^{n-q+1} h(v_i, v_i) \ge 0. \tag{7.4}$$

Proof It suffices to apply Proposition 7.6 to the restrictions to $V \in \mathcal{G}r_{n-q+1}(\mathbb{C}^n)$ of the forms in $\mathcal{W}$. By the assumption, $h|_V$ has a nonzero positive index of inertia for all $h \in \mathcal{W} \setminus \{0\}$. □

An analogous statement to Proposition 7.8 can be formulated for relatively open convex cones of Hermitian forms with positive index of inertia $\ge q$.

Proposition 7.11 *Let $\mathcal{W}$ be a relatively open convex cone in $\mathcal{P}_n$ and assume that each h in $\mathcal{W}$ has a positive index of inertia $\ge q$, for an integer $0 < q \le n$. Then, for every $V \in \mathcal{G}r_{n-q+1}(\mathbb{C}^n)$, we can find an integer $r_V > 0$ and linearly independent $v_1, \dots, v_{r_V} \in V$ such that*

$$\sum_{i=1}^{r_V} h(v_i, v_i) > 0, \quad \forall h \in \mathcal{W}. \tag{7.5}$$

Proof For every $V \in \mathcal{G}r_{n-q+1}(\mathbb{C}^n)$, the set $\mathcal{W}_V = \{h|_V \mid h \in \mathcal{W}\}$ is a relatively open convex cone of $\mathcal{P}_{n-q+1}$ such that all of its elements $h|_V$ have a nonzero positive index of inertia. The thesis follows by applying Proposition 7.8 to $\mathcal{W}|_V$. □

Proposition 7.12 *Let W be a convex cone in $\mathcal{P}_n$ such that the elements of maximal rank of W have a positive index of inertia $\ge q$ (q is an integer with $0 < q \le n$). Then, for every $V \in \mathcal{G}r_{n-q+1}(\mathbb{C}^n)$ we can find an integer $r_V > 0$ and linearly independent $v_1, \dots, v_{r_V} \in V$ such that*

$$\sum\nolimits_{i=1}^{r_V} h(v_i, v_i) \ge 0, \quad \forall h \in \mathcal{W}. \tag{7.6}$$

Proof It suffices to apply Proposition 7.11 to $\mathcal{W} + \mathring{\mathcal{P}}_n^+$ and note that (7.5) for all $h \in \mathcal{W} + \mathring{\mathcal{P}}_n^+$ implies (7.6) for all $h \in \mathcal{W}$.

Remark 7.13 The positive integer r_V of Propositions 7.11, 7.12 is the dimension of the kernel of any form of maximal rank in $\overline{\mathcal{W}_V} \cap \bar{\mathcal{P}}_{n-q+1}^-$.

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8 Notions of pseudo-concavity

In [23], it was proved that the Poincaré lemma for the tangential Cauchy–Riemann complex of locally *CR*-embeddable *CR* manifolds fails in the degrees corresponding to the indices of inertia of its scalar Levi forms of maximal rank. On the other hand, in [18] it was shown that the Lefschetz hyperplane section theorem for q-dimensional complex submanifolds generalizes to weakly q-pseudo-concave *CR* submanifolds of complex projective spaces.

This suggests to seek for suitable weakening of the pseudo-concavity conditions to allow degeneracies of the Levi form. A natural condition of weak 1-pseudo-concavity is to require that no semidefinite scalar Levi form has maximal rank. Under some genericity assumption, by using Proposition 7.12, this translates into the fact that $[\ker\mathcal{L}]$ is non-trivial. Indeed, this hypothesis implies maximum modulus and unique continuation results analogous to those for holomorphic functions of one complex variable. We expect that properties that are peculiar to holomorphic functions of *several* complex variables would generalize to *CR* functions under suitable *(weak) 2-pseudo-concavity conditions.* This motivates us to give below a tentative list of conditions, motivated partly by the discussion in Sect. 7 and partly by the results of the next sections.

Notation 8.1 If $\mathcal{V}\subset\mathcal{Z}$ is a distribution of complex vector fields on $\Omega^{\text{open}}\subset M$, we use the notation $[\ker\mathcal{L}]_{\mathcal{V}}$ for the semipositive tensors $\sum_{i=1}^{r} Z_i\otimes\bar{Z}_i$ of $[\ker\mathcal{L}]$ with $Z_i\in\mathcal{V}$.

Definition 8.1 Let $p_0\in M$. We say that M is

($\Psi^{s}_{p_0}(q)$): *strongly q-pseudo-concave* at p_0 if all $\mathcal{L}_\xi$, with $\xi\in H^0_{p_0}M\setminus\{0\}$, are nonzero and have Witt index $\geq q$;

($\Psi^{w}_{p_0}(q)$): *weakly q-pseudo-concave* at p_0 if its scalar Levi forms of maximum rank at p_0 have Witt index $\geq q$;

($\Psi^{e}_{p_0}(q)$): *essentially q-pseudo-concave* at $p_0\in M$ if, for every distribution of smooth complex vector fields $\mathcal{V}\subset\mathcal{Z}$, of rank $n-q+1$, defined on an open neighborhood U of p_0, we can find an open neighborhood U' of p_0 in U and a $\tau\in[\ker\mathcal{L}]^{n-q+1}_{\mathcal{V}}(U')$.

($\Psi^{e^*}_{p_0}(q)$): *essentially*-q-pseudo-concave* at $p_0\in M$ if, for every distribution of smooth complex vector fields $\mathcal{V}\subset\mathcal{Z}$, of rank $n-q+1$, defined on an open neighborhood U of p_0, we can find an open neighborhood U' of p_0 in U and a $\tau\in[\ker\mathcal{L}]_{\mathcal{V}}(U')$.

We drop the reference to the point p_0 when the property is valid at all points of M.

We also consider the (global) condition

($\Psi^{we}(q)$) For all $p\in M$ and $\mathcal{V}\subset\mathcal{Z}$ of rank $n-q+1$ on a neighborhood U of p, $\bigcup_{p'\in U}[\ker\mathcal{L}]_{\mathcal{V},p'}$ is a bundle with non-empty fibers and such that for every sequence $\{p_\nu\}\subset M$, converging to $p\in M$, every $\tau\in[\ker\mathcal{L}]_{\mathcal{V},p}$ is a cluster point of $\cup_\nu[\ker\mathcal{L}]_{\mathcal{V},p_\nu}$.

Recall that, according to the notation introduced on page 5, the elements of $[\ker\mathcal{L}](U')$ are different from zero at each point of U'.

Remark 8.1 If $q>1$, then $\Psi^{\star}_{p_0}(q)\Rightarrow\Psi^{\star}_{p_0}(q-1)$ for $\star=s,w,e,e^*$, and (cf. Proposition 7.6 and [21, §2])

$$\Psi^{w}(q)\Leftarrow\Psi^{s}(q)\Rightarrow\Psi^{e}(q)\Rightarrow\Psi^{e^*}(q),\quad\text{for } q\geq 1.$$

Lemma 8.2 *Assume that M is essentially q-pseudo-concave. Then, for every rank $n-q+1$ distribution $\mathcal{V}\subset\mathcal{Z}$ on an $\Omega^{\text{open}}\subset M$, we can find a global section $\tau\in[\ker\mathcal{L}]^{(n-q+1)}_{\mathcal{V}}(\Omega)$.*

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Proof By the assumption, for each $p \in \Omega$, there is an $U^{\text{open}} \subset \Omega$ with $p \in U_p$ and $\tau_p = \sum_{i=1}^{n-q+1} Z_i \otimes \bar{Z}_i \in [\ker \mathcal{L}]^{(n-q+1)}(U_p)$ with $Z_i \in \mathcal{V}(U_p)$. The global τ can be obtained by gluing together the τ_p's by a nonnegative smooth partition of unity on Ω subordinate to the covering $\{U_p\}$. □

In the same way, we can prove

Lemma 8.3 *Assume that M is essentially*-q-pseudo-concave. Then, for every rank $n-q+1$ distribution $\mathcal{V} \subset \mathcal{Z}$ on an $\Omega^{\text{open}} \subset M$, we can find a global section $\tau \in [\ker \mathcal{L}]_{\mathcal{V}}(\Omega)$.* □

Example 8.4 Let $F_{h_1,\ldots,h_r}(\mathbb{C}^m) \subset \mathcal{G}r_{h_1}(\mathbb{C}^m) \times \cdots \times \mathcal{G}r_{h_r}(\mathbb{C}^m)$ denote the complex flag manifold consisting of the r-tuples $(\ell_{h_1}, \ldots, \ell_{h_r})$ with $\ell_{h_1} \subsetneqq \cdots \subsetneqq \ell_{h_r}$, for an increasing sequence $1 \leq h_1 < \cdots < h_r < m$. Here, as usual, ℓ_h is a generic $\mathbb{C}$-linear subspace of dimension h of $\mathbb{C}^m$.

For an increasing sequence of integers $1 \leq i_1 < i_2 < \cdots i_\nu < m$, of length $\nu \geq 2$, we define the *CR*-submanifold M of $F_{i_1,i_3,\ldots}(\mathbb{C}^m) \times F_{i_2,i_4,\ldots}(\mathbb{C}^m)$ consisting of pairs $((\ell_{i_1}, \ell_{i_3}, \ldots), (\ell_{i_2}, \ell_{i_4}, \ldots))$ with $\overline{\ell}_{i_h} \subset \ell_{i_{h+1}}$ for $0 < h < \nu$. Set

$$d_0 = i_1,\ d_1 = i_2 - i_1, \ldots, d_h = i_{h+1} - i_h, \ldots d_{\nu-1} = i_\nu - i_{\nu-1},\ d_\nu = m - i_\nu.$$

This M is a minimal (i.e., $\mathcal{Z}(M) + \overline{\mathcal{Z}}(M)$, and their iterated commutators yield all complex vector fields on M), compact *CR* manifold of *CR*-dimension n and *CR*-codimension k, with

$$n = \sum_{i=0}^{\nu-1} d_i d_{i+1}, \quad k = 2 \sum_{\substack{1 \leq i < j \leq \nu \\ j-i \geq 2}} d_i d_j,$$

as was explained in [34, §3.1]. Then, with $q = \min_{1<i<\nu} d_i$, our M is essentially, but not strongly, q-pseudo-concave when $\nu \geq 3$, because the non-vanishing scalar Levi forms generate at each point a subspace of dimension $2\sum_{i=1}^{\nu-2} d_i d_{i+2} < k$.

In [34], several classes of homogeneous compact *CR* manifolds are discussed, from which more examples of essentially, but not strongly, q-pseudo-concave manifolds can be extracted.

Example 8.5 Let us consider the 11-dimensional real vector space $\mathcal{W}$ consisting of 4×4 Hermitian-symmetric matrices of the form

$$h = \begin{pmatrix} A & B \\ B^* & -A \end{pmatrix} \quad \text{with } A, B \in \mathbb{C}^{2\times 2}, \quad A = A^*, \text{trace}(A) = 0.$$

We claim that all non-singular elements of $\mathcal{W}$ have Witt index two. In fact, for an element h of $\mathcal{W}$, either $A = 0$, or A is non-degenerate. If $A = 0$, the matrix A is non-degenerate iff $\det(B) \neq 0$, and in this case, the Witt index is two as the two-plane of the first two vectors of the canonical basis of $\mathbb{C}^4$ is totally isotropic. If $A \neq 0$, a permutation of the vectors of the canonical basis of $\mathbb{C}^4$ transforms h into a Hermitian-symmetric matrix h' with

$$h' = \begin{pmatrix} C & D \\ D^* & -C \end{pmatrix},$$

for a positive definite Hermitian-symmetric $C \in \mathbb{C}^{2\times 2}$. By a linear change of coordinates in $\mathbb{C}^2$, the positive definite C reduces to the 2×2 identity matrix I_2. This yields a change of coordinates in $\mathbb{C}^4$ by which h' transforms into

$$h'' = \begin{pmatrix} I_2 & E \\ E^* & -I_2 \end{pmatrix}, \quad \text{with } E \in \mathbb{C}^{2\times 2}.$$

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For a matrix of this form, we have, for $v, w \in \mathbb{C}^2$,

$$h''\begin{pmatrix} v \\ w \end{pmatrix} = 0 \Leftrightarrow \begin{cases} v + Ew = 0, \\ E^*v - w = 0 \end{cases} \Leftrightarrow \begin{cases} v + EE^*v = 0, \\ w = E^*v \end{cases} \Leftrightarrow \begin{cases} v = 0, \\ w = 0. \end{cases}$$

Therefore, all h'' of this form are non-singular and their Witt index is independent of E and equal to two. This shows that all $h \in \mathcal{W}$ with $A \neq 0$ are non-singular with Witt index two. Thus, the set of singular matrices of $\mathcal{W}$ is

$$\left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \middle| \det(B) = 0 \right\},$$

which is the cone of the non-singular quadric of the 3-dimensional projective space.

If we take a basis $h_1, \ldots, h_{11}$ of $\mathcal{W}$, the quadric M of $\mathbb{C}^{14} = \mathbb{C}^4_z \times \mathbb{C}^{11}_w$, defined by the equations

$$\mathrm{Re}(w_i) = h_i(z, z), \ \ 1 \leq i \leq 11,$$

is a *CR* manifold of type (4, 11) which is weakly and weakly*-2-pseudo-concave, but not strongly or essentially 2-pseudo-concave.

We obtain examples of *CR* manifolds $M = \{(z, w) \in \mathbb{C}^4 \times \mathbb{C}^7 \mid \mathrm{Re}(w_i) = h_i(z, z), \ 1 \leq i \leq 7\}$, of type (4, 7) and *strongly* 2-pseudo-concave by requiring that $h_1, \ldots, h_7$ be a basis either of the subspace $\mathcal{W}'$ of $\mathcal{W}$ in which B is traceless and symmetric, or of the $\mathcal{W}''$ in which B is quaternionic.

Example 8.6 Let M be the minimal orbit of $\mathbf{SU}(p, p)$ in the complex flag manifold $F_{1,2p-2}(\mathbb{C}^{2p})$, for $p \geq 3$. Its points are the pairs (ℓ_1, ℓ_{2p-2}) consisting of an isotropic line ℓ_1 and a $(2p-2)$-plane ℓ_{2p-2} with $\ell_1 \subset \ell_{2p-2} \subset \ell_1^{\perp}$, where perpendicularity is taken with respect to a fixed Hermitian-symmetric form of Witt index p on $\mathbb{C}^{2p}$.

Then M is a compact *CR* submanifold of $F_{1,2p-2}(\mathbb{C}^{2p})$, of *CR* dimension $(2p-3)$ and *CR* codimension $(4p-4)$, which is essentially 1-pseudo-concave and, when $p > 3$, weakly and weakly*-$(p-2)$-pseudo-concave, but not essentially 2-pseudoconcave.

8.1 Convexity/concavity at the boundary and weak pseudo-concavity

Let us comment on the notion of 1-convexity/concavity at a boundary point of a domain Ω of Sect. 6 in the light of the discussion on Hermitian forms of Sect. 7.

Let ρ be a real-valued smooth function on $\Omega^{\text{open}} \subset M$ and p_0 a point of Ω with the property that, for each $i d\xi_{p_0}$ in $H^{1,1}_{p_0}(\rho)$, the restriction of $i d\xi_{p_0}$ to the space $\{Z_{p_0} \in T^{0,1}_{p_0} M \mid Z_{p_0}\rho = 0\}$ has a nonzero positive index of inertia. The positive multiples of these Hermitian-symmetric forms make a relatively open convex cone $\mathcal{W}$ in the space $\mathcal{P}_{n-1}$ of Hermitian-symmetric forms on $T^{0,1}_{p_0} M \cap \ker d\rho(p_0)$. By Proposition 7.8, we can find an $r > 0$ and $\tau_0 \in H^{1,1,(r)}_{p_0} M$ such that

$$i d\xi(\tau_0) > 0, \quad \forall \xi \in \mathscr{A}_1(\Omega), \quad \text{s.t.} \ \ d\rho(p_0) + i\xi_{p_0} \in T^{*1,0}_{p_0} M.$$

Since $H^{1,1}_{p_0}(\rho)$ is affine with underlying vector space $\{\mathcal{L}_\eta \mid \eta \in H^0_{p_0} M\}$, it follows that actually $\tau_0 \in [\ker \mathcal{L}]^{(r)}_{p_0}$. The same argument applies to the case of a nonzero negative index of inertia.

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Thus, by Lemma 6.18, the condition for $\Omega_{\rho(p_0)} = \{p \in \Omega \mid \rho(p) < \rho(p_0)\}$ to be strongly (1)-convex, or strongly (1)-concave at p_0 is that

$$\exists \tau_0 \in [\ker \mathcal{L}]_{\ker d\rho,\, p_0} \ \text{ such that } \begin{cases} dd^c\rho(\tau_0) > 0, & \text{(strongly 1-convex)},\\ dd^c\rho(\tau_0) < 0, & \text{(strongly 1-concave)}. \end{cases} \tag{8.1}$$

A glitch of the notion of strong-1-convexity (resp. -concavity) is that it is not, in general, stable under small perturbations. This can be ridden out by adding the global assumption of essential-2-pseudo-concavity of M. Set, for simplicity of notation, $\rho(p_0) = 0$ and $d\rho(p_0) \neq 0$.

Proposition 8.7 *Suppose that M is essentially 2-pseudo-concave and that $\Omega_0 = \{p \in \Omega \mid \rho(p) < 0\}$ is strongly 1-concave at $p_0 \in \partial\Omega_0$. Then,*

(1) *We can find $\tau_0 \in [\ker \mathcal{L}]^{(n-1)}_{\ker d\rho,\, p_0}$ such that $dd^c\rho(\tau_0) < 0$;*
(2) *We can find an open neighborhood U of p_0 in Ω such that at every $p' \in U$ the open set $\Omega_{\rho(p')} = \{p \in \Omega \mid \rho(p) < \rho(p')\}$ is smooth and strongly 1-concave at p'.*

9 Cauchy problem for *CR* functions—uniqueness

In this section, we discuss uniqueness for the initial value problem for *CR* functions, with data on a non-characteristic smooth initial hypersurface $N \subset M$.

Uniqueness is well understood when M is a *CR* submanifold of a complex manifold (see, for example, [41]). Let $\Omega \subset M$ be an open neighborhood of a non-characteristic point p_0 of N, such that $\Omega \setminus N$ is the union of two disjoint connected components $\Omega^{\pm}$.

Proposition 9.1 *Assume that M is a minimal CR submanifold of a complex manifold* X. *If $f \in \mathscr{O}_M(\Omega^+) \cap \mathscr{C}^0(\bar{\Omega}^+)$ and $f|_N$ vanishes on an open neighborhood of a non-characteristic point p_0 of N, then $f \equiv 0$ on Ω^+.*

We have a similar statement for *CR* distributions.

Proposition 9.2 *Assume that M is either a real-analytic CR manifold, or a CR submanifold of a complex manifold* X *that is minimal at every point. Let N be a $\mathcal{Z}$-non-characteristic hypersurface of M, such that $M \setminus N$ is the union of two disjoint connected open subsets $M_{\pm}$. Then, there is an open neighborhood U of N in M such that any CR distribution on M_+ having vanishing boundary values on N, vanishes on $U \cap M_+$.*

Proof An $f \in \mathscr{D}'(M_+)$ is *CR* if $Zf = 0$ in M_+, in the sense of distributions, for all $Z \in \mathcal{Z}(M)$. We say that f has *zero boundary value* on N if for each $p \in N$, we can find an open neighborhood U_p of p in M and a *CR*-distribution $\tilde{f} \in \mathscr{D}'(U_p)$ which extends $f|_{M_+ \cap U_p}$ and is zero on $U_p \setminus \bar{M}_+$. Note that, since N is non-characteristic, all *CR* distributions defined on a neighborhood of N admit a *restriction* to N.

The case where M is a real-analytic *CR* manifold reduces to the classical Holmgren uniqueness theorem.

In the other case, where M is $\mathscr{C}^\infty$ smooth, but is assumed to be minimal, we first choose a slight deformation N_d of N such that N_d is contained in $\overline{M^+}$ and coincides with N near p. Moreover, we can achieve that the *CR* orbit $\mathscr{O}(p, N_d)$ of p in N_d intersects $N_d \cap M^+$. Since M^+ is minimal at every point, *CR* distributions holomorphically extend to open wedges attached to M^+. In particular, this holds for the boundary value of $f|_{M_d^+}$ (M_d^+ being the side of N_d containing M^+) at any point of $N_d \cap M^+$.

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Using that wedge extension propagates along *CR* orbits, we get wedge extension from N_d at p. Examining how the wedges are constructed by analytic disk techniques, one more precisely obtains a neighborhood V of p in $\overline{M^+}$ and an open truncated cone $C \subset \mathbb{C}^n$ such that $\tilde{f}$ holomorphically extends to $W_N = \bigcup_{z \in V \cap N}(z + C)$, and f to $W^+ = \bigcup_{z \in V \cap M^+}(z + C)$. The idea is to work with analytic disks attached to (deformations of) N_d and to nearby hypersurfaces of M.

Since $\tilde{f}$ is the boundary value of f, the two extensions glue to a single function $F \in \mathscr{O}(W_N \cup W^+)$. On the other hand, F is zero on W_N (since $\tilde{f}$ vanishes near p) and thus on W^+, by the unique continuation of holomorphic functions. Finally f, being the boundary value of F, has to vanish on $N \cap M^+$. □

Remark 9.3 Thanks to the extension result proved in [27,37], see also [36], it suffices to assume that M^+ is *globally minimal*, i.e., that M^+ consists of only one *CR* orbit.

For an embedded *CR* manifold with property (H), uniqueness results can be derived from Proposition 3.3. Indeed, in this case, a *CR* function defined on a neighborhood in M of a point $p_0 \in N$ and whose restriction to N has a zero of infinite order at p_0, also has a zero of infinite order at p_0 as a function on M and then is zero on the connected component of p_0 in its domain of definition by the strong unique continuation principle.

The situation is quite different for abstract *CR* manifolds: there are examples of pseudoconvex M on which there are nonzero smooth *CR* functions vanishing on an open subset (see, for example, [40]). Here, for the pseudo-concave case, we give a uniqueness result which is similar to those of [13,21,22], but more general, because we do not require the existence of sections τ of $[\ker \mathcal{L}]^{(n)}$, i.e., we drop the rank requirement, but we assume that the initial hypersurface N is non-characteristic with respect to the subdistribution Θ of $\mathcal{Z}$, which was defined in Sect. 4.

In this context, we can slightly generalize *CR* functions by considering, for a given $\tau \in [\ker \mathcal{L}](M)$, functions f on M satisfying

$$\begin{cases} f \in L^2_{\text{loc}}(M), \quad \forall Z \in \tilde{\Theta},\, Zf \in L^2_{\text{loc}}(M) \ \text{ and } \exists \kappa_Z \in L^\infty_{\text{loc}}(M, \mathbb{R}) \\ \text{such that} \quad |(Zf)(p)| \leq \kappa_Z(p)|f(p)| \ \text{ a.e. on } M. \end{cases} \tag{9.1}$$

Condition (9.1), with $\mathcal{Z}(M)$ instead of $\tilde{\Theta}(\tau)$, naturally arises when we consider *CR* sections of a complex *CR* line bundle (see [21, §7]).

We note that the hypersurface N is non-characteristic at a point p_0 with respect to the distribution Θ if it is non-characteristic at p_0 for $\Theta(\tau)$ for some $\tau \in [\ker \mathcal{L}](M)$.

Proposition 9.4 *Let* $\Omega^{\text{open}} \subset M$ *and* $N \subset \partial\Omega$ *a smooth* Θ*-non-characteristic hypersurface in* M*. Then, there is a neighborhood* U *of* N *in* M *such that any solution* f *of* (9.1)*, which is continuous on* $\bar{\Omega}$ *and vanishes on* N*, is zero on* $U \cap \Omega$.

Proof We note that the assumption of constancy of rank in unessential and never used in the proof of [22, Theorem 4.1]. We reduce to that situation by considering the $\tilde{\Theta}$-structure on M, defined by the distribution of (4.6), after we make the following observation. Since the statement is local, we can assume that N splits M into two closed half-manifolds $M_\pm$, with $\Omega = M_-$ and $\partial\Omega = N$. A continuous solution f of (9.1) in M_- vanishing on N, when extended by 0 on M_+, defines a continuous solution $\tilde{f}$ of (9.1) in M with supp $\tilde{f} \subset \bar{M}_-$. In fact, since $L \in \tilde{\Theta}(M)$ is first order, $L\tilde{f}$ equals Lf on M_- and 0 on M_+, as one can easily check by integrating by parts and using the identity of weak and strong extensions of [15]. Hence, $\tilde{f}$ still satisfies (9.1) and vanishes on an open subset of M. By proving Carleman estimates,

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similar to those in [21, Theorem 5.2], we obtain that $\tilde{f}$ vanishes along the Sussmann leaves of $\tilde{\Theta}$ transversal to N (see [17,22]). These leaves fill a neighborhood of N in M, where $\tilde{f}$ vanishes. This proves our contention. □

Remark 9.5 Note that $\mathbb{C}^n \times \mathbb{R}^r$ is weakly pseudo-concave (but not essentially pseudo-concave). Thus, we need the genericity assumption (2.5) to get uniqueness in this case. The uniqueness for the non-characteristic Cauchy problem in the case of a single partial differential operator of [11,45] may be considered a special case of this proposition, when the *CR* dimension is one.

Uniqueness in the case where N can be characteristic for Θ, but not for $\mathcal{Z}$, will be obtained by adding a pseudo-convexity hypothesis.

First, we prove a Carleman-type estimate.

Lemma 9.6 *Let* τ *be a section of* $[\ker \mathcal{L}]$ *and* ψ *a real-valued smooth function on* M. *Then, there is a smooth real-valued function* κ *on* M *such that*

$$\|\exp(t\psi)L_0 f\|_0^2 + \sum_{i=1}^{r} \|\exp(t\psi)Z_i f\|_0^2 \geq \int (2t \cdot dd^c\psi(\tau) + \kappa)\,|f|^2 e^{2t\psi} d\mu,$$
$$\forall f \in \mathscr{C}_0^\infty(M),\ \forall t > 0. \tag{9.2}$$

Here the L^2-norms and the integral are defined by utilizing the smooth measure $d\mu$ associated with a fixed Riemannian metric on M.

Proof Let $\tau = \sum_{i=1}^{r} Z_i \otimes \bar{Z}_i$, $\sum_{i=1}^{r}[Z_i, \bar{Z}_i] = \bar{L}_0 - L_0$, with $Z_i, L_0 \in \mathcal{Z}(M)$. We will indicate by κ_1, κ smooth functions on M which only depend on $Z_1, \ldots, Z_r$. For $f \in \mathscr{C}_0^\infty(M)$, and a fixed $t > 0$, set $v = f \cdot \exp(t\psi)$. Integration by parts yields

$$\sum_{i=1}^{r} \|Z_i v - tvZ_i\psi\|_0^2 = \sum_{i=1}^{r} \|Z_i^* v - tv\bar{Z}_i\psi\|_0^2 + \int \sum_{i=1}^{r} [Z_i, \bar{Z}_i]v \cdot \bar{v}\, d\mu$$
$$+ \operatorname{Re} \int \left(\kappa_0 + \sum\nolimits_{i=1}^{r} 2t(Z_i\bar{Z}_i\psi)\right) |v|^2 d\mu,$$

where the superscript star stands for formal adjoint with respect to the Hermitian scalar product of $L^2(d\mu)$. For the second summand in the right-hand side, we have

$$\begin{aligned}
\int \sum_{i=1}^{r}[Z_i, \bar{Z}_i]v \cdot \bar{v}\, d\mu &= \int \bar{L}_0 v \cdot \bar{v}\, d\mu - \int L_0 v \cdot \bar{v}\, d\mu \\
&= -\int L_0 v \cdot \bar{v}\, d\mu - \int v \cdot \overline{L_0 v}\, d\mu - \int \kappa_1 |v|^2 d\mu \\
&= -2\operatorname{Re} \int L_0 v \cdot \bar{v}\, d\mu - \int \kappa_1 |v|^2 d\mu \\
&\geq -2\|L_0 v - tvL_0\psi\|_0 \|v\|_0 - \int (\kappa_1 + 2t \operatorname{Re} L_0\psi)|v|^2 d\mu \\
&\geq -\|L_0 v - tvL_0\psi\|_0^2 - \int (1 + \kappa_1 + 2t \operatorname{Re} L_0\psi)|v|^2 d\mu.
\end{aligned}$$

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Therefore, we obtain the estimate

$$\|L_0 v - tvL_0\psi\|_0^2 + \sum_{i=1}^{r} \|Z_i v - tvZ_i\psi\|_0^2$$
$$\geq \int \left(t[Z_i\bar{Z}_i + \bar{Z}_i Z_i]\psi - 2t(\mathrm{Re}\, L_0)\psi - \kappa_2\right)|v|^2\,\mathrm{d}\mu = \int (2tP_\tau\psi + \kappa)|v|^2\mathrm{d}\mu.$$

By Proposition 6.6, this yields (9.2). □

From the Carleman estimate (9.2), we obtain a uniqueness result *under convexity conditions*, akin to the one of [24, §28.3] for a scalar p.d.o.

Proposition 9.7 *Assume there is a section* $\tau \in [\ker\mathcal{L}]$ *and* $\psi \in \mathscr{C}^\infty(M, \mathbb{R})$ *such that*

$$\mathrm{d}\psi(p_0) \neq 0, \quad \mathrm{d}d^c\psi(\tau) > 0. \tag{9.3}$$

Then, there is an open neighborhood U *of* p_0 *in* M *with the property that any solution* f *of* (9.1) *which vanishes a.e. on* $U \cap \{p \mid \psi(p) > \psi(p_0)\}$ *also vanishes a.e. on* U. □

Remark 9.8 In fact, it suffices to require that (9.1) is satisfied by the operators $Z_1, \ldots, Z_r, L_0$.

Let Ω be an open domain in M, and $p_0 \in \partial\Omega$ a smooth point of the boundary.

Proposition 9.9 *If* Ω *is either* Θ*-non-characteristic or strictly 1-convex at* p_0 *(according to Definition 6.4), then any* f *satisfying* (9.1) *in* Ω*, and having zero boundary values on a neighborhood of* p_0 *in* $\partial\Omega$*, is 0 a.e. on the intersection of* Ω *with a neighborhood of* p_0 *in* M.

Proof With P_τ defined by (4.7), (4.8), and a real parameter s, we have

$$e^{-s\psi}P_\tau(e^{s\psi}) = s\left(\tfrac{1}{2}\sum_{i=1}^{r}(Z_i\bar{Z}_i + \bar{Z}_i Z_i)\psi - X_0\psi\right) + s^2\sum_{i=1}^{r}|Z_i\psi|^2$$
$$= s\,\mathrm{dd}^c\psi(\tau) + s^2\sum_{i=1}^{r}|Z_i\psi|^2.$$

Thus, the condition of Proposition 9.7 is satisfied for a suitable $\tau \in [\ker\mathcal{L}]$ near p_0 either when $\partial\Omega$ is Θ-non-characteristic at p_0, by taking $s \gg 1$, or, in case $\partial\Omega$ is $\Theta(\tau)$-characteristic at p_0, if $dd^c\psi(\tau)(p_0) > 0$. □

Remark 9.10 We observe that strict 1-convexity at p_0 implies that $\partial\Omega$ is $\tilde{\Theta}$-non-characteristic at p_0.

10 Cauchy problem for *CR* functions existence

In this section, we will investigate properties of *CR* functions on *CR* manifolds satisfying weak 2-pseudo-concavity assumptions.

Proposition 10.1 *Let* Ω *be an open subset of a CR manifold* M *enjoying property* $\Psi^{we}(2)$*. Assume that* p_0 *is a smooth, strongly 1-convex,* Θ*-non-characteristic point of* $\partial\Omega$*. Then, for every relatively compact open neighborhood* U *of* p_0 *in* M*, we can find an open neighborhood* U' *of* p_0 *in* U *such that*

$$|f(p)| \leq \sup_{U\cap\partial\Omega} |f|, \quad \forall p \in U' \cap \Omega, \quad \forall\, f \in \mathscr{O}_M(\Omega) \cap \mathscr{C}^2(\bar{\Omega}), \tag{10.1}$$

and strict inequality holds if f *is not a constant on* $U' \cap \Omega$.

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Proof We can assume that Ω is locally defined near p_0 by a real-valued $\rho \in \mathscr{C}^\infty(U)$:

$$U \cap \Omega = \{p \in U \mid \rho(p) < 0\}, \quad \text{and} \quad \exists\, Z \in \Theta(U) \;\; \text{s.t.} \;\; (Z\rho)(p_0) \neq 0.$$

To make local bumps of $\partial\Omega$ near p_0, we fix smooth coordinates x centered at p_0, that we can take for simplicity defined on U, and, for a nonnegative real-valued smooth function $\chi(t) \in \mathscr{C}_0^\infty(\mathbb{R})$, equal to 1 on a neighborhood of 0, set $\phi_\epsilon(p) = e^{-1/\epsilon}\chi(|x|/\epsilon)$. Then, we consider the domains

$$U_\epsilon^- = \{p \in U \mid -\phi_\epsilon(p) < \rho(p) < 0\}.$$

There is $\epsilon_0 > 0$ such that $U_\epsilon^- \Subset U$ and the points of $N''_\epsilon = \partial U_\epsilon^- \cap \Omega$ are smooth and Θ-non-characteristic for all $0 < \epsilon \leq \epsilon_0$. In fact, N''_ϵ is a small deformation of $N'_\epsilon = \{\phi_\epsilon > 0\} \cap \partial\Omega$, which is smooth and Θ-non-characteristic for $0 < \epsilon \ll 1$.

We claim that, for sufficiently small $\epsilon > 0$, the modulus $|f|$ of any function $f \in \mathscr{O}_M(U_\epsilon^-) \cap \mathscr{C}^2(\bar{U}_\epsilon^-)$ attains its maximum on N. We argue by contradiction.

If our claim is false, then for all $0 < \epsilon \leq \epsilon_0$ we can find $p_\epsilon \in N''_\epsilon$ and $f_\epsilon \in \mathscr{O}_M(U_\epsilon^-) \cap \mathscr{C}^2(\bar{U}_\epsilon^-)$ with $|f(p_\epsilon)| > |f(p)|$ for all $p \in U_\epsilon^-$. In fact, $\Psi^{we}(2)$ implies the maximum modulus principle, and therefore, the maximum of $|f_\epsilon|$ is attained on the boundary of U_ϵ^-. By Proposition 6.15, this implies that there is $\xi_\epsilon \in H^0_{M,p_\epsilon}(U_\epsilon^-)$ such that $\mathcal{L}_{\xi_\epsilon}^{N''_\epsilon} \geq 0$. By the strong-1-convexity assumption, there is $\tau_0 \in [\ker\mathcal{L}]_{d\rho^\perp, p_0}$ (see Notation 8.1) such that $dd^c\rho(\tau_0) > 0$. For $\epsilon_\nu \searrow 0$, the sequence $\{p_{\epsilon_\nu}\}$ converges to p_0. We can take a function $\tilde{\rho} \in \mathscr{C}^\infty(U)$ such that $\tilde{\rho}$ agrees to the second order with $(\rho + \phi_{\epsilon_\nu})$ at p_{ϵ_ν}, for all ν, and with ρ at p_0.

We obtain a contradiction, because τ_0 belongs to $[\ker\mathcal{L}]_{d\tilde{\rho}^\perp, p_0} = [\ker\mathcal{L}]_{d\rho^\perp, p_0}$ and therefore, by $\Psi^{we}(2)$, is a cluster point of a sequence of elements $\tau_{\epsilon_\nu} \in [\ker\mathcal{L}]_{d\tilde{\rho}^\perp, p_{\epsilon_\nu}} = [\ker\mathcal{L}]_{d(\rho+\phi_{\epsilon_\nu})^\perp, p_{\epsilon_\nu}}$, and $dd^c\tilde{\rho}(\tau_{\epsilon_\nu}) = dd^c(\rho + \phi_{\epsilon_\nu})(\tau_{\epsilon_\nu}) \leq 0$ by Proposition 6.15 and Corollary 6.8. In fact, $dd^c(\rho + \phi_{\epsilon_{\nu'}})(\tau_{\epsilon_{\nu'}}) \longrightarrow dd^c\rho(p_0)(\tau_0)$ when $\tau_{\epsilon_{\nu'}} \longrightarrow \tau_0$. □

Theorem 10.2 *Let Ω be an open subset of a CR manifold M enjoying property $\Psi^{we}(2)$ and N a relatively open subset of $\partial\Omega$, consisting of smooth, strongly 1-convex, Θ-non-characteristic points. If M is locally CR-embeddable at all points of N, then we can find an open neighborhood U of N in M such that for every $f_0 \in \mathscr{O}_N(N)$, there is a unique $f \in \mathscr{O}_M(U \cap \Omega) \cap \mathscr{C}^\infty(\overline{U \cap \Omega})$ with $f = f_0$ on N.*

Proof The result easily follows from the approximation theorem in [7] and the estimate of Proposition 10.1 □

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