A NON-COMMUTATIVE, ANALYTIC VERSION OF HILBERT'S 17-TH PROBLEM IN TYPE II_1 VON NEUMANN ALGEBRAS

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ABSTRACT. Let $Y_1, \ldots, Y_n$ be n indeterminates. For $I = (i_1, \ldots, i_p)$, $i_s \in \{1, 2, \ldots, n\}$, $s = 1, 2, \ldots, p$, let Y_I be the monomial $Y_{i_1} Y_{i_2} \cdots Y_{i_p}$. Denote by $|I| = p$. Let $\mathbb{C}_{\mathrm{an}}[Y_1, Y_2, \ldots, Y_p]$ be the ring of non-commutative series $\sum a_I Y_I$, $a_I \in \mathbb{C}$, such that $\sum |a_I| R^{|I|} < \infty$ for all $R > 0$. On $\mathbb{C}_{\mathrm{an}}[Y_1, Y_2, \ldots, Y_n]$ we have a canonical involution extending by linearity $(a_I Y_I)^* = \overline{a} Y_{I^{\mathrm{op}}}$, $a_I \in \mathbb{C}$, $I \in \mathcal{I}_n$, $I = \{i_1, i_2, \ldots, i_p\}$, $I^{\mathrm{op}} = \{i_p, i_{p-1} \ldots, i_1\}$. By $\mathbb{C}_{\mathrm{an}}^{\mathrm{sym}}[Y_1, Y_2, \ldots, Y_n]$ we denote the real subspace of $\mathbb{C}_{\mathrm{an}}[Y_1, Y_2, \ldots, Y_n]$ of series that are auto-adjoint. We say that two series p, q are cyclic equivalent if $p - q$ is a sum (possible infinite) of scalar multiples of monomials of the type $Y_I - Y_{\tilde{I}}$, where $\widetilde{I}$ is a cyclic permutation of I. We call a series q in $\mathbb{C}_{\mathrm{an}}[Y_1, \ldots, Y_p]$ a sum of squares if q is a weak limit of sums $\sum_s b_s^* b_s$, where $b_s \in \mathbb{C}_{\mathrm{an}}[Y_1, \ldots, Y_p]$.

We prove that if a series $p(Y_1, \ldots, Y_n)$ in $\mathbb{C}_{\mathrm{an}}^{\mathrm{sym}}[Y_1, \ldots, Y_n]$ has the property that $\tau(p(X_1, \ldots, X_n)) \geqslant 0$ for every M type II_1 von Neumann algebra with faithful trace τ and for all selfadjoint $X_1, X_2, \ldots, X_n$ in M, then p is equivalent to a sum of squares in $\mathbb{C}_{\mathrm{an}}[Y_1, \ldots, Y_n]$. As a corollary, it follows that the Connes embedding conjecture is equivalent to a statement on the structure of matrix trace inequalities: if $p(Y_1, \ldots, Y_n)$ in $\mathbb{C}_{\mathrm{an}}^{\mathrm{sym}}[Y_1, \ldots, Y_n]$ is such that $\operatorname{tr} p(X_1, \ldots, X_n) \geqslant 0$, for all selfadjoint matrices $X_1, \ldots, X_n$, of any size, then p should be equivalent to a sum of squares in $\mathbb{C}_{\mathrm{an}}[Y_1, \ldots, Y_n]$.

1. INTRODUCTION

The Connes embedding conjecture ([2]) states that:

Every type II_1 *factor (equivalently any type* II_1 *von Neumann algebra) can be embedded into the factor* R^ω (see [2], p. 105).

Equivalent forms of this conjecture have been extensively studied by Kirchberg ([4]), and subsequently in [7], [9], [10], [1], [6].

In this paper we prove that the Connes embedding conjecture is equivalent to a statement on the structure of the trace inequalities on matrices. To prove this, we deduce an analogue of the Hilbert 17-th problem in the context of type II_1 von Neumann algebras.

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More precisely, let $Y_1, \dots, Y_n$ be n indeterminates. Let $\mathcal{I}_n$ be the index set of all monomials in the variables $Y_1, Y_2, \dots, Y_n$

$$\mathcal{I}_n = \{(i_1, \dots, i_p) \mid p \in \mathbb{N},\, i_1, \dots, i_p \in \{1, 2, \dots, n\}\}$$

(we assume that $\emptyset \in \mathcal{I}_n$ and that $\emptyset$ corresponds to the monomial which is identically one). For $I = (i_1, \dots i_p)$ let $|\mathcal{I}| = p$ and let Y_I denote the monomial $Y_{i_1} Y_{i_2} \cdots Y_{i_p}$. For such an I define $I^{\rm op} = (i_p, \dots, i_1)$ and define an adjoint operation on $\mathbb{C}[Y_1, Y_2, \dots, Y_n]$ by putting $Y_I^* = Y_{I^{\rm op}}$.

We let $\mathbb{C}_{\rm an}[Y_1, \dots, Y_n]$ be the ring of all series

$$V = \Big\{ \sum_{I \in I_n} a_I Y_I \mid a_I \in \mathbb{C}, \Big\| \sum a_I Y_I \Big\|_R = \sum |a_I| R^{|I|} < \infty, \forall\, R > 0 \Big\}.$$

It turns out (Section 2) that V is a Fréchet space, and hence that V has a natural weak topology $\sigma(V, V^*)$. We will say that an element q in V is a sum of squares if q is in the weak closure of the cone of sum of squares

$$\sum_s p_s^* p_s, \quad p_s \in \mathbb{C}_{\rm an}[Y_1, Y_2, \dots, Y_n].$$

By $\mathbb{C}_{\rm an}^{\rm sym}[Y_1, Y_2, \dots, Y_n]$ we denote the real subspace of all analytic series that are autoadjoint.

We say that two series p, q in $\mathbb{C}_{\rm an}^{\rm sym}[Y_1, Y_2, \dots, Y_n]$ are *cyclic equivalent* if $p - q$ is a weak limit of sums of scalar multiples of monomials of the form $Y_I - Y_{\tilde{I}}$, where $I \in \mathcal{I}_n$, and $\widetilde{I}$ is a cyclic permutation of I.

Our analogue of the Hilbert's 17-th problem is the following

Theorem 1.1. *Let $p \in \mathbb{C}_{\rm an}^{\rm sym}[Y_1, \dots, Y_n]$ such that, whenever M is a separable type II_1 von Neumann algebra with faithful trace τ and $X_1, \dots, X_n$ are selfadjoint elements in M, then by substituting $X_1, \dots, X_n$ for $Y_1, \dots, Y_n$, we obtain $\tau(p(X_1, \dots, X_n)) \geqslant 0$. Then p is cyclic equivalent to a weak limit of a sum of squares in $\mathbb{C}_{\rm an}[Y_1, \dots, Y_n]$.*

As a corollary, we obtain the following statement which describes the Connes embedding conjecture strictly in terms of (finite) matrix algebras. (A similar statement has been noted in [3]).

Corollary 1.2. *The Connes embedding conjecture holds if and only if whenever $p \in \mathbb{C}_{\rm an}^{\rm sym}[Y_1, \dots, Y_n]$ is such that, whenever we substitute selfadjoint matrices $X_1, X_2, \dots, X_n$ in $M_N(\mathbb{C})$ endowed with the canonical trace tr, we have $\mathrm{tr}(p(X_1, X_2, \dots, X_n)) \geqslant 0$. Then p should be equivalent to a weak limit of sums of squares.*

To prove the equivalence of the two statements we use the fact that the Connes conjecture is equivalent to show that the set of non-commutative moments of n elements in a type II_1 factor can be approximated, in a suitable way, by moments of n elements in a finite matrix algebra.

2. Properties of the vector space $\mathbb{C}_{\rm an}[Y_1, Y_2, \dots, Y_n]$

We identify in this section the vector space $\mathbb{C}_{\rm an}[Y_1, \dots, Y_n]$ with the vector space $V = \Big\{ (a_I)_{I \in \mathcal{I}_n} \mid \sum_{I \in \mathcal{I}_n} |a_I| R^I < \infty,\, R > 0 \Big\}$. Denote by $\|(a_I)_{I \in \mathcal{I}_n}\|_R = \sum_{I \in \mathcal{I}_n} |a_i| R^I$

for $R > 0$. Clearly, $\|\cdot\|_R$ is a norm on V. In the next proposition (which is probably known to specialists, but we could not provide a reference) we prove that V is a Fréchet space. For two elements $J, K \in \mathcal{I}_n$, $J = (j_1, \dots, j_n)$, $K = (k_1, \dots, k_s)$, we denote by $K \sharp J = (j_1, \dots, j_n, k_1, \dots, k_s)$ the concatenation of J and K.

Proposition 2.1. *With the norms $\|\cdot\|_R$, V becomes a Fréchet space. Moreover, the operation $*$ defined for $a = (a_I)$, $b = (b_I)$ by*

$$(a * b)_I = \sum_{\substack{J,K \in \mathcal{I}_n \\ J \sharp K = I}} a_J a_K$$

(which in terms of $\mathbb{C}_{\mathrm{an}}[Y_1, \dots, Y_n]$ corresponds to the product of series) is continuous with respect to the norms $\|\cdot\|_R$, that is

$$\|a * b\|_R \leqslant \|a\|_R \|b\|_R, \quad \textit{for all } R > 0.$$

Proof. To prove that V is a Fréchet vector space, consider a Cauchy sequence $b^s = (b^s_I)_{I \in \mathcal{I}_n}$. Thus for all $R, \varepsilon > 0$ there exist $N_{R,\varepsilon}$ in $\mathbb{N}$ such that for all $s, t > N_{R,\varepsilon}$ we have $\|b^s - b^t\|_R < \varepsilon$. Clearly, this implies that $(b^s_I)_{I \in \mathcal{I}_n}$ converges pointwise to a sequence $(b_I)_{I \in \mathcal{I}_n}$ and it remains to prove that b belongs to V and b^s converges to b.

To do this we may assume (by passing to a subsequence, and using a typical Cantor diagonalization process) that

$$\|b^s - b^{s+1}\|_s \leqslant \frac{1}{2^s} \quad \text{for all } s \text{ in } \mathbb{N}.$$

But then for every $R > 0$, we have that

$$\begin{aligned} |b^s_I - b_I| R^{|I|} &\leqslant \sum_{\substack{s = [R]+1 \\ I \in \mathcal{I}_n}} |b^s_I - b^{s+1}_I| R^{|I|} \\ &\leqslant \sum_{\substack{s = [R]+1 \\ I \in \mathcal{I}_n}} |b^s_I - b^{s+1}_I| s^{|I|} \leqslant \sum_{s=[R]+1} \frac{1}{2^s}. \end{aligned}$$

This proves that $b \in V$ and that b^s converges to b.

For the second part of the statement we have to estimate, for all $a, b \in V$ and $R > 0$, the quantity

$$\begin{aligned} \sum_{I \in \mathcal{I}_n} \sum_{\substack{J,K \in \mathcal{I}_n \\ J \sharp K = I}} |a_J|\, |b_K| R^{|I|} &= \sum_{I \in \mathcal{I}_n} \sum_{\substack{J,K \in \mathcal{I}_n \\ J \sharp K = I}} |a_J|\, |b_K| R^{|J|+|K|} \\ &= \Bigg(\sum_{J \in \mathcal{I}_n} |a_J| R^{|J|} \Bigg) \Bigg(\sum_{K \in \mathcal{I}_n} |b_K| R^{|K|} \Bigg) = \|a\|_R \|b\|_R. \end{aligned}$$

This proves that $a * b \in V$ and that $\|a * b\|_R \leqslant \|a\|_R \|b\|_R$. □

In what follows we describe the dual V^* of the Fréchet space V.

Lemma 2.2. *V^* is identified with the space of all sequences of complex numbers*

$$\{(t_I)_{I \in c I_n} \mid \exists\, R > 0,\ \sup_{I \in \mathcal{I}_n} |t_I| R^{-|I|} < \infty\}.$$

The duality between V and V^* is realized via the pairing

$$\langle a,t\rangle = \sum_{I\in\mathcal{I}_n} a_I t_I \quad \textit{for } a\in V,\, t\in V^*,$$

which is convergent if $\sup\limits_{I\in\mathcal{I}_n} |t_I| R^{-|I|} < \infty$ *for some* R.

Proof. It is obvious that each sequence $(t_I)_{I\in\mathcal{I}_n}$ such that $\sup\limits_{I\in\mathcal{I}_n} |t_I| R^{-|I|}$ defines an element in V^*.

Conversely, if φ is a continuous linear functional on V, then there exists a semi-norm $\|\cdot\|_R$ and a positive constant $C>0$, such that

$$|\varphi((a_I)_{I\in\mathcal{I}_n})| \leqslant C \sum_{I\in\mathcal{I}_n} |a_I| R^{|I|}.$$

But then by the usual duality between ℓ^1 and ℓ^∞ and since $\{(a_I)R^{|I|})_{I\in\mathcal{I}_n} \mid (a_I)_{I\in\mathcal{I}_n} \in V\}$ is dense in $\ell^1(\mathcal{I}_n)$, it follows that there exists $(t_I)_{I\in\mathcal{I}}$ such that $\sup\limits_{I\in\mathcal{I}_n} (t_I) R^{-|I|} < \infty$ and such that $\varphi((a_I)_{I\in\mathcal{I}_n}) = \sum\limits_{I\in\mathcal{I}_n} a_I t_I$. □

By V_{sym} we consider the real subspace of V consisting of all $\{(a_I)_{I\in\mathcal{I}_n} \in V \mid a_{I^{\mathrm{op}}} = \overline{a_I},\, \forall I\in\mathcal{I}_n\}$, and by V^*_{sym} we consider the space of all real functionals on V_{sym}.

If σ is a permutation of $\{1,2,\dots,p\}$ (which we denote by $\sigma\in S_p$) and $I\in\mathcal{I}_n$, $I=(i_1,i_2,\dots,i_p)$ then by $\sigma(I)$ we denote $(i_{\sigma(1)}, i_{\sigma(2)},\dots,i_{\sigma(p)})$. By $S_{p,\mathrm{cyc}}$ we denote the cyclic permutations as S_p. We omit the index p when it is obvious.

Let $W=\{(t_I)_{I\in\mathcal{I}_n} \mid t_I = t_{\sigma(I)},\, \forall\,\sigma\in S_{\mathrm{cyc}} and\, \exists\, R>0$ such that $\sup |t_I| R^{-|I|} < \infty,\, t_I = \overline{t_{I^{\mathrm{op}}}},\, \forall\, I\in\mathcal{I}_n\}$.

Lemma 2.3. *W is a closed subspace of V^*_{sym}. Moreover, if $\varphi_s = (t^s_I)_{I\in\mathcal{I}_n} \in W$ converges to $\varphi\in W$ in the $\sigma(V^*,V)$ topology, then there exists $R>0$ such that*

$$\sup_{s,I\in\mathcal{I}_n} |t^s_I| R^{-|I|} < \infty.$$

Proof. W is a closed subspace of V^* follows immediately from the fact that W is the fixed point subspace for the actions of the finite groups $S_{p,\mathrm{cyc}}$, $p\geqslant 1$, on the components of the indices. The same is true for the third condition in the definition of W.

The second statement is a standard consequence of the Banach-Steinhaus theorem. Indeed, if $\varphi_s \to \varphi$ in the $\sigma(V^*,V)$ topology, then $(\varphi_s)_{s\in\mathbb{N}}$ forms an equicontinuous family and hence there exist a semi-norm $\|\cdot\|_R$ on V and a constant $C>0$ such that

$$|\varphi_s((a_I)_{I\in\mathcal{I}_n})| \leqslant C\|(a_I)_{I\in\mathcal{I}_n}\|_R.$$

By the preceding section this means exactly the condition in the statement. □

We note the following straightforward consequence of the Bipolar theorem ([11], [12]):

Lemma 2.4. *Let $K_m \subseteq K_2 \subseteq W$ be closed convex cones, (which will correspond to the sets K_{mat} and K_{II_1} in the next section). Let $L_2 = K_2^0$, $L_m = K_m^0$ be the corresponding polar sets (the polars are with respect V_{sym}) in V_{sym}. (In particular $W^\perp \subseteq L_2 \subseteq L_m$.)*

By the annihilator $W^{\perp}$ of the space W we mean the (relative) annihilator with respect to$V_{\rm sym}$. Likewise we do for the polar sets.

Let L_p be a $\sigma(V_{\rm sym}, V^*_{\rm sym})$ closed conex subset of L_2. Then to prove that $L_p + W^{\perp} = L_2$ is equivalent to prove that $L_p^0 \cap W = K_2$ and hence it is sufficient to verify that $L_p^0 \cap W \subseteq K_2$.

Proof. Indeed, in this situation $L_p + W^{\perp} \subseteq L_2$ and since $L^0 \cap W = (L_p + W^{\perp})^0$ it follows immediately from the Bipolar theorem that $K_2 = L_2^0 \subseteq (L_p + W^{\perp})^0 = L_p \cap W$. Moreover, if the last two closed convex sets are equal, i.e. if $K_2 = L_p^0 \cap W$, then by the Bipolar theorem we get $L_2 = K_2^0 = L_p^0 \cap W = (L_p + W^{\perp})^{00} = L_p + W^{\perp}$. □

3. The set of non-commutative moments for variables in a type II_1 von Neumann algebra

We consider the following subsets of the (real) vector space W introduced in the preceding section:

Let

$$K_{\mathrm{II}_1} = \sigma(V^*, V)\text{-closure}\{(\tau(x_I))_{I \in \mathcal{I}_n} \mid M \text{ type } \mathrm{II}_1 \text{ separable von Neumann algebra, with faithful trace } \tau,\ x_i = x_i^* \in M,\ i = 1, 2, \dots, n\}.$$

Here by x_ϕ we mean the unit of M.

We also consider the following set. Let tr be the normalized trace $\frac{1}{N}\,\mathrm{Tr}$ on $M_N(\mathbb{C})$. Then

$$K_{\rm mat} = \sigma(V^*, V)\text{-closure}\{(\mathrm{tr}(x_I))_{I \in \mathcal{I}_n} \mid N \in \mathbb{N},\ x_i = x_i^* \in M_N(\mathbb{C})\}.$$

We clearly have that $K_{\rm mat} \subseteq K_{\mathrm{II}_1} \subseteq W$ and as proven in [9], $K_{\rm mat}$, K_{II_1} are convex ($\sigma(V^*, V)$-closed or $\sigma(V^*_{\rm sym}, V_{\rm sym})$ closed sets. Let $\tilde{K}_{\mathrm{II}_1}$, $\tilde{K}_{\rm mat}$ be the (convex) cones generated by the convex sets K_{II_1} and $K_{\rm mat}$

We consider also the following subset of $V_{\rm sym} \subseteq V$

$$L_p = \overline{\mathrm{co}}^{\sigma(V, V^*)}\{a^* * a \mid a \in V\},$$

where $*$ is the operation introduced in Proposition 2.1.

Observation 3.1. In the identification of V with $\mathbb{C}_{\rm an}[Y_1, Y_2, \dots, Y_n]$, L_p corresponds to the series that are limits of sums of squares. Moreover, $W^{\perp} \cap V_{\rm sym}$ corresponds to the $\sigma(V, V^*)$-closure of the span of the selfadjoint part of monomials of the form $Y_I - Y_{\sigma(I)}$, $\sigma \in S_p$. Hence $L_p + (W^{\perp} \cap V_{\rm sym}$ corresponds to the series in $\mathbb{C}^{\rm sym}_{\rm an}[Y_1, \dots, Y_n]$ that are equivalent to a weak $(\sigma(V, V^*))$ limit of sum of squares:

$$\sum_{s \in S} b^*_{s^*} b_s, \quad b_s \in \mathbb{C}_{\rm an}[Y_1, Y_2, \dots, Y_n].$$

Moreover, $L_p \subseteq K^0_{\mathrm{II}_1} \subseteq K^0_{\rm mat}$.

Proof. Indeed the series corresponding to $a^* * a$ is

$$\sum_{I\in\mathcal{I}_n}(a^* * a)_I Y_I = \sum_I \sum_{J\sharp K} a_J^* a_K Y_{J\sharp K} = \sum_{I\in\mathcal{I}_n}\sum_{J\sharp K=I}\overline{a_{J^{\mathrm{op}}}}a_K Y_J Y_K$$

$$= \sum_{J,K\in\mathcal{I}_n}\overline{a}_J a_K Y_{J^{\mathrm{op}}}Y_K = \Big[\sum_{J\in\mathcal{I}_n}(\overline{a}_J Y_J^*)\Big]^*\Big(\sum_{K\in\mathcal{I}_n}a_K Y_K\Big).$$

Moreover, when pairing a series $\sum a_I Y_I \in \mathbb{C}_{\mathrm{an}}[Y_1,\dots,Y_n]$ with $t_I = \tau(x_I)$, $I\in\mathcal{I}_n$, $x_1,\dots,x_n\in M$, $x_i = x_i^*$, the result is $\sum a_I\tau(x_I) = \tau\Big(\sum a_I x_I\Big)$. Hence an element in L paired with an element in K_{II_1} is the trace of a sum squares and hence is positive. Thus, $L_p\subseteq K^0_{\mathrm{II}_1}$ and hence $L_p + (W^\perp\cap V_{\mathrm{sym}})\subseteq K^0_{\mathrm{II}_1}$ so $L_p^0\cap W\supseteq K_{\mathrm{II}_1}$. □

With these observations we can prove Theorem 1.1.

Proof. By the above observations and by Lemma 2.4 this amounts to prove that

$$L_p + (W^\perp\cap V_{\mathrm{sym}}) = K^0_{\mathrm{II}_1}$$

and by the previous remarks we only have to prove that

$$L_p^0\cap W\subseteq \tilde{K}_{\mathrm{II}_1}.$$

Thus we want to show that if an element $\theta = (\theta_I)_{I\in\mathcal{I}}$ with $\theta_\emptyset = 1$ belongs to L_p^0 then there exists a type II_1 von Neumann algebra M with faithful trace τ, and selfadjoint elements $x_1,\dots,x_n$ in M such that

$$\theta_I = \tau(x_I)\quad \text{for } I\in\mathcal{I}_n.$$

To construct the Hilbert space on which M acts consider the vector space

$$\mathcal{H}_0 = \mathbb{C}_{\mathrm{an}}[Z_1,\dots,Z_n]$$

where $Z_1,Z_2,\dots,Z_n$ are indetermined variables. We consider the following scalar product on $\mathcal{H}_0$:

$$\langle Z_I, Z_J\rangle = \theta_{J^{\mathrm{op}}\sharp I}$$

or more general for $(a_I)_{I\in\mathcal{I}_n}, (b_I)_{I\in\mathcal{I}_n}\in V$

$$\Big\langle \sum_{I\in\mathcal{I}_n}a_I Z_I, \sum_{J\in\mathcal{I}_n}b_J Z_J\Big\rangle = \langle b^* * a,\theta\rangle.$$

(Recall that $\langle\ ,\ \rangle$ is the pairing between V and V^* introduced in Section 2). Note that the scalar product is positive since

$$\Big\langle \sum_{I\in\mathcal{I}_n}a_I Z_I, \sum_{I\in\mathcal{I}_n}a_I Z_I\Big\rangle = \langle a^* * a,\theta\rangle,$$

which is positive since $t\in L_p^0\cap W\subseteq L_p^0$.

We let $\mathcal{H}$ to be the Hilbert space completion of $\mathcal{H}_0$ after we mod out by the elements ξ with $\langle\xi,\xi\rangle = 0$. Note that in this Hilbert space completion, we have obviously that $\sum\limits_{I\in\mathcal{I}_n}a_I Z_I$ is the Hilbert space limit after $p\to\infty$ of $\sum\limits_{\substack{I\in\mathcal{I}_n\\ |I|\leqslant p}}a_I Z_I$.

We consider the following unitary operators U_t^i, $i = 1, 2, \ldots, n$, $t \in \mathbb{R}$ acting on $\mathcal{H}_0$ isometrically and hence on $\mathcal{H}$.

For $i \in \{1, 2, \ldots, n\}$ and $q \in \mathbb{N}$ we will denote by i^q the element $(i, i, \ldots, i)$ of length q and belonging to $\mathcal{I}_n$.

The following formula defines U_t^i

$$U_t^i Z_I = \sum_{s=0}^{\infty} \frac{(it)^s}{s!} Z_{i^s \sharp I} \quad \text{for } I \in \mathcal{I}_n.$$

To check that U_t^i is corectly defined, and extendable to an unitary in $\mathcal{H}$ it is sufficient to check that

$$\langle U_t^i Z_I, U_t^i Z_J \rangle = \langle Z_I, Z_J \rangle \tag{1}$$

for all $I, J \in \mathcal{I}_n$. But we have

$$\begin{aligned}\langle U_t^i Z_I, U_t^i Z_J \rangle &= \sum_{p,q} \frac{(it)^p \overline{(it)^q}}{p!q!} \langle Z_{i^p \sharp I}, Z_{i^q \sharp J} \rangle = \sum_{p,q} \frac{(it)^p \overline{(it)^q}}{p!q!} \theta_{J^{\mathrm{op}} \sharp i^{p+q} \sharp I} \\ &= \sum_{p,q} \frac{i^{p-q} t^{p+q}}{p!q!} \theta_{J^{\mathrm{op}} \sharp i^{p+q} \sharp I}.\end{aligned}$$

Note that all the changes in summation are allowed and that all the series are absolutely convergent since $(t_I)_{I \in \mathcal{I}} \in W$ and hence there exists $R > 0$ such that $|t_I| \leqslant R^{|I|}$ (we also use the analiticy of the series $\sum\limits_{p \geqslant 0} \frac{(it)^p}{p!}$.

For a fixed k, the coefficient of t^k in the above sum is

$$t^k (\theta_{J^{\mathrm{op}} \sharp i^k \sharp I}) \sum_{p+q=k} \frac{i^{p-q}}{p!q!}$$

and this vanishes unless $k = 0$ because of the corresponding property of the exponential function: $e^{it} e^{-it} = 1$.

This proves (1) and hence that U_t^i can be extended to an isometry on $\mathcal{H}_0$ and hence on $\mathcal{H}$.

Next we check that for all $j = 1, 2, \ldots, n$,

$$U_t^j U_s^j = U_{t+s}^j \quad \text{for all } t, s \geqslant 0$$

for $I \in \mathcal{I}_n$. But

$$U_t^j U_s^j Z_I = \sum_{p,q=0}^{\infty} \frac{(it)^p (is)^q}{p!q!} Z_{j^p \sharp j^q \sharp I} = \sum_{k=0}^{\infty} \sum_{p+q=k} \frac{(it)^p (is)^q}{p!q!} Z_{j^k \sharp I}$$

and this is then equal to $U_{t+s}^j Z_I$ because of the corresponding property for the exponential function: $e^{is} e^{it} = e^{i(s+t)}$. From this it then follows that U_t^j is a one parameter group of unitaries for all $j = 1, 2, \ldots, n$.

We let M be the von Neumann algebra of $B(\mathcal{H})$ generated by all U_t^j, $j = 1, 2, \ldots, n$, $t \in \mathbb{R}$. We will verify that $Z_\emptyset$ is a cyclic trace vector for M (and hence it will be also separating, [13]).

To verify that $Z_\emptyset$ is a trace vector it sufficient to check that for all p, q and all $i_1, i_2, \dots, i_p, j_1, j_2, \dots, j_q \in \{1, 2, \dots, n\}$ we have that for all $t_1, \dots, t_p, s_1, \dots, s_q \in \mathbb{R}$

$$\langle U_{t_1}^{i_1} \cdots U_{t_p}^{i_p} \cdots U_{s_1}^{j_1} \cdots U_{s_q}^{j_q} Z_\Phi, Z_\Phi \rangle = \langle U_{s_1}^{j_1} \cdots U_{s_q}^{j_q} U_{t_1}^{i_1} \cdots U_{t_p}^{i_p} Z_\Phi, Z_\Phi \rangle. \tag{2}$$

It is immediate that the right side of (2) is equal to

$$\sum_{\substack{\alpha_1, \dots, \alpha_p = 1 \\ \beta_1, \dots, \beta_q = 1}}^{\infty} \frac{(it_1)^{\alpha_1} \cdots (it_p)^{\alpha_p} (is_1)^{\beta_1} \cdots (is_q)^{\beta_q}}{\alpha_1! \cdots \alpha_p! \beta_1! \cdots \beta_q!} \theta_{i_1^{\alpha_1} \cdots i_p^{\alpha_p} j_1^{\beta_1} \cdots j_q^{\beta_q}}.$$

Because of the cyclic symmetry of $(\theta_I)_{I \in \mathcal{I}_n}$ is then equal to the lefthand side of (2).

The sums involved are absolutely convergent since $|t_I| \leqslant R^{|I|}$, $I \in \mathcal{I}_n$ for some $R > 0$ and because of the fact that the scalar function involved are entire analytic functions.

We now prove that Z_Φ is a cyclic vector. Denote by iy_j the selfadjoint generator for the group U_t^j, $j = 1, 2, \dots, n$.

We claim that for all $I \in \mathcal{I}_n$, the vector Z_I belongs to the domain of iy_j and that $y_j Z_I = Z_{j \sharp I}$, $j = 1, 2, \dots, n$, $I \in \mathcal{I}_n$. Indeed this follows by evaluating the limit

$$\lim_{t \to 0} \frac{1}{i} \frac{1}{t} (U_t^j Z_I - Z_I) - Z_{j \sharp I}.$$

But this is equal to

$$t \left(\sum_{q=2}^{\infty} \frac{(it)^q}{q!} Z_{j^n \sharp I} \right)$$

and this converges to zero when $t \to 0$ because of the summability condition for $(t_I)_{I \in \mathcal{I}_n}$.

Indeed the square of norm of this term is

$$t^2 \sum_{p,q=2}^{\infty} \frac{(it)^p \overline{(it)^q}}{p! q!} \theta_{I^* \circ j^{n+m} \circ I}.$$

From this we deduce that Z_Φ belongs to domain of $y_{j_1} \cdots y_{j_q}$ and $y_{j_1} \cdots y_{j_q} Z_\Phi = Z_{(j_1 \cdots j_q)}$.

This implies in particular that Z_Φ is cyclic for M and also that

$$\langle y_I Z_\Phi, Z_\Phi \rangle = \theta_I, \quad I \in \mathcal{I}_n.$$

By [13] it follows that Z_Φ is also separating for M and hence that M is a type II_1 von Neumann algebra and that the vector form $\langle \cdot Z_\Phi, Z_\Phi \rangle$ is a trace on M.

By definition the elements y_j, $j = 1, 2, \dots, n$ are selfadjoint and affiliated with M.

Moreover since $\tau(m) = \langle m Z_\Phi, Z_\Phi \rangle$ is a faithfull finite trace on M and since there exist $R > 0$ such that

$$\tau(y_j^{2k}) = \theta_{j^{2k}} \leqslant R^{2k} \quad \text{for all } k,$$

it follows that y_j are bounded elements in M such that $\tau(y_I) = \theta_I$ for all $I \in \mathcal{I}_n$. This proves that $(\theta_I)_{I \in \mathcal{I}_n}$ belongs to K_{II_1} and hence that $L_p + (W^\perp \cap V_{\mathrm{sym}}) \subseteq \tilde{K}_{\mathrm{II}_1}$. This completes the proof of the theorem. □

We are now ready to prove Corollary 1.2.

Proof. With our previous notations this amounts to show that the statement $L_p + (W^\perp \cap V_{\rm sym}) = K^0_{\rm mat}$ is equivalent to the Connes embedding conjecture.

Since we already know that $L_p + (W^\perp \cap V_{\rm sym}) \subseteq \tilde{K}_{{\rm II}_1}$, this amounts to prove that the statement that $K_{\rm mat} = K_{{\rm II}_1}$ is equivalent to Connes conjecture.

The only non-trivial part is that the statement $K_{\rm mat} = K_{{\rm II}_1}$ implies the Connes conjecture. So assume that for a given type $\rm II_1$ von Neumann algebra M and for given $x_1, \dots, x_n$ in M there exists $(X^k_i)^n_{i=1}$ in $M_{N_k}(\mathbb{C})$ such that $(\mathrm{tr}(X^k_I))_{I \in \mathcal{I}_n}$ converges in the $\sigma(W, V)$ topology to $(\tau(x_I))_{I \in \mathcal{I}_n}$.

By Lemma 2.3, we know that there exists $R > 0$ such that for all $k \in \mathbb{N}$

$$|\,\mathrm{tr}(X^k_I)| \leqslant R^{|I|}$$

and hence that

$$\mathrm{tr}((X^k_j)^{2p}) \leqslant R^{2p} \quad \text{for all } j = 1, 2, \dots, n,\ p, k \in \mathbb{N}.$$

But this implies that $\|X^k_j\| \leqslant R$ for all $k \in \mathbb{N}$, $j = 1, 2, \dots, n$.

But then the map $x_j \to (X^k_j)_{k \in \mathbb{N}}$ extends to an algebra morphism from M into R^ω which preserves the trace. □

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