Weak, strong and 4 semigroup solutions of classical SDE: an example

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Abstract

We propose a notion of solution of a classical SDE which is intermediate between weak and strong and which reduces the theory of stochastic flows to the study of a particular class of semigroups. We illustrate this idea with the example of the Ornstein-Uhlenbeck flow for which the associated 4 semigroups can be explicitly calculated.

1 Introduction

The structure of the present note is the following. In section (**??**) we recall some known facts about strong and weak solutions of SDE and we refer to [KarShre88] for more information. In section (**??**) we motivate the definition of the 4–semigroup–solution of an SDE. Finally, in section (**??**) we describe the 4 semigroups canonically associated to the Ornstein-Uhlenbeck process.

The main point of the 4–semigroup–solution, which is intermediate between weak and strong solutions, is that it reduces the theory of stochastic flows to the study of a particular class of semigroups. This fact was exploited in [AcKo99b], [AcKo00b] to prove the existence of the infinite volume flow of a class of interacting particle systems by means of Hille–Yoshida type estimates. The usual existence criteria for classical or quantum flows were not applicable to this cases.

Here we deal only with scalar valued processes but the fact that the theory can be applied to arbitrary vector valued (including infinite dimesional) processes supports our hope that, combining this approach with some analytical estimates due to Röckner, one could prove existence results for stochastic flows which cannot be handled with the present techniques.

2 Weak and strong solutions of SDE

We will only consider real valued processes and all L^p–spaces considered ($1 \leq p \leq \infty$) will be complex valued). For any Hilbert space $\mathcal{H}$, we denote $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

Definition 1 *Given a probability space* $(\Omega, \mathcal{F}, P)$*, a filtration* $(\mathcal{F}_{t]})$ *in* $(\mathcal{F})$*, an* $(\mathcal{F}_{t]})$*-Brownian motion is a process* $W_t : \Omega \to \mathbb{R}$ *such that*

(i) $(W_t - W_s)$ *is a mean zero gaussian* $(\mathcal{F}_{t]})$*-adapted independent increment process with variance* $|t - s|$

(ii) $\forall\, t \in \mathbb{R}_+$ *and* $P - \forall\, \omega \in \Omega$*, the map* $t \mapsto W_t(\omega)$ *is continuous.*

Definition 2 *Given a probability space* $(\Omega, \mathcal{F}, P)$*, a filtration* $(\mathcal{F}_{t]})$ *an* $(\mathcal{F}_{t]})$*-Brownian motion* (W_t) *and two measureble functions* $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$*, a strong solution of the SDE*

$$dX_t = bdt + \sigma dW \tag{1}$$

Is a real valued stochastic proces $X \equiv (X_t)$, defined on $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_{t]})$–adapted, satisfying

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{2}$$

in the sense that the integrals exist and the identity holds.

Definition 3 *In the above notations, a weak solution of (**??**) is a quadruple*

$$X \equiv (X_t) \ ; \quad \hat{W} \equiv (\hat{W}_t) \ ; \ (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \ ; \quad (\hat{\mathcal{F}}_{t]})$$

with the following properties:

(i) *$(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ is a probability space*

(ii) *$(\hat{\mathcal{F}}_{t]})$ is a filtration in $(\hat{\mathcal{F}})$*

(iii) *X is an $(\hat{\mathcal{F}}_{t]})$–adapted, continuous trajectories, real valued process*

(iv) *$\hat{W}$ is an $(\hat{\mathcal{F}}_{t]})$-adapted Brownian motion*

(v) *X satisfies the integral equation*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)d\hat{W}_s \tag{3}$$

Thus the main difference between a strong and a weak solutions of an SDE is that, in the strong case the BM is given a priori and the solution is adaptd to the filtration generated by it and by its initial data, while in the weak case the BM is built from the solution and is adaptd to the filtration generated by it. A typical example of an equation admitting a weak solution which is not strong is:

$$dX_s = \text{ sgn } (X_s)dB_s$$

Definition 4 *Given b, σ as above, a Markovian generator L on $C^\infty(\mathbb{R}_+, \mathbb{R})$, $\Omega = C(\mathbb{R}_+; \mathbb{R})$, $\mathcal{F} =$ Borel $(\Omega), (\mathcal{F}_{t]})$ the natural filtration, a solution of the martingale problem for L is a probability measure on $(\Omega, \mathcal{F})$ such that*

$$f(X_t) - \int_0^t (Lf)(X_s)ds =: M_t \quad ; \quad \forall\, f \in C^\infty(\mathbb{R}_+, \mathbb{R})$$

is a $(\mathcal{F}_{t]}) - P$–local martingale with continuous trajectories.

It is known that, under general conditions, given a solution X of the martingale problem for L there exist two measurable functions b, σ, with b and $|\sigma|$ uniquely determined by L, such X is a weak solution of (**??**).

3 The 4–semigroup–solution

Let $W = (W_t)$ be a given Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with associated filtration $(\mathcal{F}_{t]})$ and associated L^2-space

$$\Gamma = L^2(\Omega, \mathcal{F}, P)$$

Let $X_0 = X(0)$ be the initial data of equation (**??**). We assume that X_0 is a random variable with distribution equivalent to the Lebesgue measure. Therefore X_0 can be identified to the self-adjoint multiplication operator on $L^2(\mathbb{R})$ given by

$$\mathrm{id}(x) := x \qquad ; \qquad x \in \mathbb{R} \tag{4}$$

In the same way $L^\infty(\mathbb{R})$ is identified to the algebra of multiplication by bounded measurable functions acting on $L^2(\mathbb{R})$. When ambiguities are possible we write M_f to distinguish between $f \in L^\infty(\mathbb{R})$ and the corresponding multiplication operator.

If (X_t) is a strong solution of equation (**??**), then the map

$$j_t : f \in L^\infty(\mathbb{R}) \to j_t(f) := M_{f(X_t)} \in L^\infty(\mathbb{R} \times \Omega) \subseteq \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \tag{5}$$

is a w^*–continuous Markov flow of random multiplication operators. According to the convenience one can replace the algebra $L^\infty(\mathbb{R})$ by other algebras such as $C_b(\mathbb{R})$, cylindrical functions, ...

Conversely, given j_t, the process (X_t) is uniquely determined by the relation

$$j_t(\mathrm{id}) = X_t \tag{6}$$

and by the fact that the function id, defined by (**??**), is a limit, in the strong operator operator topology on $\mathcal{B}(L^2(\mathbb{R}))$, of bounded measurable functions.

The flow equation for $f(X_t)$:

$$df(X_t) = \sigma f'(X_t)dW_t + \left(\frac{\sigma^2}{2} f''(X_t) - bX_t f'(X_t)\right) dt. \tag{7}$$

when translated in terms of the corresponding flow of multiplication operators $(j_t(f) = M_{f(X_t)})$ becomes:

$$dj_t(f) = i[p(\sigma), j_t(f)]dw_t - \frac{1}{2}[p(\sigma), [p(\sigma), j_t(f)]]dt + i[p(b_\sigma), j_t(f)]dt \tag{8}$$

where f runs in a suitable domain in $L^\infty(\mathbb{R})$ and, for any differentiable function g we use the notations:

$$[a, b] := ab - ba \tag{9}$$

$$p(g) := \frac{1}{2}\left(gp + pg\right) \tag{10}$$

$$p := \frac{1}{i}\partial_x \tag{11}$$

Equation (**??**) continues to have a meaning if we replace the multiplication operator f by an arbitrary bounded operator A on $L^2(\mathbb{R})$ for which all the commutators make sense. This gives (formally) a quantum extension of the equation of a classical diffusion flow. Formally any diffusion has a quantum extension; analytically this is false even in 1 dimension (cf. [FagMon96]). Furthermore, even at a formal level, there are several quantum extensions of a classical diffusion flow (e.g. in (**??**) one can replace $p(\sigma)$ and $p(b_\sigma)$ by $p(\sigma) + u$ and $p(b_\sigma) + v$, where u, v are arbitrary multiplication operators, without changing the classical flow). See [Fag99] Sect. 4.2 for a detailed discussion including the d-dimensional case. In the present note we only discuss the classical case for which it can be proved that the results below do not depend on the choice of this extension.

Let us introduce the notations:

$$\psi_0 := 1 \qquad ; \qquad \psi_{\chi_{[0,t]}} = e^{-\int_0^t dW_s - \frac{t}{2}} \in L^2(\Omega, \mathcal{F}, P) \tag{12}$$

where 1 denotes the constant function equal to 1 in $L^2(\Omega, \mathcal{F}, P)$. Moreover, if $\varphi, \psi \in L^2(\Omega, \mathcal{F}, P)$ are arbitrary vectors, the map

$$b \otimes B \in \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \mapsto b\langle\varphi, B\psi\rangle \in \mathcal{B}\left(L^2(\mathbb{R})\right)$$

has a unique extension to a bounded linear map denoted

$$A \in \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \mapsto \langle\varphi, A\psi\rangle \in \mathcal{B}\left(L^2(\mathbb{R})\right)$$

One can show that this map is an extension of the time zero conditional expectation $E_{0]}$ onto the σ–algebra of the initial condition X_0 of equarion (**??**), i.e. if $A = f(X_t)$ then

$$\langle\varphi, f(X_t)\psi\rangle = E_{0]}(\varphi \cdot f(X_t) \cdot \psi)$$

where, for $\omega \in \Omega$, $(\varphi \cdot f(X_t) \cdot \psi)(\omega)$ is the multiplication by: $\varphi(\omega) f(X_t((\omega))\psi(\omega)$ (replacing $X_t(\omega)$ by $X_t(x, \omega)$ – the solution of (**??**) starting at $x \in \mathbb{R}$ – one would obtain a multiplication operator on $L^2(\mathbb{R})$). With these notations one can define 4 1–parameter linear maps

$$P_{00}^t, P_{01}^t, P_{01}^t, P_{11}^t : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$$

through the prescription :

$$\begin{pmatrix} P_{00}^t(f)(X_0) & P_{01}^f(f)(X_0) \\ P_{01}^t(f)(X_0) & P_{11}^t(f)(X_0) \end{pmatrix} := \begin{pmatrix} \langle \psi_0, j_t(f)\psi_0 \rangle & \langle \psi_0, j_t(f)\psi_{\chi_{[0,t]}} \rangle \\ \langle \psi_{\chi_{[0,t]}}, j_t(f)\psi_0 \rangle & \langle \psi_{\chi_{[0,t]}}, j_t(f)\psi_{\chi_{[0,t]}} \rangle \end{pmatrix} =$$

$$= \begin{pmatrix} E_{0]}(f_1(X_t)) & E_{0]}(f_2(X_t)e^{-\int_0^t dw_s - \frac{t}{2}}) \\ E_{0]}(f_2(X_t)e^{-\int_0^t dw_s - \frac{t}{2}}) & E_{0]}(f_s(X_t)e^{-2\int_0^t dw_s - t}) \end{pmatrix} \tag{13}$$

The second identity in (**??**) shows that $P_{01}^t = P_{01}^t$ and that all the $P_{\alpha\beta}^t$ are positivity preserving. Both properties are not obvious from the first identity and in the quantum case they are not true in general.

It can be proved [AcKo99b], [AcKo00b] that $P_{00}^t, P_{01}^t, P_{01}^t, P_{11}^t$ are w^*–continuous semigroups and that they uniquely determine the classical flow $j_t(f) = f(X_t)$ in the sense that they uniquely determine all the partial scalar products (in $L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)$):

$$\langle e^{\int_\mathbb{R} g_s dW_s}, f(X_t) e^{\int_\mathbb{R} h_s dW_s} \rangle = E_{0]} \left(e^{\int_\mathbb{R} g_s dW_s} \cdot f(X_t) \cdot e^{\int_\mathbb{R} h_s dW_s} \right)$$

for any choice of $g, h \in L^2(\mathbb{R})$. By the totality of the exponential martingales these products uniquely determine the flow $f(X_t)$.

By the Hille–Yoshida theorem also the generators of these 4 semigroups

$$\begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

uniquely determine the classical flow $j_t(f) = f(X_t)$ in the same sense.

It is clear from (**??**) that P_{00}^t is the usual Markov semigroup associated to the process (X_t) and the other ones are perturbations of it. The formal generators of these semigroups are easily determined using (**??**) and the Ito formula:

$$L_{00} = \frac{1}{2}\sigma^2 \partial_x^2 - b\partial_x \tag{14}$$

$$L_{01} = L_{10} = L_{00} + \sigma \partial_x \tag{15}$$

$$L_{11} = L_{00} + \sigma\partial_x + 1 \tag{16}$$

however it should be emphasized that the sums in (??) and (??) are only formal in the sense that it may happen that the integral expressions (??) are well defined while the domains of the corresponding differential operators have zero intersections. The situation is exactly analogous to what happens in the usual Feynman–Kac or Girsanov formula. In fact the semigroups (??) are Girsanov perturbations of the basic Markov semigroup.

We sum up our conclusions in the following theorem.

Theorem 1 *A necessary condition for a process (X_t) to be a strong solution of equation (??) is that the four operators defined by (??), (??), (??) are generators of w^*–continuous semigroups (possibly on different domains).*

Remark One can prove the converse of the above statement if the following linear combinations of the above four generators:

$$\theta_0 := L_{00} = b\partial_x + \frac{1}{2}\sigma^2\partial_x^2 \qquad ; \qquad \theta_2 := \sigma\partial_x \tag{17}$$

have a common core containing a sequence of functions (necessarily twice continuously differentiable) converging to the multiplication operator by the function id, defined by (??), strongly on a core of this operator.

In view of the above result the following definition is quite natural.

Definition 5 *Given b, σ, W and $(\Omega, P, \mathcal{F}, (\mathcal{F}_{t]})$ as above, we say that the SDE (??) admits a solution in the sense of the 4 semigroups, if the four operators defined by (??), (??), (??) are generators of w^*–continuous semigroups (possibly on different domains).*

4 The 4 semigroups canonically associated to the Ornstein-Uhlenbeck process

The classical (one-dimensional) Ornstein-Uhlenbeck process is the real-valued stochastic process satisfying

$$dX_t = \sigma dW_t - bX_t dt \qquad ; \qquad X(0) = X_0$$

where σ, b are positive constants and $(W_t)_{t\geq 0}$ is the standard Wiener process on $(\Omega, \mathcal{F}, P)$. Equivalently

$$X_t = X_0 \mathrm{e}^{-bt} + \sigma \int_0^t \mathrm{e}^{-b(t-s)} dW_s.$$

The associated flow $j_t : L^\infty(\mathbb{R}) \to L^\infty(\Omega, \mathcal{F}, P)$; $t \geq 0$ is defined by:

$$j_t(f) := f(X_t)$$

and, for $f \in C^2(\mathbb{R})$ it satisfies the equation

$$df(X_t) = \sigma f'(X_t) dW_t + \left(\frac{\sigma^2}{2} f''(X_t) - bX_t f'(X_t) \right) dt. \tag{18}$$

In this case (i.e. with σ and b constants) the four operators defined by (**??**), (**??**), (**??**) are effectively generators of strongly (and not only w^*–) continuous semigroups which can be written explicitly by applying Mehler's formula to the two perturbations of L_{00}:

$$P^t_{00}(f)(x) = \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt} x + \sqrt{(1-\mathrm{e}^{-bt})/b}\, y \right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy \tag{19}$$

$$P^t_{01}(f)(x) = \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt} x + \sqrt{(1-\mathrm{e}^{-bt})/b}\, y + \sigma t \right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy \tag{20}$$

$$P^t_{10}(f) = P^t_{01}(f) \tag{21}$$

$$P^t_{11}(f)(x) = \mathrm{e}^t \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt} x + \sqrt{(1-\mathrm{e}^{-bt})/b}\, y + \sigma t \right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy. \tag{22}$$

Indeed, for a smooth f (which is bounded by assumption), the right hand sides of all these identities are differentiable in t and lead to the correct PDE which has a unique solution.

Remark. Although the semigroups (**??**), (**??**) are simple perturbations of the semigroup (**??**), it is worth noticing that the differentiation operator on L^∞ $f \mapsto \sigma f'$ is not relatively bounded with respect to the generator (**??**). Indeed, for $\sigma^2 = 2, b = 2$, there exists a sequence (f_n) of smooth functions vanishing at infinity and satisfying:

$$\|f_n\|_\infty = \pi/(2\sqrt{n}), \quad \|f'_n\|_\infty \geq \mathrm{e}^{-1} \log(1+n), \quad \|L^0_0(f_n)\|_\infty = 1/\sqrt{n}$$

Therefore the generators (**??**), (**??**) are simple but not "regular" perturbations of the generator (**??**).

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