AN INVITATION TO THE WEAK COUPLING AND LOW DENSITY LIMITS

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ABSTRACT

We describe a new approach to the weak coupling and low density limits, introduced in previous papers. We outline the basic ideas of the proof. We also explain the differences and the common points between the two limits. As a new result we establish the connection between the quantum stochastic differential equation, deduced in [11] for the low density limit and the one-particle scattering operator appearing in the generator of the semigroup obtained by Dümcke ([25]) as low density limit of the reduced evolution.

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§0 Introduction

The problem of constructing a satisfactory theory of the irreversible and dissipative quantum phenomena has a long history. The basic physical idea is that dissipation and irreversibility in the behavior of a system arise when this system interacts with some "macroscopic system" (called, according to the interpretation, **noise** or **reservoir** or **heat bath**,...): the composite system evolves according to the usual Hamiltonian dynamics and, due to the interaction, if X is an observable of the initial system and $X(t)$ its time evolved in the Heisenberg picture, then $X(t)$ will depend also on the degrees of freedom of the reservoir. If these are averaged out, one obtains a new system observable $\overline{X}(t)$ which no longer obeys a reversible evolution. The map $X \to \overline{X}(t)$ is called the **reduced dynamics**. The equations obeyed by the observables of the form $\overline{X}(t)$ are usually complicated (integro–differential equations) and give little insight because the details of the evolution of the reservoir are in principle uncontrollable and the information which one usually has on the reservoir is reduced to a few macroscopic parameters (such as temperature, relaxation times, leading frequencies, coupling constants between system–reservoir,...). One is therefore led to study the evolution $\overline{X}(t)$ in some limiting situation, in the hope that the limiting procedure will sweep away the inessential details and produce a meaningful equation.

Many such limiting procedures have been considered (weak coupling, low density, singular coupling, hydrodynamical limit,...) and they describe different physical situations. In the present paper we shall discuss the first two (it is known that the singlar coupling limit is essentially equivalent to the weak coupling limit). Motivated by some results of Friedrichs ([32]) on second order perturbation theory, van Hove ([30]) introduced the **weak coupling limit** (WCL) as a device to derive a Pauli–type master equation for the irreversible time evolution of the macroscopic observables of a large quantum system. The idea of the low density limit (LDL) originates from Grad ([33]), who considered this limit in order to derive the Boltzmann equation for the one–particle distribution function of a classical dilute gas. Throughout this paper, we use simply WCL for weak coupling limit and LDL for low density limit. Both limits have been studied in the framework of the rigorous theory of open quantum systems: the WCL by Martin and Emch ([34]), Dell'Antonio ([23a]), Davies ([20,21,22]), Pulè ([29]); the LDL by Palmer ([28]) and Dümcke ([25]). For a review on the derivation of the classical kinetic equations from Hamiltonian dynamics, see Spohn [35]. In all these papers only the reduced dynamics of a system, in interaction with some reservoir, is considered and their main results consist in proving that, in the limiting situations and in the models considered, the reduced dynamics becomes a semigroup whose generator is uniquely determined in terms of quantities related to the original Hamiltonian model. The main theoretical insight in this first phase of development was the GKSL (Gorini, Kossakowski, Sudarshan ([36]), Lindblad ([37])) theorem on the structure of bounded generators of quantum Markovian semigroups.

A more ambitious plan is to deduce a limiting equation not only for the reduced evolution of the system, but for the whole coupled system, including the reservoir. The relevance, for the physical applications, of the additional information obtained with this procedure is explained by Gardiner and Collet ([41a]), Barchielli ([41b], [41c]). The first difficulty of this program is that there is no hope that the limiting evolution will live on the Hilbert space where the approximate evolutions live, hence standard operator techniques cannot be applied. In fact, in the analogous classical situations, the approximate evolutions converge only in the probabilistic sense of **convergence in law** and, until the late eighties, there was no natural quantum analogue (or substitute) for this notion (Dümcke's proposal [26] to consider the limits of correlation kernels in the sense of [38] met some difficulties in its implementation, with the non–time–ordered correlations). Moreover, the classical analogy suggested that the limiting equations should be not ordinary, but stochastic differential equations, and until the early eighties there was no quantum analogue of these.

A turning point in this development was Hudson and Parthasarathy's introduction of quantum stochastic calculus ([31]). This allowed to produce unitary dilations (in the sense of Kümmerer [46]) of those irreversible quantum evolutions that were obtained as WCL or LDL of reduced dynamics of open quantum systems and, by doing so, to produce natural candidates for the limit equations, governing the composite (system + noise) system. In other terms the existence of these unitary dilations ([46], [47], [48], [49a], [49b], [50], [27]) and their very natural physical interpretation, suggested that if the limit of the evolution of the coupled system existed (in whatever sense), then it should not satisfy an ordinary Hamiltonian equation, but a quantum stochastic equation in the sense of Hudson and Parthasarathy. The program of synthetizing the various experiences of laser models, deductions of master equations, construction of dilations and quantum central limit theorems, was formulated by Frigerio in [40], where some of basic lines of the future developement were outlined. The theoretical status of this program was however quite unclear for several reasons:

(i) Although currently used in quantum optics (especially laser theory) since their introduction in the early sixties ([42], [43a], [43b]), the various types of quantum Brownian motions were poorly understood theoretically because the mutual compatibility of the approximations introduced for their deduction ([44]) mas not clear. Even worse, some of the limiting equations seemed to depend on the order of these approximations ([51], [52]).

(ii) It was never made precise in what sense the quantum Brownian motions had to be considered approximations of the quantum electromagnetic field.

(iii) The quantum Poisson process was introduced by Hudson and Parthasarathy in [31], and generalized by Frigerio and Maassen in [40], on a purely mathematical basis. For this new quantum noise even the phenomenological justifications, existing for the quantum Brownian motion, were missing, being replaced only by some physical intuitions of Frigerio and Maassen suggesting that, in some sense, the low density limit should be related to the quantum Poisson process just as the weak coupling limit is related to the quantum Brownian motion. The later developement are fully confirming the intuitions of Frigerio and Maassen, but the mathematical problem to be overcome even to **formulate** the LDL problem were formidable (and some of them are still open). For example: How to obtain a Poisson process from a low density limit? Which quantum Poisson process arise in this way? How to relate these processes with Palmer's and Dümcke's results on the LDL?

(iv) All the above mentioned quantum noises live in a representation of the CCR (or CAR) algebra $W(L^2(\mathbf{R}, dt\,;\, K))$ over the space of square–integrable functions of a "time variable" $t \in \mathbf{R}$ with values in a suitable Hilbert space K, and their time evolution is just the time shift (which has Lebesgue spectrum); the representation is determined by a quasi–free state with covariance operator of the form $1 \otimes Q$, Q being an operator on K. By contrast, a physical reservoir in thermal equilibrium is described (in the Boson case) by the CCR algebra $W(L^2(\mathbf{R}^d))$, $\mathbf{R}^d$ being d–dimensional physical space, and its time evolution is determined by a one–parameter unitary group on $L^2(\mathbf{R}^d)$ whose generator H (the one–particle Hamiltonian) is bounded from below: the covariance operator Q of the reference state is a function of H, say $Q = \coth(\frac{\beta}{2}(H-\mu))$. It was not clear how these two pictures were related and, in particular, if the former is in some sense a consequence of the latter. In particular, how does it happen that time is an exterior parameter in the usual physical description and a **test function parameter** in the stochastic picture?

(v) The physical meaning of the space K was not clear: in all the applications of stochastic calculus to physical models ([53], [54], [41], [20]), this space was constructed **by hands**, i.e. postulated on the basis of heuristic and phenomenological assumptions, while, if we look at the stochastic description as an approximation of a Hamiltonian description, then the structure of the space K should be deduced from the microscopic properties of the system.

The main purpose of the series of papers ([1],..., [16], [39]), was to try to answer the above mentioned problems and in particular to show that the quantum Brownian motions (QBM) and the quantum Poisson process (QPP) can be obtained as limits (in an appropriate sense) of "realistic" models of quantum reservoirs and, moreover, that the conditions which define the corresponding limiting procedures have a natural physical interpretation. Although the final equations obtained in the above mentioned papers are explicit and easy to deal with, the procedure to obtain them is quite nontrivial, involving among other things, estimates not available in the present literature on quantum field theory and a continuous feed–back between the guessing of the limiting stochastic equation and the rigorous deduction of it.

In the present paper we made the attempt to **invite** other researchers to explore the beautiful new territory which is emerging from these results and which still abounds of unexplored regions. Our goal is to outline the basic ideas and techniques of these developments by illustrating them with the simplest examples (we omit the Fermion case, the non–rotating wave approximation case, the details of the finite temperature case, the squeezing case, the problems concerning the Langevin equation, ...). Although our paper is of an expository nature, we have made the attempt to unify the multitude of technical lemmata needed in the various estimates (this attempt has been particularly successful for the **Pulè inequalities** (cf. Section 9)).

We also obtain a new result concerning the identification of the coefficients of the stochastic equation deduced in ([9]) with particular matrix elements of the T operator. This identification connects in a natural way the Alicki-Frigerio ([39]) formulation of the Dümcke generator ([25]) with the quantum stochastic differential equation obtained in [11].

A corollary of this identification is a new proof (via the Frigerio–Maassen theory of the quantum Poisson process ([40])) of the unitary of the solution of the LDL equation derived in [9]. A comparison with the original unitarity proof in [11] and its simplified version in [9], shows that sometimes a deeper physical insight can lead to a drastic simplification of the mathematical proofs.

Because of the width and the difficulty of the problem, we believe it useful to give reader a quick purely qualitative description of the present status of the above mentioned problems. We exclude the LDL from this general outline and postpone its discussion to Section 5), because this discussion requires at least the knowledge of the statement of the problem and of the main results obtained so far.

The quantum Brownian motions.

The precise sense in which the quantum Brownian motions are approximations of quantum fields is the following: one starts from a quasi–free representation of the CCR or the CAR over a Hilbert space H_1 (one–particle space) and from a 1–particle dynamics (1–parameter unitary group $S_t : H_1 \to H_1$) compatible with the given representation, in the sense that the dynamics S_t induces a 1–parameter group u_t^0 of automorphisms of the CCR (or the CAR) algebra over H_1.

Given these ingredients and a positive number λ, one defines the **collective coherent vectors** (or the **collective number vectors** in the Fermion case), and the space $\mathcal{H}_\lambda$ which is the closed linear span of the collective coherent (resp. number) vectors. The space $\mathcal{H}_\lambda$ **approximates** the space $\mathcal{H}$ of a certain QBM in the following sense: for each $\lambda > 0$ each collective coherent (resp. number) vector uniquely defines a coherent (resp. number) vector of the QBM. Moreover, if $\phi, \psi \in \mathcal{H}$ correspond respectively to $\phi_\lambda, \psi_\lambda \in \mathcal{H}_\lambda$, then $\lim_{\lambda \to 0} < \phi_\lambda, \psi_\lambda >= \langle \phi, \psi \rangle$. The detailed description of this procedure can be found in Section 3). For the moment it is important to notice that this approximation procedure involves only the quantum field and its 1–particle dynamics: it is totally independent of the interaction of the field with any reservoir.

We shall see that the situation is quite different for the quantum Poisson process.

Finally let us remark that, due to the different choice of the **collective vectors** in the Bose or Fermi case (resp. coherent and number vectors) the techniques needed to handle the two cases are quite different, even if many of the basic estimates are of the same nature (cf. Section 9)).

The inner space of the QBM

The QBM arising from the procedure described above are either the Fock or the universal invariant QBM on $L^2(\mathbf{R}, dt\,;\, K)$ where K is a Hilbert space, called **the inner space of the QBM**, uniquely determined by the microscopic Hamiltonian model. The **standard QBM** is obtained when $K = \mathbf{C}$. The explicit description of K provides an important physical insight in the approximation scheme of the quantum fields by QBM's because it shows precisely which degrees of freedom of the underlying Hamiltonian model give rise to the Hilbert space K and which physical parameters give rise to the parameters characterizing the various quantum noises. We consider this determination of the correct K to be a substantial addition to the existing physical literature whose nontriviality is shown by the fact that all the previous justifications of the QBM, based on heuristic considerations, did not lead to the correct choice of K.

A remarkable fact is that the inner space K is, at least in the simplest cases of **linear interactions** independent of the statistics (Bose or Fermi) of the original quantum field.

The role of the interaction

The above mentioned results apply to a simple class of models, characterized by the fact that the interaction system–reservoir is **linear** in the field operators and satisfies a **rotating wave approximation** condition (cf. condition (1.8)). If either of these assumptions is dropped, then more complicated noises arise. The case of models without rotating wave approximation, but with a system Hamiltonian with discrete spectrum, has been dealt in ([6]) where it is proved that to each frequency of the system Hamiltonian it corresponds an independent copy of a QBM of the type described in Section 3) (however the copies of QBM corresponding to different frequencies are not isomorphic).

The case of a continuous spectrum system Hamiltonian has not yet been dealt with rigorously, one might expect a continuous tensor product of QBM parametrized by the spectrum of the system Hamiltonian, however we cannot do a definite conjecture because, even in the simple case of the reduce dynamics, this is a long standing open problem.

Much more subtle is the role played by a nonlinear dependence of the interaction on the fields. In [8] it was proved that, for a polynomial interaction of degree $p > 1$, the term by term limit of the matrix elements of the iterated series exists but, even when one can interchange the limit and the summation (this in the case for $p = 2$), the evolution is neither unitary nor stochastic (cf. Sections 6) and 7) below for more details).

This led Frigerio (unpublished) to conjecture that in case of nonlinear interactions, one should consider **interaction dependent collective vectors**. The correctness of this conjecture has been recently proved in [16] where the techniques developed in [4] for linear Fermion WCL together with the nonlinear estimates of [8] are modified so to control the WCL of a quadratic Bose model.

§1 . Statement of the problem

Let H_0 and H_1 Hilbert spaces interpreted respectively as the system and the one particle reservoir space . Let $W(H_1)$ be the Weyl C^*-algebra on H_1, i.e. the closure of the linear space spanned by the set $\{W(f) : \ f \in H_1\}$ for the unique C^*-norm on it (cf. [45]); let H be a self-adjoint bounded below operator on H_1 and $\beta > 0$ and μ be real numbers interpreted as inverse temperature and chemical potential respectively. Let the fugacity z be given by $z = e^{\beta\mu}$ and define

$$Q_z \ := \ \left(1 + ze^{-\beta H}\right) \cdot \left(1 - ze^{-\beta H}\right)^{-1} \ = \coth(\frac{1}{2}\beta(H-\mu)) \tag{1}$$

and suppose that, for each z in a an interval $[0, Z]$, Q_z is a self-adjoint operator on a domain $\mathcal{D}$, independent on z. Denote φ_{Q_z} the mean zero gauge invariant quasi-free state on $W(H_1)$ with covariance operator Q_z, i.e.

$$\varphi_{Q_z}(W(f)) \ = \ \exp\left(-\frac{1}{2} < f, Q_z f >\right) \quad , \quad \forall f \in H_1 \tag{2}$$

and let $\{\mathcal{H}_{Q_z}, \pi_{Q_z}, \Phi_{Q_z}\}$ be the GNS - triple of $\{W(H_1), \varphi_{Q_z}\}$, so that

$$< \Phi_{Q_z}, \pi_{Q_z}(W(f))\Phi_{Q_z} > = \ \varphi_{Q_z}(W(f)) \tag{3}$$

We shall write W_{Q_z} for $\pi_{Q_z} \circ W$. The Fock representation corresponds to the case $Q_z = 1$, which corresponds to the limiting case $\beta = \infty$ (or $z = 0$). In this case the GNS representation will be simply denoted $\{\mathcal{H}, \pi, \Phi\}$. Let S_t^1 be a unitary group on $B(H_1)$ (the one particle free evolution of the reservoir) and suppose that

$$S_t^1 \cdot Q_z \ = \ Q_z \cdot S_t^1 \quad , \quad \forall t \geq 0 \tag{4}$$

where the equality is meant on $\mathcal{D}$. Typically we shall choose $S_t^1 = \exp(itH)$. This implies that the second quantization of S_t^1, denoted $W(S_t^1)$, leaves φ_{Q_z} invariant hence it is implemented, in the GNS representation, by a unitary 1-parameter group $V_t^{(z)}$ whose generator $H_R^{(z)}$ is called the free Hamiltonian of the reservoir. As in [1],... , [16] we assume that there exists a non zero subspace K of H_1 (in all the examples it is a dense subspace) such that

$$\int_{\mathbf{R}} | < f, S_t g > | dt \ < \infty \quad , \quad \forall f, g \in K \tag{5}$$

Moreover , we suppose that $Q_z K \subseteq K$ and $e^{itH} K \subseteq K$ for all $t \in \mathbf{R}$. For example, for the free Bose gas, $H_1 = L^2(R^d)$ for some $d \geq 3$, H is the Laplacian on $\mathbf{R}^d$ and K can be choosen to be $L^1 \cap L^\infty(\mathbf{R}^d)$. Let be given a self-adjoint operator H_S on the system space H_o, called the system Hamiltonian. The total free Hamiltonian is defined to be

$$H_0^{(z)} := H_S \otimes 1 + 1 \otimes H_R^{(z)} \tag{6}$$

where, $H_R^{(z)} := d\Gamma(H)$. The interaction Hamiltonians V_λ considered in [1], ... , [16] are of the type

$$V_\lambda := \begin{cases} i\lambda(D \otimes (A^+(g))^p - c.c.), & \text{for WCL} \\ i(D \otimes A^+(g_0)A(g_1) - c.c), & \text{for LDL} \end{cases} \tag{7}$$

$_{(1.7b)where,}$ g,g$_0$, $g_1 \in K \subset H_1$ and D is a bounded operator on H_0 (we make this assumption for sake of simplicity, but our techniques apply, with minor modifications to a large class of unbounded operators, including creation, annihilation and number operators) and p is a positive integer. Notice that in the WCL case we have the factor λ (coupling constant) but in the LDL case this is replaced by the factor 1. In other words, we consider the LDL only for scattering type interactions (by this we mean an interaction which preserves the natural **Z**-grading on the reservoir space induced by the generalized number operator) and the WCL only for exchange type interactions, i.e. of the form (1.7a) . Of course, we could consider scattering type interaction also for the WCL but, in the present paper, we shall not deal with this.

The WCL corresponds to letting the coupling constant $\lambda \to 0$, while time is scaled as t/λ^2. The LDL corresponds to letting the density n in the state φ_{Q_z} tends to zero, time being scaled as $\bar{n}t/n$, $\bar{n}$ being a rescaled density to be held fixed. The density n is the limit as the volume $V \to +\infty$ of the expectation value of the number operator N_V in finite volume in the state $\varphi_{Q_z}^{(V)}$ divided by V. It becomes asymptotically proportional to the fugacity z in the limit as $z \to 0$; for example, for the free Bose gas in three space dimensions, with $H = -\frac{1}{2}\Delta$, one has

$$n = n(\beta, z) = z \int_{\mathbf{R}^3} (\exp(\beta k^2/2) - z)^{-1} d^3k$$

With a suitable choice of $\bar{n}$, the LDL is the limit $z \to 0$, time being scaled as t/z. <u>*REMARK*</u> There is a physical reason why the LDL should go with an interaction which preserves the (generalized) number operator. Scaling a parameter means that it is a controllable parameter. If the interaction were not to commute with number, then the density could take any value in the course of time. Of course, there could be nothing wrong in considering number type interaction in the weak coupling limit (with nonzero temperature!) as was done by Davies ([20]).

Moreover in the WCL case we shall assume that

$$\exp(itH_S) \cdot D \cdot \exp(-itH_S) \;=\; e^{-i\omega t} D \quad \text{for some} \;\; \omega > 0 \tag{8}$$

(**rotating wave approximation**) and for the LDL case, we shall assume that

$$\exp(itH_S) \cdot D \cdot \exp(-itH_S) \;=\; D \tag{9}$$

and that g_0 and g_1 have disjoint energy spectra, i.e.

$$< g_0, S_t g_1 > \;=\; 0 \quad , \quad \forall t \in \mathbf{R} \tag{10}$$

The physical meaning of these assumptions will be explained in Section 5). In fact as shown in [6], [10] and [14], none of the conditions is necessary but the basic ideas of the problem, as well as of the proofs of the main results, can be understood even if one limits oneself to these cases.

Denote for each $D \in B(H_0)$,

$$D_0 \;=\; D \quad , \quad D_1 \;=\; -D^+ \tag{11}$$

With these notations, the Hamiltonian of the total system can be expressed in

$$H_\lambda^{(z)} := H_S \otimes 1 + 1 \otimes H_R^{(z)} + V_\lambda \tag{12}$$

and the wave operator at time t is defined by

$$U_t \;:=\; \exp\left(itH_0^{(z)}\right) \cdot \exp\left(-itH_\lambda^{(z)}\right) \tag{13}$$

Therefore we have the formal identity

$$\frac{d}{dt} U_t \;=\; \frac{1}{i} V_\lambda(t) U_t \quad ; \quad U_0 \;=\; \mathbf{1} \tag{14}$$

where

$$V_\lambda(t) := \exp\left(itH_0^{(z)}\right) V_\lambda \exp\left(-itH_0^{(z)}\right) = \begin{cases} i\lambda \sum_{\varepsilon \in \{0,1\}} D_{1-\varepsilon} \otimes A^\varepsilon (S_t g)^p, & \text{WCL} \\ i \sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}), & \text{LDL} \end{cases} \tag{15}$$

where, $S_t = e^{it\omega} S_t^1$ in the WCL case and $S_t = S_t^1$ in the LDL case. It is well known that the solution of (1.14) is given by the iterated series

$$U_t = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n V_\lambda(t_1) \cdots V_\lambda(t_n) \tag{16}$$

which, at least for $p \leq 2$, is convergent on the domain of the vectors of the form

$$u \otimes \Phi_{Q_z}(x \int_{S/x^2}^{T/x^2} S_u f du) \tag{17}$$

where $u \in H_o$, $f \in K$, $S, T \in \mathbf{R}$,

$$x := \begin{cases} \lambda \ , & \text{WCL} \\ z^{1/2} \ , & \text{LDL} \end{cases} \tag{18}$$

and

$$\Phi_{Q_z}(x \int_{S/x^2}^{T/x^2} S_u f du) := W_{Q_z}(x \int_{S/x^2}^{T/x^2} S_u f du) \Phi_{Q_z} \tag{19}$$

is a collective coherent vector in the sense of [2] (resp. collective number vector in the sense of [4]).

From Lemma (3.2) of [2], we know that the assumption (1.5) implies that the sesquilinear form $(\cdot|\cdot) : K \times K \longrightarrow \mathbf{C}$ defined by

$$(f|g) := \int_{\mathbf{R}} < f, S_t g > dt \quad , \quad f, g \in K \tag{20}$$

defines a pre-scalar product on K. We denote $\{K, (\cdot|\cdot)\}$, or simply K, the completion of the quotient of K by the zero $(\cdot|\cdot)$ -norm elements . In [20],

... , [29] it was shown that for all normal initial states φ of the system and some interactions, essentially all of the type (1.7), under some additional conditions , the limit

$$\lim_{x \to 0} \varphi \otimes \varphi_{Q_z} \left(U^+_{t/x^2} (X \otimes 1) U_{t/x^2} \right) \tag{21}$$

exists and is equal to

$$\varphi(T_t(X))$$

where $\{T_t\}_{t\geq 0}$ is a quantum Markovian semigroup. The techniques introduced in the series papers ([1], ... , [15]) allowed to control the limits, as $x \to 0$ of expressions of the form

$$\psi \otimes \varphi_{Q_z}\Big(\mathbf{1} \otimes W(f_x)\; U_{t/x^2}\; \mathbf{1} \otimes W(f_x')\Big) \tag{22}$$

and

$$\psi \otimes \varphi_{Q_z}\Big(\mathbf{1} \otimes W(f_x)\; U_{t/x^2}(X \otimes 1)U^+_{t/x^2}\; \mathbf{1} \otimes W(f_x')\Big) \tag{23}$$

where ψ is any normal linear functional on the system space and

$$f_x := x \int_{S/x^2}^{T/x^2} S_u f du \quad , \quad f \in K \tag{24}$$

(in the Fermi case, instead of collective coherent vectors we use collective number vectors which are defined in an analogous way). In (1.22) $x = \lambda$ (WCL) or $z^{1/2}$ (LDL) and, in the WCL case φ_{Q_z} is any quasi-free state. In the control of the limit, as $x \to 0$, of the expressions (1.21) and (1.21a), the basic difference between the low density and the weak coupling case is that in the low density case the wave operator $U_{t/z}$ depends on z (the fugacity) only through the rescaling of the time ($t \mapsto t/z$), while in the weak coupling case the wave operator $U^{(\lambda)}_{t/\lambda^2}$ depends on λ (the coupling constant) in its very structure. On the other hand, the reservoir state is independent of λ in the weak coupling limit, but dependent of z in the low density limit.

Both in the WCL and LDL cases, the first step in our investigation will be to control the limit of expressions of the form

$$\lim_{x\to 0} < u \otimes W(x \int_{S/x^2}^{T/x^2} S_u f du)\Phi, U_{t/x^2} \cdot v \otimes W(x \int_{S'/x^2}^{T'/x^2} S_u f' du)\Phi > \tag{25}$$

in the Fock case. Our experience with several models suggests that the most difficult step consists in controlling this limit and understanding the equation (usually stochastic) it satisfies. Once this is done, the passage to the finite temperature case and to the Langevin equation (i.e. the limit (1.21a)) is by no means trivial, but the difficulties are mainly of a technical nature. A possible exception is the fact that in the LDL case, the limit (1.21) can be controlled for all $t \in \mathbf{R}$ in the Fock case, while the same limit in the finite temperature case and the limit (1.21a) can be controlled only for t in a certain interval. It is not yet clear if this difficulty is of a fundamental nature or a purely technical one. In the present paper we shall:

-state the main results obtained up to now both in the WCL and LDL;

-outline the main strategy of the proofs;

-give the basic idea about how control the limit (1.23) both in the WCL and the LDL cases (we shall restrict ourselves to the Boson case);

-discuss the differences and the common points between the two kinds of limit (WCL and LDL);

-explain why in the Fock case one can hope to obtain both the WCL and the LDL limit for the same "collective coherent vectors (cf. Section 7) of the present paper).

-identify the unitary operator, driving the SDE deduce in [9] (and [11]) for the LDL in the Fock case, with a matrix element of the $T-$operator of the Hamiltonian system consisting of the system and one reservoir particle.

§2 The meaning of the low density limit in the Fock case

From (1.1) it is clear that the Fock case corresponds to $z = 0$. But, since the LDL means that we consider the limit $z \to 0$, the following problem naturally arises: what does "low density" mean in the Fock case?

In order to answer this question let us recall an argument due to Palmer ([28]) and consider the LDL interaction (1.7b) in the case $z \neq 0$. This means that, in (1.7b), the creation and annihilation operators are referred to the cyclic representation of the state Φ_{Q_z}, defined by (1.1) and (1.2). Denoting $\Gamma_{Q_z}(H_1)$ this space and $\Gamma(H_1)$ the corresponding Fock space, we know that there is a unitary isomorphism of

$$\Gamma_{Q_z}(H_1) \longrightarrow \Gamma(H_1) \otimes \Gamma(H_1^\iota) \tag{26}$$

where H_1^ι is the conjugate Hilbert space of H_1 (cf. [4]). Under this isomorphism, one has the correspondences:

$$\Phi_{Q_z} \longrightarrow \Phi_F \otimes \Phi_F^\iota \tag{27}$$

and

$$W_{Q_z}(f) \longrightarrow W(Q_+ f) \otimes W(Q_- f) \tag{28}$$

$$A_{Q_z}(f) \longrightarrow A(Q_+ f) \otimes 1 + 1 \otimes A^+(Q_- f) \tag{29}$$

where,

$$Q_+ := \sqrt{\frac{Q_z + 1}{2}} \ , \qquad\qquad Q_- = \iota \sqrt{\frac{Q_z - 1}{2}} \tag{30}$$

Notice that

$$Q_z = \frac{1 + ze^{-\beta H}}{1 - ze^{-\beta H}} \tag{31}$$

So, expanding in z, we obtain:

$$\frac{Q_z + 1}{2} = \frac{1}{1 - ze^{-\beta H}} = 1 + ze^{-\beta H} + z^2 e^{-2\beta H} + O(z^3) \tag{32}$$

and

$$\frac{Q_z - 1}{2} = \frac{ze^{-\beta H}}{1 - ze^{-\beta H}} = ze^{-\beta H} + z^2 e^{-2\beta H} + O(z^3) \tag{33}$$

Therefore,

$$Q_+ = 1 + \frac{1}{2} ze^{-\beta H} + o(z) \ , \qquad Q_- = \iota \ z^{1/2} e^{-\frac{\beta}{2} H} + o(z) \tag{34}$$

so that, in the correspondence (2.4),

$$A_{Q_z}(f) \longrightarrow A(f)\otimes 1+\frac{z}{2}\cdot A(e^{-\beta H}f)\otimes 1+z^{1/2}\cdot 1\otimes A^+(\iota\, e^{-\beta H/2}f)+o(z) \quad (35)$$

and naturally

$$A^+_{Q_z}(f) \longrightarrow A^+(f)\otimes 1+\frac{z}{2}\cdot A^+(e^{-\beta H}f)\otimes 1+z^{1/2}\cdot 1\otimes A(\iota\, e^{-\beta H/2}f)+o(z) \quad (36)$$

Thus, in the canonical representation, one has:

$$A^+_{Q_z}(f)A_{Q_z}(g) \longrightarrow A^+(f)A(g)\otimes 1+z^{1/2}\cdot\big(A^+(f)\otimes A^+(\iota\; e^{-\beta H/2}g)+A(g)\otimes A^+(\iota\; e^{-\beta H/2}f)\big)+$$

$$+\frac{z}{2}\left(A^+(f)A(e^{-\beta H}g)+A(g)A^+(e^{-\beta H}f)\right)\otimes 1+o(z^{3/2}) \quad (37)$$

The $z^{1/2}$-terms are dealt with techniques similar to the WCL the z terms will give a very simple contribution in the limit. Therefore the only qualitative new feature is given by the term $A^+(f)A(g)\otimes 1$, which acts trivially on the conjugate Fock space. This is qualitatively new with respect to the WCL case because the term $A^+(f)A(g)$ corresponds to a finite interaction, i.e. not vanishing for $z\to 0$. These considerations lead to study the interaction

$$i(D^+\otimes A^+(f)A(g)-D\otimes A^+(g)A(f)) \quad (38)$$

in the Fock space. Motivated by the analogy with the weak coupling limit Accardi and Lu ([9]) were led to consider the same collective coherent vectors as in the WCL limit, i.e. to study the limit

$$< u\otimes W(x\int_{S/x^2}^{T/x^2} S_u f du)\Phi, U_{t/x^2}v\otimes W(x\int_{S'/x^2}^{T'/x^2} S_u f' du\Phi > \quad (39)$$

With an appropriate generalization of the techniques developed for the WCL limit, in [9] it was proved (cf. also the following Sections) that the limit (2.12) as $x\to 0$, exists for all $t\in\mathbf{R}$ and satisfies a QSDE driven by a pure number process (cf. Theorem (II) of Section 4)). In view of the difference between the interactions in the WCL and LDL (cf. (1.7a), (1.7b)), it might seem quite surprising that the limit (2.12) exists in the LDL case. The qualitative reason why this should happen is that the finiteness (i.e. independence on x) of the LDL interaction will be compensated by the fact that the interaction is of second degree. This balance is very delicate and, due to its importance (both technical and conceptual), the entire Section 7) will be devoted to explain it.

Later results [10], [12] showed, however, that the natural extension of the above mentioned procedure to the finite temperature case and the Langevin equation leads to a breakdown of unitarity in the limit evolution. These results, as in the WCL nonlinear case, maturated the conviction that the analogy between the WCL and the LDL case is a touchy point as far as the collective vectors are concerned and that the choice of the collective coherent vectors (1.18b) should be replaced by a more subtle one. However, since the problem of the new general choice is still open, and since in any case we expect that the techniques needed to deal with the more general case will be modifications of the techniques already developed, we confine our discussion to the simpler case.

§3 Convergence of the collective process

In this Section we show that, at a purely kinematical level, i.e. with $t = 0$ in (1.23) and with the given choice of the collective vectors, the low density limit coincides with the weak coupling limit and the limiting process is the Fock Brownian motion on $L^2(\mathbf{R}, dt; K)$ where K is the Hilbert space described after (1.19). This will give the limit space on which our limit processes live.

In order to formulate our result, let us recall from [2] the definition of Boson Quantum Brownian Motion:

<u>*DEFINITION*(3.1)</u> Let $\mathcal{K}$ be a Hilbert space, T an inteval in $\mathbf{R}$. Let $Q \geq 1$ be a self-adjoint operator on $\mathcal{K}$ with domain $\mathcal{D}(Q)$ and let

$$\{\mathcal{H}_Q, \pi_Q, \Psi_Q\} \tag{40}$$

denote the GNS representation of the CCR over $L^2(T, dt; \mathcal{K})$ with respect to the quasi-free state φ_Q on $W(L^2(T, dt; \mathcal{K}))$ characterized by

$$\varphi_Q(W(\xi)) = e^{-\frac{1}{2}<\xi, 1\otimes Q\xi>} \quad ; \quad \xi \in \mathcal{D}(Q) \subset L^2(T, dt; \mathcal{K}) \tag{41}$$

The stochastic process, in the sense of [18]

$$\Big\{\mathcal{H}_Q \ , \ A(\chi_{(s,t]} \otimes f) \ , \ A^+(\chi_{(s,t]} \otimes f) \ ; \ (s,t] \subseteq T \ , f \in \mathcal{K}\Big\} \tag{42}$$

on the domain of coherent vectors, where $A(\cdot)$, $A^+(\cdot)$ denote respectively the annihilation and creation fields in the representation (3.1) , is called the **Boson Q-Quantum Brownian Motion** on $L^2(T, dt; \mathcal{K})$. The **1-Quantum Brownian Motion** will be called the **Fock Brownian Motion** . In this case the space $\mathcal{H}_Q = \mathcal{H}_1$ is isomorphic to the Fock space over $L^2(T, dt;\ \mathcal{K})$ and denoted $\Gamma(L^2(T, dt;\ \mathcal{K}))$.

<u>*LEMMA*(3.2)</u> For each $f, f' \in K, S, S', T, T' \in \mathbf{R}$, one has

$$\lim_{x\to 0} < x\int_{S/x^2}^{T/x^2} S_u f du, x\int_{S'/x^2}^{T'/x^2} S_u f' du >=< \chi_{[S,T]}, \chi_{[S',T']} >_{L^2(\mathbf{R})} \cdot \int_{\mathbf{R}} < f, S_t f' > dt$$

$$=:< \chi_{[S,T]} \otimes f, \chi_{[S',T']} \otimes f' >_{L^2(\mathbf{R}, dt; K)} \tag{43}$$

PROOF See Lemma (3.2) of [2].

LEMMA (3.3) In the notations of Definition (3.1), for each $n \in \mathbf{N}$, $\{f_k\}_{k=1}^n \subset K$, $\{S_k, T_k\}_{k=1}^n \subset \mathbf{R}$, $\{\alpha_k\}_{k=1}^n \subset \mathbf{R}$,

$$\lim_{x\to 0} < \Phi_{Q_z}, W(\alpha_1 x \int_{S_1/x^2}^{T_1/x^2} S_u f_1 du) \cdots W(\alpha_n x \int_{S_n/x^2}^{T_n/x^2} S_u f_n du)\Phi_{Q_z} >$$

$$=< \Psi, W(\alpha_1 \chi_{[S_1,T_1]} \otimes f_1) \cdots W(\alpha_n \chi_{[S_n,T_n]} \otimes f_n)\Psi > \tag{44}$$

Moreover the convergence is uniform for $\{\alpha_k\}_{k=1}^n, \{S_k, T_k\}_{k=1}^n$ in a bounded set of $\mathbf{R}$, where $W(\psi \otimes f)$ ($\psi \in L^2(\mathbf{R}, dt), f \in K$) are Weyl operators of the Fock Brownian motion on $L^2(\mathbf{R}, dt; K)$, Ψ is the vacuum (moreover, in LDL case, Fock vacuum) of $\Gamma(L^2(\mathbf{R}, dt; K))$. *PROOF* See Lemma (2.1) of [9].

In the following we shall use the notation, similar to (1.18b),

$$\Psi(\xi \otimes f) := W(\xi \otimes f)\Psi \ , \qquad \xi \in L^2(\mathbf{R}) \ \text{ and } \ f \in K \tag{3.6}$$

§4 The main results

Before entering into technical details, it is useful to have at least an idea of the results achieved so far. This will also help to clarify one's ideas about further developements.

In this section we list, without proofs, the main results corresponding to the weak coupling and low density limits. The basic ideas of the proofs will be given in the following sections

<u>*THEOREM(I.)*</u> (WCL: linear and Fock case) In the notations of Lemma (2.1), for all $u, v \in H_0$, $f, f \in K$; S, T, $S', T' \in \mathbf{R}$ the limit, as $\lambda \to 0$, of

$$< u \otimes \Phi \left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du \right), U^{(\lambda)}_{t/\lambda^2} v \otimes \Phi \left(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du \right) > \tag{45}$$

exists and is equal to:

$$< u \otimes \Psi \left(\chi_{[S,T]} \otimes f \right), U(t) v \otimes \Psi \left(\chi_{[S',T']} \otimes f \right) > \tag{46}$$

where $U(t)$ is the (unitary) solution of the QSDE

$$dU(t) = \left\{ D \otimes dA^+_t(g) - D^+ \otimes dA_t(g) - (g|g)_- D^+ D \otimes 1 dt \right\} U(t) \tag{47}$$

with the initial condition

$$U(0) = 1 \tag{48}$$

and where

$$(f|g)_- := \int_{-\infty}^0 < f, S_t g > dt \tag{49}$$

<u>*PROOF*</u> See Theorem (II.) of [2].

In order to state the result in the LDL case we introduce the new simbol:

$$\|g\|^2_- := \int_{\infty}^0 | < g, S_t g > | dt \tag{50}$$

<u>*THEOREM(II.)*</u> (LDL: Fock case) In the Fock case, if

$$16 \|D\|^2 \cdot \max(\|g_0\|_-, \|g_1\|_-) < 1 \tag{51}$$

then for all $t \in \mathbf{R}$, the limit of (2.12), as $z \to 0$, exists and is equal to

$$< u \otimes W(\chi_{[S,T]} \otimes f)\Psi, U(t)v \otimes W(\chi_{[S',T']} \otimes f')\Psi > \tag{52}$$

where $U(t)$ satisfies the pure jump QSDE:

$$U(t) = 1 + \int_0^t \sum_{\varepsilon \in \{0,1\}} (D_1(\varepsilon) \otimes dN_s(g_\varepsilon, g_{1-\varepsilon}) + D_2(\varepsilon) \otimes dN_s(g_\varepsilon, g_\varepsilon))U(s) \tag{53}$$

where, by definition,

$$N_t(f, g) := N(\chi_{[0,t]} \otimes |f >< g|) \tag{54}$$

is the number process on the Fock space over $L^2(\mathbf{R}, dt; K)$ and $N(\chi_{[0,t]} \otimes |f >< g|)$ is characterized by the property:

$$< \Phi(\xi_1 \otimes h_1), dN(\chi_{[0,t]} \otimes |f >< g|)\Phi(\xi_2 \otimes h_2) >= \bar{\xi}_1(t) \cdot \xi_2(t) \ < h_1, f >< g, h_2 > dt \tag{55}$$

$$D_1(\varepsilon) := (1 - T_\varepsilon)^{-1} D_\varepsilon = \sum_{n=0}^{\infty} T_\varepsilon^n D_\varepsilon \tag{56}$$

$$D_2(\varepsilon) := (g_{1-\varepsilon}|g_{1-\varepsilon})_- D_\varepsilon D_{1-\varepsilon}(1 - T_\varepsilon)^{-1} \tag{57}$$

and

$$T_\varepsilon := (g_\varepsilon|g_\varepsilon)_-(g_{1-\varepsilon}|g_{1-\varepsilon})_- D_\varepsilon D_{1-\varepsilon} \tag{58}$$

Moreover, $U(t)$ is unitary. *PROOF* See Theorems (6.1) and (6.2) of [9].

REMARK: Contrarily to what happens in the WCL case, the proof the unitarity of the limit procees $U(t)$ in LDL case is quite nontrivial. Moreover the equation (4.9) was deduced in a purely mathematical way in [9]. It is a remarkable fact that its coefficients are in fact matrix elements of the T-operator on the one particle space. This connection will be explained in Section 5).

The result for the WCL limit Fock case with nonlinear type interaction, i.e. $p > 1$ in (1.7a) is the following

THEOREM(III.) (WCL: nonlinear case) For all $u, v \in H_0$, $f, f \in K$; $S, T, S', T' \in \mathbf{R}$ the limit, as $\lambda \to 0$, of

$$< u \otimes \Phi\left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du\right), U_{t/\lambda^2}^{(\lambda)} v \otimes \Phi\left(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du\right) > \tag{59}$$

exists **formally** and is equal to:

$$< u \otimes \Psi \left(\chi_{[S,T]} \otimes f\right), U(t) v \otimes \Psi \left(\chi_{[S',T']} \otimes f\right) > \tag{60}$$

where $U(t)$ is equal to

$$\exp\Big(-tD^+D \cdot \int_{-\infty}^0 p! < g, S_t g >^p dt\Big) \otimes 1 \tag{61}$$

Here "formally means that we expand $U^{(\lambda)}_{t/\lambda^2}$ in (4.14) using the iterated series, consider term-by-term limit of the series and finally we resum the limiting series. For $p = 2$, the limit is not only formal but also exact.

<u>*REMARK*</u> The fact that in this case the limit is not unitary and has no stochastic part provided the first indication that the choice of the "collective vectors" should depend on the interaction. This opened a new line of investigation. A first result in this direction is in [16].

The reason why for $p > 2$, we have a weaker result is that, in this case, the uniform estimate of Section 11) below which allows to exchange the limit (as $\lambda \to 0$) and the sum (for $n = 0, 1, \cdots$), is not available. <u>*PROOF*</u> See

Theorem (4.1) of [8].

<u>*THEOREM(IV.)*</u> (WCL: finite temperature case) For all $u, v \in H_0$, $f, f \in K$; $S, T, S', T' \in \mathbf{R}$ the limit, as $\lambda \to 0$, of

$$< u \otimes \Phi_Q \left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du\right), U^{(\lambda)}_{t/\lambda^2} v \otimes \Phi_Q \left(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du\right) > \tag{62}$$

exists and is equal to:

$$< u \otimes \Psi_Q \left(\chi_{[S,T]} \otimes f\right), U(t) v \otimes \Psi_Q \left(\chi_{[S',T']} \otimes f\right) > \tag{63}$$

where $\Psi_Q(\psi \otimes f)$ ($\psi \in L^2(\mathbf{R}, dt)$, $f \in K$) are the coherent states of the Q-quantum Brownian motion over $L^2(\mathbf{R}, dt; K)$, and $U(t)$ is the (unitary) solution of the QSDE

$$dU(t) = \Big\{D \otimes dA_t^+(g) - D^+ \otimes dA_t(g) - \Big[(g|g)^-_{\frac{1}{2}(Q+1)} D^+D \otimes 1 + (g|g)^-_{\frac{1}{2}(Q-1)} DD^+ \otimes 1\Big] dt\Big\} U(t) \tag{64}$$

with initial condition $U(0) = 1$ and where

$$(g|g)^{-}_{\frac{1}{2}(Q\pm 1)} := \int_{-\infty}^{0} < g, S_t \frac{1}{2}(Q \pm 1) g > dt \tag{65}$$

<u>*PROOF*</u> See Theorem (1.7) of [3].

<u>*THEOREM(V.)*</u> (LDL: finite temperature case) In the finite temperature case, if (4.7) holds and t is small, then the limit, as $z \to 0$, of

$$< u \otimes \Phi_{Q_z}\left(z^{1/2}\int_{S/z}^{T/z} S_u f du\right), U_{t/z} v \otimes \Phi_{Q_z}\left(z^{1/2}\int_{S'/z}^{T'/z} S_u f' du\right) > \tag{66}$$

exists and is equal to

$$< u \otimes \Psi_Q\left(\chi_{[S,T]} \otimes f\right), U(t) v \otimes \Psi_Q\left(\chi_{[S',T']} \otimes f\right) > \tag{67}$$

where $U(t)$ satisfies the QSDE

$$U(t) = 1 + \int_0^t \sum_{\varepsilon\in\{0,1\}} (D_1(\varepsilon) \otimes dN_s(g_\varepsilon, g_{1-\varepsilon}) + D_2(\varepsilon) \otimes dN_s(g_\varepsilon, g_\varepsilon) + D_3(\varepsilon) ds) U(s) \tag{68}$$

where, $D_1(\varepsilon)$ and $D_2(\varepsilon)$ are the same as in the Fock case

$$D_3(\varepsilon) := \sum_{n=1}^{\infty} M(\varepsilon, n-1)(D_\varepsilon D_{1-\varepsilon})^n \tag{69}$$

and

$$M(\varepsilon, n) := \int_{-\infty}^{0} dt_1 \cdots \int_{-\infty}^{0} dt_{2n+1} < g_{1-\varepsilon}, S_{t_1} g_{1-\varepsilon} >< g_\varepsilon, S_{t_2} g_\varepsilon > \cdots$$

$$< g_{1-\varepsilon}, S_{t_{2n-1}} g_{1-\varepsilon} >< g_\varepsilon, S_{t_{2n}} g_\varepsilon >< g_{1-\varepsilon}, S_{t_{2n+1}} g_{1-\varepsilon} > \overline{< g_\varepsilon, S_{t_{2n+1}+\cdots+t_2} e^{-\frac{1}{2}\beta H} g_\varepsilon >} \tag{70}$$

<u>*PROOF*</u> See Theorem (7.1) of [12].

<u>*REMARK*</u> The presessce of dt-term in (4.23) shows that in general $U(t)$ is not unitary. This discovery led us to look for a choice of the collective

coherent vectors different from (1.18b). At moment, although several condidates have been proposed and are under examination, this research has not yet come to an end (cf. the Remark after Theorem (III.)).

$\underline{THEOREM(VI.)}$ (WCL: finite temperature, Langevin case) For all $u, v \in H_0$, $f, f \in K$; $S, T, S', T' \in \mathbf{R}$ and $X \in B(H_0)$ the limit, as $\lambda \to 0$, of

$$< u \otimes \Phi_Q \left(\lambda \int_{S/\lambda^2}^{T/\lambda^2} S_u f du \right), U^{(\lambda)}_{t/\lambda^2}(X \otimes 1) U^{(\lambda)*}_{t/\lambda^2} v \otimes \Phi_Q \left(\lambda \int_{S'/\lambda^2}^{T'/\lambda^2} S_u f' du \right) > \tag{71}$$

exists for t small and is equal to:

$$< u \otimes \Psi_Q \left(\chi_{[S,T]} \otimes f \right), U(t)(X \otimes 1)U(t)^* v \otimes \Psi_Q \left(\chi_{[S',T']} \otimes f \right) > \tag{72}$$

where Ψ_Q is mean-zero gauge invariant quasi free state with covariance $1 \otimes Q$ and $U(t)$ is the (unitary) solution of the QSDE (4.9) with initial condition $U(0) = 1$. $\underline{PROOF}$ See Theorem (4.1)' of [3].

$\underline{THEOREM(VII.)}$ (LDL: Fock, Langevin case) For all $u, v \in H_0$, $f, f \in K$; $S, T, S', T' \in \mathbf{R}$ and $X \in B(H_0)$ the limit, as $z \to 0$, of

$$< u \otimes \Phi_{Q_z} \left(z \int_{S/z^2}^{T/z^2} S_u f du \right), U_{t/z^2}(X \otimes 1) U^*_{t/z^2} v \otimes \Phi_{Q_z} \left(\int_{S'/z^2}^{T'/z^2} S_u f' du \right) > \tag{73}$$

exists and is equal to:

$$< u \otimes \Psi \left(\chi_{[S,T]} \otimes f \right), U(t)(X \otimes 1)U(t)^* v \otimes \Psi \left(\chi_{[S',T']} \otimes f \right) > \tag{74}$$

where Ψ is the Fock vacuum and $U(t)$ is the (unitary) solution of the QSDE (4.19) with initial condition $U(0) = 1$. $\underline{PROOF}$ See Theorem (4.7) of [10].

Several other results concerning other cases: (dropping the rotating wave approximation, Fermion reservoir, squeezing initial state, ...,), have been obtained (cf. [4], [6], [7], [14] and [15]), but we shall not list them here.

§5 The low density limit and its connection with scattering theory

While in the WCL the situation can be considered clearly understood, at least in its basic qualitative features, and we expect that the main progresses in this direction will concern the specific models of interest for physical applications, on the lines of what has already been done for the linear and quadratic models, in the LDL, despite some important achivements, the problem is not yet closed.

The first problem to be solved was: how to substantiate mathematically Frigerio and Maassen's intuition that **in some sense** the Poisson process should be obtained as low density limit of **something**. Palmer's argument, explained in detail in Section 2) above, showed that even if physically the LDL makes sense only at finite temperature, the basic new mathematical feature (with respect to the WCL), i.e. the presence of a term in the interaction not vanishing with the fugacity, involved only the Fock space. It was then natural as a first level of attack to the problem, to single out this term of the interaction and to try and extract the maximum information from it. This was done in [9] and the result is that effectively (and a posteriori quite surprisingly) one obtains, in the LDL for this model, an SDE driven by pure number (zero intensity Poisson) processes.

Contrarily to what happens in the WCL, the operator coefficients of the number processes in the SDE obtained in [9] were related in a highly non-trivial way to the operator coefficients of the original Hamiltonian. Consequently:

(i) The proof of the unitarity of the solution of the SDE had to be obtained with direct calculations, any direct reference to the general theory of [31] being impossible (cf. [9] and the even more complicated proof given in the first LDL paper [11]).

(ii) No physical interpretation of the coefficients of the number processes could be given in [9].

In order to overcome these drawbacks, the second problem to be solved was to clarify the connection between these coefficients and the generator of the reduced evolution in the LDL case, which had been previously found by Dümcke (with completely different techniques based on the BBGKY hierarchy and scattering theory) and which had a direct physical interpretation, being related to the 1-particle scattering operator of the orginal Hamiltonian system.

A first step towards the solution of this problem was done by Alicki and Frigerio in [39], who recast the Dümcke generator in a form that could be directly related to the general theory of [25], and therefore more suitable for the comparison with the results of [9]. In the remaining of this Section we shall complete the analysis of [9] by showing that the coefficients of SDE deduced in [9] can in fact be identified with some particular matrix elements of the 1-particle T-operator of the original Hamiltonian system. A first fall out of this result is that now the laborious direct proofs of the unitarity condition given in [9] become useless, since we can obtain them as a corollary of the general unitarity theorem of [40]. The last, and mathematically most difficult, problem, which is still open at the moment, is to obtain the LDL in its full generality. The solution of this problem will probably require a synthesis of all the techniques developed up to now to deal with the LDL, the linear WCL and the nonlinear WCL.

In the remaining of this Section we shall concentrate ourselves on the connection between the coefficients of the SDE deduced in [9] and the T-operator. In order to do that in full generality, and to explain the physical meaning of the various simplifying assumptions introduced in the model studied in [25], we shall slightly generalize this model. For the sake of definiteness, we shall take $H_1 = L^2(\mathbf{R}^3)$ and $H = -\frac{1}{2}\Delta$.

Note that, because of number conservation for interactions of scattering type, the closed subspace of $H_0 \otimes \mathcal{H}_R$ generated by vectors of the form $u \otimes A^+(f)\Phi \quad (u \in H_0,\ f \in H_1)$, which is naturally isomorphic to $H_0 \otimes H_1$, is globally invariant under the time evolution operator $\exp[i(H_S \otimes 1 + 1 \otimes H_R + V)t]$, and the restriction of the time evolution operator to this subspace corresponds to an evolution operator on $H_0 \otimes H_1$ given by

$$\exp[i(H_S \otimes 1 + 1 \otimes H + V_1)t], \tag{75}$$

where

$$V_1 = i(D \otimes |g_0\rangle\langle g_1| - \text{c.c.}) \tag{76}$$

Dümcke's results ([25]) tell us that the reduced evolution of observables in $\mathbf{B}(H_0)$ is completely determined, in the LDL, by the scattering operator for the evolution (5.1) on $H_0 \otimes H_1$ and by the temperature of the reservoir. This corresponds to the physical intuition that particles of a dilute gas should scatter independently, one at a time, on the system. The relevant operators are the Møller wave operators

$$\Omega_{\pm} = s - \lim_{t\to\pm\infty} \exp[-i(H_S \otimes 1 + 1 \otimes H + V_1)t] \exp[i(H_S \otimes 1 + 1 \otimes H)t] \quad (77)$$

the T operator

$$T = V_1 \Omega_+ \quad (78)$$

and the S operator

$$S = \Omega_-^* \Omega_+ \quad (79)$$

Under rather general assumptions, which are satisfied in the present case, S is unitary.

In the following we shall take advantage of the simple form (5.2) of the interaction V_1 to compute the T and the S operator explicitly; this will allow us to make contact between our Theorems (II.) and (VII.) and Dümcke's result ([25]).

In Section 1) and 2) we have assumed that $[H_S,\ D] = 0$ (cf. (1.9)). In order to better understand the physical meaning of this assumption, let us generalize it by assuming that, for some $\omega \geq 0$, one has $[H_S,\ D] = -\omega D$ (rotating wave approximation). Define now the group $\{S_t : t \in \mathbf{R}\}$ of unitary operators on H_1 by

$$S_t := \exp[i(H - \omega_0 P_0 - \omega_1 P_1)t]; \quad t \in \mathbf{R} \quad (80)$$

where ω_0, ω_1 are real numbers with

$$\omega = \omega_0 - \omega_1 \ , \qquad \omega_0,\ \omega_1 > 0 \quad (81)$$

and where P_0, P_1 are mutually orthogonal projections commuting with H and such that

$$P_\varepsilon g_\varepsilon = g_\varepsilon \quad ; \quad \varepsilon = 0, 1 \ ; \quad (82)$$

this construction is possible since we have assumed (cf. (1.10)) that

$$\langle g_0,\ \exp[-iHt] g_1 \rangle = 0 \quad \forall t \in \mathbf{R} \quad (83)$$

We denote by $\{e_j :\ j = 0, 1, \ldots\}$ a complete orthonormal set of eigenvectors of H_S in H_0, with

$$H_S\ e_j = \mu_j e_j;\ j = 0, 1, \ldots \quad (84)$$

and denote by $\{e_{j'j};\ j, j' = 0, 1, \dots\}$ the corresponding matrix units

$$e_{j'j} = |e_{j'}\rangle\langle e_j| \quad ; \quad j, j' = 0, 1, \dots \tag{85}$$

For the sake of definiteness, we shall assume that $\omega \geq 0$ and that μ_0, μ_1 are negative, with

$$\mu_0 = -\omega_0 \leq \mu_1 = -\omega_1 < 0 \tag{86}$$

Denote by H' the infinitesimal generator of S_t:

$$H' = H - \omega_0 P_0 - \omega_1 P_1 \tag{87}$$

(more generally, we might take $\mu_0 = \omega^* - \omega_0$, $\mu_1 = \omega^* - \omega_1$, $H' = H - \omega_0 P_o - \omega_1 P_1 + \omega^* 1$, but this would only make the notation heavier, without adding to understanding). The operator $1 \otimes H'$ in $H_0 \otimes H_1$ may be interpreted as **effective total unperturbed energy operator** in place of $H_S \otimes 1 + 1 \otimes H$, since, for all real s, we have

$$V_1(s) := \exp[-i(H_S \otimes 1 + 1 \otimes H)s] \cdot V_1 \cdot \exp[i(H_S \otimes 1 + 1 \otimes H)s] =$$

$$= \exp[-i1 \otimes H's] \cdot V_1 \cdot \exp[i1 \otimes H's] = i \sum_{\varepsilon \in \{0,1\}} D_\varepsilon \otimes |S_{-s} g_\varepsilon\rangle\langle S_{-s} g_{1-\varepsilon}| \tag{88}$$

where, as in the previous sections, $D_0 = D,\ D_1 = -D^+$. It is clear from (5.3) – (5.5) and (5.14) that the same scattering operators $\Omega_\pm$, T and S will be obtained if $H_0 \otimes 1 + 1 \otimes H$ is replaced by $1 \otimes H'$ as total unperturbed energy operator; so we shall make this replacement in the rest of this section. In this interpretation, the total unperturbed energy is carried by the reservoir particle, g_ε carrying an additional (negative) energy $-\omega_\varepsilon = \mu_\varepsilon$ corresponding to the energy levels of the system with which the reservoir particle interacts.

So we see that (1.9) can be satisfied by means of a redefinition of H_S as 0 and H as H', which does not alter the relevant operators. We introduce some generalizations of the notation in Section 2). For the sake of definiteness, we shall assume that

$$H_1 = L^2(\mathbf{R}^3), \quad H = -\frac{1}{2}\Delta \tag{89}$$

and for any f, $g \in K$ and $E \in \mathbf{R}$, let

$$(g|f)(E) = \int_{\mathbf{R}} \exp[-iEt]\langle g,\ S_t f\rangle \mathrm{d}t \tag{90}$$

so that $(g|f)$ corresponds to the special case $E = 0$. In particular, with the choice (5.14a) and for $g = g_\varepsilon$, we have

$$(g_\varepsilon|f)(E) = \int_{\mathbf{R}} \exp[-i(E+\omega_\varepsilon)t]\langle g_\varepsilon, \exp[iHt]f\rangle dt =$$

$$= \int_{\mathbf{R}^3} d^3k\ 2\pi\delta\left(E - \frac{1}{2}k^2 + \omega_\varepsilon\right)\overline{\hat{g}_\varepsilon(k)}\hat{f}(k) \tag{91}$$

where $\hat{f}(k)$ denotes the (unitary) Fourier transform of f evaluated at k.

Note that (5.16) vanishes for $E \leq -\omega_\varepsilon$. With the interpretation of $1 \otimes H'$ as effective total unperturbed energy operator, we may interpret

$$(Gf)_\varepsilon(E) := [2\pi(g_\varepsilon|g_\varepsilon)(E)]^{-1/2}(g_\varepsilon|f)(E)$$

$$= \frac{\int_{\mathbf{R}^3} d^3k\delta\left(E - \frac{1}{2}k^2 + \omega_\varepsilon\right)\overline{\hat{g}_\varepsilon(k)}\hat{f}(k)}{\left[\int_{\mathbf{R}^3} d^3k\delta\left(E - \frac{1}{2}k^2 + \omega_\varepsilon\right)|\hat{g}_\varepsilon(k)|^2\right]^{1/2}} \quad \text{if } (g_\varepsilon|g_\varepsilon)(E) > 0 \tag{92}$$

$$(Gf)_\varepsilon(E) := 0 \quad \text{if } (g_\varepsilon|g_\varepsilon)(E) = 0 \tag{93}$$

as the component of f on g_ε carrying total energy E (decomposed as the sum of $-\omega_\varepsilon = \mu_\varepsilon$ and $\frac{1}{2}k^2$).

In analogy with the notation of Section 4), define also

$$(g|f)_-(E) := \int_{-\infty}^{0} \exp[-iEt]\langle g, S_t f\rangle\ \mathrm{d}t \tag{94}$$

$$T_\varepsilon(E) := (g_\varepsilon|g_\varepsilon)_-(E)(g_{1-\varepsilon}|g_{1-\varepsilon})_-(E)D_\varepsilon D_{1-\varepsilon} \tag{95}$$

(again, the definitions of Section 4 correspond to the special case $E = 0$). Note that for all real E one has

$$|(g|g)_-(E)| \leq \int_{-\infty}^{0} |\langle g, S_t g\rangle|\ \mathrm{d}t = \|g\|_-^2$$

so that, under the assumption that

$$16\|D\|^2 \max(\|g_0\|_-, \|g_1\|_-) < 1 \tag{96}$$

(see Theorem (II.)), we have the convergent geometric series

$$\sum_{n=0}^{\infty} T_\varepsilon(E)^n = (1 - T_\varepsilon(E))^{-1} \tag{97}$$

Define also

$$R_{00}(E) = (g_1|g_1)_-(E) \;\; D_0 D_1 (1 - T_0(E))^{-1} \tag{98}$$

$$R_{01}(E) = (1 - T_0(E))^{-1} D_0 \tag{99}$$

$$R_{10}(E) = (1 - T_1(E))^{-1} D_1 \tag{100}$$

$$R_{11}(E) = (g_0|g_0)_-(E) \; D_1 D_0 (1 - T_1(E))^{-1} \tag{101}$$

<u>$LEMMA(5.1)$</u>. With the above assumptions and notations we have, for all $u, \in H_0, f \in k$,

$$T(u \otimes f) = \frac{i}{2\pi} \sum_{\varepsilon,\varepsilon'=0,1} \int_{-\omega_\varepsilon}^{\infty} R_{\varepsilon'\varepsilon}(E) u \otimes g_{\varepsilon'} (g_\varepsilon|f)(E) \quad \mathrm{d}E \tag{102}$$

<u>$PROOF$.</u> We may expand T as an iterated series

$$T = s - \lim_{t\to\infty} \sum_{n=0}^{\infty} (-i)^n \int_{0=s_0\le s_1 \le \ldots \le s_n \le t} \!\!\!\!\ldots \int V_1(s_0) V_1(s_1) \ldots V_1(s_n) ds_1 \ldots ds_n$$

where $V_1(s)$ is given by (1.16). Performing the change of variables $t_j = s_{j-1} - s_j : j = 1, \ldots, n$, and taking the limit as $t \to \infty$ we obtain

$$T(u \otimes f) = i \sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1\}^{n+1}} (\prod_{j=1}^{n} \delta_{1-\varepsilon_{j-1},\varepsilon_j}) D_{\varepsilon_0} D_{\varepsilon_1} \ldots D_{\varepsilon_n} u \otimes g_{\varepsilon_0}$$

$$\cdot \int_{-\infty}^{0} \mathrm{d}t_1 \ldots \int_{-\infty}^{0} \mathrm{d}t_n (\prod_{j=1}^{n} \langle g_{\varepsilon_j}, \; S_{t_j} g_{\varepsilon_j} \rangle) \langle S_{t_1+\ldots+t_n} g_{1-\varepsilon_n}, f \rangle$$

Using (5.20) we obtain

$$\langle S_{t_1+\ldots+t_n} g_{1-\varepsilon_n}, f \rangle = \frac{1}{2\pi} \int \mathrm{d}E \exp[-iE(t_1 + \ldots + t_n)] (g_{1-\varepsilon_n}|f)(E)$$

Hence

$$T(u \otimes f) = \frac{i}{2\pi} \sum_{n=0}^{\infty} \sum_{\varepsilon\in\{0,1\}^{n+1}} (\prod_{j=1}^{n} \delta_{1-\varepsilon_{j-1},\varepsilon_j}) D_{\varepsilon_0} D_{\varepsilon_1} \ldots D_{\varepsilon_n} u \otimes g_{\varepsilon_0} \cdot$$

$$\cdot \int \mathrm{d}E(\prod_{j=1}^{n} \int_{-\infty}^{0} \mathrm{d}t_j \exp[-iEt_j]\langle g_{\varepsilon_j}, S_{t_j} g_{\varepsilon_j}\rangle)(g_{1-\varepsilon_n}|f)(E) \tag{103}$$

Due to the factor $(\prod_{j=1}^{n} \delta_{1-\varepsilon_{j-1},\varepsilon_j})$, the summation $\sum_{\varepsilon\in\{0,1\}^{n+1}}$ reduces to a summation over the four possible values of the pair $(\varepsilon'\,,\,\varepsilon) := (\varepsilon_0\,,\,1-\varepsilon_n)$. Then one can identify the convergent geometric series of the form (5.21) appearing in (5.24); and (5.23) follows.

In order to introduce the S operator, it is convenient to use the following notation. Any bounded operator A on $H_0 \otimes H_1$ can be expressed through a matrix–valued integral kernel $A_{j'j}(k'\,,\,k)$: $j\,,\,j' = 0, 1\,, \ldots\,;\, k,\, k' \in \mathbf{R}^3$, such that, for all $u,\, u' \in H_0,\, f,\, f' \in H_1$ one has

$$\langle u' \otimes f'\,,\, Au \otimes f\rangle$$

$$= \sum_{j,j'} \int_{\mathbf{R}^3} \mathrm{d}^3\, k \int_{\mathbf{R}^3} \mathrm{d}^3\, k'\, \overline{\langle e_{j'}, u'\rangle} \overline{\hat{f}'(k')} A_{j'j}(k', k) \langle e_j, u\rangle \hat{f}(k) \tag{104}$$

In general, this kernel will be a distribution rather than an ordinary function of $k,\, k'$. With this notation, we have the following well–known result of scattering theory:

$$(S-1)_{j'j}(k'\,,\,k) = -2\pi i \delta\left(\frac{1}{2}k'^2 - \frac{1}{2}k^2 + w_{j'} - w_j\right) T_{j'j}(k', k) \tag{105}$$

<u>*LEMMA*(5.2)</u>. With the above assumptions and notations, define

$$S_{\varepsilon'\varepsilon}(E) = \delta_{\varepsilon'\varepsilon} 1 + [(g_{\varepsilon'}|g_{\varepsilon'})(E)]^{1/2} R_{\varepsilon'\varepsilon}(E) [(g_{\varepsilon}|g_{\varepsilon})(E)]^{1/2} \tag{106}$$

Then, for all $u,\, u' \in H_0,\, f,\, f' \in K$, one has, in the notations (5.17a), (5.17b)

$$\langle (u' \otimes f'),\ S(u \otimes f)\rangle = \langle (u' \otimes f'),\ (u \otimes f)\rangle +$$

$$+ \sum_{\varepsilon,\varepsilon'=0,1} \int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \langle u', [S_{\varepsilon'\varepsilon}(E) - \delta_{\varepsilon'\varepsilon} 1] u\rangle \overline{(Gf')_{\varepsilon'}(E)} (Gf)_\varepsilon(E) \tag{107}$$

<u>*PROOF.*</u> A straightforward application of Lemma (5.1) gives

$$T_{j'j}(k'k) = i \sum_{\varepsilon,\varepsilon'=0,1} \langle e_{j'}, R_{\varepsilon',\varepsilon}\left(\frac{1}{2}k^2 - \omega_\varepsilon\right) e_j\rangle \hat{g}_{\varepsilon'}(k') \overline{\hat{g}_\varepsilon(k)} \tag{108}$$

Inserting (5.29) into (5.27) and using (5.16), (5.17) one obtains (5.28); the only difficulty is to keep track of powers of 2π.

REMARK(5.3). Equation (5.28) is the representation of the operator S, which commutes with the unperturbed energy operator $1 \otimes H'$, as a matrix–valued multiplication operator in the direct integral decomposition of $H_0 \otimes H_1$ into fibers corresponding to $H' = E$. Unitary of S amounts to the fact that, for every E, the operator $S(E) \in \mathbf{B}(H_0) \otimes M(2, \mathbf{C})$ given by

$$S(E) = \sum_{\varepsilon,\varepsilon'=0,1} S_{\varepsilon'\varepsilon}(E) \otimes e_{\varepsilon'\varepsilon} \tag{109}$$

($e_{\varepsilon'\varepsilon} : \varepsilon, \varepsilon' = 0, 1$ being matrix units of $M(2, \mathbf{C})$) is unitary. $S(E)$ may be regarded as the S operator on the energy shell of total energy E. *REMARK*(5.4).

The stochastic differential appearing in Theorem (II.) (Equation (4.9)) can be readily seen to be

$$\sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon'\varepsilon}(0) \otimes dN_s(g_{\varepsilon'}, g_\varepsilon) = dN_s(S(0) - 1; 0) \tag{110}$$

in the notation of [40]. This is the quantum Poisson process of zero intensity corresponding to the S operator on the energy shell of total energy $E = 0$. Then unitarity of the solution $U(t)$ of equation (4.9), which has been proved in [9], follows from unitarity of $S(0)$ and the general theory of [40].

The notation (5.28), (5.30) allows us to give a particularly transparent form to the generator $L^{(LDL)}$ of the semigroup obtained by Dümcke [25] in the low density limit: the following Proposition may be regarded as a rigorous version of some hand–waving arguments in [40]. *PROPOSITION*(5.5).

With the above assumptions and notations, the explicit form of $L^{(LDL)}$ is given by

$$L^{(LDL)}(x) = \frac{1}{2\pi} \sum_{\varepsilon,\varepsilon'=0,1} \int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \exp[-\beta(E + \omega_\varepsilon)] \cdot (S_{\varepsilon'\varepsilon}(E)^{+} X S_{\varepsilon'\varepsilon}(E) - \delta_{\varepsilon'\varepsilon} X) \tag{111}$$

for all X in $\mathbf{B}(H_0)$. *PROOF.* It has been shown by Dümcke in [25] that

$$L^{(LDL)}(X) = \Psi(X) + Z^{+}X + XZ \ ; \ X \in \mathbf{B}(H_0), \tag{112}$$

where the completely positive map Ψ on $\mathbf{B}(H_0)$ is given by

$$\Psi(X) = 2\pi \sum_{w \in Sp[H_S,\,\cdot\,]} \int_{\mathbf{R}^3} \mathrm{d}^3k \int_{\mathbf{R}^3} \mathrm{d}^3k' \delta\left(\frac{1}{2}k'^2 - \frac{1}{2}k^2 + w\right)$$

$$\cdot \exp[-\beta k^2/2] T_w^+(k',k) X T_w(k',k) \tag{113}$$

and where the element Z of $\mathbf{B}(H_0)$ is given by

$$Z = -i \int_{\mathbf{R}^3} \mathrm{d}^3k \exp[-\beta k^2/2] T_0(k,k) \tag{114}$$

In the above formulas, $T_w(k',k)$ is defined by

$$T_w(k',k) = \sum_{j,j':w_{j'}-w_j=w} T_{j'j}(k',k) e_{j'j} \tag{115}$$

Then we can insert the explicit expression (5.29) of $T_{j'j}(k',k)$ and perform the integrations over k, k', recalling that from (5.16), (5.9) we have

$$\int_{\mathbf{R}^3} \mathrm{d}^3k \delta\left(E - \frac{1}{2}k^2 + \omega_\varepsilon\right) \overline{\hat{g}_{\varepsilon'}(k)} \cdot \hat{g}_\varepsilon(k) = \delta_{\varepsilon'\varepsilon}(2\pi)^{-1}(g_\varepsilon|g_\varepsilon)(E) \tag{116}$$

With the use of (5.27), we obtain

$$\Psi(X) = \frac{1}{2\pi} \sum_{\varepsilon,\varepsilon'=0,1} \int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \exp[-\beta(E+\omega_\varepsilon)] \cdot [S_{\varepsilon'\varepsilon}(E)^+ - \delta_{\varepsilon'\varepsilon}1] X [S_{\varepsilon'\varepsilon}(E) - \delta_{\varepsilon'\varepsilon}1]$$

$$Z = \frac{1}{2\pi} \sum_{\varepsilon=0,1} \int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \exp[-\beta(E+\omega_\varepsilon)][S_{\varepsilon\varepsilon}(E) - 1] \tag{117}$$

Combining (5.33), (5.37) and (5.38) the result follows. <u>*REMARK*(5.6).</u>

The semigroup with generator (5.32) admits a unitary dilation in terms of a quantum Poisson process, see [40]. We introduce the Hilbert space

$$\mathcal{K} := L^2((-\omega_0,\infty), dE) \oplus L^2((-\omega_1,\infty), dE) \tag{118}$$

(note that (5.39) is not the same K as in the rest of the paper) and the von Neumann subalgebra M of $L^\infty(\mathbf{R}, dE;\ M(2,\mathbf{C}))$ of functions $E \mapsto \{Y_{\varepsilon'\varepsilon}(E) : \varepsilon'\varepsilon = 0,1\}$ such that $Y_{\varepsilon'\varepsilon}(E) = 0$ for $E < -\omega_\varepsilon$.

M acts on $\mathcal{K}$ in the obvious way. Define a positive linear functional μ on M by

$$\mu(Y) := \frac{1}{2\pi}\sum_{\varepsilon=0,1}\int_{-\omega_\varepsilon}^{\infty} \mathrm{d}E \exp[-\beta(E+\omega_\varepsilon)]Y_{\varepsilon\varepsilon}(E) = \langle \xi,\ Y\xi\rangle_{\mathcal{K}} \tag{119}$$

where $\xi \in \mathcal{K}$ is given by

$$\xi(E) := (2\pi)^{-1/2}\{\chi_{(-\omega_0,\infty)}(E)\exp\left[-\frac{\beta}{2}(E+\omega_0)\right]\oplus\chi_{(-\omega_1,\infty)}(E)\exp\left[-\frac{\beta}{2}(E+\omega_1)\right]\} \tag{120}$$

The function $E \mapsto S(E)$, $S(E)$ given by (5.30), can be regarded as an element S of $\mathbf{B}(H_0) \otimes M$. With this notation, the generator (5.32) can be rewritten in the form of [40] as follows:

$$L^{(LDL)}(X) = (id \otimes \mu)(S^+(X\otimes 1)S - (X \otimes 1)) \tag{121}$$

Then it follows from [40] that, for all $X \in \mathbf{B}(H_0)$ and $t \in \mathbf{R}^+$,

$$\exp[L^{(LDL)}t](X) = \mathbf{E}[U^+(t)(X\otimes 1)U(t)] \tag{122}$$

where $U(t)$ is the (unitary) solution of the $QSDE$

$$dU(t) = dN_t(S-1;\xi)U(t)\ ,\ U(0) = 1 \tag{123}$$

$\underline{REMARK(5.7)}$. In the finite temperature case and with a suitable choice of collective vectors (not the collective coherent vectors of Section 4)), the matrix elements of $U_{t/z}$ should converge, in the limit as $z \to 0$, to matrix elements of the solution $U(t)$ of a QSDE of a form similar to (5.44). The main conceptual difficulty in order to verify this conjecture consists in the individuation of the collective vectors with which to form the matrix elements. Once this is done, it is possible to expect that a combination of all the techniques developed up to now (both for the WCL and the LDL) should lead, after some hard work, to the desired result. The only surprises, arising from the further mathematical developements should concern (as it was for WCL) the resulting space of "test function".

For the moment we only wish to explain why, in contrast with the Fock case where only $S(0)$ appears, here we have continuously many values for the total energy E, and an integration over E weighted by the Boltzmann factor $\exp[-\beta(E+\omega_\varepsilon)]$.

In the equilibrium state at strictly positive temperature $T = 1/\beta$, particles of all energies are present, with numbers which become proportional to $z\exp[-\beta k^2/2]$ in the limit as $z \to 0$. The total energy of a particle with momentum k and of type ε ($\varepsilon \in \{0,1\}$) is redefined to be $E = \frac{1}{2}k^2 - \omega_\varepsilon$, to keep into account the energies of the energy levels of the system on which the reservoir particles scatter. This allows to describe scattering of a particle on the system by saying that a scattering particle changes its type from ε to $\varepsilon' = 1-\varepsilon$, while its total energy remains unchanged: $\frac{1}{2}k^2 - \omega_\varepsilon = \frac{1}{2}k'^2 - \omega_{\varepsilon'}$. At the same time, the system performs a transition under the action of the operator $D_{\varepsilon'}$. The particle type $\varepsilon \in \{0,1\}$ remains a quantum degree of freedom, interacting with the quantum system. On the other hand, the total energy E becomes a **classical** variable (with continuous spectrum), since the uncertainty relation $\Delta E\Delta(t/z) \geq \hbar$ involving energy and rescaled time t/z is no longer a restriction in the limit as $z \to 0$. This gives rise to the peculiar structure of M as an algebra of 2×2 matrices whose entries are functions of E. Each value E of total energy contributes to $L^{(LDL)}$ with the Boltzmann factor $\exp[-\beta k^2/2] = \exp[-\beta(E+\omega_\varepsilon)]$ for an incoming particle of type ε.

§6. Main strategy of the proof

In the folllowing we list the main steps in the strategy of the proof of the above results. We have formulated these steps so to include all the situations in which these results could be obtained (Boson and Fermion, WCL and LDL, linear and nonlinear interaction, with or without rotating wave approximation, ...). Of course the techniques of the proof depend heavily on the specific situation, however the following steps seem to be a common factor.

-step (1). Expand U_{t/x^2} in the iterated series

$$U_{t/x^2} = \sum_{n=0}^{\infty} (-i)^n \int_0^{t/x^2} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_\lambda(t_1) \cdots V_\lambda(t_n) \tag{124}$$

-step (2). Expand $V_\lambda(t_1) \cdots V_\lambda(t_n)$ and find a sum of many tems of the form

$$x_n \cdot D_{\varepsilon_1} \cdots D_{\varepsilon_n} \otimes A^{\varepsilon_1}(S_{t_1} g) \cdots A^{\varepsilon_N}(S_{t_n} g) \tag{125}$$

where N is equal to pn and, p is the degree of the interaction, supposed to be a homogeneous polynomial in the creation and annihilation operators. Moreover,

$$x_n := \begin{cases} x^n, & \text{WCL} \\ 1, & \text{LDL} \end{cases} \tag{126}$$

-step (3). By normal ordering of the product (6.2) we find a sum of terms of the form

$$\prod_{h=1}^{m} < S_{t_{p_k}} G, S_{t_{q_k}} G' > \cdot A^+(S_{t_{\alpha_1}} g^{(1)}) \cdots A^+(S_{t_{\alpha_\nu}} g^{(\nu)}) \cdot A(S_{t_{\beta_1}} g_{(1)}) \cdots A(S_{t_{\beta_\mu}} g_{(\mu)}) \tag{127}$$

where, $G, G', g^{(1)}, \cdots, g^{(\nu)}, g_{(1)}, \cdots, g_{(\mu)}$ are equal to g in WCL case and to g_0 or g_1 in LDL case.

-step (4). Separate the negligible (type II) terms from the relevant ones (type I). In all cases (with the only exception of LDL finite temperature case), the type I terms are those terms of the form (6.4) in which all the time pairs (t_{p_k}, t_{q_k}), in the product for $h = 1, \cdots, m$, are **time consecutive** in the sense that:

$$p_k = q_k - 1 \tag{6.4a}$$

for each $k = 1, \cdots, m$. All the other terms are called type II.

-step (5). Apply various generalizations of the Pulè inequality to prove that a sum of order $n!$ terms of matrix elments with respect to appropriate "collective vectors of products of the form (6.4) is estimated by

$$C^m/(n-m)! \qquad t \in [0,T] \tag{128}$$

where C is a constant depending only on g, g_0, g_1 and m is the number of scalar products arising from exchanges of creators and annihilators. We shall see that the basic difference between WCL and LDL is that in the former case, $m \leq \frac{n}{2}$ so that (6.5) is always less than or equal to $C^m/[n/2]!$, while in the latter case m can be very close to n (e.g. equal to $n-1$).

-step (6). Use (6.5) to prove the uniform convergence (in x) of the iterated series so that we can exchange $\lim_{x\to 0}$ and $\sum_{n=0}^{\infty}$

-step (7). Compute the explicit limit of the non negligible (type I) terms and show that the limit of negligible (type II) terms tends to zero as $x \to 0$.

-step (8). Compute the limit of

$$\lim_{x\to 0} \frac{d}{dt} < u \otimes W(x \int_{S/x^2}^{T/x^2} S_u f du)\Phi, U_{t/x^2}^{(x)} v \otimes W(x \int_{S'/x^2}^{T'/x^2} S_u f' du)\Phi > \tag{129}$$

-step (9). By means of step (8) prove that the limit, as $x \to 0$, of

$$< u \otimes W(x \int_{S/x^2}^{T/x^2} S_u f du)\Phi, U_{t/x^2}^{(x)} v \otimes W(x \int_{S'/x^2}^{T'/x^2} S_u f' du)\Phi > \tag{130}$$

satisfies an integral equation.

-step (10). Apply step (9) to identify this equation with a QSDE.

Notice that it is by no means true that the $n-$th term of the iterated series of (6.7) converges to the $n-$th term of the iterated series solution of the QSDE. This is the reason why a good deal of quantum stochastic calculus intuition is needed in order to guess the correct form of the limiting equation.

-step (11). Reduce the finite temperature case to the Fock case.

-step (12). The Langevin equation. This requires some new techniques with respect to the U_t-evolution.

However our experience is that the bsic difficulties, both technical and conceptual, are those concerning the limits of the matrix elements of the $U_{t/x^2}^{(x)}$ operator. Once the correct equation for the limit unitary evolution $U(t)$ has benn individuated, it is usually a question of time to get the convergence of the corresponding Langevin equation.

§7. Why can one hope to obtain both the weak coupling and the low density limits with the same type of coherent vectors

In Section 4) we said that we shall consider the WCL and LDL for the same type of coherent vectors even if the interactions are different, now let us outline the basic idea of how can one hope to obtain both the WCL and the LDL with the same type of coherent vectors.

Both in the LDL and the WCL limit, the $n-th$ iterated integral of the perturbation series is of the form

$$x_n \int_0^{t/x^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n V(t_1) \cdots V(t_n) \tag{131}$$

where, V is not the same as (1.7) but has the following form:

$$V = \begin{cases} i(D \otimes (A^+(g))^p - c.c.) & \text{for WCL} \\ i(D \otimes A^+(g_0)A(g_1) - c.c.) & \text{for LDL} \end{cases} \tag{132}$$

and x_n is defined in (6.3) (i.e. the λ factors have been taken out of V and put into x_n).

Using the explicit form of V and neglecting the system operators D, D^+ (inessential for our considerations) we are reduced to compute matrix elements in some collective coherent vectors of the form (1.18b), of expressions of the form

$$x^n \int_0^{t/x^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n (A^{\varepsilon_1}(S_{t_1}g))^p \cdots (A^{\varepsilon_n}(S_{t_n}g))^p \qquad \text{WCL} \tag{133}$$

$$\int_0^{t/x^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n A^+(S_{t_1}g_{\varepsilon_1})A(S_{t_1}g_{1-\varepsilon_1}) \cdots A^+(S_{t_n}g_{\varepsilon_n})A(S_{t_n}g_{1-\varepsilon_n}) \qquad \text{LDL} \tag{134}$$

Bringing the products in the integrals (7.3) and (7.4) to normally ordered form we find that

$$(A^{\varepsilon_1}(S_{t_1}g))^p \cdots (A^{\varepsilon_n}(S_{t_n}g))^p \tag{135}$$

and

$$A^+(S_{t_1}g_{\varepsilon_1})A(S_{t_1}g_{1-\varepsilon_1}) \cdots A^+(S_{t_n}g_{\varepsilon_n})A(S_{t_n}g_{1-\varepsilon_n}) \tag{136}$$

have both the form:

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} \prod_{h=1}^m < G, S_{t_{q_h}-t_{p_h}} G' > (A^+ \cdots)_{r_1} (A \cdots)_{r_2} \tag{137}$$

where $(A^+ \ \cdots)_r$ (resp. $(A \ \cdots)_r$) denotes a product of r creation (resp. annihilation) operators of the form $A^+(S_{t_j} g_\varepsilon)$ (resp. $A(S_{t_j} g_\varepsilon)$; G, G' are equal to g in the WCL case and g_0 or g_1 in the LDL case. Moreover (7.7) means that there are exactly m scalar products which have been produced by application of the CCR so that exactly m creators and m annihilators

$$r_1 + r_2 = \begin{cases} pn - 2m, & \text{for WCL} \\ 2n - 2m, & \text{for LDL} \end{cases} \tag{138}$$

With the change of variables

$$x^2 t_j = s_j, \quad j = 1, \cdots, n \tag{139}$$

in (7.7) and applying the adjoints of the creators on the coherent vectors on the left:

$$W(x \int_{S/x^2}^{T/x^2} S_u f du)\Phi \tag{140}$$

and the annihilators on the coherent vectors on the right:

$$W(x \int_{S'/x^2}^{T'/x^2} S_u f' du)\Phi \tag{141}$$

for each product $(A^+ \ \cdots)_{r_1}$, $(A \ \cdots)_{r_2}$ we obtain the following factor

$$\prod_{h=1}^{r_1} x \int_{S/x^2}^{T/x^2} < S_u f, S_{s_{\alpha_h}/x^2} G > du \prod_{h=1}^{r_2} x \int_{S'/x^2}^{T'/x^2} < S_{s_{\beta_h}/x^2} G', S_u f' > du \tag{142}$$

with G, G' as explained above. Notice that the limit of the expressions

$$\int_{S/x^2}^{T/x^2} < S_u f, S_{s_{\alpha_h}/x^2} G > du \tag{143}$$

and

$$\int_{S'/x^2}^{T'/x^2} < S_{s_{\beta_h}/x^2} G', S_u f' > du \tag{144}$$

exist always because of (1.6), so the quantity

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} x_n \int_0^{t/x^2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{h=1}^m < G, S_{t_{q_h}-t_{p_h}} G' >$$

$$< W(x \int_{S/x^2}^{T/x^2} S_u f du)\Phi, (A^+ \cdots)_{r_1} (A \cdots)_{r_2} W(x \int_{S'/x^2}^{T'/x^2} S_u f' du)\Phi > \quad (145)$$

has the form

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} x^{r_1+r_2-2n} \cdot x_n \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \prod_{h=1}^m < G, S_{(s_{q_h}-s_{p_h})/x^2} G' > \cdot C(x) \quad (146)$$

where $C(x)$, which is a product of integrals of the form (7.11), (7.12), tends to some limit as $x \to 0$.

Now for each integral in (7.14) of the form

$$\cdots \int_0^{s_{p_h-1}} ds_{p_h} \cdots \int_0^{s_{q_h-1}} < G, S_{(s_{q_h}-s_{p_h})/x^2} G' > ds_{q_h} \int_0^{s_{q_h}} ds_{q_h+1} \cdots \quad (146)$$

(7.14)

withthechangeofvariables

$(s_{q_h} - s_{p_h})/x^2 = t_{q_h}$(147)one gets

$$\cdots x^2 \cdot \int_0^{s_{p_h-1}} ds_{p_h} \cdots \int_{-s_{p_h}/x^2}^{(s_{q_h-1}-s_{p_h})/x^2} < G, S_{t_{q_h}} G' > dt_{q_h} \int_0^{x^2 t_{q_h}+s_{p_h}} ds_{q_h+1} \cdots \quad (148)$$

as $x \to 0$ the integral (7.16) converges to a limit which is either

$$\cdots x^2 \cdot \int_0^{s_{p_h-1}} ds_{p_h} \cdots \int_{-\infty}^0 < G, S_{t_{q_h}} G' > dt_{q_h} \int_0^{s_{q_h-1}} ds_{q_h+1} \cdots \quad (149)$$

or zero. In this sense we say that the scalar product (7.14a) **produces a factor** x^2. For the WCL linear case ($p = 1$) and the LDL case, in the product $\prod_{h=1}^m < G, S_{(s_{q_h}-s_{p_h})/x^2} G' >$ the pairs of indices (s_{p_h}, s_{q_h}) are all mutually different, therefore this product produces a factor x^{2m}.

On the other hand, for the WCL nonlinear case ($p \geq 2$), because of the choice of the interaction (7.2), we can have more than one annihilation or creation at the same time and therefore, it is possible that for some $1 \le h \neq h \le m$, $p_h = p_{h'}$ and $q_h = q_{h'}$, in this case one has

$$\cdots \int_0^{s_{p_h-1}} ds_{p_h} \cdots \int_0^{s_{q_h-1}} < G, S_{(s_{q_h}-s_{p_h})/x^2} G' >< G, S_{(s_{q_{h'}}-s_{p_{h'}})/x^2} G' > ds_{q_h} \int_0^{s_{q_h}} ds_{q_h+1} \cdots$$

$$= \cdots x^2 \cdot \int_0^{s_{p_h-1}} ds_{p_h} \cdots \int_{-s_{p_h}/x^2}^{(s_{q_h-1}-s_{p_h})/x^2} < G, S_{t_{q_h}} G' >^2 dt_{q_h} \int_0^{x^2 t_{q_h}+s_{p_h}} ds_{q_h+1} \cdots \tag{150}$$

So the product $\prod_{h=1}^m < G, S_{(s_{q_h}-s_{p_h})/x^2} G' >$ gives only a factor $x^{2m'}$ for some $m' \le m$. Obviously by the above explaination, one can take m' as the cardinality of the set $\{q_h\}_{h=1}^m$ i.e. the maximum number of q_h different among themselves. Summing up we conclude that (7.14) is of the following form

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} x^{r_1+r_2-2n} \cdot x_n \cdot x^{2m_0} C_1(x) \tag{151}$$

where x_n is given by (6.3), $C_1(x)$ tends to some limit as $x \to 0$ and

$$m_0 = \text{card } \{q_1, \cdots, q_m\} = \begin{cases} m, & \text{for LDL and WCL linear cases} \\ m', & \text{for WCL nonlinear cases} \end{cases} \tag{152}$$

Since in both cases the product $\prod_{h=1}^m < G, S_{(s_{q_h}-s_{p_h})/x^2} G' >$ produces the factor x^{2m_0} we see that the power of x in (7.18) becomes

-In the LDL case,

$$x^{r_1+r_2-2n} \cdot x_n \cdot x^{2m_0} = x^{2n-2m-2n+2m} = 1 \tag{153}$$

-In the WCL linear case

$$x^{r_1+r_2-2n} \cdot x_n \cdot x^{2m_0} = x^{n+(n-2m)-2n+2m} = 1 \tag{154}$$

For this reason one can obtain the limit both in LDL and WCL linear cases.

In the WCL nonlinear case again a factor x^{2m_0} is produced, however in this case:

$$x^{r_1+r_2-2n} \cdot x_n \cdot x^{2m_0} = x^{n+(pn-2m)-2n+2m'} = x^{(p-1)n-2m+2m'} \tag{155}$$

and since at each time index we have at most p creators and m' labels the different time indices of creators, it follows that $m \le m'p$. Therefore,

$$(p-1)n - 2m + 2m' \ge (p-1)n - 2m + 2m/p = (p-1)n - 2m\frac{p-1}{p} \tag{156}$$

This shows that if $2m < pn$ then the right hand side of (7.23) is strictly greater than

$$(p-1)n - pn\frac{p-1}{p} = 0$$

So also in the WCL nonlinear case that the limit (7.18) exists but in this case only the pure scalar product terms ($pn = 2m$) may give a non-zero contribution. This is a typical central limit effect (cf. [19] where ideas and techniques borrowed from the WCL and LDL have been used to prove quantum central limit theorems).

§8. The role of the interaction

The first difference between LDL and WCL is that in the LDL case the wave operator at time t depends on x (keep in mind that in LDL case $x = z^{1/2}$) only via the time scaling $t \to t/x^2$, while in the WCL limit the interaction itself is multiplied by x, which gives a factor x^n in the $n-th$ term of the iterated series. In the notation (6.3) for x_n and (1.15) for $V(t)$, this series is

$$U_{t/x^2} = \sum_{n=0}^{\infty}(-1)^n x_n \int_0^{t/x^2} dt_1 \dots \int_0^{t_{n-1}} dt_n V(t_1)\dots V(t_n) \tag{157}$$

Both in the WCL and the LDL case we shall consider the limits of the matrix elements of U_{t/x^2} in the same family of vectors (collective coherent vectors) i.e.

$$W(x\int_{S/x^2}^{T/x^2} S_u f du)\Phi$$

In the Fock case, if the interactions V were the same for the WCL and the LDL, the existence of the limit in the LDL case would imply that the corresponding limit in the WCL case is identically zero.

Notice that this is not necessarily true in the finite temperature case. In fact in that case the vacuum vector will depend on $x = z^{1/2}$ in the LDL case but will not depend on $x = \lambda$ in the WCL case. Therefore in this case both limits might exist.

The crucial role, in the Fock case, is played by the difference in the interactions which, in the LDL case, is of number type, i.e. of the form:

$$i(D \otimes A^+(g_0)A(g_1) - c.c.)$$

and in the WCL case of the form

$$i(D \otimes (A^+(g))^p - c.c.)$$

Notice that we have obtained a nonzero WCL in the case $p > 1$, therefore, for the argument given above, the limit cannot exist (for this interaction) in the LDL Fock case.

This shows that the number preserving form of the interaction and not the degree (2 versus 1) is the essential factor. However before that we want to explain why the number preserving character of the interaction is crucial and for a generic interaction of d degree one should not expect the existence of both the WCL and the LDL. If we show that this interaction has a WCL limit then by the argument in this Section it will be impossible that it has also a LDL.

§9. On the sum $\sum_{m,\{p_h,q_h\}_{h=1}^m}$

In this Section we shall illustrate two basic differences among the terms of the iterated series for the WCL and LDL case. In Sections 11), 14) and 15) we shall prove that these differences lay at the root of the fact that, in the QSDE obtained in the LDL case, the coefficients of the noises are power series of products of the coefficients of the original Hamiltonian system weighted with appropriate scalars, while in the linear WCL case the coefficients of the creation and annihilation noises are the same as those of the creation and annihilation terms in the original Hamiltonian system.

In order to explain this differences we must go back to the products (7.6), (7.7)and study the common normally ordered form (7.8). The crucial difference between the products (7.6) and (7.7) is that, in (7.6) (i.e. in the WCL case)

i) For $p = 1$, all the indices t_j which appear under the product $(A^+\cdots)_{r_1}(A\cdots)_{r_2}$, are different from all the t_{p_h}, t_{q_h} appearing in the product $\prod_{h=1}^m$.

ii) No t_{p_h}, t_{q_h} are equal to some t_{q_k}, t_{p_k} if $k \neq h$.

In (7.7) however (i.e. in the LDL case) neither i) nor ii) above are true. As a consequence of the failure of i) and ii), the change of variables like (7.15), which involves $(t_{q_h} - t_{p_h})/x^2$, will affect not only the $\prod_{h=1}^m$ integral but also the integrals of the form of (7.11) and (7.12).

More precisely: the failure of i) is related to the fact that we consider a nonlinear (degree ≥ 2) interaction. The failure of ii) depends more on the specific form of the interaction. For example, for the interaction $i(D^+\otimes A^2(g)$ - c.c.) no t_{p_h} can be equal to some t_{q_k} with $k \neq h$ (however it could be equal to some t_{p_k}) but property i) fails. While for the interaction $i(D^+\otimes A^+(g_0)A(g_1)$ - c.c.) (or more generally, for any interaction which involves a product of both creation and annihilation operators) both properties i) and ii) fail.

To illustrate the origin of the failure of properties i) and ii), let us consider the case $n = 3$. Then

$$A^+(S_{t_1}g_0)A(S_{t_1}g_1)A^+(S_{t_2}g_1)A(S_{t_2}g_0)A^+(S_{t_3}g_0)A(S_{t_3}g_1)$$

$$= A^+(S_{t_1}g_0) < S_{t_1}g_1, S_{t_2}g_1 >< S_{t_2}g_0, S_{t_3}g_0 > A(S_{t_3}g_1) + \text{something} \quad (158)$$

In this case, looking at the first and the second scalar product we have the situation:

$$(t_1, t_2) = (t_{p_1}, t_{q_1}); \quad (t_2, t_3) = (t_{p_2}, t_{q_2}) \quad (159)$$

so that $t_{q_1} = t_{p_2}$. Moreover the operators $A^+(S_{t_1}g_0)$ and $A(S_{t_3}g_1)$ will act on the collective coherent vectors so that i) can not hold.

Summing up for the pair of indices t_{p_h}, t_{q_h}, i.e. those pairs corresponding to scalar products arising from the commutators of creation and annihilation operators, we have the following possible situations:

$$\{p_1, \dots, p_m\} \cap \{q_1, \dots, q_m\} = \emptyset \tag{160}$$

$$\text{card } \{p_1, \dots, p_m\} = \quad \text{card } \{q_1, \dots, q_m\} = m \tag{161}$$

For interactions of the form

$$i(D \otimes A^+(g)^p - c.c.) \tag{162}$$

(9.3) holds but (9.4) holds only for $p = 1$. For interactions of the form

$$i(D \otimes A^+(g_0)A(g_1) - c.c.) \tag{163}$$

only (9.4) is true.

Moreover since m is the number of scalar products which appear in the normally ordered form and each scalar product uses exactly one creation and one annihilation operator, we deduce that for interactions of the form (9.5) with $p = 1$, $m \leq n/2$; for interactions of the form (9.5) with $p > 1$, $m' \leq n/2$ (m' is the cardinality of the time index set $\{q_h\}_{h=1}^m$); but for interactions of the form (9.6) (i.e. LDL case), $m \leq n - 1$ so that m can be very near to n and this implies, as we shall see in more detail in Section 11), that in order to assure the uniform convergence of the series we need a bound of the norm of D in the LDL case, contrarily to what happens in the WCL case.

Now let us see the difference in the behaviour of the index set $\{p_h, q_h\}_{h=1}^m$ in the two cases.

For each fixed $n \in \mathbf{N}$ and $\varepsilon \in \{0, 1\}^n$, denote by $j_1 < \cdots < j_k$ the set

$$\{j \in \{1, \cdots, n\} : \varepsilon(j) = 1\}$$

In the WCL case the inidces j_h are characterized by the property that in time t_{j_h} $(h = 1, \cdots, k)$ one has a creator and at all other times an annihilation operator. This shows that, in the WCL case, we have $\{p_h\}_{h=1}^m \subset \{1, \cdots, n\} \setminus \{j_h\}_{h=1}^m$ and $\{q_h\}_{h=1}^m \subset \{j_h\}_{h=1}^m$ (one can assume that the set is ordered), which is an improvement of (9.3), but this is not necessary for the LDL case since for each time we have one creation and one annihilation operator.

Moreover, since $\{p_h\}_{h=1}^m$ labels the annihilators which are used to produce scalar product with the creators, labeled by $\{q_h\}_{h=1}^m$, one has $p_h < q_h$ for any $h = 1, \cdots, m$. This implies that in the WCL linear case (*i.e.* $p = 1$), one has the identity:

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} = \sum_{m=0}^{[n/2]} \sum_{1\le q_1<\cdots<q_m\le n,\{q_h\}_{h=1}^m\subset\{j_h\}_{h=1}^k} \sum_{\substack{\{p_h\}_{h=1}^m\subset\{1,\cdots,n\}\setminus\{j_h\}_{h=1}^k,|\{p_h\}_{h=1}^m|=m \\ p_h<q_h,h=1,\cdots,m}} \tag{164}$$

In the nonlinear case ($p \ge 2$), we can produce at most $[pn/2]$ scalar products, i.e. $0 \le m \le [pn/2]$ and moreover some p_h (q_h) can be equal to some other $p_{h'}$ ($q_{h'}$), therefore we can only conclude that

$$1 \le q_1 \le \cdots \le q_m \le n, \quad |\{p_h\}_{h=1}^m| \le m \tag{165}$$

It follows that in the WCL nonlinear case,

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} = \sum_{m=0}^{[pn/2]} \sum_{1\le q_1\le\cdots\le q_m\le n,\{q_h\}_{h=1}^m\subset\{j_h\}_{h=1}^k} \sum_{\substack{\{p_h\}_{h=1}^m\subset\{1,\cdots,n\}\setminus\{j_h\}_{h=1}^k,|\{p_h\}_{h=1}^m|\le m \\ p_h<q_h,h=1,\cdots,m}} \tag{166}$$

In the LDL case (9.3) is not true and $0 \le m \le n-1$, but (9.4) is true so that one has

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} = \sum_{m=0}^{n-1} \sum_{1\le q_1<\cdots<q_m\le n} \sum_{\substack{1\le p_1,\cdots p_m\le n,|\{p_h\}_{h=1}^m|=m \\ p_h<q_h,h=1,\cdots,m}} \tag{167}$$

§10. Pulè inequalities

The prototype of the inequalities we are going to consider in this Section is in Lemma (3) of [29]. For this reason we call them Pulè inequalities. They play a basic role in the proof of the uniform estimate.

For $t \in \mathbf{R}_+$ and any natural integer n, we denote

$$\Delta_t^{(n)} := \{(t_1, \cdots, t_n) \in \mathbf{R}^n : \ t \geq t_1 \geq \cdots \geq t_n \geq 0\} \tag{168}$$

With this notation the most general Pulè inequality used up to now in the weak coupling or low density problem is the following: <u>*LEMMA* (10.1)</u> Let $f : \mathbf{R} \longrightarrow \mathbf{R}_+$ be a positive integrable symmetric (i.e. $f(t) = f(-t)$) function. Let $n, m \in \mathbf{N}$, $1 \leq m \leq n-1$ and let $2 \leq q_1 < \cdots < q_m \leq n$; $1 \leq p_1 \leq \cdots \leq p_m \leq n-1$; $p_h < q_h$, $h = 1, \cdots, m$.

For $\{p_h, q_h\}_{h=1}^m$ as above, let $\mathcal{S}_m$ denote the group of permutations of m elements and let

$$\mathcal{S}_m^1 := \{\sigma \in \mathcal{S}_m \ : \ p_h < q_{\sigma(h)}, \ h = 1, \cdots, m\} \tag{169}$$

Then, for any function $F \in L^1_{\text{loc}}(\mathbf{R}^{n-m})$, one has

$$\lambda^{-2m} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{\sigma \in \mathcal{S}_m^1} \prod_{h=1}^m f(\frac{t_{q_{\sigma(h)}} - t_{p_h}}{\lambda^2}) F(t_1, \cdots, \widehat{t_{q_1}}, \cdots, \widehat{t_{q_m}}, \cdots, t_n)$$

$$\leq \Big[\int_{-\infty}^0 f(t)dt\Big]^m \int_{\Delta_t^{(n-m)}} d\tau_1 \cdots \widehat{d\tau_{q_1}} \cdots \widehat{d\tau_{q_m}} \cdots d\tau_n F(\tau_1, \cdots, \widehat{\tau_{q_1}}, \cdots, \widehat{\tau_{q_m}}, \cdots, \tau_n) \tag{170}$$

<u>*PROOF*</u> In our notations and using the symmetry of f, the left hand side of (10.3) becomes

$$\lambda^{-2m} \sum_{\sigma \in \mathcal{S}_m^1} \int_{\sigma \Delta_t^{(n)}} dt_1 \cdots dt_n \prod_{h=1}^m f(\frac{t_{q_h} - t_{p_h}}{\lambda^2}) F(t_1, \cdots, \widehat{t_{q_1}}, \cdots, \widehat{t_{q_m}}, \cdots, t_n) \tag{171}$$

where, for each fixed $q_1, \cdots, q_m$ as above, the action of $\mathcal{S}_m$ on $\mathbf{R}^n$ is defined by

$$\sigma(t_1, \cdots, t_{q_1}, \cdots, t_{q_m}, \cdots, t_n) = (t_1, \cdots, t_{q_{\sigma(1)}}, \cdots, t_{q_{\sigma(m)}}, \cdots, t_n) \tag{172}$$

Clearly since F does not depend on the q-variables, it remains unchanged under the action of σ. But since $\sigma\Delta_t^{(n)} \cap \sigma'\Delta_t^{(n)} = \emptyset$ for $\sigma \neq \sigma'$, (10.4) is equal to

$$\lambda^{-2m} \int_{\bigcup_{\sigma \in \mathcal{S}_m^1} \sigma\Delta_t^{(n)}} dt_1 \cdots dt_n \prod_{h=1}^{m} f(\frac{t_{q_h} - t_{p_h}}{\lambda^2}) F(t_1, \cdots, \widehat{t_{q_1}}, \cdots, \widehat{t_{q_m}}, \cdots, t_n) \tag{173}$$

Now let us introduce the change of variables

$$x_h := (t_{q_h} - t_{p_h})/\lambda^2 \quad ; \qquad h = 1, \cdots, m \tag{174}$$

$$y_j := t_j \quad \text{for} \quad j \in \{1, \cdots, n\} \setminus \{q_1, \cdots, q_m\} \tag{175}$$

Because of our conditions $q_j \neq q_k$ for $j \neq k$ and therefore the cardinality of the set of the indices j in (10.7) is exactly $n - m$. Moreover, since $(t_1, \cdots, t_n) \in \Delta_t^{(n)}$ and the indices j in (10.7) are chosen in the complement of the set $\{q_1, \cdots, q_m\}$ it follows that

$$j < j' \implies t \geq y_j \geq y_{j'} \geq 0 \tag{176}$$

Equivalently, by relabeling the indices j, we can assume that $(y_1, \cdots, y_{n-m}) \in \Delta_t^{(n-m)}$. Moreover since $t_{p_h} \geq t_{q_h}$, the variables x_h are in $\mathbf{R}_-$ for each λ. Therefore, after the change of the variables (10.7), (10.8) and above mentioned relabeling of the y_j, the integral (10.6) is majorized by

$$\int_{\Delta_t^{(n-m)}} F(y_1, \cdots, y_{n-m}) dy_1 \cdots dy_{n-m} \int_{\mathbf{R}_-^m} \prod_{h=1}^{m} f(x_h) dx_1 \cdots dx_m \tag{177}$$

which is precisely the right hand side of (10.3).

The general form (10.2) of the Pulè inequality will be used in the deduction of the Langevin equation for the low density limit and in the more general approach to the low density limit suggested in [10]. In the weak coupling limit the following two simpler forms of the inequality will be sufficient.

COROLLARY (10.2) In the notations and assumptions of Lemma (10.1), suppose that F is a symmetric function of $n - m$ variables and let

$$\mathcal{S}_m^0 := \{\sigma \in \mathcal{S}_m \ : \ p_{\sigma(h)} < q_h, \ h = 1, \cdots, m\} \tag{178}$$

Then

$$\lambda^{-2m}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \sum_{\sigma\in\mathcal{S}_m^0}\prod_{h=1}^m f(\frac{t_{p_{\sigma(h)}}-t_{q_h}}{\lambda^2})F(t_1,\cdots,\widehat{t_{q_1}},\cdots,\widehat{t_{q_m}},\cdots,t_n)$$

$$\leq \Big[\int_{-\infty}^0 f(t)dt\Big]^m \int_{\Delta_t^{(n-m)}} d\tau_1\cdots\widehat{d\tau_{q_1}}\cdots\widehat{d\tau_{q_m}}\cdots d\tau_n F(\tau_1,\cdots,\widehat{\tau_{q_1}},\cdots,\widehat{\tau_{q_m}},\cdots,\tau_n) \tag{179}$$

<u>*PROOF*</u> We have only to remark that, due to symmetry of F the left hand side of (10.12) is equal to

$$\lambda^{-2m}\sum_{\sigma\in\mathcal{S}_m^0}\int_{\sigma\Delta_t^{(n)}} dt_1\cdots dt_n\prod_{h=1}^m f(\frac{t_{p_h}-t_{q_h}}{\lambda^2})F(t_1,\cdots,\widehat{t_{q_1}},\cdots,\widehat{t_{q_m}},\cdots,t_n) \tag{180}$$

the remaining part of the proof is the same as in Lemma (10.1).

<u>*REMARK*</u> Corollary (10.2) is used in the low density and weak coupling limit with $F(t_1,\cdots,\widehat{t_{q_1}},\cdots,\widehat{t_{q_m}},\cdots,t_n)=1$; in this case the bound (10.10) becomes

$$\lambda^{-2m}\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n \sum_{\sigma\in\mathcal{S}_m^o}\prod_{h=1}^m f(\frac{t_{p_{\sigma(h)}}-t_{q_h}}{\lambda^2}) \ \leq \frac{t^{n-m}}{(n-m)!}\Big[\int_{-\infty}^0 f(t)dt\Big]^m \tag{181}$$

These inequalities allow to obtain, for a sum of card$(\mathcal{S}_m^0)$ terms, the same estimate that (6.5) gives for each single term. Since

$$\text{card}(\mathcal{S}_m^0)\sim\ m!$$

and m can be $[n/2]$ in the weak coupling and $n-1$ in the low density case, the improvement is quite substantial.

§11. The uniform estimate

In the present section, we apply the Pulè inequality to obtain a uniform estimate (in x) for the matrix elements of (7.4) (WCL) and (7.5) (LDL) in the same family of collective coherent vectors (7.10a), (7.10b).

We know from the analysis of Section 7) that it is sufficient to deduce a bound for expressions of form (7.14). In this Section we shall prove this estimate in the LDL case and WCL case with linear interaction ($p = 1$).

Notice that in (7.14), the quantity $C(x)$ is a $r_1 + r_2$ factor's product and each factor has the form of (7.11) and (7.12). Moreover, from (7.9), if we denote

$$C := 1 \bigvee \max_{F=f,f';\ G=g,g_0,g_1} \int_{-\infty}^{\infty} | < G, S_t F > | dt \tag{182}$$

then the module of $C(x)$ in (7.14) is majorized by $C^{2(n-m)}$. Therefore the module of (7.14) is majorized by

$$\sum_{m,\{p_h,q_h\}_{h=1}^m} x^{r_1+r_2-2n} \cdot x_n \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \prod_{h=1}^m | < G, S_{(s_{q_h}-s_{p_h})/x^2} G' > | \cdot C^{2(n-m)} \tag{183}$$

and in any case, from (6.3), (7.20) and (7.21), one has

$$x^{r_1+r_2-2n} \cdot x_n = x^{-2m} \tag{184}$$

The sum $\sum_{m,\{p_h,q_h\}_{h=1}^m}$ in (11.2) is given by (9.7) and (9.10) respectively. Notice that, if in the two summations over $\{p_h\}_{h=1}^m$ and over $\{q_h\}_{h=1}^m$, we neglect the restrictions

$$\{q_h\}_{h=1}^m \subset \{j_h\}_{h=1}^k \tag{185}$$

and

$$\{p_h\}_{h=1}^m \subset \{1, \cdots, n\} \setminus \{j_h\}_{h=1}^k \tag{186}$$

then (11.2) is majorized by

$$\sum_m{}' \sum_{1 \le q_1 < \cdots < q_m \le n} \sum_{\substack{1 \le p_1, \cdots, p_m \le n,\ p_h < q_h, h=1,\cdots,m \\ |\{p_h\}_{h=1}^m| = m}} C^{2(n-m)} x^{-2m} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n$$

$$\prod_{h=1}^m | < G, S_{(s_{q_h}-s_{p_h})/x^2} G' > | \tag{187}$$

where, the only difference between the WCL (linear) and the LDL case is that in the summation $\sum'_m$

- $m = 0, 1, \cdots, n/2$ in the WCL case;
- $m = 0, 1, \cdots, n-1$ in the LDL case.

Notice that for each fixed m, $1 \le q_1 < \cdots < q_m \le n$ and function f,

$$\sum_{\substack{1\le p_1,\cdots,p_m\le n,\ p_h<q_h, h=1,\cdots,m \\ |\{p_h\}_{h=1}^m|=m}} f(s_{q_h} - s_{p_h}) =$$

$$= \sum_{1\le p_1,\cdots,p_m\le n,\ p_h<q_h, h=1,\cdots,m} \sum_{\sigma\in\mathcal{S}_m^0} f(s_{q_h} - s_{p_{\sigma(h)}}) \tag{188}$$

where, $\mathcal{S}_m^0$ is defined in (10.11). Summing up, if we denote

$$f(t) := \max_{G,G'=g,g_0\cdot g_1} |< G, S_t G' >| \tag{189}$$

then f is a positive, symmetric L^2 function and (11.6) is dominated by

$$\sum_m{}' \sum_{1\le q_1<\cdots<q_m\le n} \sum_{1\le p_1,\cdots,p_m\le n,\ p_h<q_h, h=1,\cdots,m} C^{2(n-m)} x^{-2m} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \prod_{h=1}^m f(s_{q_h} - s_{p_h}) \tag{190}$$

Applying Corollary (10.2) with $F = 1$ to (11.9), one gets the majorization of (11.6) by

$$\sum_m{}' \sum_{1\le q_1<\cdots<q_m\le n} \sum_{1\le p_1,\cdots,p_m\le n,\ p_h<q_h, h=1,\cdots,m} C^{2(n-m)} \frac{t^{n-m}}{(n-m)!} \cdot \Big(\int_\infty^0 f(t)dt\Big)^m$$

$$\le \sum_m{}' \binom{n}{m}^2 \cdot \frac{(C^2 t)^{n-m}}{(n-m)!} \cdot \Big(\int_\infty^0 f(t)dt\Big)^m \tag{191}$$

where, the factor $\binom{n}{m}^2$ comes from the sums

$$\sum_{1\le q_1<\cdots<q_m\le n} \sum_{1\le p_1,\cdots,p_m\le n,\ p_h<q_h, h=1,\cdots,m}$$

and the other factors from the Pulè inequality.

In WCL (linear) case, $0 \le m \le n/2$ and therefore the right hand side of (11.10) (so (11.6)) is dominated by

$$C_0^n/[n/2]! \tag{192}$$

where constant C_0. But in LDL case, m runs from 0 to n-1, i.e. it might be near n, so that $n-m$ might be very small. For this reason we need that the quantity $\int_{\infty}^{0} f(t)dt$ is small. More precisely, denoting

$$\|g\|_{-}^{2} := \int_{-\infty}^{0} f(t)dt \tag{193}$$

(in view of (11.8) this notation is compatible with (4.6)) and supposing, without loss of generality, that

$$\|g\|_{-} \neq 0$$

the right hand side of (11.10) is less or equal than

$$\sum_{m=0}^{n-1} \|g\|_{-}^{2m} \cdot \frac{(C^2t)^{n-m}}{(n-m)!} \cdot 4^n = (4 \cdot \|g\|_{-}^{2})^n \sum_{m=0}^{n-1} \frac{(C^2t)^{n-m}}{(n-m)! \cdot \|g\|_{-}^{2(n-m)}} \tag{194}$$

So putting $k = n - m$, (11.13) becomes

$$(4 \cdot \|g\|_{-}^{2})^n \sum_{k=1}^{n} \frac{(C^2t)^k}{k! \cdot \|g\|_{-}^{2k}} \leq (4 \cdot \|g\|_{-}^{2})^n \cdot \exp(\frac{C^2t}{\|g\|_{-}^{2}}) \tag{195}$$

Summing up, we have proved the following uniform estimate:

$\underline{THEOREM(11.1)}$ The quantity

$$| < u \otimes \Phi(x \int_{S/x^2}^{T/x^2} S_u f du), \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n V(t_1) \cdots V(t_n) v \otimes \Phi(x \int_{S'/x^2}^{T'/x^2} S_u f' du) > | \tag{196}$$

is majorized by

$$C_n := \begin{cases} C^n/[n/2]! \ , & \text{WCL} \\ (\|g\|_{-}^{2} \cdot 8\|D\|)^n \cdot \exp(C^2t/\|g\|_{-}^{2}), & \text{LDL} \end{cases} \tag{197}$$

where, C is a constant.

§12. Absence of the dA, dA^{+}, dt terms in the low density limit: Fock case

In this Section we explain why in the LDL Fock limit we should expect that the SDE for $U(t)$ does not contain terms in dA, dA^{+}, dt.

By inspection of (7.7), we see that, in all the terms appearing in the $n-th$ term of the iterated series, the first operator is always a creator and the last one always annihilator. This implies that the normally ordered form of the $n-th$ term of the iterated solution of U_t cannot contain a purely scalar term (i.e. a term in which all pairs of creators and annihilators have seen used to produce a scalar product). As already remarked in Section 6) it is in general not true that the $n-th$ term of the iterated series converges to the $n-th$ terms of the iterated solution of the SDE. However if we write the solution of a given SDE

$$dU(t) = F_{\alpha} dM_{\alpha}(t) U(t) \ ; \quad U(0) = 1$$

where the driving noises are of the form

$$dA_t^{+}(g), \quad dA_t(g), \quad dN_t(g), \quad dt$$

and the F_{α} are operators in the initial space, in terms of the iterated series

$$U(t) = 1 + \sum_{n=1}^{\infty} U^{(n)}(t) \tag{198}$$

$$U^{(n+1)}(t) = \int_0^t F_{\alpha} dM_{\alpha}(s) U^{(n)}(s) \ ; \quad U^{(0)}(t) = 1 \tag{199}$$

then, the explicit expression for $U^{(n)}(t)$ will be a sum of iterated stochastic integrals of the form

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} F_{\alpha_1} F_{\alpha_2} \cdots F_{\alpha_n} dM_{\alpha_1}(t_1) \cdots dM_{\alpha_n}(t_n) \tag{200}$$

over all possible indices α_j.

We say that a term of the form (12.3) is a pure dt term if

$$dM_{\alpha_1}(t_1) \cdots dM_{\alpha_n}(t_n) = dt_1 \cdots dt_n \tag{201}$$

with this terminology the following statement is true: The sum of the matrix elements of all the purely $dt-$terms in the iterated series expansion of the

SDE is always equal to the limit of the sum of the matrix elements of all the purely scalar terms in the normally ordered form of the interaction. This is due to the fact that the coefficient of this terms cannot contain the test functions f, f' of the two coherent vectors with respect to which we are considering the matrix elements, while the matrix element of any non scalar term will always produce a scalar product of some g_ε with f or f'.

Since we have seen in Section 7) that in the Fock LDL case there are no purely scalar terms in the normally ordered form of any term of the iterated series, it follows that the equation for $U(t)$, supposing it exists, cannot contain a $dt-$term.

On the other hand, if $U(t)$ is unitary, then if it contains some dA and dA^+ terms, then it must contain also some $dt-$terms. Therefore also the presence of dA and dA^+ terms, in the equation for $U(t)$, is ruled out. This also explains why there is hope that the limit process in LDL **finite temperature** is unitary: in this case in fact we obtain $dt-$term but not dA and dA^+ terms.

§13. The negligible terms: Fock case

As already seen in Section 7), both in the WCL and the LDL case, after the normal ordering, the matrix element in some collective coherent vectors of the $n-th$ term of the iterated series is a sum of expressions of the form

$$x^{2n-2m}\int_{\Delta^{(n)}_{t/x^2}} dt_1\cdots dt_n \prod_{n=1}^{m} < G, S_{t_{q_h}-t_{p_h}}G' > \cdot C(x) \tag{202}$$

where, for each $n\in\mathbf{N}$ and $t\in\mathbf{R}$

$$\Delta^{(n)}_t := \{(t_1,\ldots,t_n)\in\mathbf{R}^n : t\geq t_1\geq t_2\geq\ldots\geq t_n\} \tag{203}$$

and $C(x)$ is a product of the terms with the forms (7.11) and (7.12).

In the limit $x\to 0$ some of the terms (13.1) will vanish and some not. In the Fock case (but not in the finite temperature one!) the criterium to distinguish the relevant terms from the neglible ones is the same both in the WCL and the LDL case and does not depend on the interaction. Namely: the neglible terms (called terms of type II) are those in which, in the product

$$\prod_{h=1}^{m} < g, S_{t_{q_h}-t_{p_h}}g >$$

there exists some index h such that

$$q_h - p_h > 1 \tag{204}$$

In other terms, the only surviving terms in the limit (called terms of type I), are those corresponding to the products

$$\prod_{h=1}^{m} < g, S_{t_{q_h}-t_{q_h-1}}g > \tag{205}$$

arising from the communtators

$$[A(S_{t_{q_h-1}}g), A^+(S_{t_{q_h}})]$$

Let us first explain how one can prove the terms of type II tends to zero as $x\to 0$. To this goal notice that (13.1) is majorized in modules by a constant to the power n times a integral with the form

$$\cdots \int_0^{t_{q_h}-2} dt_{q_h-1} x^{-2} \int_0^{t_{q_h}-1} dt_{q_h} | < G, S_{(t_{q_h}-t_{p_h})/x^2} G' > | \int_0^{t_{q_h}} dt_{q_h+1} \cdots \tag{206}$$

With the change of variable

$$(t_{q_h} - t_{p_h})/x^2 = \tau_h$$

we obtain

$$\cdots \int_0^{t_{q_h}-2} dt_{q_h-1} \int_{-t_{p_h}/x^2}^{(t_{q_h-1}-t_{p_h})/x^2} d\tau_h | < G, S_{\tau_h} G' > | \int_0^{\lambda^2 \tau_h + t_{p_h}} dt_{q_h+1} \cdots \tag{207}$$

Notice that in the $d\tau_h$-integral in (13.6) goes from $-t_{p_h}/x^2$ to $(t_{q_h-1} - t_{p_h})/x^2$ which is < 0 a.e. if the condition (13.3) holds.

Due to the integrability of $\tau \to < G, S_\tau G' >$, it follows that

$$\lim_{\lambda \to 0} \int_{-t_{p_j}/x^2}^{(t_{q_h-1}-t_{p_h})/x^2} d\tau_h | < G, S_{\tau_h} G' > | = 0 \tag{208}$$

almost everywhere (in t_{q_h-1} and t_{p_h}) and, since

$$\left| \int_{-t_{p_h}}^{(t_{q_h-1}-t_{p_h})/x^2} d\tau_h < G, S_{\tau_h} G' > \right| \le \int_{-\infty}^{+\infty} | < G, S_t G' > | dt \tag{209}$$

we have that

$$\lim_{\lambda \to 0} \int_{\Delta^{(n)}_{t/x^2}} dt_1 \dots dt_n \prod_{h=1}^{m} < G, S_{t_{q_h} - t_{p_h}} G' > = 0 \tag{210}$$

by dominated convergence.

§14. The limit of the type I terms

Now let us explain how one can control the explicit form of the limit of type I terms as $x \to 0$. We begin to consider the integral (13.1) in the LDL case, which is the most difficult since several indices q_α can be equal among themselves. Notice that each fixed m and $\{p_h, q_h\}_{h=1}^m$ determine a unique decompose of the set $\{q_h\}_{h=1}^m$ into maximal increasing chains with respect to the property $q_\beta = q_{\beta+1} - 1$, i.e. a decomposition

$$\{q_h\}_{h=1}^m = \{q_1, \cdots, q_\alpha\} \cup \cdots \cup \{q_\beta, \cdots, q_m\} \tag{211}$$

with the properties:

$$q_1 = q_2 - 1, \cdots, q_{\alpha-1} = q_\alpha - 1, \quad \text{but} \quad q_\alpha < q_{\alpha+1} - 1 \tag{212}$$

$$\cdots$$

$$q_{\beta-1} < q_\beta - 1, \quad \text{but} \quad q_\beta = q_{\beta+1} - 1, \cdots, q_{m-1} = q_m - 1 \tag{213}$$

The WCL linear case corresponds to a particular decomposition

$$\{q_1, \cdots, q_m\} = \{q_1\}, \ \{q_2\}, \cdots, \{q_m\} \tag{214}$$

We rewrite the integarl (13.1) to the following form

$$\cdots x^{-2m} \int_0^{t_{q_1-1}} dt_{q_1} < G, S_{(t_{q_1}-t_{q_1-1})/x^2} G' > \int_0^{t_{q_1}} dt_{q_1+1} < G, S_{(t_{q_1+1}-t_{q_1})/x^2} G' > \cdots$$
$$\int_0^{t_{q_\alpha-1}} dt_{q_\alpha} < G, S_{(t_{q_\alpha}-t_{q_\alpha-1})/x^2} G' > \int_0^{t_{q_\alpha}} dt_{q_\alpha+1} \cdots$$
$$\cdots \int_0^{t_{q_\beta-1}} dt_{q_\beta} < G, S_{(t_{q_\beta}-t_{q_\beta-1})/x^2} G' > \int_0^{t_{q_\beta}} dt_{q_\beta+1} < G, S_{(t_{q_\beta+1}-t_{q_\beta})/x^2} G' >$$
$$\cdots \int_0^{t_{q_m-1}} dt_{q_m} < G, S_{(t_{q_m}-t_{q_m-1})/x^2} G' > \int_0^{t_{q_m}} dt_{q_m+1} \cdots C(x) \tag{215}$$

where, $C(x)$ is given by (14.14) below. With the change of variables

$$(t_{q_h} - t_{q_h-1})/x^2 = \tau_h \tag{216}$$

the integral (14.5) becomes

$$\cdots \int_{-t_{q_1-1}/x^2}^0 d\tau_1 < G, S_{\tau_1} G' > \int_{-t_{q_1-1}/x^2-\tau_1}^0 d\tau_2 < G, S_{\tau_2} G' > \cdots$$
$$\int_{-t_{q_1-1}/x^2-\tau_1-\cdots-\tau_{\alpha-1}}^0 d\tau_\alpha < G, S_{\tau_\alpha} G' > \int_0^{t_{q_1-1}+x^2(\tau_1+\cdots+\tau_\alpha)} dt_{q_\alpha+1} \cdots C(x) \tag{217}$$

For simplicity we shall discuss only the first chain of integral since the other ones are dealt with the same idea. In (14.7) there are two types of integrals: the chain

$$\int_{-t_{q_1-1}/x^2}^{0} d\tau_1 < G, S_{\tau_1} G' > \int_{-t_{q_1-1}/x^2-\tau_1}^{0} d\tau_2 < G, S_{\tau_2} G' > \cdots$$

$$\cdots \int_{-t_{q_1-1}/x^2-\tau_1-\cdots-\tau_{\alpha-1}}^{0} d\tau_\alpha < G, S_{\tau_\alpha} G' > \tag{218}$$

and the term

$$\cdots \int_{0}^{t_{q_1-1}+x^2(\tau_1+\cdots+\tau_{\alpha-1})} dt_{q_\alpha+1} \cdots \tag{219}$$

Now, as $x \to 0$, (14.8a) tends to

$$\int_{-\infty}^{0} d\tau_1 < G, S_{\tau_1} G' > \int_{-\infty}^{0} d\tau_2 < G, S_{\tau_2} G' > \cdots \int_{-\infty}^{0} d\tau_\alpha < G, S_{\tau_\alpha} G' > \tag{220}$$

and (14.8b) tends to

$$\int_{0}^{t_{q_1-1}} dt_{q_\alpha+1} \cdots \tag{221}$$

Therefore, both in the WCL and the LDL cases, from the integral of the product $\prod_{h=1}^{m} < G, S_{(t_{q_h}-t_{q_h-1})/x^2} G' >$ we obtain the quantity

$$\prod_{h=1}^{m} \int_{-\infty}^{0} < G, S_t G' > dt \tag{222}$$

Notice that in the WCL case, $G = G' = g$ so (14.10) is simply the form

$$(\int_{-\infty}^{0} < g, S_t g > dt)^m \tag{223}$$

But in the LDL case G, G' are equal to g_0 or g_1, more precisely, by (1.16) we know that for each fixed $n \in \mathbf{N}$ and $\varepsilon \in \{0,1\}^n$, if $\varepsilon(q_h) = 0$ then $G = G' = g_0$, otherwise $G = G' = g_1$. So if we introduce the notation

$$m_1 := \sum_{h=1}^{m} \varepsilon(q_h) \tag{224}$$

then (14.10) is equal to

$$(\int_{-\infty}^{0} < g_0, S_t g_0 > dt)^{m-m_1} \cdot (\int_{-\infty}^{0} < g_1, S_t g_1 > dt)^{m_1} \tag{225}$$

Now let see what happens to $C(x)$, i.e. to the product

$$\prod_{h=1}^{r_1} \int_{S/x^2}^{T/x^2} < S_u f, S_{s_{\alpha_h}/x^2} G > du \cdot \prod_{h=1}^{r_2} \int_{S'/x^2}^{T'/x^2} < S_{s_{\beta_h}/x^2} G', S_u f' > du \tag{226}$$

which comes from the action of the annihilators on the coherent vectors. Since $\{\alpha_h\}_{h=1}^m$ label the remaining creators, i.e. $\{\alpha_h\}_{h=1}^m \subset \{1, \cdots, n\} \setminus \{q_h\}_{h=1}^m$, we know that the change of variables (14.5) does not affect the first product in (14.14). Therefore, after the change of variables

$$v_h := u - s_{\alpha_h}/x^2, \quad h = 1, \cdots, r_1 \tag{227}$$

it tends to

$$\prod_{h=1}^{r_1} \chi_{[S,T]}(s_{\alpha_h}) \cdot (f|G) \tag{228}$$

where, $G = g$ for WCL case and g_0 or g_1 for LDL case.

In the WCL case, the second product is also simple, since $\{\beta_h\}_{h=1}^{r_2}$ label the remaining annihilators and therefore $\{\beta_h\}_{h=1}^{r_2} \cap \{q_h\}_{h=1}^m = \emptyset$. This implies that the change of variables (14.5) does not affect the second product (notice that now we consider only the WCL case) in (14.14). So by the change of variables

$$v_h := u - s_{\beta_h}/x^2, \quad h = 1, \cdots, r_2 \tag{229}$$

it tends to

$$\prod_{h=1}^{r_2} \chi_{[S',T']}(s_{\beta_h}) \cdot (g|f') \tag{230}$$

But in the LDL case, the second product is not so simple since the change of variables (14.5) influnce the variables $\{\beta_h\}_{h=1}^{r_2}$. More precisely, because of $q_\alpha < q_{\alpha+1} - 1$, ... , (see (14.2)) and considering only the terms of type I ($p_h = q_h - 1$ for any $h = 1, \cdots, m$), one knows that $A(S_{t_{q_\alpha}} g_{\varepsilon(q_\alpha)})$ is not used to produce scalar product, i.e. q_α is equal to some β_h. Thus with the change of variables (14.5), this t_{β_h} becomes equal to

$$t_{q_1-1}/x^2 + (\tau_1 + \cdots + \tau_{\alpha-1}) \tag{231}$$

It follows that the integral

$$\int_{S'/x^2}^{T'/x^2} < S_{t_{q_\alpha}/x^2} G', S_u f' > du \tag{232}$$

is equal to

$$\int_{S'/x^2}^{T'/x^2} < S_{t_{q_1-1}/x^2+(\tau_1+\cdots+\tau_{\alpha-1})} G', S_u f' > du \tag{233}$$

and gives the limit

$$\chi_{[S',T']}(t_{q_1-1}) \cdot (G'|f') \tag{234}$$

A similar phnomenoma happens in the right end points of each of the q-subsets ($\{q_1, \cdots, q_\alpha\}$, $\cdots$, $\{q_\beta, \cdots, q_m\}$). Thus, we obtain the following

THEOREM(14.1) For each $n \in \mathbf{N}$ the limit (13.1) exists both in the WCL and the LDL cases and their explicit form can be formulated (cf. Theorem (6.1) in [2] and Lemma (3.4) in [9]).

Putting together Theorem (14.1), Section 12) and the uniform estimate one has the following

COROLLARY(14.2) Limit (1.21) exists both in the WCL and the LDL cases (in the LDL case we need the condition (4.7)) and their explicit form can be formulated (cf. Theorem (5.4) in [2] and Theorem (5.1) in [9]).

§15. Deduction of the SDE: the basic strategy

In this Section we give a qualitative outline of the basic ideas which allow to deduce an equation for the limit (2.12). First of all, we would like to remark that the discussion in this section is only necessary in the WCL linear and the LDL cases. The result in the WCL nonlinear case is trivial since (with the given coherent vectors) the limit is only a pure scalar product term.

As already remarked in Section 6) this equation cannot be deduced by inspection from the explicit form of the limit (2.12) because, as it will be clear a posteriori, the n–th term of the iterated series does not converge to the n–th term of the iterated solution of the SDE. We begin our discussion for a general polynomial interaction. First define a vector $G_x(t)$ of the system space by:

$$< u \otimes \Phi_x(x \int_{S/x^2}^{T/x^2} S_u f du), U_{t/x^2} v \otimes \Phi_x(x \int_{S'/x^2}^{T'/x^2} S_u f' du) >=:< u, G_x(t) > \tag{235}$$

Our goal is to find an integral equation (in weak form) for $G_x(t)$.

Differentiating (in t) the left hand side of (15.1) we obtain

$$< u \otimes \Phi_x(x \int_{S/x^2}^{T/x^2} S_u f du), -ix^{-2} V_\lambda(t/x^2) U_{t/x^2} v \otimes \Phi_x(x \int_{S'/x^2}^{T'/x^2} S_u f' du) > \tag{236}$$

where, $V_\lambda(t)$ is defined in (1.15), i.e. $V_\lambda(t/x^2)$ is a sum of terms of the form

$$i \cdot y \cdot C \otimes P(t) \tag{237}$$

$x = y = \lambda$ in the WCL case and $x = z^{1/2}$, $y = 1$ in LDL case, $C = D$ or D^+ and $P(t)$ is a product of creators and annihilators in the test functions $S_{t/x^2} g_j$ (for some $g_j \in K \subset H_1$) where j runs in a finite set ($g_j = g$ in the WCL case $g_j = g_0, g_1$ in the LDL case).

Now we bring C to the left hand side of the scalar product and we let all the creators on the left of the product $P(t)$ act on the coherent vector $\Phi_x(x \int_{S/x^2}^{T/x^2} S_u f du)$.

As a result, we find that (15.2) is expressed as a sum of terms of the form:

$$\prod_{j=1}^{h} x \int_{S/x^2}^{T/x^2} < S_u f, S_{t/x^2} g_j >$$

$$< y \cdot C^{+}u \otimes \Phi_x(x \int_{S/x^2}^{T/x^2} S_u f du), x^{-2}(1 \otimes Q(t))U_{t/x^2}v \otimes \phi'_x) \tag{238}$$

where h is the number of creators on the left of the product $P(t)$ and $Q(t)$ is a product only of annihilation operators.

Bringing $1 \otimes Q(t)$ to the right of U_{t/x^2} one replaces in (15.4) by a sum of 2 terms: one of the form

$$\prod_{j=1}^{h} x \int_{S/x^2}^{T/x^2} < S_u f, S_{t/x^2} g_j >$$

$$< y \cdot C^{+}u \otimes \Phi_x(x \int_{S/x^2}^{T/x^2} S_u f du), x^{-2}[1 \otimes Q(t), U_{t/x^2}]v \otimes \Phi_x(x \int_{S'/x^2}^{T'/x^2} S_u f' du) > \tag{239}$$

where, $[\cdot, \cdot]$ denotes the commutator and another of the form

$$\prod_{j=1}^{d} x \int_{S'/x^2}^{T'/x^2} < S_{t/x^2} g'_j, S_u f' > \prod_{j=1}^{h} x \int_{S/x^2}^{T/x^2} < S_u f, S_{t/x^2} g_j >< y \cdot C^{+}u, x^2 G_x(t) > \tag{240}$$

Moreover, in the WCL case for each time we have either one creator or one annihilator, therefore $h = 0$ in (15.5) and $h = 1, d = 0$ or $h = 0, d = 1$ in (15.6). On the other hand, in the LDL case, due to the number type interaction, the first operator is always a creator, so $h = 1$ in (15.5) and $h = d = 1$ in (15.6). This implies that both in the WCL and the LDL cases,

$$x^h \cdot y \cdot x^{-2} = x^{-1} \tag{241}$$

The form of the terms (15.5), (15.6) shows again the connection between the choice of the collective vectors and the interaction. In fact in the WCL case with linear interaction we have that

$$d + h = 1 \; ; \quad x = y = \lambda \tag{242}$$

hance the factor $\lambda^{d+h} \cdot y \cdot \lambda^{-2}$ is equal to 1.

In the LDL case we have that

$$d + h = 2 \; ; \quad x = z, \quad y = 1 \tag{243}$$

so that again $z^{d+h} \cdot y \cdot z^{-2} = 1$.

In both cases therefore the expression (15.6) is equal to

$$\prod_{j=1}^{d}\int_{S'/x^2}^{T'/x^2} < S_{t/x^2}g'_j, S_u f' > \prod_{j=1}^{h}\int_{S/x^2}^{T/x^2} < S_u f, S_{t/x^2}g_j >< C^+u,\ G_x(t) > \tag{244}$$

and this is easy to control since

$$\lim_{x\to 0} < C^+u, G_x(t) >=< C^+u, G(t) > \tag{245}$$

$$\lim_{x\to 0}\int_{S/x^2}^{T/x^2} du < S_u f, S_{t/x^2}G >= \chi_{[S,T]}(t)\cdot (f|G) \tag{246}$$

and

$$\lim_{x\to 0}\int_{S'/x^2}^{T'/x^2} du < S_{t/x^2}G', S_u f' >= \chi_{[S',T']}(t)\cdot (G'|f') \tag{247}$$

for any $G, G' \in K$

The crucial step in the deduction of the equation both in the WCL and the LDL case is then to prove that also the expression (15.5) is a sum (maybe infinite) of terms of the form (15.6) plus something that goes to zero as $x \to 0$.

The basic reason why this happens is the following: expanding U_{t/x^2} and using the iterated series, (15.5) becomes a series whose n-th term is

$$\prod_{j=1}^{h} x\int_{S/x^2}^{T/x^2} < S_u f, S_{t/x^2}g_j > \int_{\Delta^{(n)}_{t/x^2}} < C^+u \otimes \Phi_x(x\int_{S/x^2}^{T/x^2} S_u f du),$$

$$x^{-1}[1\otimes Q(t), V(t_1)\cdots V(t_n)]v \otimes \Phi_x(x\int_{S'/x^2}^{T'/x^2} S_u f' du) > \tag{247}$$

(15.11)

$Expanding the commutator in (15.11) gives rise to a sum of terms in which the commutators$

$[1\otimes Q(t), V(t_j)]$(248)appear. Each of these terms will produce some scalar product of the form

$$< S_{t/\lambda^2}g_h, \quad S_{t_j}g_k > \tag{249}$$

so that, if $j \neq 1$, we always obtain a term of type II in the sense of Section 12), i.e. something that goes to zero. Thus the commutator in (15.11) is equal to (up to something that goes uniformly to zero)

$$[1\otimes Q(t), V(t_1)]V(t_2)\cdots V(t_n) \tag{250}$$

Replacing, in (15.11), the commutator term by the expression (15.14) one obtains, in the weak coupling limit, a multiple of $< C^+u, G_x(t) >$ plus something which tends to zero.

In the LDL case, the situation is more difficult. In fact, since now the interaction is of number type, the commutator in (15.14) is equal to a scalar product times an annihilator. Bringing the annihilator to the right of $V(t_2)\cdots V(t_n)$ we obtain two terms with the following forms:

$$V(t_2)\cdots V(t_n)\cdot \text{annihilator} \tag{251}$$

and

$$[C \otimes \text{annihilator}, V(t_2)\cdots V(t_n)] \tag{252}$$

In (15.15) we can apply the annihilator on the right to the coherent vector and get a multiple of $< C \cdot C^+u, G_x(t) >$. But (15.16) has again the form of (15.14). So in the iterated series of U_{t/x^2}, for each $n \in \mathbf{N}$, we have to repeat the procedure $n-1$ times. This explains again why in the LDL case the coefficients of the limiting SDE are expressed as series.

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