Probability Measures and CAN Operators

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Abstract— **For a probability measure μ on $\mathbb{R}^d$ with finite moments of all orders, we can define the creation operator $a^+(j)$, the annihilation operator $a^-(j)$, and the neutral operator $a^0(j)$ for each coordinate $1 \le j \le d$. These operators are used to characterize several properties of probability measures**

I. Creation, Annihilation, and Neutral Operators

LET μ be a probability measure on $\mathbb{R}^d$ with finite moments of all orders, namely, for any nonnegative integers $i_1, i_2, \dots, i_d$,

$$\int_{\mathbb{R}^d} |x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}|\, d\mu(x) < \infty,$$

where $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Let $F_0 = \mathbb{R}$ and for $n \ge 1$ let F_n be the vector space of all polynomials in $x_1, x_2, \dots, x_d$ of degree $\le n$. Then we have the inclusion chain

$$F_0 \subset F_1 \subset \dots \subset F_n \subset \dots \subset L^2(\mu).$$

Next, define $G_0 = \mathbb{R}$ and for $n \ge 1$ define G_n to be the orthogonal complement of F_{n-1} in F_n. Then the spaces G_n, $n \ge 0$, are orthogonal. Define a real Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} G_n \quad \text{(orthogonal direct sum)}.$$

For each $n \ge 0$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto G_n. Let X_j, $1 \le j \le d$, be the multiplication operator by x_j. Accardi and Nahni[3] have recently observed that for any $1 \le j \le d$ and $n \ge 0$

$$X_j G_n \perp G_k, \quad \forall\, k \ne n-1, n, n+1,$$

where $G_{-1} = \{0\}$ by convention. Then they used this fact to obtain the following *fundamental recursion equality:*

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n, \quad n \ge 0, \tag{1}$$

where $P_{-1} = 0$ by convention. When $d = 1$, this equality reduces to the well-known recursion formula:

$$x P_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \omega_n P_{n-1}(x), \tag{2}$$

where $P_n(x)$'s are orthogonal polynomials with respect to μ, $P_n(x)$ is a polynomial of degree n with leading coefficient 1, and $\{\alpha_n, \omega_n\}$'s are the Jacobi-Szegö parameters of μ.

Now, for each $n \ge 0$ and $1 \le j \le d$, define three operators by

$$D_n^+(j) = P_{n+1} X_j P_n \,:\, G_n \longrightarrow G_{n+1},$$
$$D_n^-(j) = P_{n-1} X_j P_n \,:\, G_n \longrightarrow G_{n-1},$$
$$D_n^0(j) = P_n X_j P_n \,:\, G_n \longrightarrow G_n.$$

Using these operators, we can define for each $1 \le j \le d$ three densely defined linear operators $a^+(j)$, $a^-(j)$, and $a^0(j)$ from $\mathcal{H}$ into itself by

$$a^+(j)|_{G_n} = D_n^+(j), \quad n \ge 0,$$
$$a^-(j)|_{G_n} = D_n^-(j), \quad n \ge 0,$$
$$a^0(j)|_{G_n} = D_n^0(j), \quad n \ge 0.$$

The operators $a^+(j)$, $a^-(j)$, and $a^0(j)$ are called *creation, annihilation*, and *neutral operators*, respectively. The collection

$$\{\mathcal{H},\, a^+(j),\, a^-(j),\, a^0(j) \,|\, 1 \le j \le d\}$$

is called the *interacting Fock space* of the probability measure μ.

For convenience, we will use the term "CAN operators" to call the creation, annihilation, and neutral operators. By using the multiplication and CAN operators, we can rewrite the fundamental recursion equality in Equation (1) as the equality in the next theorem.

Theorem I.1: For each $1 \le j \le d$, the following equality holds:

$$X_j = a^+(j) + a^-(j) + a^0(j). \tag{3}$$

II. Polynomially Symmetric and Factorizable Measures

Definition II.1: A probability measure μ on $\mathbb{R}^d$ is said to be *polynomially symmetric* if

$$\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}\, d\mu(x) = 0$$

for all nonnegative integers $i_1, i_2, \dots i_d$ with $i_1 + i_2 + \dots + i_d$ being an odd integer.

Note that if μ is a symmetric measure with finite moments of all orders, then it is polynomially symmetric. However, the converse is not true.

The next theorem has been proved in our paper [1].

Theorem II.2: A probability measure μ on $\mathbb{R}^d$ with finite moments of all orders is polynomially symmetric if and only if $a^0(j) = 0$ for all $j = 1, 2, \dots, d$.

Definition II.3: A probability measure μ on $\mathbb{R}^d$ is said to be *polynomially factorizable* if

$$\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}\, d\mu(x) = \int_{\mathbb{R}^d} x_1^{i_1}\, d\mu(x) \int_{\mathbb{R}^d} x_2^{i_2}\, d\mu(x) \cdots \int_{\mathbb{R}^d} x_d^{i_d}\, d\mu(x)$$

for all nonnegative integers $i_1, i_2, \dots i_d$.

Obviously, if μ is a product measure with finite moments of all orders, then it is polynomially factorizable. However, the converse is not true.

The next theorem follows from Theorem 4.10 in our paper [1].

Theorem II.4: Let μ be a probability measure on $\mathbb{R}^d$ with finite moments of all orders. Then μ is polynomially factorizable if and only if for any $i \neq j$, the operators in $\{a^+(i), a^-(i), a^0(i)\}$ commute with the operators in $\{a^+(j), a^-(j), a^0(j)\}$.

III. Probability Measures on the Real Line

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Let V be the vector space of all polynomials in x and let V_n be its subspace consisting of all polynomials of degree $\leq n$.

Let $F_n = V_n/\sim$. Here the equivalence relation $\sim$ is given by μ-almost everywhere, namely, $f \sim g$ if $f = g$ holds μ-a.e.

Assumption: In this section all linear operators $T : V \to V$ are assumed to satisfy the condition that $T(V_n) \subset V_n$ for all $n \geq 0$, namely, all subspaces $V_n, n \geq 0$, are invariant under T.

A. Probability Measures on $\mathbb{R}$ with Finite Supports

Observe that if a probability measure μ on $\mathbb{R}$ is supported by m distinct points, then

$$F_j = V_j, \quad j = 0, 1, 2, \dots, m-1,$$
$$F_j = V_{m-1}, \quad j = m, m+1, \dots .$$

The following theorem can be easily verified.

Theorem III.1: Suppose μ is a probability measure on $\mathbb{R}$ supported by m distinct points. Then the following equalities hold:

(1) $\mathrm{Tr}\,(a_\mu^0|_{V_k}) = \mathrm{Tr}\,(a_\mu^0|_{V_{m-1}})$ for all $k \geq m-1$.
(2) $\mathrm{Tr}\,([a_\mu^-, a_\mu^+]|_{V_k}) = 0$ for all $k \geq m-1$.

Definition III.2: Two linear operators S and T from V into itself are called *trace equivalent* on V, denoted by $S \overset{t}{\sim} T$ on V, if

$$\mathrm{Tr}\,(S|_{V_k}) = \mathrm{Tr}\,(T|_{V_k}), \quad \forall\, k \geq 0.$$

They are called *trace equivalent* on V_n, denoted by $S \overset{t}{\sim} T$ on V_n, if

$$\mathrm{Tr}\,(S|_{V_k}) = \mathrm{Tr}\,(T|_{V_k}), \quad \forall\, 0 \leq k \leq n.$$

The next theorem is from our paper [2]. It characterizes those measures supported by finitely many points in $\mathbb{R}$ in terms of the CAN operators.

Theorem III.3: Let $m \geq 1$ be a fixed integer. Let a^0 and $a^{-,+}$ be two linear operators from V_{m-1} into itself. Then there exists a probability measure μ on $\mathbb{R}$ supported by m distinct points such that $a^0 \overset{t}{\sim} a_\mu^0$ and $a^{-,+} \overset{t}{\sim} [a_\mu^-, a_\mu^+]$ on V_{m-1} if and only if the following conditions hold:

(1) The spaces V_k, $0 \leq k \leq m-2$, are invariant under a^0 and $a^{-,+}$.
(2) $\mathrm{Tr}\,(a^{-,+}|_{V_k}) > 0$ for all $0 \leq k \leq m-2$.
(3) $\mathrm{Tr}\,(a^{-,+}|_{V_{m-1}}) = 0$.

B. Probability Measures on $\mathbb{R}$ with Infinite Supports

Let μ be a probability measure on $\mathbb{R}$ with infinite support, namely, the support of μ contains infinitely many points. In this case, we have

$$F_n = V_n, \quad \forall\, n \geq 0.$$

The next theorem has been proved in our paper [2].

Theorem III.4: Let a^0 and $a^{-,+}$ be two linear operators from V into itself. Then there exists a probability measure μ on $\mathbb{R}$ with infinite support such that $a^0 \overset{t}{\sim} a_\mu^0$ and $a^{-,+} \overset{t}{\sim} [a_\mu^-, a_\mu^+]$ on V if and only if the following conditions hold:

(1) The spaces V_n, $n \geq 0$, are invariant under a^0 and $a^{-,+}$.
(2) $\mathrm{Tr}\,(a^{-,+}|_{V_n}) > 0$ for all $n \geq 0$.

Let Ξ denote the set of all trace equivalent classes of ordered pairs $(a^0, a^{-,+})$ of linear operators from V into itself satisfying either one of the following conditions (a) and (b):

(a) $\mathrm{Tr}\,(a^{-,+}|_{V_n}) > 0$ for all $n \geq 0$.
(b) There exists m such that
  (1) $\mathrm{Tr}\,(a^0|_{V_k}) = \mathrm{Tr}\,(a^0|_{V_{m-1}})$ for all $k \geq m-1$,
  (2) $\mathrm{Tr}\,(a^{-,+}|_{V_k}) > 0$ for all $0 \leq k \leq m-2$,

(3) $\mathrm{Tr}\,(a^{-,+}|_{V_k}) = 0$ for all $k \geq m-1$.

Theorem III.5: There is a one-to-one correspondence between the set Ξ and the set of all probability measures on $\mathbb{R}$ with finite moments of all orders.

C. Probability Measures on $\mathbb{R}$ with Compact Supports

Theorem III.6: A probability measure μ on $\mathbb{R}$ with finite moments of all orders has compact support if and only if the following two sequences of real numbers are bounded:

(1) $\mathrm{Tr}\,(a^0_\mu|_{F_n}) - \mathrm{Tr}\,(a^0_\mu|_{F_{n-1}})$, $n \geq 1$.
(2) $\mathrm{Tr}\,(a^{-,+}_\mu|_{F_n})$, $n \geq 1$.

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