Models of time

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1 How old is an electron?

The notion of time is baffling mankind since many centuries. We attribute an age to many things: living beings, archaeological findings, rocks, stars, and even universe. But we are less confident when we try to attribute an age to a single atom, or to an electron or to a photon. We speak of lifetimes of elementary particles, but we cannot distinguish an "old" electron from a "young" one. So, to answer questions like: "does time pass for an electron?", we have to clarify our ideas on time.

Philosophical knowledge is based on descriptions. This has some advantages but it is not easy to decide if the meaning attributed by different people to the same definition is the same or not. For example suppose one asks you to interpret the following definition of time:

"... The parts of time have their being from the coupling or continuation through the indivisible present instant, given that it be always other and other, from its parts other and other succeed each other and always exist. ..."

and to frame it into an historical context by attributing it to either:

(i) Heiddeger or (ii) Lacan or (iii) Henry of Gent

I wonder how many would guess that the correct attribution is to Henry of Gent (1279) (the original statement being:

"... Partes temporis habere esse ex copulatione seu continuatione ad instans indivisible praesens, licet illud semper sit aliud et aliud ex partes eius succedunt et semper sunt aliae et aliae ...").

Saint Augustine (in the *Confessioni*) expresses more clearly the same idea:

"... Past no longer exsists, future not yet exsists. But if present would remain always present and never fade away in the past, there would not be time, but ethernity. ..."

In the classical physics of Galilei and Newton space and time are containers with an onthological autonomy:

"... Time, absolute, true, mathematical, in itself and by its own nature without any relation with anything external, flows uniformly ..." (*Principia mathematica*).

According to Kant space and time are not objects but forms of the human knowledge: we do not "know" space and time, but our knowledge is organized in a space–time manner:

"... The idea of time does not originate from senses, but is presupposed by them ... Time is not something objective and real. It is neither substance, nor accident, nor relation, but a necessary subjective condition, due to the nature of human mind, to coordinate within itself all perceptible things according to a fixed law. ... (from: *De mundis sensibilis atque intelligibilis forma et principiis*).

Lucrezio had a similar point of view: *tempus item per se non est.*

This is an appealing point of view but leaves the following question open: how to speak of space and time before the emergence of human conscience in the Universe?

The attempt to reconcile the time of physics with the psychological time is usually attributed to Bergson (time as interior duration), but his known statement: *time is an invention or nothing* is (surprisingly, due to the fact the two thinkers considered their ideas on time in mutual disagreement) near to the well known and widely quoted passage from Einstein's condolence letter Michele Besso's sister (marzo 1955): "... For us practicing physicists, the distinction between past, present and future is only an illusion, even if a tenacious one ...."

The artistic arbitrariness in Borges' statement:

"... time is a river that sweeps me away, but I am the river; it is a tiger that tears me to pieces, but I am the tiger; it is a fire that devours me, but I am the fire.

should be compared with the empty arbitrariness in Hawking's statement:

"... The concept imaginary time is the fundamental concept on the basis of which the mathematical model has to be formulated; ordinary time would be in this case a derived model that we invent – as a part of a mathematical model – with the goal of describing subjective impressions about the universe" (*Halley Lectures*, 1989).

The list of metaphorae used to described time could continue indefinitely and in some sense they all confirm the famous saying of St. Augustine according to which time belongs to that class of concepts that everybody believes they are clear but nobody can explicitly define. This impossibility has been even theorized Paul Ricoeur in the three volume work "Temps et récit" [Ricoe99] whose main thesis is that the nature of time cannot be object of rational thinking, but only of a "poetic resolution" through the production of histories, tales and novels,through which we acquire an indirect comprehension of the notion of "time" and of our existence in it.

A similar purely esthetic attitude, even if not so explicit, but with a slightly encyclopedic character and with original combinations of text and images can be found in [Priest74]. The philosophical and theological aspects of the debate on time are developed in parallel with the scientific aspects in the book by Castagnino and Sanguineti [CaSa00].

In this paper we will look at time from the point of view of mathematics. Scientific knowledge is based on a different kind of metaphorae, called definitions, models and procedures (protocols). Scientists do not illude themselves to overcome completely the intrinsic ambiguity of language, but they try to limit it with the help of models. In some sense contemporary scientists think in terms of models.

A model is a simple example of an axiomatic theory: usually the term "model" is referred to a very specific situation and the term "axiomatic theory" to a wider enterprise such as the unification of different contexts, however the logical structure of the two are the same: to define a context (axioms) and to draw consequences from it according to the rules of logic.

The maturity of a science is measured by the degree in which it succeeds in condensating its knowledge in mathematical models, or equivalently axioms, and in deducing its procedures from them. The multiplicity of possible models is an healthy antidote to the illusions of certitude.

Scientific activity oscillates among the three poles of:

(i) inventing new models;

(ii) deducing observable consequences from them;

(iii) verifying experimentally these consequences.

In the present paper we will play this game with the notion of time. Emphasis will be on incompleteness (in the sense explained below): even on an extremely basic and fundamental level our ideas on time are not sufficiently precise to fix a unique class of mathematical models. In the first part of this paper this idea will be illustrated with simple geometrical models; in the second part we will concentrate on time reflections and discuss the various attempts which have been made in the physical and mathematical literature to substantiate this notion.

The goal of an axiomatic theory of time should be that of challenging clever physicists to discover empirically observable differences between different mathematical models of time. The examples discussed below show that this challenge might not be a trivial one. For example, according to the principle of "topological relativity", discussed in section (4) below it is not possible to distinguish experimentally between the western tradition of linear time and the Indian tradition of circular time. In particular, as shown in section (4), circular time does not necessarily imply eternal recurrence of history.

In Greek philosophy change and movement was related to imperfection. Perfection was related to immobility. Thus perfect objects, like fixed stars, don't feel the flow of time. Contemporary science has shifted its idea of perfection from immobility (like fixed stars) to elementarity. When we try to describe our idea of the flow of time, i.e. of irreversibility, independently of mathematical formulae, unavoidably we end up in describing a situation in which a multiplicity of interacting systems (i.e. a complex system) are separated into non interacting systems (i.e. a simple system). To reconstruct the complex system from the simple ones may not be logically impossible but is surely much more complicated than the converse operation. In fact in order to break many different interactions, it is simply required to create a single interaction which dominates them all but, to reconstruct them one has to act individually on each broken tie. For example, to put and keep a gas in a box is a relatively simple operation if one can act collectively on the gas as a whole, but if, after opening the box and allowing expansion, the gas is mixed with another volume of gas made of the same molecules, to reproduce exactly the original configuration is a task of practically (although not logically) impossible complexity. In this sense the difference between life and death is the same as the difference between a set of individuals and an organization.

The above example shows that the notions of existence, identity, system, motion, space, time, signal..., are strictly related: if some of them are assumed as primitive, the other ones can be introduced as derived.

The question whether some of them have to be considered as "objectively primitive" with respect to the other ones is interesting and has a long history.

For example, Aristotle takes motion as primitive notion and defines time as the measure of motion. In a world without conscience, hence without knowledge, motion is still conceivable. From this point of view Aristotle is nearer to scientific mentality than Kant in the sense that his approach allows a more objective, human independent, definition of time. For Newton, time and space (more generally – state) are primitive notions and "motion" is defined as change of state in time.

In special relativity one assumes as primitive much more complex notions such as:

(i) the notion of signal

(ii) the notion of event

(iii) the finiteness of the speed of light

and, having payed this price, one can achieve a complete "space–time democracy". These notions are used to distinguish between "space differences" and "time differences" among events. Two events which cannot be connected by a signal are defined to be simultaneous: if they are different, their difference is of space type. Two events that can be connected by a signal are always different and their difference is of time type.

Thus in relativity we can distinguish between space and time diversity, hence motion can be introduced as a measure of time diversity. However, since time reflection is admitted as a *physical transformation*, in relativity the distinction between past and future is relative to the reference frame.

Any model of time depends on the notion of "system" but, from an holistic point of view the notion of system itself is a quite anthropomorphic notion: if two systems interact, what does it mean to distinguish between them? The boundary is necessarily arbitrary and the separation depends on the choice of some scales of magnitude ("large" distances, "weak" interactions, ...) which might be quite natural for human beings but from a non anthropocentric point of view are not privileged.

If we accept the existence of elementary particles, then we can give an (approximate) definition of decay as decomposition into (approximately) non interacting elementary constituents. But from a more sophisticated, field theoretical point of view,which includes self–interactions, the "elementary" particles are simply manifestations of the field and they too can decay. In fact, from this point of view, one should not speak of "decays" but simply of "transformations" from one manifestation of the field to another one.

Transformations among elementary particles may be reversible, but what does this precisely means? This question is nontrivial because of the necessity to distinguish between the reversibility of the time evolution and the "time reversal symmetry", which in many theories such as nonrelativistic classical and quantum mechanics can be defined in dependently of any specific time evolution (cf. section (12)).

2 Time as a 1–dimensional connected continuum

Any model translates one's intuition of a physical object or phenomenon. Abstraction of properties leads to the construction of a mathematical model. Then one tries to go back to the original intuition, i.e. to check to what extent this is reflected by the model.

The obstruction to this procedure is that there may be, and in general there are, many inequivalent models. In the case of time this leads to the following alternative:

(i) our intuition is incomplete: physical time has more properties than those specified by the model

(ii) all models are present in nature

Here there is an historical asymmetry between space and time: in fact it is now commonly accepted that there exist many simultaneous models of space, while many models of time can exist but not "simultaneously". The postulate of homogeneity is an extrapolation from local to global, but since all our perceptions are local both in space and in time, the way to match together a multiplicity of local perceptions into a single global picture necessarily introduces some elements which go beyond experimental evidence.

In the following of this section we shall discuss some consequences of the axiom which underlies most of contemporary scientific models of time:

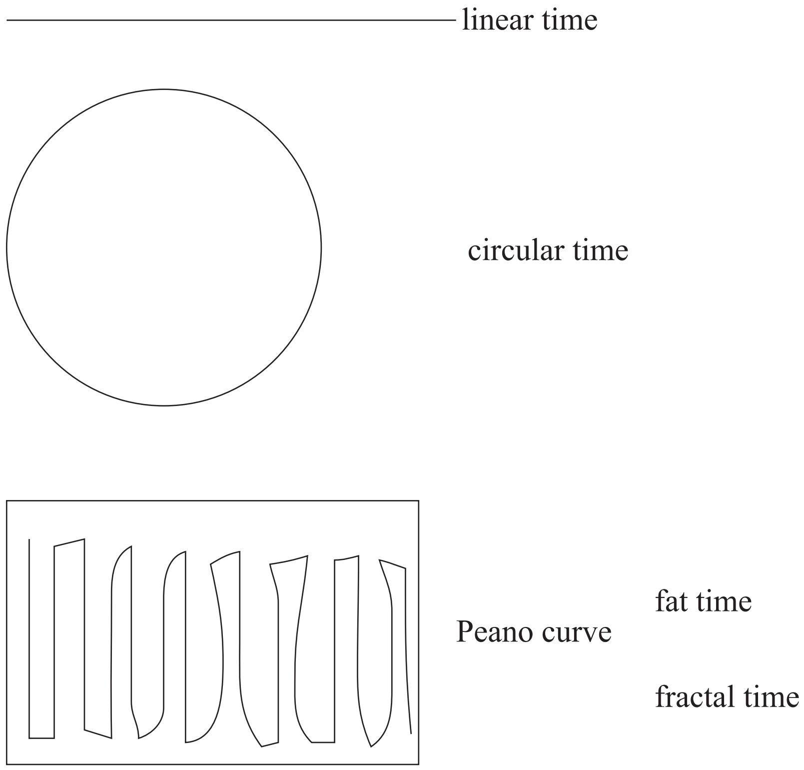
(A1.) **Time is a 1–dimensional connected continuum**.

This axiom excludes the existence of time quanta (cronons,... [CaRe78]). Discrete models of time have been investigated in the mathematical, physical, psychological , ... literature. We emphasize that all the mathematical models described in the present section continue to be meaningful also in the case of discrete time. However our point in this paper is that, even keeping a conservative view of time as a one dimensional connected continuum, still there is a lot of space for non trivial inequivalent possibilities.

The continuum, like the infinite, is not accessible to human experiments in actual form, but only in its potential form. If one believes in the time–energy indeterminacy principle, even this potentiality is questionable from a physical point of view because the measure of extremely small periods of time would imply extremely large fluctuations in energy.

According to Hilbert the goal of an axiomatization of a physical theory is: *... to formulate the physical requirements so that the mathematical model is uniquely determined...* (at least up to isomorphism). This requirement is usually called *completeness* in logic.

It is clear that axiom (A1) is far from complete, in fact there exist many 1–dimensional connected continua! For example the real line $\mathbb{R}$ (linear time) and the circle (circular time). The Peano curve or any other fractal curve provide additional models which satisfy axiom (A1) but which suggest different intuitive images of time (fat time, fractal time, ...).



The introduction of a unit of time is equivalent to the introduction of an action, on this continuum, of the positive rational numbers (multiplication).

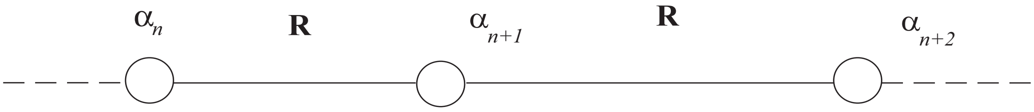
The archimedean nature of this action is implicitly postulated when one identifies time with the real line, but this is clearly an additional axiom and non archimedean models of time are certainly possible from a mathematical point of view and not necessarily implausible from the physical point of view, as shown by the example in the following section.

3 A non archimedean model of time

The following construction was inspired by a discussion with Professor E. Brieskorn in which he explained to me a model of extended real line due to Hausdorff ($\sim$ 1903). The model that follows is less sophisticated than Hausdorff's but it helps getting an intuition of how incomplete Axiom (A1) is. The construction goes as follows: for each $n \in \mathbb{Z}$ fix an homeomorphism u_n between the real line $\mathbb{R}$ and the open interval $(n, n+1)$. Now fix an arbitrary sequence (α_n) of increasing ordinals and associate to each $n \in \mathbb{Z}$ the corresponding α_n. This construction gives a one–to–one correspondence

$$u : \hat{\mathbb{R}} := \cup_n \{(n, n+1)\} \cup (\cup_n \{\alpha_n\}) =: \cup_n (\mathbb{R})_n \cup (\cup_n \{\alpha_n\}) \to \mathbb{R}$$

and we can define a topology on $\hat{\mathbb{R}}$ so that the map u is an homeomorphism. Moreover we can use this map to transport on $\hat{\mathbb{R}}$ the usual order structure on $\mathbb{R}$.



The usual multiplication on $\mathbb{R}$ can be extended to $\hat{\mathbb{R}}$ by the prescriptions:

$$\lambda \cdot x = (\lambda x) \quad \text{if} \quad x \in (\mathbb{R})_n \quad \text{for some} \quad n$$

$$\lambda \alpha_n = \alpha_n \qquad \text{if} \qquad \lambda > 0$$

$$\lambda \alpha_n = \alpha_{n+1} \qquad \text{if} \qquad \lambda < 0$$

$$\lambda \alpha_n = 0 \qquad \text{if} \qquad \lambda = 0$$

This extension is such that the multiplication by positive numbers is continuous but the time reflection $t \to -t$ is not Thus $\hat{\mathbb{R}}$ is connected but the action of $\mathbb{R}$ on $\hat{\mathbb{R}}$ is non–archimedean.

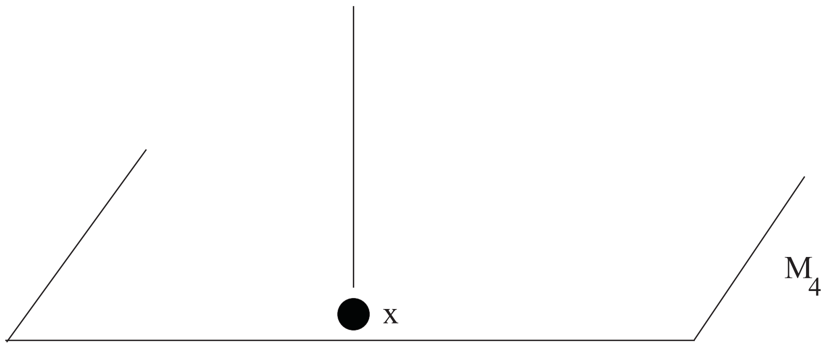
Recently, in the attempt to model an intrinsic asymmetry in the time evolution of physical systems, S. Wickramasekara [Wickr01], introduced the topology on $\mathbb{R}$, generated by the basic open sets $[a, b)$. In this topology multiplication by positive real numbers is continuous, but time reflection ($t \to -t$) is discontinuous (e.g. $a + 1/n$ converges to a in this topology but $-a - 1/n$ cannot be in any interval of the form $[-a, -a + \epsilon)$ with $\epsilon > 0$ and therefore it doesn't converge to $-a$ in this topology). In this topology past and future are not symmetric because the basic open intervals include the left point but not the right one and the discontinuity of time reflection reflects this asymmetry.

In the above discussed non archimedean model, past and future are symmetric and the discontinuity of time reflection comes from the fact that the "special points" α_n represent "singularities of time" in the sense that each of them plays the role of "infinite past" for one interval and of "infinite future" for another one. This interpretation reflects the idea of the simultaneous existence of infinitely many time flows (parallel universes). A more traditional interpretation may regard each interval as an "era" and the time singularity α_n represents the "big crunch" for the n–th era and the "big bang" for the $n + 1$–th. The non archimedean character of the model reflects the incommunicability between different epochs.

4 Time in classical physics

For classical Hamiltonian mechanics time is an external parameter, i.e. strictly speaking it is not an observable of the theory, while observables are sections in the cotangent bundle. For Einstein they are sections in the tangent bundle so that the basic observables are restricted to the kinematical observables of classical physics: positions (space), time, their conjugate observables, velocities and energy, and functions of them. In the concrete models one has to further specify which class of functions are allowed (measurable, smooth, analytic, compact support, rapidly decaying, ...): different choices lead to different theories. One can conceive more general bundles whose fiber includes the tangent space and some new, non kinematical degree of freedom (such as spin, color,...).

Classical theories (as opposed to quantum) are characterized by the universal compatibility of all the observables. A maximal family of independent observables (i.e. such that no one of them is expressible as a function of the remaining ones) defines the state space, or more precisely a representation of it (cf. [Ac81c] or [AcRe96] for a detailed analysis of this concept). In classical theories we can distinguish two main attitudes. One, which is typical of classical relativity, could be named "space–time democracy" and according to it the universe is a fiber bundle with a 4–dimensional manifold M_4 as a basis

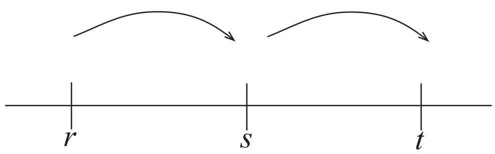


Another one, which is at the basis of non relativistic physics, can be named "time supremacy" and according to it:

– the universe is a fiber bundle with a 1–dimensional manifold as a basis
– the fiber is the state space S
– the time evolution $T(s,t)$ is parallel transport from one fiber to another

The mathematical model of this scenario should be a fiber bundle with basis $\mathbb{R}$ and fiber a space S. The curves on $\mathbb{R}$ are the ordered pairs (s,t). They form a groupoid for the multiplication

$$(s,t)(r,s) = (r,t)$$

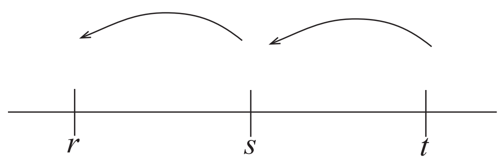


The dynamical evolution is a parallel transport

$$(s,t) \to T(s,t) : S_s \to S_t$$

The existence of two time orientations gives rise to 2 multiplications

$$(s,t) : s \to t \quad ; \qquad (s,t) : t \to s$$



$$(r, s)(s, t) = (r, t)$$

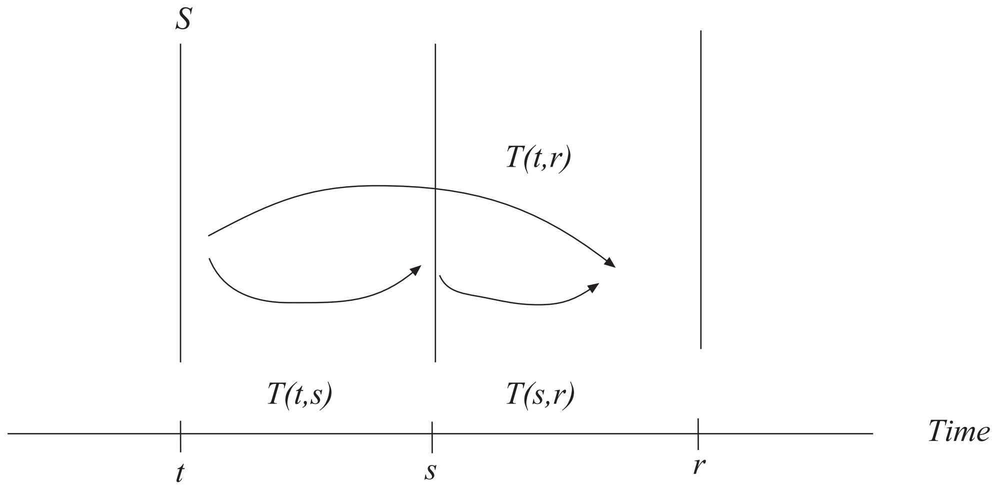
To use motion as a measure of time only means that the parallel transport is given a priori. To use time as order of motion means that we have a priori decided the direction of time.

Definition 1 *A temporal evolution is determined by the assignment, for each element of family of unordered pairs* $\{s, t\} \subseteq T \times T$*, of a map*

$$T_{s,t} : S_s \to S_t$$

Moreover it is required that, if the pair $\{s, u\}$ is in the family and maps $T_{s,t} : S_s \to S_t$ and $T_{t,u} : S_t \to S_u$

$$T_{t,u} T_{s,t} = T_{s,u} : S_s \to S_u \tag{1}$$



The idea that time orders motion is expressed by the requirement that:

(i) the base manifold is ordered and parallel transport (evolution) respects the order

$$r < s < t \Rightarrow T(s, t)T(r, s) = T(r, t)$$

Suppose we want to take seriously Aristotle: "... This is time in reality: the number of motion according to the before and the after ..."
i.e. it is motion that gives the direction of time, then we might argue as follows. A "curve" on the time manifold is given by an unordered pair $\{s,t\}$. However the associated parallel transport must go from one fiber to another. In other terms: the motion must have a direction. Suppose that

$$T_{\{s,t\}} : S_s \to S_t \tag{2}$$

then we say by definition that

$$s < t$$

In other words, if the dynamics (motion) $T(s,t)$ is physically realized, then we say that $s < t$ (s is in the past of t). Thus the orientation of the curve $\{s,t\}$ is given by the fact that the parallel transport $T_{\{s,t\}}$ maps the fiber at s to the fiber at t and not conversely. In conclusion: the Aristotelean view that motion orders time is expressed by the requirement that we use parallel transport to induce an order on the base manifold

$$T(s,t)T(r,s) = T(r,t) \Rightarrow r < s < t$$

This leads to the following

Definition 2 *The Aristotelean future of $s \in T$ is the set of all $t \in T$ such that the parallel transport (2) is well defined. Similarly we say that $s \in T$ is in the past of $t \in T$.*

Definition 3 *$\{T, \{T_{s,t}\}$ is called linearly ordered if no $s \in T$ belongs to its proper future. It is called without origin if the proper past of each $s \in T$ is non empty. It is called connected that $s \in T$ is in the past of some $t \in T$ and in the future of some $t' \in T$. It is called circular if every $s \in T$ belongs to its proper future.*

In a circular time recurrence (of a state with respect to a given evolution) can be defined canonically. In a linearly ordered time, in order to define recurrence (of a state with respect to a given evolution) we need to fix an identification of the different fibers of the basic time manifold. Without it would be meaningless to speak of "the same state at different times". The minimal requirement on such an identification is that, for any $t \in \mathbb{R}$ it is given a one–to–one map $j_t : S_t \to S$, where S_t is the fiber at t and S is a fixed space.

For human beings this identification map is given by the "memory", but from a logical point of view this is non canonical and therefore, as already noticed by Heraclitus, the notion of "the same state at different times" is problematic.

Given such an identification, we say that a state $x_t \in S_t$ is "the same" as the state $x_s \in S_s$ if

$$j_t(x_t) = j_s(x_s) \in S$$

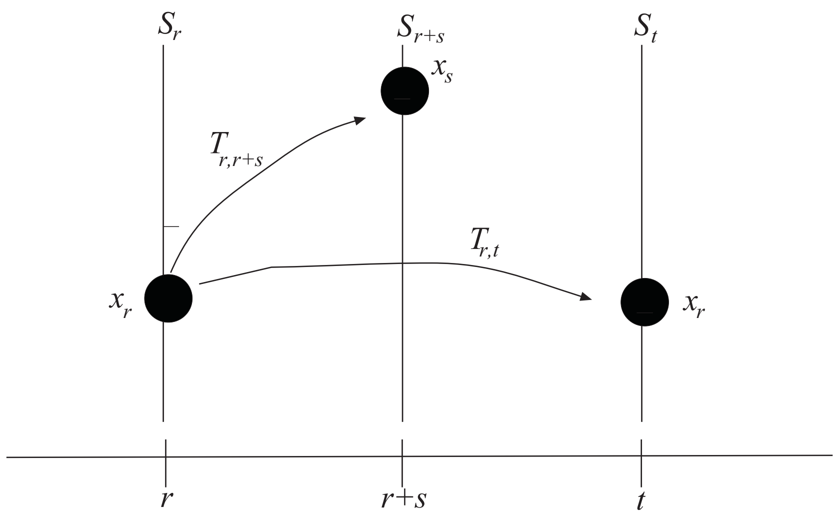
and that a state $x_s \in S_s$ is "recurrent with respect to the dynamics $\{T_{s,t}\}$ if it is "the same" as the state $T_{s,t}x_s \in S_t$ Furthermore, given such an identification, one can use the structure of additive (semi)group on $\mathbb{R}$ to define time homogeneity of an evolution by:

$$T(r,s) = T(r+\tau, s+\tau) \ ; \qquad \forall\, \tau$$

where the $=$ symbol means that "the same states" , on the fibers S_r and $S_{r+\tau}$ are mapped into "the same states", on the fibers S_s and $S_{s+\tau}$.

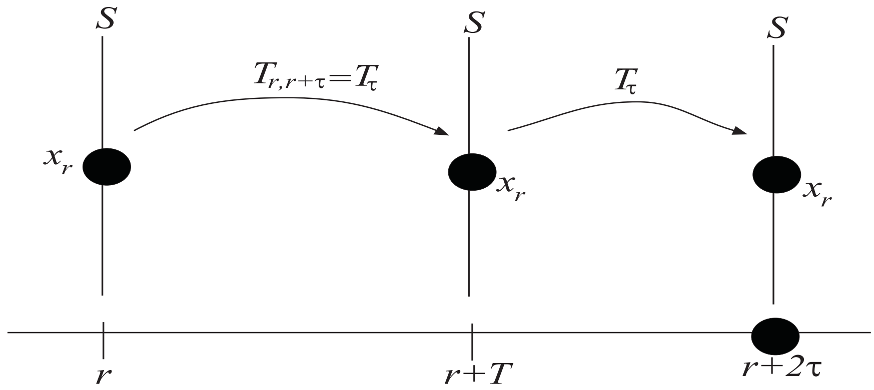
One easily sees that recurrence of a state x_s in the interval $[s, s+\tau]$ plus time homogeneity implies cyclicity of this state with period τ because

$$T(s+n\tau, s+(n+1)\tau)x_s = T(s, s+\tau)x_s = x_s$$



conversely: suppose time is circular with period τ

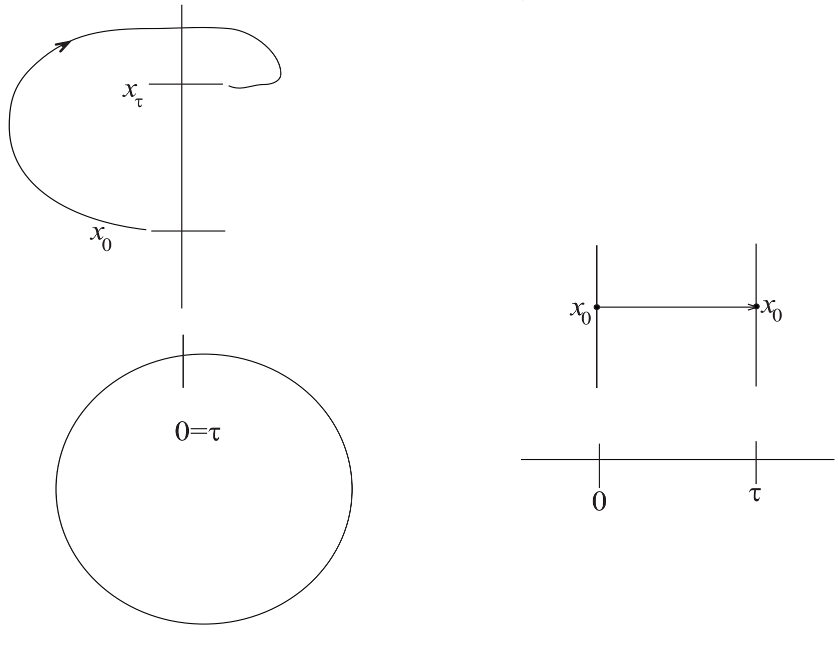
$$t + \tau \equiv t \qquad \text{on the time manifold}$$



but that the initial state x_r is "wandering", i.e. its orbit has no loops, then after a time cycle the state (say of the universe) cannot be the original one,

$$x_\tau = T_\tau x_0 \neq x_0$$

hence it is not possible to prove that a time cycle has been closed.



Ciclicity without recurrence

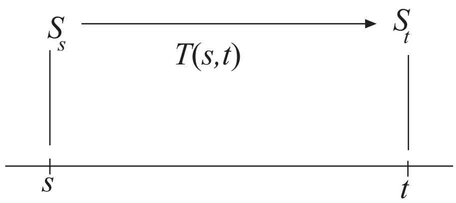
Recurrence without ciclicity

Summing up we can now formulate the "topological relativity" of time as follows: it is not possible to distinguish experimentally between:

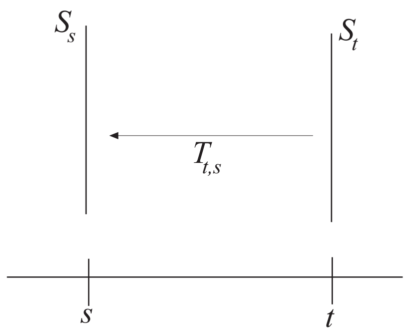
(i) circular time and wandering state

(ii) linear time and no recurrence

The point of view (i) would lead to a cylinder as the basic model of "flat" (i.e. without matter) space–time rather than the $\mathbb{R}^4$ of special relativity. The theory of Lorentz transformations can be extended to such a space but, again, if we insist on the usual topology for the circle, time reflection will introduce a discontinuity. In any case from both the mathematical and the conceptual point of view, this is an interesting possibility to investigate. An interesting discussion of cyclic and linear time in physics and chemistry is in the Di Meo's monograph [DiMeo96]. Finally let us consider the connection between reversibility and invertibility of the evolution.



Invertibility of $T(s,t)$ means that $T(s,t)^{-1}$ exists



but it can be of the form $T(s',t')$ only if time is circular. Thus we conclude that, if motion measures time and if $T(s,t): \ S_s \to S_t$ is physically realized, then if time is not circular,

$$T(t,s) : S_t \to S_s$$

is not a physical object even if it exists as a mathematical object.

Notice that: Reversibility $\neq$ Recurrence

5 Reversibility and homogeneity

An homogeneous reversible system is one whose dynamics is homogeneous and invertible, i.e.:

$$(T^{-t}) \qquad T^{-1} \text{ exists}$$

Reversible homogeneous discrete systems cannot go to any form of (dynamical or thermodynamical) equilibrium in finite time because

$$T^{n+1}x = T^n x \Leftrightarrow Tx = x$$

The same is true for continuous systems because

$$T^{t+\varepsilon}x = T^t x \Leftrightarrow T^\varepsilon x = x$$

However non time homogeneous systems can go to equilibrium in finite time because relations such as

$$T(n, n+1) \cdot T(n-1, n) \dots T(0,1)x = T(n-1, n) \dots T(0,1)x$$

$$T(s,t)T(0,s)x = T(0,s)x$$

mean only that there exists x_s such that

$$T(s,t)x_s = x_s \; ; \qquad \forall\, s > t$$

This identity is problematic because $T(s,t) : S_s \to S_t$ so we have to give a meaning to the notion of "being the same state on different time fibers".

6 Arrows of time

Definition. If there is an observable A and an expectation (mean) value (i.e. a probability measure) $\langle A \rangle$ such that the map

$$t \mapsto \langle T(s,t)[A] \rangle$$

is monotonic, then we speak of an "arrow of time".

In the cosmological arrow A is the distance of galaxies: the expansion of the universe gives a arrow of time. Also contraction also would be an arrow of time. Thus, in agreement with the point of view of Aristotle, the real arrow comes from change, i.e. from motion.

In the thermodynamical arrow A is entropy and time flows in the direction of degrading energy. The thermodynamical arrow creates conflicting intuitions because suggests that complex structures should degrade to simpler ones which is apparently in contrast with the biological or historical evolution. This contrast might be only apparent because a global increase of entropy is not in contradiction with a local decrease.

A more subtle arrow of time emerged from the stochastic limit of quantum theory and is discussed in section (1.18) of [AcLuVo02]. This has to do with the fact that the quantum transport coefficients (or generalized susceptivities), deduced from the stochastic limit of an Hamiltonian evolution in the forward time direction, is the complex conjugate of the coefficient deduced from the backward time evolution. Since the imaginary part of the quantum transport coefficient describes an energy shift, it follows that a red shift in the forward direction of time should become a blue shift in the backward direction. A possible astrophysical interpretation of this fact, based on the analysis of the Pioneer 6 data was discussed in [AcLaLuRi95].

7 Time reversal: axiomatic approach

The usual mathematical model of the set of states of a physical system is given by the convex set

$$\mathcal{S} := \mathcal{S}(\mathcal{A})$$

of normal states of a von Neumann algebra $\mathcal{A}$. If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on a Hilbert space, we will use the notation $\mathcal{S}(\mathcal{H})$ instead of $\mathcal{S}\left(\mathcal{B}(\mathcal{H})\right)$.

The following definition is often adopted. We will see that not all interesting cases are included in it because antiautomorphisms are excluded.

Definition 4 *A symmetry of a quantum system is a weakly continuous affine bijection of the set $\mathcal{S}$ of states.*

A time reversal is an involutory symmetry $\mathcal{T}$ of $\mathcal{S} = \mathcal{S}(\mathcal{A})$ such that

$$\mathcal{T}^2 = \mathcal{T} \tag{3}$$

If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the structure of the symmetries of a quantum system is described by the following theorem due to Wigner (cf. [CaDeViLe97] for an elegant proof):

Theorem 1 *A map u of $\mathcal{S}(\mathcal{H})$ into itself is an affine bijection if and only if it is of the form*

$$u(\rho) = U\rho U^{-1} \textit{ for all } \rho \textit{ in } \mathcal{S}(\mathcal{H}) \tag{4}$$

where U is a unitary or antiunitary operator in $\mathcal{H}$, determined by u up to a phase factor. Any operator U, satisfying (4) will be said to implement the symmetry u.

Theorem 2 *Let $\mathcal{T}$ be a time reversal on $\mathcal{S} = \mathcal{K}(\mathcal{H}_S)$. Then there exists a unitary or antiunitary operator T on $\mathcal{H} = \mathcal{H}_S$ such that*

$$\mathcal{T}(\rho) = T\rho T^{-1} \textit{ for all } \rho \textit{ in } \mathcal{K}(\mathcal{H}) \tag{5}$$

Proof. It follows from Theorem (1) that a symmetry $\mathcal{T}$ is involutory if and only if any unitary or antiunitary operator T implementing it in the sense of (4), i.e.

$$\mathcal{T}(\rho) = T\rho T^{-1} \text{ for all } \rho \text{ in } \mathcal{S}(\mathcal{H}) \tag{6}$$

satisfies

$$\rho = \mathcal{T}^2(\rho) = T^2\rho T^{-2} \text{ for all } \rho \text{ in } \mathcal{S}(\mathcal{H}) \tag{7}$$

and therefore there must exist $\lambda \in \mathbb{C}$ such that $T^2 = \lambda$. For a unitary operator T this implies that $T^2 = \lambda T_0^2$ where $T_0^2 = \pm 1$ and λ is a complex number of modulus 1.

Remark. The following Lemma shows that the phase ambiguity is reduced for antiunitary operators.

Lemma 1 *An antiunitary operator T is such that $T^2 = \lambda$ with $\lambda \in \mathbb{C}$, then $\lambda = \pm 1$.*

Proof. Since $T^2 = \lambda$ and T^2 is unitary, λ must have modulus 1. Moreover $\forall\, \psi \in \mathcal{H}$, the identity $T^2\psi = \lambda\psi$ and antiunitarity imply that

$$TT^2\psi = T\lambda\psi = \overline{\lambda}T\psi$$

On the other hand

$$TT^2\psi = T^2T\psi = \lambda T\psi$$

It follows that $\lambda = \overline{\lambda}$ and therefore $T^2 = \pm 1$.

8 Examples

Example (1) Complex conjugation in L^2–spaces is the basic example of antiunitary operator.

Lemma 2 *The complex conjugation in $L^2(S, \lambda)$ is defined by:*

$$C\psi := \overline{\psi}$$

C is a antiunitary operator satisfying $C^2 = 1$.

Proof.

$$\forall\, \psi \in \mathcal{S}(\mathcal{A}),\ \forall\, \lambda \in \mathbb{C}$$

$$C(\lambda\psi) = (\lambda\psi)^- = \overline{\lambda\psi} = \overline{\lambda} C\psi$$

$$\langle C\psi_1, C\psi_2\rangle = \int (C\psi_1)^-(C\psi_2) d\lambda = \int \psi_1 \overline{\psi}_2 d\lambda = \langle \psi_2, \psi\rangle$$

Define

$$\mathcal{T}\rho = C\rho C$$

The dual map of $\mathcal{T}$ is defined by

$$Tr\mathcal{T}(\rho)X = Tr\rho\mathcal{T}'(x)\ ; \quad x \in \mathcal{A}$$

Since

$$Tr\mathcal{T}(\rho)X = TrC\rho Cx = Tr\rho CXC$$

it follows that

$$\mathcal{T}'(x) = CXC \tag{8}$$

Lemma 3

$$C^* = C$$

Proof. Since C is antiunitary $C^*C = 1$. But also $C^2 = 1$. Hence $C^* = C$.

Lemma 4 *$\mathcal{T}'$, defined by (8) is an antilinear *–automorphism.*

Proof. $\forall\, x \in \mathcal{A}$

$$\mathcal{T}'(\lambda x) = C\lambda XC = \overline{\lambda}CXC = \overline{\lambda}\mathcal{T}'(x)$$

$$\mathcal{T}'(x)\mathcal{T}'(y) = CXC^2yC = CXYC = \mathcal{T}'(x)$$

$$\mathcal{T}'(x)^* = (CXC)^* = CX^*C = \mathcal{T}'(X^*)$$

Remark.

For integral spin time reversal is implemented by complex conjugtion.

Example (2) Let $\mathcal{A} = M(2, \mathbf{C})$ and

$$\tau\colon \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

denote the usual transposition. Notice that

$$\tau x = Jx^*J \quad ; \quad J\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix}$$

Example (3) Let

$$\tau_o\colon \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Then

$$\tau x = Kx^*K$$

$$K\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_2 \end{pmatrix}$$

Remark. For Pauli matrices

$$\tau\colon \sigma_1, \sigma_2, \sigma_3 \longrightarrow -\sigma_1, -\sigma_2, -\sigma_3$$

thus for spin 1/2 (in general any half integer spin), time reversal is implemented by K because, in analogy with classical angular momentum, you want spin (and orbital angular momentum) to change sign under time reversal.

Example (4). Let

$$\tau_2\colon \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}$$

Then

$$\tau x = Lx^*L$$

$$L\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_1 \\ \overline{z}_2 \end{pmatrix}$$

Notice that now $L^2 = -1$.

9 Time reflections in von Neumann algebras

Wigner's theorem, discussed in section (7) has been extended to more general von Neumann algebras according to the following lines.

Definition 5 *A Jordan $*$–automorphism of $\mathcal{A}$ is a linear map τ of $\mathcal{A}$ in itself satisfying*

$$\tau(x^*) = \tau(x)^* \quad ; \quad \forall x \in \mathcal{A} \tag{9}$$

$$\tau^2 = id \tag{10}$$

$$\tau(xy + yx) = \tau(x)\tau(y) + \tau(y)\tau(x) \tag{11}$$

One first proves that the dual of a weakly continuous affine bijection of the set $\mathcal{S}$ of normal states of a von Neumann algebra $\mathcal{A}$ is a Jordan $*$–automorphism. The one defines

Definition 6 *A time reflection in $\mathcal{A}$ is an involutive Jordan $*$–automorphism of $\mathcal{A}$*

The following extension of Wigner's theorem is due to Kadison.

Theorem 3 *Let α be a Jordan $*-$automorphism of $\mathcal{A}$ then there exist a maximal central projection $z \in \mathcal{A}$ such that;*

$$\alpha(xy) \;=\; \alpha(x)\alpha(y)z + \alpha(y)\;\alpha(x)\;(1 - z) \tag{12}$$

Moreover if

$$\alpha^2 = id$$

then

$$\alpha(z) = z$$

(this is not true in general).

In particular,

Corollary 1 *If $\mathcal{A}$ is a factor, then a Jordan $*-$automorphism is either a $*-$automorphism or a $*-$anti-automorphism.*

In particular (Stormer), if $\mathcal{A}$ is a factor and τ is implemented, then it is implemented by a unitary or anti-unitary hermitian operator.

Theorem 4 *If $A = \mathcal{B}(\mathcal{H})$ every involutive Jordan $*-$automorphism is given by*

$$\tau\ x = Jx^*J$$

where J is antilinear isometric and $J^2 = 1$, or

$$\tau\ x = -Kx^*K$$

where K is antilinear isometric $K^2 = -1$

The possible use of general involutory automorphisms in von Neumann algebras to describe time reflections has been extensively discussed in [Maje84b], [Maje84a], [Maje83]. A discussion more oriented towards the time reversibility of the Markov property is in [AcMo99].

10 Time reversibility and time reversal invariance

In general it is assumed that for any system S there exists an involutory symmetry $\mathcal{T}$ of $\mathcal{S}$ which can be interpreted as the time reversal operation.

The problem to distinguish which additional conditions should be satisfied by an involutive symmetry of a given state space to guarantee the uniqueness of the time reversal requires a deep investigation. Usually these additional conditions are expressed

(i) either in terms of privileged observables such as positions, momenta,...

(ii) or in terms of a given, privileged, dynamics.

The basic example of the attitude (i) is given by the following:

Definition 7 *The time reversed state $\tilde{\rho}$ of a state ρ of a system S of n strctureless particles is defined by*

$$Tr[F(\{X_j\},\{P_j\})\tilde{\rho}] = Tr[F(\{X_j\},\{-P_j\})\rho] \tag{13}$$

Corollary. The time reversed $\tilde{\rho}$ of a state ρ is unique whenever the set of functions F is large enough to separate the states (basic example: the Weyl algebra of the CCR).
Proof. Clear.

Definition 8 *The map*

$$\mathcal{T} : \rho \to \tilde{\rho}$$

is called time reversal.

Corollary. $\mathcal{T}$ is an affine map such that

$$\mathcal{T}^2 = id \Leftrightarrow \mathcal{T}^{-1} = \mathcal{T}$$

Proof. Clear.

Definition 9 *A dynamics, or time evolution of a physical system, is a* 2*–parameter family of symmetries* u_{t,t_0} *of its set of states.*

(i) For each fixed t*,* t_0*,* u_{t,t_0} *is an affine map of the convex set* $\mathcal{S}$ *into itself, i.e.:*

$$u_{t,t_0}(\alpha\rho' + (1-\alpha)\rho'') = \alpha u_{t,t_0}(\rho') + (1-\alpha)u_{t,t_0}(\rho'')$$

for all ρ'*,* ρ'' *in* $\mathcal{S}$ *and for all* α *in* $(0,1)$*.*

(ii) For all states ρ *in* $\mathcal{S}$*, all observables* A*, all Borel subsets* I *of* $\mathbb{R}$ *and all initial times* t_0*, the functions*

$$t \mapsto Tr[E_A(I)u_{t,t_0}(\rho)]$$

are continuous.

(iii) For all $t_0 < t_1 < t_2$*, we have*

$$u_{t_2,t_0} = u_{t_2,t_1}u_{t_1,t_0}$$

The dynamics u_{t,t_0} *is called homogeneous if*

(iv) For all $t > t_0$ *and all real* s*,*

$$u_{t,t_0} = u_{t+s,t_0+s}$$

The dynamics u_{t,t_0} *is called reversible if*

(v) u_{t,t_0} *is a one–to–one map of* $\mathcal{S}$ **onto** *itself.*

The dynamics of conservative systems are usually reversible. The notion of time reversal invariance is different from that of reversibility.

Definition 10 *The time evolution* u_t *is called time reversal invariant if*

$$\mathcal{T}u_t\mathcal{T}u_t\rho = \rho \text{ for all } \rho \text{ and for all } t \tag{14}$$

Note that time reversal invariance implies that u_t has the everywhere defined inverse $\mathcal{T}u_t\mathcal{T}$, even if the time evolution had been originally defined for positive time only. So, time reversal invariance implies reversibility (but not the other way around).

11 Evolutions

Let H be the Hamiltonian of a system. We will always suppose that

$$H \geq 0$$

The Schrödinger evolution of state ψ is

$$\psi_t = e^{-itH}\psi_0 = |\psi_t >$$

hence the evolution of the corresponding density matrix is given by

$$|\psi_t >< \psi_t| = e^{-itH}|\psi_0 >< \psi_0|e^{itH}$$

Therefore an arbitrary density matrix evolve according to

$$\rho \to e^{-itH}\rho e^{itH}$$

The dual evolution is defined by

$$Tr\rho_t A = Tr|e^{-itH}\psi_0 >< \psi_0|e^{itH}A| = \langle\psi_0, e^{itH}Ae^{-itH}\psi_0\rangle$$

Thus the Heisenberg evolution of an observable A is

$$A \to e^{itH}Ae^{-itH}$$

Let A be a reflection invariant observable:

$$\mathcal{T}A = A \; ; \quad t \leq 0$$

$$\mathcal{T}u_{-t}\mathcal{T}A = \mathcal{T}u_{-t}A = \mathcal{T}e^{itH}Ae^{-itH} = Te^{itH}Ae^{-itH}T = e^{TitHT}Ae^{-TitHt} =$$

If T is anti unitary this is equal to

$$e^{-itTHT}Ae^{itTHT}$$

and, if T is unitary, this is equal to

$$e^{itH}Ae^{-itH}$$

12 Time reversal in the Schrödinger representation

In the representation of $\mathcal{H}$ as $L^2(\mathbb{R}^{3n}, dx)$, it is immediately seen that (13) holds if we set

$$\mathcal{T}\rho := \tilde{\rho} = C\rho C \quad (= C\rho C^{-1}) \ , \quad \rho \in \mathcal{K}(\mathcal{H}) \tag{15}$$

where C is the natural complex conjugation on $L^2(\mathbb{R}^{3n}, dx)$:

$$C\psi := \overline{\psi}$$

indeed, we have

$$CX_jC = X_j \ , \qquad C(-i\hbar\nabla_j)C = i\hbar\nabla_j \tag{16}$$

It follows that the map $\mathcal{T}$ transforming ρ into $\tilde{\rho}$ is a *symmetry*, implemented by an anti unitary operator, and satisfying $\mathcal{T}^2 = 1$ (the identity map). If we define the time reversed evolution $\tilde{u}_t$ by

$$\tilde{u}_t = \mathcal{T}u_{-t}\mathcal{T} \ ; \quad \forall\, t \leq 0 \tag{17}$$

we obtain

$$\tilde{u}_t\rho = \exp\left[-\frac{i}{\hbar}\tilde{T}h\right]\rho\exp\left[\frac{i}{\hbar}\tilde{T}h\right] \tag{18}$$

where

$$\tilde{H} = CHIC \tag{19}$$

Note that a Hamiltonian of the form

$$-\Delta + V \tag{20}$$

satisfies $CHC = H$, so that $\tilde{u}_t = u_t$; in words, the evolution is time reversal invariant. In fact, let us consider the Scarödinger equation

$$i\frac{\partial\psi(q,t)}{\partial t} = H\psi(q,t) \tag{21}$$

where

$$H = -\frac{\Delta}{2} + V(q) \tag{22}$$

$q = (q_1, ..., q_n)$ and $V(q)$ is a real-valued function.

Theorem 5 *If $\psi(q,t)$ is a solution of the Schrödinger equation (1) then its time reciprocal wave function*

$$\psi'(q,t) = \bar{\psi}(q,-t) \tag{23}$$

is also a solution of the same Schrödinger equation

$$i\frac{\partial\psi'(q,t)}{\partial t} = H\psi'(q,t) \tag{24}$$

Proof. One gets from (1)

$$-i\frac{\partial\psi(q,-t)}{\partial t} = i\frac{\partial\psi(q,-t)}{\partial(-t)} = H\psi(q,-t) \tag{25}$$

Now by making the complex conjugation in (5) we get (4).

If we define $\tilde{u}_t$ as in (17), we have, using also $\mathcal{T}^{-1} = \mathcal{T}$,

$$\tilde{u}_t\rho = \exp\left[-\frac{i}{\hbar}\,\tilde{H}t\right]\rho\exp\left[\frac{i}{\hbar}\,\tilde{H}t\right] \tag{26}$$

where

$$\tilde{H} = \sigma T^{-1}HT \tag{27}$$

and where

$$\sigma \quad = \quad -1 \text{ if } T \text{ is unitary} \tag{28}$$

$$\sigma \quad = \quad +1 \text{ if } T \text{ is antiunitary} \tag{29}$$

If we require that both H and $\tilde{H}$ are bounded from below and unbounded from above, we are forced to assume that T is antiunitary, in agreement with the example (15). We shall assume in general that T is antiunitary.

13 Time reflection and positivity of the spectrum

.

Suppose that, in a algebraic set up, time reflection is implemented by a self-adjoint involution

$$\tau(A) = KA^*K \quad ; \quad K^2 = 1$$

Since

$$KU_tK = U_{-t}$$

it follows that, if K is unitary then, deriving the evolution, one has

$$KHK = -H$$

Therefore if τ is an automorphism, then the spectrum of H cannot be positive. Hence τ, and therefore K, must be antilinear.

14 Antilinear antiautomophisms

Definition 11 *Let $\mathcal{A}$ be a $*$–algebra. An antilinear antiautomorphism of $\mathcal{A}$ is a map*

$$\rho_0 : \mathcal{A} \to \mathcal{A}$$

satisfying

$$\rho_0(a+b) = \rho_0(a) + \rho_0(b) \qquad \textit{(additivity)}$$

$$\rho_0(\lambda a) = \overline{\lambda}\rho_0(b) \qquad \textit{(anti–linearity)}$$

$$\rho_0(ab) = \rho_0(b)\rho_0(a)$$

The following simple remark shows that there is a one–to–one correspondence between linear *–automorphisms and antilinear *–antiautomorphisms.

Lemma 5 *Let $u \in Aut\ (\mathcal{A})$ be a linear *–automorphism. Then*

$$u^*(x) := u(x^*)\ ; \quad x \in \mathcal{A}$$

*Is an anti–linear *–antiautomorphism of $\mathcal{A}$. Conversely, if u^* is an antilinear *–antiautomorphism of $\mathcal{A}$, then*

$$u(x) := u^*(x^*)$$

*is a linear *–automorphism of $\mathcal{A}$.*

Proof. u^* is clearly antilinear

$$u^*(xy) = u(y^*x^*) = u(y^*)u(x^*) = u^*(y)u^*(x)$$

$$(u^*(x))^* = (u(x^*))^* = u(x) = u^*(x^*)$$

Conversely, if u^* is an antilinear *–antiautomorphism, then u is linear and

$$u(xy) = u^*((xy)^*) = u^*(y^*x^*) = u^*(x^*)u^*(y^*) = u(x)u(y)$$

$$(u(x))^* = (u^*(x^*))^* = u^*(x) = u(x^*)$$

15 Anti–states

An anti–linear map, ρ_0 does not map states into states. This justifies the following

Definition 12 *Let $\rho_0 : \mathcal{A} \to \mathcal{A}$ be an antilinear map. A state φ on $\mathcal{A}$ is called ρ_0–invariant (or anti–invariant) if*

$$\varphi \circ \rho_0 = \overline{\varphi} \tag{30}$$

where, by definition

$$\overline{\varphi}(a) := \overline{\varphi(a)} = \varphi(a^*) \tag{31}$$

We have seen that, for anti–automorphisms ρ_0, the natural notion of invariance is

$$\varphi \circ \rho_0 = \overline{\varphi}$$

If φ is a state on $\mathcal{A}$, $\overline{\varphi}$ is defined by

$$\overline{\varphi}(a) = \overline{\varphi(a)} = \varphi(a^*)$$

hence it is not a state.

Definition 13 *An anti–linear positive functional φ on $\mathcal{A}$ such that*

$$\varphi(1) = 1$$

is called an anti–state.

Since φ is an anti–state if and only if $\overline{\varphi}$ is a state, all the notions and constructions for states are extended to anti states.

16 Automorphisms with anti–invariant states

Lemma 6 *Let R be an anti–unitary such that $R^2 = 1$ (so that $R^* = R$) and define*

$$RaR =: \rho_0(a) \tag{32}$$

*Then ρ_0 is an antilinear *–automorphism. Suppose moreover that*

$$R\Phi = \Phi \tag{33}$$

and define

$$\langle \Phi, a\Phi \rangle =: \varphi(a)$$

Then

$$\varphi \circ \rho_0 = \overline{\varphi}$$

Proof.

$$\varphi(\rho_0(a)) = \langle \Phi, RaR\Phi \rangle = \langle RR\Phi, RaR\Phi \rangle =$$

$$= \langle aR\Phi, R\Phi \rangle = \langle a\Phi, \Phi \rangle = \langle \Phi, a^*\Phi \rangle = \varphi(a^*) = \overline{\varphi}(a)$$

That ρ_0 is an automorphism is clear because

$$\rho_0(a)\rho_0(b) = (RaR)(RbR) = RaR^2bR = RabR = \rho_0(ab)$$

Finally ρ_0 is a $*$–automorphism because $R^* = R$.

17 Motivations for the definition of time reversal

Let (u_t^0) be a free evolution. Solving the Schrödinger equation in interaction representation

$$\partial_t U_t = -iH_I(t)U_t \tag{34}$$

for the interaction H_I

$$u_t^0(H_I) = H_I(t)$$

with initial condition $U_0 = 1$ and for *positive times* t, we obtain a cocycle for the free evolution (u_t^0). Following [AcKo00b] (section (1.1.29)), let us show that there is a unique way to extend this cocycle to negative times so that, composing this cocycle with the free evolution, one obtains a 1–parameter automorphism group on all the real line. This extension is called *the time reflected cocycle*.

Theorem 6 *Let $\mathcal{A}$ be an algebra, (u_t^0) a 1–parameter automorphism group on $\mathcal{A}$ and $(U_t)_{t\geq 0}$ a 1–parameter family of unitary operators in $\mathcal{A}$ such that the 1–parameter family*

$$u_t := U_t^* u_t^0(\cdot)U_t =: j_{0,t}u_t^0 \ ; \qquad t \geq 0 \tag{35}$$

is a 1–parameter semigroup of automorphisms of $\mathcal{A}$. Then there exists a unique 1–parameter automorphism group (u_t) $(t \in \mathbb{R})$ on $\mathcal{A}$ coinciding with (35) for positive values of t.

Moreover u_t has the form, for $a \in \mathcal{A}$

$$u_t(a) = \begin{cases} j_{0,t} u_t^0(a) = U_t^* \cdot u_t^0(a) \cdot U_t \ ; \ t \geq 0 \\ u_t^0 j_{0,-t}^{-1}(a) = u_t^0(U_{-t}) \cdot u_t^0(a) \cdot u_t^0(U_{-t})^* = u_t^0(U_{-t} a U_{-t}^*) \ ; \ t \leq 0 \end{cases} \tag{36}$$

Proof. By assumption each u_t, with $t \geq 0$, is invertible hence, if an (u_t) as in the thesis exists, it is uniquely defined by

$$u_{-t} = (u_t)^{-1} \quad ; \qquad t \geq 0 \tag{37}$$

Now let us check that $(u_t)^{-1}$ is given by the right hand side of (36). For each $a \in \mathcal{A}$ and $t \geq 0$, one has

$$u_{-t} u_t(a) = u_{-t}^0(U_t [U_t^* u_t^0(a) U_t] U_t^*) = u_{-t}^0 u_t^0(a) = a$$

This implies that, if $0 \leq s < t$, then

$$u_{-s} u_t = u_{-s} u_s u_{t-s} = u_{t-s} \tag{38}$$

and similarly if $s > t$. Since (u_t) is a semigroup for $t \geq 0$ (hence also for $t \leq 0$, due to (37)), (38) implies that u_t is a 1–parameter automorphism group and this ends the proof.

Notice that the solution of equation (34) for $t \geq 0$ is

$$\vec{T} e^{-i \int_0^t H_I(s) ds} = U_t \tag{39}$$

Now suppose that there exists an anti–automorphism satisfying

$$\rho_0(H_I(s)) = H_I(-s)$$

Then since ρ_0 is an antilinear antiautomorphism, we have, with $t \geq 0$:

$$\rho_0(U_t) = \overleftarrow{T} \exp i \int_0^t H_I(-s) ds = \overleftarrow{T} e^{-\int_0^{-t} H_I(\sigma) d\sigma} = \overleftarrow{T} e^{i \int_{-t}^0 H_I(s) ds} = \tag{40}$$

$$= (U_{-t})^*$$

Proceeding more constructivelylet, for $t \leq 0$, u_t^0 be the backward free evolution. Then if U_t is the evolution in interacting representation, we must have because of Theorem (6):

$$U_t = u_t^0(U_{-t})^* = u_t^0(\vec{T}e^{-i\int_0^{-t} H_I(s)ds})^* = u_t^0(\overleftarrow{T}e^{+i\int_0^{-t} H_I(s)ds})$$
$$= \overleftarrow{T}e^{i\int_0^{-t} u_t^0(H_I(s))ds} = \overleftarrow{T}e^{i\int_0^{-t} H_I(s+t)ds} = \overleftarrow{T}e^{i\int_t^0 H_I(\sigma)d\sigma}$$

With the change of variables $\sigma := s + t$, we obtain

$$U_t = \overleftarrow{T}e^{i\int_t^0 H_I(\sigma)ds} \ ; \qquad t \leq 0$$

comparing this with (40) we conclude that, for $t \leq 0$, one has

$$\rho_0(U_{-t}) = u_t^0(U_{-t})^* \tag{41}$$

This identity motivates the definition introduced in the following section.

18 Time reflections in local algebras

Let be given a von Neumann algebra $\mathcal{A}$ with a time localization. This means a triple $\{\mathcal{A}, \mathcal{A}_0, (u_t^0)\}$ such that:

$$u_t^0 \in \text{Aut } (\mathcal{A}) \tag{42}$$

$$\mathcal{A}_t := u_t^0(\mathcal{A}_0) \tag{43}$$

$$\mathcal{A} = \bigvee_{t\in\mathbb{R}} \mathcal{A}_t \tag{44}$$

There exists at most one additive map ρ_0 satisfying

$$\rho_0(A(t_1)\dots A(t_n)) = A(-t_n)^* \dots A(-t_1)^* \tag{45}$$

for any choice of $t_1, \dots, t_n$ (not necessarily ordered). Any map with this property satisfies, for any $A, B \in \mathcal{A}$:

$$\rho_0(\lambda A) = \overline{\lambda}\rho_0(A)$$

$$\rho_0(A^*) = \rho_0(A)^*$$

$$\rho_0(AB) = \rho_0(B)\rho_0(A)$$

The first two properties are clear. The third one follows from

$$\rho_0(A(t_1)\dots A(t_n)\cdot B(s_1)\dots B(s_m)) = B(-s_m)^* \dots B(-s_1)A(-t_n)^* \dots A(-t_1)^*$$

Definition 14 *An anti–automorphism $\rho_0 : \mathcal{A} \to \mathcal{A}$ satisfying (45) will be called a time reflection with respect to the time localization $\{\mathcal{A}, \mathcal{A}_0, (u_t^0)\}$. If $\mathcal{A}$ is generated by the $\mathcal{A}_t$'s topologically, then ρ_0 is required to be continuous.*

Lemma 7 *Let ρ_0 and u_t^0 be as above and let φ be an u_t^0–invariant state. Then $\varphi \circ \rho_0 = \overline{\varphi}$.*

Proof. For any

$$F(\{A(s)\}) \in \mathcal{A}_{[0,t]}$$

one has

$$\varphi(\rho_0(F(\{A(s)\})) = \varphi(F(\{A(-s)\})^*) = \varphi(u_{2s}^0[F(\{A(-s)\})^*])$$
$$= \varphi(F(\{A(s)\})^*) = \overline{\varphi}(F(\{A(s)\}))$$

$\varphi \circ \rho_0 = \overline{\varphi}$.

In the following we study the existence of time reflections on special algebras. In fact, in section (x.) we will prove that such an anti–automorphism can be explicitly constructed for any mean zero gauge invariant Gaussian field with standard free evolution.

19 The adjoint and the time reversed of an Markov semigroup

The following discussion abstracts a general scenario which is realized in several concrete examples of physical interest in the stochastic limit of quantum theory [AcLuVo02], [AcKo00b], [AcIm02a], [AcImLu02a], [AcImKo02a]. Let be given:

(i) an algebra $\mathcal{A}$, a state φ on $\mathcal{A}$, and a unitary in the algebra $U_t \in Un(\mathcal{A})$

(ii) an antiautomorphism $\rho_0 \in$ Antiaut $(\mathcal{A})$ of $\mathcal{A}$ leaving φ anti–invariant

$$\rho_a(ab) = \rho_0(b)\rho_0(a) \ ; \qquad \rho_0(a^*) = \rho_0(a)^* \ ; \quad \rho_0 - \text{antilinear}$$

$$\varphi \circ \rho_0 = \overline{\varphi} \tag{46}$$

(iii) Let $\mathcal{A}$ be realized on a Hilbert space $\mathcal{H}$ and

$$\varphi(\cdot) = \langle \Phi, (\cdot)\Phi \rangle$$

Then we have

$$\begin{aligned}\varphi(X^*U_t^*YU_t) = \langle X\Phi, U_t^*YU_t\Phi\rangle = \varphi(X^*U_t^*YU_t) = \overline{\varphi}(\rho_0(X^*[U_t^*YU_t])) = \\ = \overline{\varphi}(\rho_0(U_t)\rho(Y)\rho_0(U_t^*)\cdot\rho_0(X^*)) = \varphi(\rho_0(X)\rho_0(U_t)\rho_0(Y^*)\rho_0(U_t^*)) \\ = \langle\rho_0(X^*)\Phi, \rho_0(U_t)\rho_0(Y^*)\rho_0(U_t^*)\Phi\rangle = \varphi(\rho_0(X)\rho_0(U_t)\rho_0(Y^*)\rho_0(U_t))\end{aligned} \tag{47}$$

Suppose moreover that the following conditions are satisfied:

(iv) X, Y are such that

$$\rho_0(X) = X^* \ ; \quad \rho_0(Y) = Y^*$$

(recall that ρ_0 is antilinear)

(v) for some 1–parameter group (u_t^0) of automorphisms of $\mathcal{A}$ leaving φ invariant

$$\rho_0(U_t) = u^0_{-t}(U_t)^* \qquad ; \qquad t \geq 0$$

(vi) $u_t^0(X) = X$; $u_t^0(Y) = Y$; $\forall\, t$.

Then the identity (47) becomes, for $t \geq 0$:

$$\langle X\Phi, U_t^*YU_t\Phi\rangle = \langle X\Phi, u^0_{-t}(U_t)Yu^0_{-t}(U_t^*)\Phi\rangle$$

equivalently, introducing the notations

$$U_{[0,t]} := U_t \ ; \quad U_{[-t,0]} := u^0_{-t}(U_t^*) = u^0_{-t}(U_t)^*$$

$$\varphi(X^*U^*_{[0,t]}YU_{[0,t]}) = \varphi(X^*U^*_{[-t,0]}YU_{[-t,0]})$$

In case of a Markovian structure compatible with φ, i.e. X, Y are in the time zero algebra and

$$\varphi \circ E_{0]} = \varphi \circ E_{[0} = \varphi$$

we obtain

$$\varphi_0(X^*_{0]}E(U^*_{[0,t]}YU_{[0,t]})) = \varphi_0(X^*E_{[0}(U^*_{[-t,0]}YU_{[-t,0]}))$$

Thus, introducing the notations

$$P^t(Y) := E_{0]}(U^*_{[0,t]}YU_{[0,t]})$$

$$P^t_{\text{rev}}(Y) := E_{[0}(U^*_{[-t,0]} Y U_{[-t,0]})$$

we obtain the duality

$$\varphi_0(X^* P^t(Y)) = \varphi_0(X^* P^t_{\text{rev}}(Y)) \tag{48}$$

Notice the difference between this duality and the usual duality for Markov semigropus

$$\varphi_0(X^* P^t(Y)) = \varphi_0(P^t_+(X)^* Y) \tag{49}$$

It is well known that the adjoint semigroup P^t_+ exists under severely restrictive conditions while, as shown above, the existence of P^t_{rev} only requires the time reflection invariance of φ which is a much weaker condition. In fact in the above construction we have also used the invariance of the "time zero algebra" under time reflection. This is automatically satisfied in many concrete models, however it is not difficult to modify the above construction so to include also a non trivial action of the time reflection of the "time zero algebra".

Concrete examples of time reversed semigroups, coming from stochastic limit, are discussed in [AcIm02a], [AcImLu02a]. For a proof of the fact that the existence of the adjoint Markov semigroup is a characteristic of equilibrium situations cf. Theorem (1.41) in [AcKo00b] and the discussion following it where it is emphasized that the above conclusion strongly depends on the existence of "sufficiently many allowed transitions among the atomic levels". In presence of forbidden transitions the physical situation is much richer and the mathematical situation is much more complex and should be discussed case by case.

20 Field algebras

Theorem 7 *Let a_k, a^+_k, be a Boson field and $\langle\cdot\rangle_0$ a Gaussian mean zero gauge invariant state on it. Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a sufficiently good function.*

Then there exists a unique Boson field $a(t,k)$, $a^+(t,k)$ and a state $\langle\cdot\rangle$ on it such that the map

$$a(t,k) \mapsto e^{-it\omega_k} a_k = u^0_t(a_k)$$

extends to an isomrophism in the sense of correlators. Moreover the map

$$a(t,k) \mapsto a(t+s,k)$$

extends to a 1–parameter automorphism group u^0_t of the $a(t,k)$–field algebra

Proof (idea). Denote $\mathcal{A}(\omega)$ a field algebra generated by the

$$e^{-it\omega_k} a_k \quad ; \qquad \forall\, t, k$$

by fixing a space of test functions. Then the correlations kernels defined in terms of the elements of $\mathcal{A}(\omega)$ with the state $\langle\cdot\rangle_0$, automatically satisfy the compatibility conditions of the reconstruction theorem of [AcFrLe81].

The existence of (u_t^0) follows from the fact that the correlators will depend only on the time differences $t - s$.

Definition 15 *The algebra generated by the fields $a(t,k)$ in Theorem (7) is called the free field algebra.*

21 Existence of a time reflection on the field algebra

We will prove that on the field algebra there exists a unique map ρ_0 such that

(i) ρ_0 is an anti–automorphism, i.e.

$$\rho_0(AB) = \rho_0(B)\rho_0(A)$$

$$\rho_0(A)^* = \rho_0(A)^*$$

(ii) If $A_0 \in \mathcal{A}_0$ (time zero algebra) and u_s^0 is the free evolution, then $\forall\, s \in \mathbb{R}$

$$\rho_0(u_s^0(A_0)) = u_{-s}^0(A_0)^*$$

(in particular the time zero algebra is left invariant)

Let be given a Boson field with a free evolution $a(t,k) = e^{-it\omega_k} a_k$ and let $A(S_t g)$, $\quad A^+(S_t f)$ denote the corresponding evolutions of the smeared fields.

$$a(-t,k)^* = [e^{-i(-t)\omega_k} a_k]^* = [e^{it\omega_k} a_k]^* = e^{-it\omega_k} a_k^+$$

Then an anti–automorphism ρ_0, as in section (18), exists and it is characterized by:

$$\rho_0(a(t,k)) = a(-t,k)^* = e^{-it\omega_k} a_k^+$$

Equivalently the map ρ_0 can be obtained in 2 steps:

(i) exchange of a and a^+

(ii) replacement of ω_k by $-\omega_k$.

The formal proof should be given uses Theorem (7) and the fact that the set of correlators is left invariant by the above replacements.

In terms of test functions we see that for an arbitrary test function F

$$[a(t,k), a^+(t',k')] = e^{-it\omega_k} e^{it'\omega_{k'}} \delta(k-k') = e^{-i(t-t')\omega_k} \delta(k-k')$$

$$\rho_0(\langle a, F\rangle) = \rho_0\left(\int dt \int dk \overline{F(t,k)} e^{it\omega_k} a_k\right) = \int dt \int dk F(t,k) e^{it\omega_k} a_k^+$$
$$= \int ds \int dk (F(-s,k)) e^{is\omega_k} a_k^+ = \langle a^+, F\rangle$$

22 ρ_0–invariant Gaussian states

For Gaussian states φ everything is reduced to the pair correlations

$$\varphi(A(g)A^+(f)) = \langle g, Qf\rangle$$

Assuming that the one–particle free evolution commutes with the covariance:

$$S_t Q = Q S_t$$

one has

$$\varphi(A_f(s)A_g^+(t)) = \langle f, QS_{t-s}g\rangle$$
$$\varphi(A^+(t)A(s)) = \varphi([A^+(t), A(s)]) + \varphi(A(s)A^+(t))$$
$$= \langle g, S_{t-s}f\rangle + \langle g, QS_{t-s}f\rangle = \langle g, (1+Q)S_{t-s}f\rangle$$

Therefore

$$\varphi(A_g(-t)A_f^+(-s)) = \langle g, QS_{(-s)-(-t)}f\rangle = \langle g, QS_{t-s}f\rangle = \overline{\langle f, QS_{t-s}g\rangle} = \varphi(A_f(s)A_g^+(t))$$

Therefore

$$\varphi \circ \rho_0 = \overline{\varphi}$$

i.e. any gauge invariant Gaussian state is ρ_0–invariant.

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