The unitarity conditions for the square of white noise

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Abstract.
The renormalized stochastic differentials of the square of white noise are defined in Boson Fock space representation. The linear independence and the Itô multiplication table of these differentials are proved. The module form of the associated quantum stochastic evolution is given and unitarity conditions for its solution are obtained.

1. Introduction

The present paper brings to a conclusion a line of research carried through the past ten years with the contributions of several people and which has its roots in the development of the stochastic limit of quantum theory (cf. [AcLuVo02] for a general exposition) which led, in the early 1990's, to an extension of both classical (Itô) [Ito51] and quantum (Hudson–Parthasarathy) [HuPa84c] stochastic calculus. The new approach was called *white noise approach to stochastic calculus* [AcLuVo99] because it is based on Hida white noise analysis (cf. [Hida92], [Kuo96]) and on the observation that both classical and quantum stochastic calculus can be reduced to the analysis of the first power of the standard quantum white noise b_t, b_t^+ plus the normally ordered second power $b_t^+ b_t$.

Given this result the question whether is it possible to develop a stochastic calculus for the higher powers of white noise, i.e. a "nonlinear stochastic calculus", naturally arose and the programme of developing such a calculus was first proposed in [AcLuVo95b].

The difficulty of the problem resides in the fact that, since white noise is an operator valued distribution, the definition of its powers requires a renormalization procedure. This difficulty was well known in the physical literature where one can find many attempts of giving a meaning to the powers of local quantum fields (which are nothing but multiparameter white noises).

A similar problem also arises in classical probability with the attempts to extend stochastic calculus beyond the frame of semi-martingales. (cf. for example [RuGrVa01]).

Starting from 1995 several papers were devoted to the attempt of realizing the program formulated in [AcLuVo95b] in the simplest non linear situation, i. e. the "renormalized square of white noise" (RSWN) and several partial results were obtained. These attempts were based on the original proposal, in [AcLuVo95b], to renormalize the Itô table inside the space of the first order white noise (i.e. the usual Fock space over $L^2(\mathbb{R})$), to write the equations in the normally ordered form and to interpret the higher powers of the fields as densely defined sesquilinear forms. However they met serious difficulties to achieve the fundamental goal of the theory, i.e. the unitarity conditions: the results of [AcLuOb96], [AcBou01b], [AcBou01a], [AcBou01c], [AcBou00a], clearly indicated that the formulation of these conditions would be deeper and more complex than the corresponding result for linear white noise, proved in the early '80's by Hudson and Parthasarathy.

The situation changed with the paper [AcLuVo99] where a new renormalization technique was introduced based on the following ideas:

(i) instead of the cut-off-and-take-limits procedure, used in the physical literature, and instead of renormalizing the Ito table as done in [AcLuVo95b], one directly renormalizes the commutation relations themselves, and then looks for a Hilbert space representation of them.

(ii) instead of subtracting infinite constants, use the known identity $\delta(t)^2 = c\,\delta(t)$ (cf. [AcLuOb96] for a proof).

This allowed to explicitly construct an analogue of the Fock representation for the RSWN (at the moment no such construction is available for the higher order renormalized powers of white noise, which now constitutes the boundary of the present theory). This result was the staring point for several developments.

First Sniady [Snia99] extended the construction to the free case and produced the first evidence of the existence of algebraic obstructions preventing the possibility to produce a single Fock representation for the combination of the first and the second power of white noise.

Almost simultaneously Accardi and Skeide [AcSk99b] recognized that the representation space of the RSWN coincided with the "finite difference Fock space" which was introduced 10 years before, in a completely different context, and starting from completely different motivation, by Feinsilver and Boukas [Fei87],[Bou91a].

Finally Accardi, Franz and Skeide in [AcFrSk00] identified the Lie algebra of the RSWN to a current algebra over a central extension of the Lie algebra of $SL(2;\mathbb{R})$ and, using the Schürmann representation theorem for independent increment processes on $*$–bialgebras [Schu93], classified all the representations of this current algebra, enjoying a certain irreducibility property (for a new class of representations, cf. the forthcoming paper [AcAmFr03]). Moreover they concretely realized the RSWN basic integrators as simple sums of first order integrators, i.e of Hudson-Parthasarathy type.

In other words: the renormalized square of white noise (RSWN) can be realized on a usual Fock space and the corresponding stochastic differentials can be expressed as sums of usual, first order, white noises acting on that space.

Another result of the paper, it was to realize that the one–parameter family of classical processes, generated by the second order Weyl-Poisson operators $(B_t + B_t^+ + \lambda B_t^+ B_t)$ with respect to the vacuum vector, can be identified with the three non standard classes (i.e neither Gaussian nor Poisson) arising in the Meixner classification theorem (cf. [Ac01c] for more details on this point).

However, in their first order representation, the three basic integrators of the RSWN integrators (cf. (2.7)-(2.9)) have not a closed Itô table and this fact made it impossible to prove, in the paper [AcFrSk00], the main result of the theory, i.e. the unitarity condition. In fact the only thing one can get, from a naive application of the Hudson–Parthasarathy first order Itô table to the RSWN, is an infinite chain of coupled nonlinear operator equations on whose solution still now nothing is known.

This made clear that, for the solution of the problem, the simple application of the known formulae of first order stochastic calculus were not sufficient and new ideas and techniques were needed. For these reasons, in the three years after the paper [AcFrSk00], several alternative attacks to the unitarity problem were developed.

Accardi, Hida and Kuo [AcHiKu01] tried to develop a direct approach to the Itô table of the RSWN, i.e without using the first order representation (2.7)-(2.9). To this goal they introduced two quadratic forms, the Hida derivative and its formal adjoint, in the representation space of the RSWN. However, as remarked by Accardi, Boukas and Kuo [AcBou01f], the introduction of this operator makes the future increments of the basic integrators, linearly dependent over the past and this creates difficulties in the deduction of the unitarity conditions.

The starting point of the paper [AcBou01e] was the remark that, through the representation theorem of [AcFrSk00], the RSWN Ito algebra can be represented as a proper sub–Ito algebra of a first order Ito algebra on an appropriate representation space. This suggested the idea that, if one were able to determine the *structure coefficients* of this Ito algebra, in the sense of [AcPa85] section 6, then an infinite dimensional extension of the arguments in sections 6 and 10 of the paper [AcFaQu89] might lead to the desired unitarity conditions.

These structure coefficients were determined in the paper [AcBou01e] combining the technique of "normal ordering in the $sl(2,\mathbb{R})$–Lie algebra with an intensive use of the symbolic program "Mathematica" and lead to a finite set of conditions for unitarity.

The basic property that emerged from the calculation of these structure coefficients is that, even if the Ito algebra of the RSWN is infinite dimensional, its Ito table is locally finite, in the sense that, in the product of any two stochastic differentials, only a finite number of coefficients is nonzero.

In the present paper we use the above result to define a new multiplication on the coefficient algebra, which is associative because of Theorem (2.1) of [AcQu88] (a direct proof of associativity seems to be very difficult due to the combinatorics involved). The combination of this new multiplication with the module approach to stochastic calculus allows to find a closed simple form of the unitarity condition (cf. Theorem 1 which is our main result).

The greater complexity of the unitarity condition should not be a surprise due to the intriniscally nonlinear character of the problem. In fact the general conjecture that emerges from the results of of [AcFrSk00] is that the calculus associated with the higher (≥ 2) powers of white noise coincides with the calculus associated to a certain class of independent increment processes. In the case of second powers this class has been identified with the three non standard classes, i.e. neither Gaussian nor Poisson, in Meixner's classification theorem. For the higher powers this is an important open challenge for quantum probability.

2. The SWN and normal order in $\rho^+(U(sl(2,\mathbb{R})))$

Definition 1. *The SWN Lie algebra is the three-dimensional simple Lie algebra with basis B^+, M, B^- satisfying the commutation relations*

$$[B^-,B^+]=M,\ [M,B^+]=2B^+,\ [M,B^-]=-2B^- \tag{2.1}$$

with involution

$$(B^-)^*=B^+,\ M^*=M \tag{2.2}$$

It was shown in [AcFrSk00] that the mapping ρ^+ defined by

$$\rho^+(M)\,e_n=(2n+2)\,e_n \tag{2.3}$$

$$\rho^+(B^+)\,e_n=\sqrt{(n+1)(n+2)}\,e_{n+1} \tag{2.4}$$

$$\rho^+(B^-)\,e_n=\sqrt{n(n+1)}\,e_{n-1} \tag{2.5}$$

where e_n, $n = 0, 1, 2, \cdots$ is any orthonormal basis of $l_2(\mathbb{N})$, defines a representation of the SWN Lie algebra on $l_2(\mathbb{N})$.

The proof of the following Lemma can be found in [AcBou01e].

Lemma 1. *For all* $n, k, l, m = 0, 1, 2, ...$

$$\rho^+(B^{+^n} M^k B^{-^l})\, e_m = \theta_{n,k,l,m}\, e_{n+m-l} \tag{2.6}$$

where

$$\theta_{n,k,l,m} = H(n + m - l) \sqrt{\frac{m - l + n + 1}{m + 1}}\, 2^k (m - l + 1)_n (m + 1)^{(l)} (m - l + 1)^k \tag{2.7}$$

$$H(x) = \left\{ \begin{array}{lll} 1 & \textit{if } x \geq 0 & \\ & & \textit{is the Heaviside function} \\ 0 & \textit{if } x < 0 & \end{array} \right\}$$

$$0^0 = 1, \ \ (B^+)^n = (B^-)^n = N^n = 0, \quad \textit{for } n < 0$$

and "factorial powers" are defined by

$$x^{(n)} = x(x-1)\cdots(x-n+1)$$
$$(x)_n = x(x+1)\cdots(x+n-1)$$
$$(x)_0 = x^{(0)} = 1.$$

Moreover, for $\alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, ...\}$

$$B^{+^\alpha} M^\beta B^{-^\gamma} B^{+^a} M^b B^{-^c} = \tag{2.8}$$

$$\sum_{\lambda=0}^{\gamma} \sum_{\rho=0}^{\gamma-\lambda} \sum_{\sigma=0}^{\gamma-\lambda-\rho} \sum_{\omega=0}^{\beta} \sum_{\epsilon=0}^{b} c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}\, B^{+^{a+\alpha-\gamma+\lambda}} M^{\omega+\sigma+\epsilon} B^{-^{\lambda+c}} \tag{2.9}$$

where

$$c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon} = \tag{2.10}$$

$$\binom{\gamma}{\lambda}\binom{\gamma-\lambda}{\rho}\binom{\beta}{\omega}\binom{b}{\epsilon} 2^{\beta+b-\omega-\epsilon} S_{\gamma-\lambda-\rho,\sigma}\, a^{(\gamma-\lambda)} (a+\lambda-1)^{(\rho)} (a-\gamma+\lambda)^{\beta-\omega} \lambda^{b-\epsilon} \tag{2.11}$$

and $S_{\gamma-\lambda-\rho,\sigma}$ *are the Stirling numbers of the first kind.*

By (2.6) if $n = l$ then the orthonormal basis vectors e_m are eigenvectors of the self-adjoint operators $B^{+^n} M^k B^{-^n}$ with eigenvalues $\theta_{n,k,n,m}$.

We fix the representation ρ^+ of the universal enveloping algebra $U(sl(2;\mathbb{R}))$ of $sl(2;\mathbb{R})$ so that we realize $U(sl(2;\mathbb{R}))$ as acting on $l_2(\mathbb{N})$ with the action given by (2.6).

3. The SWN stochastic differentials

Let $\mathcal{K} = l_2(\mathbb{N})$. The Boson Fock space $\Gamma = \Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$ (see [AcLuVo99], [Chebo00],[Par92]) can be defined as the Hilbert space completion of the linear span of the exponential vectors $\psi(f)$ under the inner product

$$< \psi(f), \psi(g) >= e^{<f,g>} \tag{3.1}$$

where $f, g \in L^2(\mathbb{R}_+, \mathcal{K})$.

For $f \in L^2(\mathbb{R}_+, \mathcal{K})$ and an adjointable linear operator F on $L^2(\mathbb{R}_+, \mathcal{K})$ the annihilation, creation and conservation operators $A(f)$, $A^\dagger(f)$ and $\Lambda(F)$ respectively are defined on the exponential vectors of Γ by

$$A(f)\psi(g) =< f, g > \psi(g) \tag{3.2}$$

$$A^\dagger(f)\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\psi(g + \epsilon f) \tag{3.3}$$

$$\Lambda(F)\psi(g) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\psi(e^{\epsilon F} g) \tag{3.4}$$

where F must be such that the exponential $e^{\epsilon F}$ is defined. In what follows F will be unbounded of the form $\rho^+(B^{+^n} M^k B^{-^l})$ and the exponential will be defined, for ϵ sufficiently close to zero, by writing such an operator as the sum of its real and imaginary part and defining the exponential of each one with the use of the spectral resolution theorem for functions of self-adjoint operators.

By (3.2)-(3.4)

$$A(f)^* = A^\dagger(f),\ A^\dagger(f)^* = A(f),\ \Lambda(F)^* = \Lambda(F^*). \tag{3.5}$$

It was shown in [AcFrSk00] that the quantum stochastic differentials of B^+, M, and B^- are connected with those of A, $A^\dagger$, and Λ by

$$dM_t = d\Lambda_t(\rho^+(M)) + dt \tag{3.6}$$

$$dB_t^+ = d\Lambda_t(\rho^+(B^+)) + dA_t^\dagger(e_0) \tag{3.7}$$

$$dB_t^- = d\Lambda_t(\rho^+(B^-)) + dA_t(e_0) \tag{3.8}$$

where on the right hand side of the above we have used the notation

$$dX_t(y) = X(\chi_{[t,t+dt]}y). \tag{3.9}$$

The Itô multiplication table for $dA_t^\dagger$, $d\Lambda_t$, and dA_t is

$\cdot$	$dA_t^\dagger(f_1)$	$d\Lambda_t(F_1)$	$dA_t(f_1)$	dt
$dA_t^\dagger(f_2)$	0	0	0	0
$d\Lambda_t(F_2)$	$dA_t^\dagger(F_2 f_1)$	$d\Lambda_t(F_2 F_1)$	0	0
$dA_t(f_2)$	$< f_2, f_1 > dt$	$dA_t(F_1^* f_2)$	0	0
dt	0	0	0	0

The Itô multiplication table for dB_t^+, dM_t, and dB_t^- is not closed. In fact it is clear that the basic differentials are $d\Lambda_{n,k,l}(t)$, $dA_m(t)$ and $dA_m^\dagger(t)$ defined in Definition 2 below (and of course dt).

Definition 2. *For $n, k, l, m \in \{0, 1, ...\}$*

$$d\Lambda_{n,k,l}(t) = d\Lambda_t(\rho^+(B^{+^n} M^k B^{-^l})) \tag{3.10}$$

$$dA_m(t) = dA_t(e_m) \tag{3.11}$$

$$dA_m^\dagger(t) = dA_t^\dagger(e_m) \tag{3.12}$$

The following proposition was proved in [AcBou01e]:

Proposition 1. *For $\alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, ...\}$*

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, d\Lambda_{a,b,c}(t) = \sum c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}\, d\Lambda_{a+\alpha-\gamma+\lambda,\omega+\sigma+\epsilon,\lambda+c}(t) \tag{3.13}$$

$$d\Lambda_{\alpha,\beta,\gamma}(t)\, dA_n^\dagger(t) = \theta_{\alpha,\beta,\gamma,n}\, dA_{\alpha+n-\gamma}^\dagger(t) \tag{3.14}$$

$$dA_m(t)\, d\Lambda_{a,b,c}(t) = \theta_{c,b,a,m}\, dA_{c+m-a}(t) \tag{3.15}$$

$$dA_m(t)\, dA_n^\dagger(t) = \delta_{m,n}\, dt \tag{3.16}$$

where $\theta_{\alpha,\beta,\gamma,n}$, $\theta_{c,b,a,m}$ and $c_{\beta,\gamma,a,b}^{\lambda,\rho,\sigma,\omega,\epsilon}$ are as in Lemma 1 and $\sum$ denotes the finite sum

$$\sum_{\lambda=0}^{\gamma}\sum_{\rho=0}^{\gamma-\lambda}\sum_{\sigma=0}^{\gamma-\lambda-\rho}\sum_{\omega=0}^{\beta}\sum_{\epsilon=0}^{b}$$

All other products are equal to zero.

Proposition 2. *The stochastic differentials of Definition 2 are linearly independent in the sense that the weak equality*

$$\sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\, d\Lambda_{\alpha,\beta,\gamma}(t) + \sum_m D_{-,t,m}\, dA_m(t) + \sum_m D_{+,t,m}\, dA_m^\dagger(t) + D_0(t)\, dt = 0 \tag{3.17}$$

implies that the coefficient processes vanish on the exponential domain of $\mathcal{H} \otimes \Gamma$ i.e on the linear span of $\{u \otimes \psi(f) : u \in \mathcal{H}, f \in \mathcal{K}\}$. Here $D_0 = \{D_0(t)\}_{t\geq 0}$, $D_1 = \{D_{1,t,\alpha,\beta,\gamma} : \alpha, \beta, \gamma = 0, 1, ...\}_{t\geq 0}$, $D_- =$

$\{D_{-,t,m} : m = 0,1,...\}_{t\geq 0}$ and $D_+ = \{D_{+,t,m} : m = 0,1,...\}_{t\geq 0}$ are bounded operator processes acting on the system Hilbert space $\mathcal{H}$.

Proof. Let $\Omega = \psi(0)$ be the normalized ground state exponential vector which is sent to zero by the A and Λ noise processes. Applying the left hand side of (3.17) to $u \otimes \Omega$ and taking the inner product with $v \otimes \Omega$ where $u, v \in \mathcal{H}$ we get

$$(3.18) \quad < D_0(t)u \otimes \Omega, v \otimes \Omega >= 0 \Rightarrow < D_0(t)u, v >= 0 \Rightarrow D_0 = 0$$

by the arbitrariness of t, u and v. Eliminating the dt term from (3.17) and applying the new (3.17) to $u \otimes \Omega$ we obtain

$$\begin{gathered}\sum_m D_{+,t,m}\, dA_m^\dagger(t)u \otimes \Omega = 0 \Rightarrow \sum_m D_{+,t,m}u \otimes \chi_{[t,t+dt]}e_m = 0 \\ \Rightarrow < D_{+,t,n}u \otimes \chi_{[t,t+dt]}e_n, \sum_m D_{+,t,m}u \otimes \chi_{[t,t+dt]}e_m >= 0 \text{ (for all } n) \\ \Rightarrow ||D_{+,t,n}u \otimes \chi_{[t,t+dt]}e_n||^2 = 0 \text{ (since } < e_n, e_m >= \delta_{n,m}) \\ \Rightarrow ||D_{+,t,n}u||^2\, dt = 0 \Rightarrow ||D_{+,t,n}u||^2 = 0 \Rightarrow D_+ = 0\end{gathered}$$

by the arbitrariness of t, u and n. Equation (3.17) now becomes

$$(3.19) \qquad \sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\, d\Lambda_{\alpha,\beta,\gamma}(t) + \sum_m D_{-,t,m}\, dA_m(t) = 0.$$

Taking the adjoint of (3.19) we have

$$(3.20) \qquad \sum_{\alpha,\beta,\gamma} D^*_{1,t,\alpha,\beta,\gamma}\, d\Lambda_{\gamma,\beta,\alpha}(t) + \sum_m D^*_{-,t,m}\, dA_m^\dagger(t) = 0$$

and by repeating the above argument we find that $D_- = 0$ and so we are reduced to

$$\begin{gathered}\sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\, d\Lambda_{\alpha,\beta,\gamma}(t) = 0 \\ \Rightarrow \sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\, d\Lambda_{\alpha,\beta,\gamma}(t)\, dA_n^\dagger(t) = 0 \text{ for all } n \\ \Rightarrow \sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\theta_{\alpha,\beta,\gamma,n}\, dA^\dagger_{\alpha+n-\gamma}(t) = 0 \text{ by Proposition 1} \\ \Rightarrow \sum_{\alpha,\beta,\gamma} D_{1,t,\alpha,\beta,\gamma}\theta_{\alpha,\beta,\gamma,m+\gamma-\alpha}\, dA_m^\dagger(t) = 0 \text{ where } m = \alpha + n - \gamma \\ \Rightarrow D_{1,t,\alpha,\beta,\gamma}\theta_{\alpha,\beta,\gamma,m+\gamma-\alpha} = 0 \text{ for all } \alpha, \beta, \gamma, m, t \text{ as shown before} \\ \Rightarrow D_{1,t,\alpha,\beta,\gamma} = 0 \Rightarrow D_1 = 0.\end{gathered}$$

□

Definition 3. *Let T be a set and let $\{dM_\alpha(t) : \alpha \in T\}$ be a self-adjoint family of stochastic differentials. We say that this family has a closed,* *constant, finite Itô table if, for any* $\alpha, \beta \in T$ *there exists a family of complex numbers* $\{C^{\gamma}_{\alpha,\beta} : \gamma \in T\}$ *such that*

$$dM_{\alpha}(t)\, dM_{\beta}(t) = \sum_{\gamma} C^{\gamma}_{\alpha,\beta}\, dM_{\gamma}(t) \tag{3.21}$$

where all but finitely many of the $C^{\gamma}_{\alpha,\beta}$ *are zero.*

The SWN stochastic differentials of (3.10)-(3.12) are a special case of the following:

Definition 4. *For* $x \in \{B^{+\alpha} M^{\beta} B^{-\gamma} : \alpha, \beta, \gamma = 0, 1, ...\}$ *and* $\lambda, \mu \in \mathbb{R}$

$$dM_t(x, \lambda) = d\Lambda_t(\rho^+(x)) + \lambda\, dt \tag{3.22}$$

$$dB^+_t(x, \mu, e_m) = d\Lambda_t(\rho^+(x)) + \mu\, dA^{\dagger}_t(e_m) \tag{3.23}$$

$$dB^-_t(x, \mu, e_m) = d\Lambda_t(\rho^+(x^*)) + \mu\, dA_t(e_m) \tag{3.24}$$

where $(B^{+\alpha} M^{\beta} B^{-\gamma})^* = B^{+\gamma} M^{\beta} B^{-\alpha}$.

Proposition 3. *In the notation of Definition 4*

$$dB^+_t(x_1, \mu_1, e_m)\, dB^+_t(x_2, \mu_2, e_n) = dB^+_t(x_1 x_2, \Theta(x_1, n)\mu_2, e(x_1, n)) \tag{3.25}$$

$$dB^+_t(x_1, \mu, e_m)\, dM_t(x_2, \lambda) = dM_t(x_1 x_2, 0) \tag{3.26}$$

$$dB^+_t(x_1, \mu_1, e_m)\, dB^-_t(x_2, \mu_2, e_n) = dM_t(x_1 x_2^*, 0) \tag{3.27}$$

$$dM_t(x_1, \lambda)\, dB^+_t(x_2, \mu, e_n) = dB^+_t(x_1 x_2, \Theta(x_1, n)\mu, e(x_1, n) \tag{3.28}$$

$$dM_t(x_1, \lambda_1)\, dM_t(x_2, \lambda_2) = dM_t(x_1 x_2, 0) \tag{3.29}$$

$$dM_t(x_1, \lambda_1)\, dB^-_t(x_2, \mu, e_n) = dM_t(x_1 x_2^*, 0) \tag{3.30}$$

$$dB^-_t(x_1, \mu_1, e_m)\, dB^+_t(x_2, \mu_2, e_n) = dB^+_t(x_1^* x_2, \Theta(x_1^*, n)\mu_2, e(x_1^*, n)) + \tag{3.31}$$

$$dB^-_t(x_2^*, \Theta(x_2^*, n)\mu_1, e(x_2^*, m)) - dM_t(x_2, -\mu_1 \mu_2 \delta_{n,m}) \tag{3.32}$$

$$dB^-_t(x_1, \mu_1, e_m)\, dM_t(x_2, \lambda) = dB^-_t(x_2^* x_1, \Theta(x_2^*, m)\mu_1, e(x_2^*, m)) \tag{3.33}$$

$$dB^-_t(x_1, \mu_1, e_m)\, dB^-_t(x_2, \mu_2, e_n) = dB^-_t(x_2 x_1, \Theta(x_2, m)\mu_1, e(x_2, m)) \tag{3.34}$$

where for $x = B^{+\alpha} M^{\beta} B^{-\gamma}$ *we have used the notation* $\Theta(x, n) = \theta_{\alpha,\beta,\gamma,n}$ *and* $e(x, n) = e_{\alpha+n-\gamma}$, *with* $\theta_{\alpha,\beta,\gamma,n}$ *as in (2.7). Multiplication by* dt *gives zero.*

Proof. We will only prove equation (3.31). The proof of the remaining equations is similar. By Definition 4 and Proposition 1 we have

$$
\begin{gathered}
dB^-_t(x_1,\mu_1,e_m)\,dB^+_t(x_2,\mu_2,e_n) = \\
(d\Lambda_t(\rho^+(x_1^*)) + \mu_1\,dA_t(e_m))(d\Lambda_t(\rho^+(x_2)) + \mu_2\,dA^\dagger_t(e_n)) = \\
d\Lambda_t(\rho^+(x_1^*x_2)) + \mu_2\,d\Lambda_t(\rho^+(x_1^*))\,dA^\dagger_t(e_n) + \\
\mu_1\,dA_t(e_m)\,d\Lambda_t(\rho^+(x_2)) + \mu_1\mu_2\,dA_t(e_m)\,dA^\dagger_t(e_n) = \\
d\Lambda_t(\rho^+(x_1^*x_2)) + \mu_2\,dA^\dagger_t(\rho^+(x_1^*)e_n) + \\
\mu_1\,dA_t(\rho^+(x_2^*)e_m) + \mu_1\mu_2\delta_{n,m}\,dt = \\
d\Lambda_t(\rho^+(x_1^*x_2)) + \mu_2\Theta(x_1^*,n)\,dA^\dagger_t(e(x_1^*,n)) + \\
\mu_1\Theta(x_2^*,m)\,dA_t(e(x_2^*,m)) + \mu_1\mu_2\delta_{n,m}\,dt = \\
dB^+_t(x_1^*x_2,\Theta(x_1^*,n)\mu_2,e(x_1^*,n)) + dB^-_t(x_2^*,\Theta(x_2^*,n)\mu_1,e(x_2^*,m)) - \\
d\Lambda_t(\rho^+(x_2)) + \mu_1\mu_2\delta_{n,m}\,dt = \\
dB^+_t(x_1^*x_2,\Theta(x_1^*,n)\mu_2,e(x_1^*,n)) + dB^-_t(x_2^*,\Theta(x_2^*,n)\mu_1,e(x_2^*,m)) - \\
dM_t(x_2,-\mu_1\mu_2\delta_{n,m})
\end{gathered}
$$

□

By Propositions 1 and 3 the families of the basic stochastic differentials defined in Definitions 2 and 4 are examples of families of differentials satisfying the condition of Definition 3.

4. Module form of the SWN Itô table

Definition 5. *Let $\mathcal{D}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H})$ denote, respectively, the spaces of adjointable operators on $\mathcal{K}$ and bounded linear operators on $\mathcal{H}$. The tensor product $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$ is an inner product module with $\mathcal{B}(\mathcal{H})$-valued inner product defined on elementary tensors by*

$$(a\otimes\xi|b\otimes\eta) = a^*b < \xi,\eta > \tag{4.1}$$

On $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$ we define linear operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ by

$$\mathcal{A}(a\otimes\xi) = a\otimes A(\xi) \tag{4.2}$$

$$\mathcal{A}^\dagger(a\otimes\xi) = a\otimes A^\dagger(\xi) \tag{4.3}$$

while on $\mathcal{B}(\mathcal{H})\otimes\mathcal{D}(\mathcal{K})$ we define a linear operator $\mathcal{L}$ by

$$\mathcal{L}(a\otimes T) = a\otimes\Lambda(T). \tag{4.4}$$

Lemma 2. *In the notation of Definition 5*

$$\mathcal{A}(a \otimes \xi)^* = \mathcal{A}^\dagger(a^* \otimes \xi) \tag{4.5}$$

$$\mathcal{A}^\dagger(a \otimes \xi)^* = \mathcal{A}(a^* \otimes \xi) \tag{4.6}$$

$$\mathcal{L}(a \otimes T)^* = \mathcal{L}(a^* \otimes T^*) \tag{4.7}$$

Proof. To prove (4.7) we notice that by (4.4) and (3.5)

$$\mathcal{L}(a \otimes T)^* = (a \otimes \Lambda(T))^* = a^* \otimes \Lambda(T^*) = \mathcal{L}(a^* \otimes T^*). \tag{4.8}$$

The proof of (4.5) and (4.6) is similar.

□

Definition 6. *For $\alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, ...\}$ let $\{D_{\alpha,\beta,\gamma}\}$, $\{E_{a,b,c}\}$ be families of operators in $\mathcal{B}(\mathcal{H})$ and let $D = \sum_{\alpha,\beta,\gamma} D_{\alpha,\beta,\gamma} \otimes \rho^+(B^{+^{\alpha}} M^{\beta} B^{-^{\gamma}})$ and $E = \sum_{a,b,c} E_{a,b,c} \otimes \rho^+(B^{+^{a}} M^{b} B^{-^{c}})$. We define the $\circ$-product $D \circ E$ of D and E by*

$$D \circ E = \sum_{\alpha,\beta,\gamma} \sum_{a,b,c} \sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b} D_{\alpha,\beta,\gamma} E_{a,b,c} \otimes \rho^+(B^{+^{a+\alpha-\gamma+\lambda}} M^{\omega+\sigma+\epsilon} B^{-^{\lambda+c}}) \tag{4.9}$$

where $\sum$ and $c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b}$ are as in Proposition 1 and Lemma 1 respectively. We also define linear operators r and l on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{D}(\mathcal{K})$ with values in the space of linear operators on $\mathcal{B}(\mathcal{H}) \otimes \mathcal{K}$ by

$$r(D)T = \sum_{n,\alpha,\beta,\gamma} D_{\alpha,\beta,\gamma} \theta_{\alpha,\beta,\gamma,n-\alpha+\gamma} T_{n-\alpha+\gamma} \otimes e_n \tag{4.10}$$

$$l(D)T = \sum_{n,\alpha,\beta,\gamma} T_{n+\alpha-\gamma} \theta_{\gamma,\beta,\alpha,n+\alpha-\gamma} D_{\alpha,\beta,\gamma} \otimes e_n \tag{4.11}$$

where $T = \sum_n T_n \otimes e_n \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{K}$, $n \in \{0, 1, ...\}$ and θ is as in Lemma 1.

The operator $I = \sum_{a,b,c} I_{a,b,c} \otimes \rho^+(B^{+^{a}} M^{b} B^{-^{c}})$, where $I_{a,b,c} = \delta_{a,0}\delta_{b,0}\delta_{c,0}1$, is the $\circ$-product identity i.e $I \circ D = D \circ I = D$ for all operators D as in Definition 6. It is also a $\circ$-product unitary operator i.e $I \circ I^* = I^* \circ I = I$. Here 1 denotes the identity operator in $\mathcal{B}(\mathcal{H})$. The operator $W = W_0 \otimes \rho^+(B^{+^{0}} M^{0} B^{-^{0}}) = W_0 \otimes \rho^+(id)$, where W_0 is a unitary system space operator and $\rho^+(id)\, e_m = e_m$, is also a $\circ$-product unitary operator.

Definition 7. *For $X_t, Y_t \in \{A_t, A_t^\dagger, \Lambda_t\}$ and $a, b \in \mathcal{B}(\mathcal{H})$ we define*

$$d(a \otimes X_t) = a \otimes dX_t \tag{4.12}$$

$$(a \otimes dX_t)(b \otimes dY_t) = ab \otimes dX_t\, dY_t \tag{4.13}$$

where $dX_t\, dY_t$ is computed with the use of the Itô table for dA_t, $dA_t^\dagger$, and $d\Lambda_t$.

Proposition 4. *Let*

$$D_+ = \sum_n D_{+,n} \otimes e_n \tag{4.14}$$

$$D_- = \sum_m D_{-,m} \otimes e_m \tag{4.15}$$

$$D_1 = \sum_{\alpha,\beta,\gamma} D_{1,\alpha,\beta,\gamma} \otimes \rho^+(B^{+^\alpha} M^\beta B^{-^\gamma}) \tag{4.16}$$

$$E_1 = \sum_{a,b,c} E_{1,a,b,c} \otimes \rho^+(B^{+^a} M^b B^{-^c}) \tag{4.17}$$

where $n, m, \alpha, \beta, \gamma, a, b, c \in \{0, 1, 2, ...\}$ and $D_{+,n}, D_{-,m}, D_{1,\alpha,\beta,\gamma}, E_{1,a,b,c} \in \mathcal{B}(\mathcal{H})$. Then in the notation of (3.9) and Definitions 5 and 6

$$d\mathcal{A}_t(D_-)\, d\mathcal{A}_t^\dagger(D_+) = (D_-^* | D_+)\, dt \tag{4.18}$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{L}_t(E_1) = d\mathcal{L}_t(D_1 \circ E_1) \tag{4.19}$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{A}_t^\dagger(D_+) = d\mathcal{A}_t^\dagger(r(D_1)D_+) \tag{4.20}$$

$$d\mathcal{A}_t(D_-)\, d\mathcal{L}_t(E_1) = d\mathcal{A}_t(l(E_1)D_-) \tag{4.21}$$

All other products of stochastic differentials (including dt) are equal to zero.

Proof. By Proposition 1 and Definitions 5, 6, and 7

$$d\mathcal{A}_t(D_-)\, d\mathcal{A}_t^\dagger(D_+) = \sum_{m,n} D_{-,m} D_{+,n} \otimes dA_t(e_m) dA_t^\dagger(e_n)$$
$$= \sum_{m,n} D_{-,m} D_{+,n} \otimes \delta_{m,n} dt = \sum_n D_{-,n} D_{+,n} \otimes 1 dt = (D_-^* | D_+)\, dt,$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{L}_t(E_1) =$$
$$\sum_{\alpha,\beta,\gamma,a,b,c} \sum D_{1,\alpha,\beta,\gamma} E_{1,a,b,c} \otimes d\Lambda_t(\rho^+(B^{+^\alpha} M^\beta B^{-^\gamma})) d\Lambda_t(\rho^+(B^{+^a} M^b B^{-^c})) =$$
$$\sum_{\alpha,\beta,\gamma,a,b,c} \sum c^{\lambda,\rho,\sigma,\omega,\epsilon}_{\beta,\gamma,a,b} D_{1,\alpha,\beta,\gamma} E_{1,a,b,c} \otimes d\Lambda_t(\rho^+(B^{+^{a+\alpha-\gamma+\lambda}} M^{\omega+\sigma+\epsilon} B^{-^{\lambda+c}}))$$
$$= d\mathcal{L}_t(D_1 \circ E_1)$$

$$d\mathcal{L}_t(D_1)\, d\mathcal{A}_t^\dagger(D_+) =$$
$$\sum_{\alpha,\beta,\gamma,n} D_{1,\alpha,\beta,\gamma} D_{+,n} \otimes d\Lambda_t(\rho^+(B^{+^\alpha} M^\beta B^{-^\gamma})) dA_t^\dagger(e_n) =$$
$$\sum_{\alpha,\beta,\gamma,n} D_{1,\alpha,\beta,\gamma} D_{+,n} \theta_{\alpha,\beta,\gamma,n} \otimes dA_t^\dagger(e_{\alpha+n-\gamma}) =$$
$$\sum_{\alpha,\beta,\gamma,n} D_{1,\alpha,\beta,\gamma} D_{+,n-\alpha+\gamma} \theta_{\alpha,\beta,\gamma,n-\alpha+\gamma} \otimes dA_t^\dagger(e_n)$$
$$= d\mathcal{A}_t^\dagger(r(D_1)D_+)$$

$$
\begin{gathered}
d\mathcal{A}_t(D_-)\, d\mathcal{L}_t(D_1) = \\
\sum_{a,b,c,m} D_{-,m} E_{1,a,b,c} \otimes dA_t(e_m) d\Lambda_t(\rho^+(B^{+^a} M^b B^{-c})) = \\
\sum_{a,b,c,m} D_{-,m} E_{1,a,b,c} \theta_{c,b,a,m} \otimes dA_t(e_{c+m-a}) = \\
\sum_{a,b,c,m} D_{-,a+m-c} E_{1,a,b,c} \theta_{c,b,a,a+m-c} \otimes dA_t(e_m) = \\
d\mathcal{A}_t(l(E_1)D_-)
\end{gathered}
$$

By the Itô table for dA, $dA^\dagger$, and $d\Lambda$ all other products are equal to zero.

□

Proposition 5. *The "module" stochastic differentials dt, $d\mathcal{A}_t$, $d\mathcal{L}_t$ and $d\mathcal{A}_t^\dagger$ are linearly independent, in the sense that*

$$D_0\, dt + d\mathcal{A}_t^\dagger(D_+) + d\mathcal{L}_t(D_1) + d\mathcal{A}_t(D_-) = 0 \tag{4.22}$$

implies that

$$D_0 = D_+ = D_1 = D_- = 0 \tag{4.23}$$

where D_0 is a bounded system operator and D_+, D_1, D_- are as in (4.14)-(4.16),

Proof. By (4.14)-(4.16) and (4.12), (4.22) implies

$$D_0\, dt + \sum_n D_{+,n}\, dA_n^\dagger(t) + \sum_{a,b,c} D_{1,a,b,c}\, d\Lambda_{a,b,c}(t) + \sum_n D_{-,m}\, dA_m(t) = 0 \tag{4.24}$$

which, by Proposition 2, implies

$$D_0 = D_{+,n} = D_{1,\alpha,\beta,\gamma} = D_{-,m} = 0 \tag{4.25}$$

for all $n, m, \alpha, \beta, \gamma$. Thus

$$D_0 = D_+ = D_1 = D_- = 0 \tag{4.26}$$

□

5. Module form of the SWN unitarity conditions

Let D_0, $D_{+,n}$, $D_{-,n}$, $D_{1,a,b,c}$ be for each n, a, b, c bounded operators on the system space $\mathcal{H}$ identified with their ampliation to the tensor product of $\mathcal{H}$ and the noise Fock space Γ. In view of Definitions 4 and 5 and Lemma 2, the quantum stochastic differential equation

(5.1) $dU_t = (D_0\, dt + \sum_n D_{+,n}\, dA_n^{\dagger}(t) + \sum_{a,b,c} D_{1,a,b,c}\, d\Lambda_{a,b,c}(t) +$
$\sum_n D_{-,n}\, dA_n(t))U_t,\ U_0 = 1$

with adjoint

(5.2) $dU_t^* = U_t^*(D_0^*\, dt + \sum_n D_{+,n}^*\, dA_n(t) + \sum_{a,b,c} D_{1,c,b,a}^*\, d\Lambda_{a,b,c}(t) +$
(5.3) $\sum_n D_{-,n}^*\, dA_n^{\dagger}(t)),\ U_0^* = 1$

can be written in module form as

(5.4) $dU_t = (D_0\, dt + d\mathcal{A}_t^{\dagger}(D_+) + d\mathcal{L}_t(D_1) + d\mathcal{A}_t(D_-))U_t,\ U_0 = 1$

with adjoint

(5.5) $dU_t^* = U_t^*(D_0^*\, dt + d\mathcal{A}_t(D_+^*) + d\mathcal{L}_t(D_1^*) + d\mathcal{A}_t^{\dagger}(D_-^*)),\ U_0^* = 1$

where D_+, D_1, and D_- are as in Proposition 4. Under suitable summability assumptions on its coefficients (details will appear elsewhere) equation (5.1) admits a unique solution.

Theorem 1. *The solution of the module quantum stochastic differential equation (5.3) is unitary if and only if*

(5.6) $$\Re D_0 = -\tfrac{1}{2}\,(D_-^*|D_-^*)$$

(5.7) $$D_+ = -r(W)D_-^*$$

(5.8) $$D_1 = W - I$$

where 1 *is the identity operator in* $\mathcal{B}(\mathcal{H}) \otimes \mathcal{K}$*,* I *is the* $\circ$*-product identity,* W *is a* $\circ$*-product unitary operator such that* $r(W)r(W^*) = r(W^*)r(W) = 1$ *(then also* $l(W^*)l(W) = l(W)l(W^*) = 1$*),* D_- *is arbitrary and* $\Re D_0$ *denotes the real part of* D_0 *. In this case equation (5.3) takes the form*

(5.9) $$dU_t = ((-\tfrac{1}{2}\,(D_-^*|D_-^*) + iH)\, dt + d\mathcal{A}_t(D_-) + d\mathcal{A}_t^{\dagger}(-r(W))D_-^*) + d\mathcal{L}_t(W - I))U_t$$

where H *is any self-adjoint operator.*

Proof. By the definition of unitarity

(5.10) $$U(t)\, U^*(t) = U^*(t)\, U(t) = 1\,,\ U(0) = U^*(0) = 1$$

which is equivalent to

$$(5.11)\quad d\,(U(t)U^*(t)) = dU(t)\,U^*(t) + U(t)\,dU^*(t) + dU(t)\,dU^*(t) = 0$$

and

$$(5.12)\quad d(U^*(t)U(t)) = dU^*(t)\,U(t) + U^*(t)\,dU(t) + dU^*(t)\,dU(t) = 0.$$

By Proposition 4, (5.3) and (5.4), (5.11) is equivalent to

$$(5.13)\quad (D_0 + D_0^* + (D_+|D_+))\,dt + d\mathcal{A}_t^\dagger(D_-^* + D_+ + r(D_1^*)D_+) + d\mathcal{L}_t(D_1 + D_1^* + D_1^* \circ D_1) + d\mathcal{A}_t(D_+^* + D_- + l(D_1)D_+^*) = 0$$

while (5.10) is equivalent to

$$(5.14)\quad (D_0 + D_0^* + (D_-^*|D_-^*))\,dt + d\mathcal{A}_t^\dagger(D_+ + D_-^* + r(D_1)D_-^*) + d\mathcal{L}_t(D_1 + D_1^* + D_1 \circ D_1^*) + d\mathcal{A}_t(D_+^* + D_- + l(D_1^*)D_-) = 0.$$

By Proposition 5, (5.12) and (5.13) are equivalent to

$$(5.15)\quad D_0 + D_0^* + (D_+|D_+) = 0$$
$$(5.16)\quad D_-^* + D_+ + r(D_1^*)D_+ = 0$$
$$(5.17)\quad D_1 + D_1^* + D_1^* \circ D_1 = 0$$
$$(5.18)\quad D_+^* + D_- + l(D_1)D_+^* = 0$$
$$(5.19)\quad D_0 + D_0^* + (D_-^*|D_-^*) = 0$$
$$(5.20)\quad D_+ + D_-^* + r(D_1)D_-^* = 0$$
$$(5.21)\quad D_1 + D_1^* + D_1 \circ D_1^* = 0$$
$$(5.22)\quad D_+^* + D_- + l(D_1^*)D_- = 0$$

For D_-, D_+, D_1 as in (5.5)-(5.7) the validity of (5.18) is obvious while (5.14) follows from (5.18) and the fact that $(D_+|D_+) = (D_-^*|D_-^*)$. Conditions (5.20) and (5.16) can be written respectively as $(I + D_1) \circ (I + D_1)^* = I$ and $(I + D_1)^* \circ (I + D_1) = I$ both of which are true by the $\circ$-unitarity of W. Condition (5.19) is straightforward from (5.6) while (5.15) can be written as $D_-^* = -r(W^*)D_+$ which is the same as (5.6) since $r(W)r(W^*)D_+ = D_+$. Finally (5.17) is the adjoin t of (5.15) and (5.21) is the adjoint of (5.19) since, by Definition 6, $l(D_1^*)D_- = (r(D_1)D_-^*)^*$ and $l(D_1)D_+^* = (r(D_1^*)D_+)^*$.

□

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