A stochastic limit approach to the fractional quantum Hall effect

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Abstract. We study three diffent approximations of the usual many–body model for the fractional quantum Hall effect (FQHE) consisting in a two–dimensional electron gas (2DEG) in a uniform electric and magnetic field.

For each of them we consider the stochastic limit and we find the quantum Langevin equation and the generators of the master equation. This allows us to calculate the explicit form of the conductivity and the resistivity tensors.

The relevance of the levels beyond the lowest Landau's one is explained, and a *fine tuning* relation between the electric and the magnetic fields is obtained from the stochastic resonance principle. It is shown that this relation could be at the origin of the quantization of the resistivity tensor.

1 Introduction: the single electron problem

In the past years a big effort has been made by many physicists in order to propose a definitive model for the FQHE. However, despite many proposal, no definitive model for the FQHE is available in the literature up to this moment. In 1983 Laughlin, [Lau1] proposed a trial wave function, ψ_L, which describes an incompressible fluid and minimizes, in a certain electron density regime, the energy of the system.

In these notes we propose a model of $N < \infty$ charged interacting particles belonging to a two dimensional layer contained in the (x, y)-plane and subjected to a uniform electric field along y, $\underline{E} = E\hat{j}$, and to an uniform magnetic field $\underline{B}$ along z, $\underline{B} = B\hat{k}$.

The hamiltonian of the system can be written as

$$H^{(N)} = H_0^{(N)} + \lambda(H_c^{(N)} + H_B^{(N)}) \tag{1}$$

where $H_0^{(N)}$ is the sum of N contributions:

$$H_0^{(N)} = \sum_{i=1}^{N} H_0(i) \tag{2}$$

where $H_0(i)$ describes the minimal coupling of the electrons with the field:

$$H_0 = \frac{1}{2m}\left(\underline{p} + \frac{e}{c}\underline{A}(r)\right)^2 + e\underline{E}\cdot\underline{r} \tag{3}$$

while $H_c^{(N)}$ is the canonical Coulomb interaction between charged particles:

$$H_c^{(N)} = \frac{1}{2}\sum_{i\neq j}^{N}\frac{e^2}{|\underline{r}_i - \underline{r}_j|} \tag{4}$$

$H_B^{(N)}$ is the interaction of the charges with the background, which will be considered in detail u the following.

The strategy we will use can be summarized in the following steps:

1) we first solve the eigenvalue equation for the single charge hamiltonian (**??**):

$$H_0\psi_{np}(\underline{r}) = \mathcal{E}_{np}\psi_{np}(\underline{r}) \qquad , \qquad n \in \mathbf{N},\ p \in \mathbf{Z} \tag{5}$$

where the double index is due to the fact that, as we will see, two quantum numbers are necessary to fix the eigenstate;

2) secondly, we use the set $\{\psi_{np}\}$ to rewrite $H^{(N)}$ in second quantization, introducing fermion creation and annihilation operators;

3) we propose different approximations of the original hamiltonian, obtaining different models. We also introduce the background hamiltonian as a possible candidate for the interaction;

4) we compute the stochastic limit of the various models

5) from the stochastic limit we deduce the related Langevin and master equations

6) we use these equations to deduce an expression of the electron current and, as a consequence, of the conductivity and resistivity tensors.

As we will see in some details, this last point will suggest that the hypothesis which makes of the lowest Landau level (LLL) the only relevant energy level, is too restrictive so that also the contributions of the levels mixing *must* be considered. Also, we will show that in order to get an electron current orthogonal to both the electric and the magnetic fields $\vec{E}$ and $\vec{B}$, we must tune these fields in an appropriate way.

In order to solve equation (**??**), we fix the so-called Landau gauge:

$$\underline{A} = -B(y, 0, 0) \tag{6}$$

In this gauge the hamiltonian becomes

$$H_0 = \frac{1}{2m}\left[\left(p_x - \frac{eB}{c}\,y\right)^2 + p_y^2\right] + eEy$$

which, commutes with p_x, the momentum operator along x,

$$[p_x, H_0] = 0$$

This suggests to look for eigenstates of H_0 which are also eigenstates of p_x:

$$H_0\psi(\underline{r}) = \varepsilon\psi(\underline{r}) \tag{7}$$

$$\psi(\underline{r}) = Ce^{ikx}\varphi(y) \tag{8}$$

where C is a normalization constant, which will be fixed by the geometry of the system.

Using this factorization, equation (7) can be rewritten as

$$\left(\frac{1}{2m}p_y^2 + \frac{1}{2}m\omega^2(y-y_0)^2\right)\varphi(y) = \varepsilon'\varphi(y) \tag{9}$$

where we have defined the following quantities:

$$\omega = \frac{eB}{mc} \tag{10}$$

$$\varepsilon' = \varepsilon - \frac{\hbar^2 k^2}{2m} + \frac{1}{2}m\omega^2 y_0^2 \tag{11}$$

$$y_0 = \frac{1}{m\omega^2}\left(\hbar k\omega - eE\right) \tag{12}$$

It is clear now that $\varphi(y)$ and ε' are respectively an eigenstate of the harmonic oscillator and its corresponding eigenvalue. If we fix the geometry of the system by requiring periodic boundary condition on x, $\psi(-L_x/2, y) = \psi(L_x/2, y)$, for almost all y, we also conclude that the momentum k along x, cannot take arbitrary values but must be quantized. In particular, if the system is infinitely extended along y, then all the possible values of k are:

$$k = \frac{2\pi}{L_x}p\ , \qquad p \in \mathbf{Z} \tag{13}$$

Normalizing the wave functions (**??**) in the strip $[-L_x/2, L_x/2] \times \mathbf{R}$, we finally get:

$$\varepsilon_{np} = \hbar\omega(n+1/2) - \frac{eE}{2m\omega^2}\left(eE - \frac{4\hbar\omega\pi p}{L_x}\right) \tag{14}$$

$$\psi_{np}(\underline{r}) = \frac{e^{i\frac{2\pi p x}{L_x}}}{\sqrt{L_x}}\varphi_n(y - y_0^{(p)}) \tag{15}$$

where φ_n is just the n–th eigenstate of the one-dimensional harmonic oscillator, while y_0 is given in (**??**) with k fixed as in (**??**).

Equation (**??**) shows that the wave function $\psi_{np}(\underline{r})$ factorizes in a x–dependent part, which is labelled by the quantum number p, and a part, depending only on y, which is labelled by both n **and** p due to the presence of $y_0^{(p)}$ in the argument of the function φ_n.

It may be interesting to remark that when $E = 0$ the model collapses to the one of a simple harmonic oscillator, see [BMS] for instance and an infinite degeneracy in p of each Landau level (n fixed) appears.

2 Second quantized models

Following the canonical procedure we can write the N–electron hamiltonian

$$H := H^{(N)} = H_0^{(N)} + \lambda H_c^{(N)} + \lambda H_B^{(N)} \tag{16}$$

in the following way:

$$H = \sum_{\alpha} \varepsilon_\alpha a_\alpha^+ a_\alpha + \frac{\lambda e^2}{2} \sum_{\alpha\beta\alpha'\beta'} V_{\alpha\beta\alpha'\beta'} a_\alpha^+ a_\beta^+ a_{\alpha'} a_{\beta'} \tag{17}$$

where, if we neglect the interaction with the background, the $V_{\alpha\beta\alpha'\beta'}$ are scalar coefficients given by:

$$V_{\alpha\beta\alpha'\beta'} := \langle \psi_\alpha(\underline{r}_1)\psi_\beta(\underline{r}_2), \frac{1}{|\underline{r}_1 - \underline{r}_2|}\, \psi_{\alpha'}(\underline{r}_2)\psi_{\beta'}(\underline{r}_1)\rangle = \frac{1}{2\pi}\int \frac{d^2k}{|\underline{k}|}\, \hat{V}_{\alpha\beta'}(\underline{k})\hat{V}_{\beta\alpha'}(-\underline{k}), \tag{18}$$

$$\hat{V}_{\alpha\beta'}(\underline{k}) := \int \overline{\psi_\alpha(\underline{r})} e^{i\underline{k}\cdot\underline{r}} \psi_{\beta'}(\underline{r}) d^2r \tag{19}$$

In (17) a_α, a_α^+ are fermionic operators

$$\{a_\alpha, a_\beta\} = \{a_\alpha^+, a_\beta^+\} = 0 \qquad \{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta} \tag{20}$$

and we have also introduced the simplifing notation:

$$\alpha := (n_\alpha, p_\alpha) \tag{21}$$

It is clear that ε_α appearing in (**??**) is exactly the single electron energy related to (n_α, p_α) and is given in formula (**??**).

As we have already mentioned, in the hamiltonian (**??**) the background term is missing, the reason being that we will introduce it back in the form of a field interacting with the electrons.

First approximation

This is the result of the very crude approximation which consists in assuming:

- first, that the various $\hat{V}_{\alpha\beta}(k)$ in (**??**) take their maximum values in correspondence of a certain set of values of the momentum $\underline{k}$, $\{\underline{k}_l\}_{l\in I}$;

- secondly, that a cutoff $g(k)$ must be introduced in the $\underline{k}$ integral defining $V_{\alpha\beta\alpha'\beta'}$;

- finally we introduce the interaction with the background field $b(k)$ by postulating that the coefficients $V_{\alpha\beta\alpha'\beta}$ depend linearly on $b(k)$ (this is a *quasi–dipole approximation* in the sense of [AcLu]):

$$V_{\alpha\beta\alpha'\beta'} \simeq \frac{1}{2\pi} \sum_{l\in I} \hat{V}_{\alpha\beta'}(\underline{k}_l)\hat{V}_{\beta\alpha'}(-\underline{k}_l) \int \frac{d^2k}{|\underline{k}|} \simeq \sum_{l\in I} \hat{V}_{\alpha\beta'}(\underline{k}_l)\hat{V}_{\beta\alpha'}(-\underline{k}_l) \int d^2k g(\underline{k}) b^+(\underline{k})$$

In these hypotheses the surviving term in λH_B will be the free hamiltonian for the background operators $b(\underline{k})$ which we assume to be of the form

$$\lambda H_B = H_R := \int \omega(\underline{k}) b^+(\underline{k}) b(\underline{k}) d^2k \tag{22}$$

Defining

$$D_{\alpha\beta\alpha'\beta'} := \sum_{l\in I} \hat{V}_{\alpha\beta'}(\underline{k}_l)\hat{V}_{\beta\alpha'}(-\underline{k}_l), \tag{23}$$

the form of the complete hamiltonian will be

$$\begin{aligned} H &= \sum_{\alpha} \varepsilon_\alpha a_\alpha^+ a_\alpha + \int \omega(\underline{k}) b^+(\underline{k}) b(\underline{k}) d^2k + \frac{\lambda e^2}{2} \sum_{\alpha\beta\alpha'\beta'} (D_{\alpha\beta\alpha'\beta'} a_\alpha^+ a_\beta^+ a_{\alpha'} a_{\beta'} b^\dagger(g) + h.c.) \\ &= H_S + H_R + \lambda H_I \end{aligned} \tag{24}$$

The cutoff term $g(\underline{k})$ must satisfy the analiticity criterium discussed in [AcLuVo00]. A particular choice for $g(\underline{k})$, which looks physically reasonable, is a Yukawa–like function

$$g(\underline{k}) = \frac{e^{-\mu|\underline{k}|}}{|\underline{k}|}$$

which is natural to describe a screened potential.

Remark 1. The usual dipole approximation is recovered if, in (**??**), the only relevant contribution comes from $\underline{k}_l = 0$. In this case we have, see (**??**),

$$\hat{V}_{\alpha\beta'}(\underline{0}) = \langle \psi_\alpha, \psi_{\beta'} \rangle = \delta_{\alpha\beta'} \tag{25}$$

so that $D_{\alpha\beta\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\beta\alpha'}$. Therefore the interaction hamiltonian in (**??**) becomes

$$H_I = \frac{e^2}{2} \sum_{\alpha\beta} b^+(g) a^+_\alpha a^+_\beta a_\beta a_\alpha + \text{h.c.} \tag{26}$$

which commutes with H_S. For this reason the stochastic limit of this model would be trivial since no energy difference would appear in $H_I(t)$ (cf. Section (3) below). This will be made more explicit in the following, and explains why we must go beyond the dipole approximation.

Second approximation

In the Hamiltonian (**??**) all the Landau levels are involved in the summations. Following some of the old ideas on FQHE we try now to single out those levels which are likely to include the dominating contributions.

The simplest choice consists in restricting the summations in (**??**) only to $n_\alpha = n_\beta = n_{\alpha'} = n_{\beta'} = 0$, that is to consider only the effects of the lowest Landau level (LLL).

However, if we decide to stay in the LLL, something more detailed can be said. In particular the integrals defining $\hat{V}_{\alpha\beta}$ can be computed exactly, using standard gaussian integrals and, if $n_\gamma = n_\mu = 0$, we find

$$\hat{V}_{\gamma\mu}(\underline{k}) = \frac{(-1)^{p_\mu - p_\gamma} \sin(k_x L_x/2)}{\frac{k_x L_x}{2} + \pi(p_\mu - p_\gamma)} \cdot \exp\left\{ \frac{i}{2} k_y (y_0^{(p_\gamma)} + y_0^{(p_\mu)}) - \frac{m\omega}{h\hbar} (y_0^{(p_\gamma)} - y_0^{(p_\mu)})^2 - \frac{\hbar k_y^2}{4m\omega} \right\} \tag{27}$$

Notice that the first term is always defined and it is simply equal to 1 when $k_x L_x = 2\pi(p_\mu - p_\gamma)$. Although clearly over simplified formula, (**??**) can be used to deduce some useful of indications on the orders of magnitude of the terms involved. Due to the presence of $e^{-\frac{\hbar}{4m\omega} k_y^2}$ in (**??**) we can reasonably assume that the main contributions to $V_{\alpha\beta\alpha'\beta'}$ comes from $k_y \simeq 0$. Moreover, the other gaussian appearing in (**??**) achieves its maximum value for $p_\gamma = p_\mu$. In other words, for $p_\gamma = p_\mu$ we get:

$$\hat{V}_{\gamma\mu}(\underline{k}) \simeq \hat{V}_{\gamma\mu}(k_x, 0) \simeq \frac{\sin(k_x L_x/2)}{\frac{k_x L_x}{2}} \leq 1 \tag{28}$$

because for $|k_x|$ increasing $\hat{V}_{\gamma\mu}(\underline{k})$ certainly decreases. Therefore we conclude that, if $n_\gamma = n_\mu = 0$, then

$$\hat{V}_{\gamma\mu}(\underline{k}) \simeq \delta_{p_\mu p_\gamma}$$

However this approximation is too rough because it leads to the same result discussed in Remark 1 above, i.e. that the stochastic limit of the model is trivial. For this reason we have to look for a better approximation for $\hat{V}_{\gamma\mu}(\underline{k})$, also in view of its role in the Hamiltonian (**??**) below. To this goal we proceed in the following way:

- as before we take $\hat{V}_{\gamma\mu}(k) \simeq \hat{V}_{\gamma\mu}(k_x, 0)$;
- since

$$\exp\left\{-\frac{m\omega}{4\hbar}\,(y_0^{(p_\gamma)} - y_0^{(p_\mu)})^2\right\} = \exp\left\{-\frac{\pi^2 h}{m\omega L^2 x}\,(p_\gamma - p_\mu)^2\right\}$$

  we put

$$e^{-\frac{m\omega}{4\hbar}\,(y_0^{(p_\gamma)} - y_0^{(p_\mu)})^2} \simeq \delta_{p_\gamma p_\mu} + c_{-1}(\delta_{p_\gamma, p_\mu+1} + \delta_{p_\gamma, p_\mu-1}) \tag{29}$$

  where

$$c_{-1} = e^{-\frac{\pi^2\hbar}{m\omega L_x^2}} \tag{30}$$

  In this way equation (**??**) is refined in the following way

$$\hat{V}_{\gamma\mu}(\underline{k}) \simeq \frac{(-1)^{p_\mu - p_\gamma}\sin(k_x L_x/2)}{\frac{k_x L_x}{2} + \pi(p_\mu - p_\gamma)}\, e^{-\frac{\hbar}{4m\omega}k_y^2} \cdot (\delta_{p_\mu p_\gamma} + c_{-1}(\delta_{p_\gamma p_\mu+1} + \delta_{p_\gamma, p_\mu-1})) \tag{31}$$

It is clear now that what we are doing is nothing but developping a perturbative approach in the parameter c_{-1}, which is certainly small under some conditions on $\vec{B}$.

Now we can proceed as for the first approximation, introducing a cutoff $g(\underline{k})$ and a background field $b(\underline{k})$. After some easy computations we get

$$H = \sum_\alpha \varepsilon_\alpha a_\alpha^+ a_\alpha + \int \omega(\underline{k}) b^+(\underline{k}) b(\underline{k}) d^2k + \frac{\lambda e^2}{2} \sum_{\alpha\beta} \sum_{i=-1}^{1} (D_{\alpha\beta,i} b^+(g) + \text{h.c.}) \tag{32}$$

where the following electronic operators have been defined

$$D_{\alpha\beta,0} := a_\alpha^+ a_\beta^+ a_\beta a_\alpha$$

$$D_{\alpha\beta,\pm1} := 2c_{-1} a_k^+ a_\beta^+ a_\beta a_{\alpha\pm1} \tag{33}$$

Now it is evident that it is exactly the presence of the two contributions proportional to c_{-1} that makes the model non-trivial in the sense of the stochastic limit. We will return on this point in the next section.

Third approximation

This model originates from a natural consideration: if the role of the background is relevant, it is natural to use as an interaction the Fröhlich hamiltonian.

$$H_{eb} = \int \psi^\dagger(\underline{r})\psi(\underline{r})\tilde{F}(\underline{r}-\underline{r}')\phi(\underline{r}')d^2rd^2r' \tag{34}$$

where $\psi(\underline{r})$ and $\phi(\underline{r}')$ are respectively, the electron and the bosonic fields, while $\tilde{F}$ is a form factor. This is true also in view of the many important similarities between the FQHE and the superconductivity. Now, expanding $\phi(\underline{r})$ in plane waves and $\psi(\underline{r})$ in terms of the eigenstates $\psi_\alpha(\underline{r})$, see (**??**), introducing

$$g_{\alpha\beta}(\underline{k}) := \frac{1}{\sqrt{(2\pi)^3}}\frac{\hat{V}_{\alpha\beta}(\underline{k})}{\sqrt{2\omega(\underline{k})}} \tag{35}$$

and taking $\tilde{F}(\underline{r}) = \delta(\underline{r})$, as it is usually done in the literature [Strocchilibro], we can write

$$H_{eb} = \sum_{\alpha\beta} a_\alpha^+ a_\beta(b(g_{\alpha\beta}) + b^+(\overline{g_{\beta\alpha}})) \tag{36}$$

which is of a form very similar to (**??**) except that the quartic term in the fermions has been replaced by a quadratic one and the form factors $g_{\alpha\beta}$ depend on the level indices (α,β).

To get the complete hamiltonian we have to add to H_{eb} to $H_0 = H_{0,e} + H_{0,R}$, where

$$H_{0,e} = \sum_\alpha \varepsilon_\alpha a_\alpha^+ a_\alpha \tag{37}$$

and

$$H_{0,R} = \int \omega(\underline{k})b^+(\underline{k})b(\underline{k})d^2k \tag{38}$$

Therefore

$$H = H_{0,e} + H_{0,R} + \lambda e^2 H_{ep} \tag{39}$$

This last hamiltonian is simpler than all those proposed so far and it has the advantage that H_{eb} is a very well studied operator!

It is also interesting that this model, as the first one in (**??**), can be considered both in the LLL or in its complete generality. However, also in this case, we will show that if we project this model in the LLL, the results will not be physically reasonable while the situation will change drastically outside of the LLL!

3 The stochastic limit for the models and their related generators

In this section we will consider separately the stochastic limit for each of the approximations introduced so far.

First approximation

The starting point is the hamiltonian (**??**) together with the commutation relations:

$$[b(\underline{k}), b^+(\underline{k}')] = \delta(\underline{k} - \underline{k}') \quad [b(\underline{k}), b(\underline{k}')] = [b^+(\underline{k}), b^+(\underline{k}')] = 0$$

$$\{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta} \quad \{a_\alpha, a_\beta\} = \{a_\alpha^+, a_\beta^+\} = 0$$

$$[a_\alpha^\varepsilon \otimes 1\!\!1_R, 1\!\!1_e \otimes b^\varepsilon(\underline{k})] = 0 \tag{40}$$

$1\!\!1_R$ and $1\!\!1_e$ are respectively the identity operator in the algebras of the radiation field and of the electrons; a_α^ε is either a_α or a_α^+, similarly $b^\varepsilon(\underline{k})$ stands for $b(\underline{k})$ or $b^+(\underline{k})$. Calling now

$$H_I = \frac{e^2}{2} \sum_{\alpha\beta\alpha'\beta'} (D_{\alpha\beta\alpha'\beta'} b^+(g) a_\alpha^+ a_\beta^+ a_{\alpha'} a_{\beta'} + \text{h.c.}) \tag{41}$$

and

$$H_0 = H - \lambda H_I = H_{0,e} + H_{0,R} \tag{42}$$

we easily find that

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} = \frac{e^2}{2} \sum_{\alpha\beta\alpha'\beta'} (D_{\alpha\beta\alpha'\beta'} a_\alpha^+ a_\beta^+ a_{\alpha'} a_{\beta'} b^+(g e^{it(\omega + \varepsilon_{\alpha\beta\alpha'\beta'})}) + \text{h.c.}) \tag{43}$$

where

$$\varepsilon_{\alpha\beta\alpha'\beta'} = \varepsilon_\alpha + \varepsilon_\beta - \varepsilon_{\alpha'} - \varepsilon_{\beta'} \tag{44}$$

For notational simplicity, it is convenient to introduce a multi–index

$$A := (\alpha\beta\alpha'\beta') = ((n_\alpha, p_\alpha), (n_\beta, p_\beta), (n_{\alpha'}, p_{\alpha'}), (n_{\beta'}, p_{\beta'})) \tag{45}$$

and an electron operator

$$P_A = P_{\alpha\beta\alpha'\beta'} := a_\alpha^+ a_\beta^+ a_{\alpha'} a_{\beta'} \tag{46}$$

With these definitions we can write $H_I(t)$ as

$$H_I(t) = \frac{e^2}{2} \sum_A (D_A P_A b^\dagger (g e^{it(\omega+\epsilon_A)}) + \text{h.c.}) \tag{47}$$

The time evolution of the wave operator $U_t^{(\lambda)}$ satisfies the Schrödinger equation in interaction representation:

$$\partial_t U_t^{(\lambda)} = -i\lambda H_I(t) U_t^{(\lambda)} \tag{48}$$

Its time rescaled version ($t \to t/\lambda^2$) is

$$\partial_t U_{t/\lambda^2}^{(\lambda)} = -i/\lambda H_I(t/\lambda^2) U_{t/\lambda^2}^{(\lambda)} \tag{49}$$

whose integral form is

$$U_{t/\lambda^2}^{(\lambda)} = \mathbb{1} - \frac{i}{\lambda} \int_0^t H_I(t'/\lambda^2) U_{t'/\lambda^2}^{(\lambda)} dt' \tag{50}$$

We see that the rescaled Hamiltonian

$$\frac{1}{\lambda} H_I(t/\lambda^2) = \frac{e^2}{2} \sum_A D_A P_A \frac{1}{\lambda} b^+ \left(e \frac{it}{\lambda^2} (\omega + \varepsilon_A) g_A \right) + \text{ h.c.} \tag{51}$$

depends from the rescaled fields

$$b_{A,\lambda}(t) = \frac{1}{\lambda} b(e^{i \frac{t}{\lambda^2} (\omega+\varepsilon_A)} g_A) \tag{52}$$

and, according to the *stochastic golden rule* [AcLuVo00] these converge (in the sense of correlators) to a quantum white noise

$$b_A(t) = \lim_{\lambda \to 0} \frac{1}{\lambda} b(g e^{i \frac{t}{\lambda^2} (\omega+\varepsilon_A)}) \tag{53}$$

characterized by the following commutation relations

$$[b_A(t), b_B(t')] = [b_A^+(t), b_B^+(t')] = 0 \tag{54}$$

$$[b_A(t), b_B(t')] = \gamma^{(A)} \delta_{\varepsilon_A, \varepsilon_B} \delta(t - t') \tag{55}$$

where

$$\gamma^{(A)} = \int_{-\infty}^{+\infty} d\tau \int d^2k |g(\underline{k})|^2 e^{i\tau(\omega(\underline{k})+\varepsilon_A)} \tag{56}$$

which is finite due to the analiticity condition satisfied by $g(\underline{k})$.

Denoting η_0 the vacuum of the master fields $b_A(t)$ one has:

$$b_A(t)\eta_0 = 0 \quad \forall A, \ \forall t \tag{57}$$

The limit hamiltonian is then

$$H_I^{(ls)}(t) = \frac{e^2}{2} \sum_A (D_A P_A b_A^+(t) + \text{h.c.}) \tag{58}$$

In this sense we say that $H_I^{(ls)}(t)$ is the "stochastic limit" of $H_I(t)$ in (**??**).

The stochastic limit of the wave operator satisfies the equation of motion

$$\partial_t U_t = -i H_I^{(ls)}(t) U_t \tag{59}$$

or, in integral form,

$$U_t = 1\!\!1 - i \int_0^t H_I^{(ls)}(t') U_{t'} dt' \tag{60}$$

Finally, for any observable X of the system and $1\!\!1_R$ as above, we denote

$$\tilde{X} := X \otimes 1\!\!1_R$$

Thus the stochastic limit of the (Heisenberg) time evolution of $\tilde{X}$ (in the interaction picture) is:

$$j_t(\tilde{X}) = U_t^+ \tilde{X} U_t \tag{61}$$

Since the $b_A(t)$ are quantum white noises, equation (**??**), and the corresponding equation for $j_t(X)$, are singular equations and to give them a meaning we bring them in normal form.

The normally ordered form of the evolution equation for the stochastic limit (**??**) of the Heisenberg evolution of a system observable, is called *the quantum Langevin equation*. One can prove that, in our case, it has the form

$$\partial_t j_t(X) = \frac{e^2}{2} \sum_A \{\overline{D}_A j_t([P_A^+, \tilde{X}]\Gamma_-^{(A)}) - D_A j_t(\Gamma_-^{(A)\dagger}[P_A, \tilde{X}])\} +$$

$$+ \frac{ie^2}{2} \sum_A \{\overline{D}_A j_t([P_A^+, \tilde{X}]) b_A(t) + D_A b_A^+(t) j_t([P_A, \tilde{X}])\} \tag{62}$$

where

$$\Gamma_-^{(A)} := \frac{e^2}{2} \sum_B D_B P_B \gamma_-^{(B)} \delta_{\varepsilon_A, \varepsilon_B} \tag{63}$$

where the *transport coefficients* (or *generalized susceptivities*) $\gamma_-^{(B)}$ are given by the formula

$$\gamma_-^{(A)} := \int_{-\infty}^{0} d\tau \int d^2k |g(\underline{k})|^2 e^{i\tau(\omega(\underline{k})+\varepsilon_A)} \tag{64}$$

and have a fundamental physical meaning because their real parts define the *inverse lifetimes* (*line broadening*) and their imaginary part, the energy shift due to the interaction.

Moreover, comparing (**??**) and (**??**) and using the formula

$$\int_{-\infty}^{0} d\tau e^{i\tau\Omega} = \pi\delta(\Omega) - iP.V.\frac{1}{\Omega} \tag{65}$$

we see that

$$\text{Re }\gamma_-^{(A)} = \pi \int d^2k |g(k)|^2 \delta(\omega(k) + \varepsilon_A) = \gamma^{(A)}$$

which shows that the real part of the susceptivity $\gamma_-^{(A)}$ coincides with the covariance $\gamma^{(A)}$ of the master field $b_A(t)$. The same formula so shows that $\gamma^{(A)} = \text{Re }\gamma_-^{(A)}$ is concentrated on the surface $\omega(k) = -\varepsilon_A$, that, due to (**??**) and (**??**), is on a surface defined by a set of integer numbers. The special role played by integer numbers is also apparent if one remarks that, equation (**??**), the only non commuting operators are those for which $\varepsilon_A = \varepsilon_B$. This means that, since $A = (\alpha\beta\alpha'\beta')$ and $B = (\gamma\delta\gamma'\delta')$, only if

$$\begin{aligned} &n_\alpha + n_\beta + n_{\gamma'} + n_{\gamma'} - n_{\alpha'} - n_{\beta'} - n_\gamma - n_\delta = \\ &= \frac{eE2\pi}{m\omega^2 L_x} \left(p_\gamma + p_\delta + p_{\alpha'} + p_{\beta'} - p_\alpha - p_\beta - p_{\gamma'} - p_{\delta'}\right) \end{aligned} \tag{66}$$

then the corresponding $b_A(t)$ and $b_B^+(t)$ don't commute. This gives a constraint on the possible values of the indices and, as we will see, produces some limitations on the relevant values of E and B. Moreover, if we work in the LLL, then the *lhs* above is identically zero, so that equation (**??**) turns out to be really a constraint on the possible values of the quantum numbers p_i.

Finally, the master equation is obtained simply by taking the mean value of the Langevin equation (**??**) in the state η_0. Using (**??**) we get

$$\langle \partial_t j_t(\tilde{X}) \rangle_{\eta_0} = \frac{e^2}{2} \sum_A \{ \overline{D}_A \langle j_t([P_A^+, \tilde{X}]\Gamma_-^{(A)}) \rangle_{\eta_0} - D_A \langle j_t(\Gamma^{(A)\dagger}[P_A, \tilde{X}]) \rangle_{\eta_0} \} \tag{67}$$

This last equation, together with (**??**), gives the explicit form of the generator:

$$L(\tilde{X}) = \frac{e^4}{4} \sum_{A,B} \overline{D}_A D_B \delta_{\varepsilon_A, \varepsilon_B} \{ \gamma_-^{(A)} [P_A^+, \tilde{X}] P_B - \overline{\gamma_-^{(A)}} P_A^+ [P_B, \tilde{X}] \} \tag{68}$$

This expression is hard to be managed. In fact, if no further assumption is made, the sum above is extended to 16 indices, restricted to 8 if we project the model in the LLL and then to 7 because of the $\delta_{\varepsilon_A,\varepsilon_B}$.

In the attempt to physical implications of the model we will discuss, in the remaining of this section two different approximations which produce simpler generators. Since the strategy is identical to the one followed here, we will only mention the main steps without giving all the details.

Second approximation

We recall that this model is restricted to the LLL and it is defined by the hamiltonian in (**??**). The commutation rules are again those in (**??**), H_0 coincides with the hamiltonian in (**??**), while $H_I^{(t)}$ is given by:

$$H_I(t) = e^{iH_0t}H_Ie^{-iH_0t} = \frac{1}{2}\sum_{\alpha\beta,i}(D_{\alpha\beta,i}b^+(ge^{i(\omega+\Omega_i)t}) + \text{h.c.}) \tag{69}$$

where

$$\Omega_0 = 0 \quad \Omega_{\pm1} = \varepsilon_\alpha - \varepsilon_{\alpha\pm1} = \mp\frac{hc}{L_x}\frac{E}{B} \tag{70}$$

From (**??**) we see that the rescaled fields still have the expression (**??**) with Ω_i replacing ε_A. Therefore the stochastic limit is obtained in the same way as for the previous model and gives the following interaction hamiltonian

$$H_I^{(ls)} = \frac{e^2}{2}\sum_{\alpha\beta,i}(D_{\alpha\beta,i}b_i^+(t) + \text{h.c.}) \tag{71}$$

where $\{b_i(t)\}$ satisfy the commutation relations (**??**), (**??**), with the multi indices A, B replaced by the single indices i, j and the susceptivities $\gamma^{(A)}$ (resp. $\gamma_-^{(A)}$) replaced by $\gamma^{(i)}$ (resp. $\gamma_-^{(i)}$) and $\gamma_-^{(i)}$ and $\gamma^{(i)}$ are defined by

$$\gamma_-^{(i)} = \int_{-\infty}^{0} d\tau \int d^2k|g(\underline{k})|^2e^{i\tau(\omega(\underline{k})+\Omega_i)} \tag{72}$$

$$\gamma^{(i)} = \int_{-\infty}^{+\infty} d\tau \int d^2k|g(\underline{k})|^2e^{i\tau(\omega(\underline{k})+\Omega_i)} \tag{73}$$

The same considerations as for the previous model can be repeated, mutatis mutandis, also now. In particular, for instance, Re $\gamma_-^{(i)}$ is concentrated on the surface $\omega(\underline{k}) = -\Omega_i$, and for this reason it is easy to check that, for canonical choices of $\omega(\underline{k})$ as $\omega(\underline{k}) = |\underline{k}|$ or $\omega(\underline{k}) = \sqrt{|\underline{k}|^2 + m^2}$, then

$$\text{Re } \gamma_-^{(0)} = \text{Re } \gamma_-^{(-1)} = 0 \tag{74}$$

The Langevin equation, associated to the Schrödinger equation (**??**), is now

$$\partial_t j_t(\tilde{X}) = \frac{ie^2}{2} \sum_{\alpha\beta,i} \{b_i^+(t) j_t([D_{\alpha\beta,i}, \tilde{X}]) +$$

$$j_t([D^+_{\alpha\beta,i}, \tilde{X}]) b_i(t)\} + \frac{e^2}{2} \sum_{\alpha\beta,i} \{j_t([D^+_{\alpha\beta,i}, \tilde{X}]\Gamma_-^{(i)} - \Gamma_-^{(i)\dagger}[D_{\alpha\beta,i}, \tilde{X}]\} \tag{75}$$

where $\tilde{X} = X \otimes 1_R$ and

$$\Gamma_-^{(i)} = \frac{e^2}{2} \sum_{\alpha\beta} D_{\alpha\beta,i} \gamma_-^{(i)} \tag{76}$$

The master equation is the mean value of the Langevin equation (**??**) in the state η_0. Using equation (**??**) we get

$$\langle \partial_t j_t(\tilde{X}) \rangle_{\eta_0^{(\xi)}} = \frac{e^2}{2} \sum_{\alpha\beta,i} \langle j_t([D^+_{\alpha\beta,i}, \tilde{X}]\Gamma_-^{(i)} - \Gamma_-^{(i)+}[D_{\alpha\beta,i}, \tilde{X}]) \rangle_{\eta_0^{(\xi)}} \tag{77}$$

Therefore, using (**??**), the form of the generator is

$$L(\tilde{X}) = \frac{e^4}{4} \sum_{\alpha\beta_i\alpha'\beta',i} (\gamma_-^{(i)}[D^+_{\alpha\beta,i}, \tilde{X}] D_{\alpha'\beta',i} - \overline{\gamma_-^{(i)}} D^+_{\alpha'\beta',i}[D_{\alpha\beta,i}, \tilde{X}]) \tag{78}$$

This expression is easier than the one in (**??**). In particular, since the model lives in the LLL, we have to sum only over five indices.

As we can see, the contribution in (**??**) with $i = 0$ is the zeroth order in C_{-1}, while the two terms for $i = \pm 1$ correspond to the (linear) perturbation contributions.

Finally the expression above can be further simplified using conditions (**??**).

Third approximation

The interaction picture Hamiltonian for this model is:

$$H_{ep}(t) = e^2 \sum_{\alpha\beta} a_\alpha^+ a_\beta (b(g_{\alpha\beta} e^{-it(\omega - \varepsilon_{\alpha\beta})}) + b^+(\overline{g}_{\beta\alpha} e^{it(\omega - \varepsilon_{\beta\alpha})})) \tag{79}$$

and its stochastic limit, obtained by applying the same stochastic golden rule, used for the rescaled Hamiltonian (**??**), is

$$H_{ep}^{(ls)}(t) = e^2 \sum_{\alpha\beta} (a_\alpha^+ a_\beta b_{\alpha\beta}(t) + \text{ h.c.}) \tag{80}$$

where $b_{\alpha'\beta'}(t')b_{\alpha\beta}^+(t)$, satisfy the commutation relations (**??**), (**??**), with the generalized susceptivities $\gamma_-^{(A)}$ (resp. $\gamma^{(A)}$) replaced by:

$$G_-^{\alpha\beta\alpha'\beta'} = \int_{-\infty}^0 d\tau \int d^2k g_{\alpha\beta}(\underline{k}) \overline{g_{\alpha'\beta'}(\underline{k})} e^{i\tau(\omega(\underline{k})-\epsilon_{\alpha\beta})}, \tag{81}$$

$$G^{\alpha\beta\alpha'\beta'} = \int_{-\infty}^\infty d\tau \int d^2k g_{\alpha\beta}(\underline{k}) \overline{g_{\alpha'\beta'}(\underline{k})} e^{i\tau(\omega(\underline{k})-\epsilon_{\alpha\beta})}. \tag{82}$$

The appeareance of $\delta_{\varepsilon_{\alpha\beta},\varepsilon_{\alpha'\beta'}}$ in $[b_{\alpha\beta}(t), b_{\alpha'\beta'}^+(t')]$ shows the relevance of the integer numbers also for this model. This point will play an important role in the computation of the conductivity tensor.

A simple consequence of the commutation relations for $b_{\alpha\beta}(t)$ and of the time consecutive principle is the following commutation rule

$$[b_{\alpha\beta}(t), U_t] = -i\Gamma_-^{\alpha\beta} U_t \tag{83}$$

The normally ordered form, of the Langevin equation, satisfied by $j_t(\tilde{X}) = U_t^+ \tilde{X} U_t$, is:

$$\partial_t j_t(\tilde{X}) = e^2 \sum_{\alpha\beta} \{ j_t([a_\alpha^+ a_\beta, \tilde{X}]\Gamma_-^{\alpha\beta} - \Gamma_-^{\alpha\beta}[a_\beta^+ a_\alpha, \tilde{X}])\} + ie^2 \sum_{\alpha\beta} \{ b_{\alpha\beta}^+(t) j_t([a_\beta^+ a_\alpha, \tilde{X}]) +$$
$$+ j_t([a_\alpha^+ a_\beta, \tilde{X}]) b_{\alpha\beta}(t)\} \tag{84}$$

where

$$\Gamma_-^{\alpha\beta} := \sum_{\alpha'\beta'} \delta_{\varepsilon_{\alpha\beta},\varepsilon_{\alpha'\beta'}} a_{\beta'}^+ a_{\alpha'} G_-^{\alpha\beta\alpha'\beta'} \tag{85}$$

The master equation is obtained by taking the mean value of (**??**) in η_0. This gives

$$\langle \partial_t j_t(\tilde{X}) \rangle_{\eta_0^{(\xi)}} = e^2 \sum_{\alpha\beta} \langle j_t([a_\alpha^+ a_\beta, \tilde{X}]\Gamma_-^{\alpha\beta} - \Gamma_-^{\alpha\beta+}[a_\beta^+ a_\alpha, \tilde{X}])\rangle_{\eta_0^{(\xi)}} \tag{86}$$

and from this we find for the generator

$$L(\tilde{X}) = e \sum_{\alpha\beta\alpha'\beta'} \delta_{\varepsilon_{\alpha\beta},\varepsilon_{\alpha'\beta'}} \{ [a_\alpha^+ a_\beta, \tilde{X}] a_{\beta'}^+ a_{\alpha'} G_-^{\alpha\beta\alpha'\beta'} - a_{\alpha'}^+ a_{\beta'} [a_\beta^+ a_\alpha, \tilde{X}] \overline{G_-^{\alpha\beta\alpha'\beta'}} \} \tag{87}$$

The expressions for $L(\tilde{X})$ obtained above will be the starting point for our successive analysis.

4 The current operator in second quantization

This section is devoted to the analysis of the current operator in second quantization. The current is proportional to the sum of the velocities of the electrons:

$$\vec{J}_\Lambda(t) = \alpha_C \sum_{i=1}^{N} \frac{d}{dt} \vec{R}_i(t) \tag{88}$$

see [BVEDB94]. Here Λ is the two–dimensional region corresponding to the physical layer, α_c is a constant which takes into account the electron charge, the value of the surface of the physical device and other physical quantities, and $\vec{R}_i(t)$ is the position operator for the i–th electron. Moreover N is the number of electrons contained in Λ. Defining

$$\vec{X}_\Lambda(t) = \sum_{i=1}^{N} \vec{R}_i(t) \ , \tag{89}$$

we have

$$\vec{J}_\Lambda(t) = \alpha_c \dot{\vec{X}}_\Lambda(t) \ . \tag{90}$$

Since $\vec{X}_\Lambda(t)$ is a sum of single-electron operators its expression in second quantization is given by

$$\vec{X}_\Lambda = \sum_{\gamma\mu} \vec{x}_{\gamma\mu} a_\gamma^+ a_\mu \tag{91}$$

where

$$\vec{x}_{\gamma\mu} = \langle \psi_\gamma, \vec{R}\psi_\gamma \rangle = \int \psi_\gamma(\underline{r})\underline{r}\psi_\mu(\underline{r}) d^2 r \tag{92}$$

Recall that $\psi_\gamma(\underline{r})$ are exactly the single electron wave functions found in Section 1, while a_α and a_α^+ satisfy the CAR (20).

The next step consists in computing the matrix elements in (**??**).

This can be done exactly, due to the easy expressions for $\psi_\gamma(\underline{r})$, even without restricting the analysis to the LLL. In fact the two components of $\vec{x}_{\gamma\mu}$ in (**??**) have the form:

$$x_{\gamma\mu}^{(1)} = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} x e^{2\pi i (p_\mu - p_\gamma)x/L_x} dx \cdot \int_{-\infty}^{+\infty} \overline{\varphi_{n_\gamma}(y - y_0^{(p\gamma)})} \varphi_{n_\mu}(y - y_0^{(p_\mu)}) dy \tag{93}$$

$$x_{\gamma\mu}^{(2)} = \frac{1}{L_x} \int_{-L_x/2}^{L_x/2} e^{2\pi i (p_\mu - p_\gamma)x/L_x} dx \cdot \int_{-\infty}^{+\infty} y \, \overline{\varphi_{n_\gamma}(y - y_0^{(p_\gamma)})} \varphi_{n_\mu}(y - y_0^{(p_\mu)}) dy \tag{94}$$

and the integrations in x can be easily performed. The integration in y can be performed by making use of the following formulas (cf. [Grad] and [Cohen]):

$$\int_{-\infty}^{+\infty} dxe^{-x^2} H_m(x+y)H_n(x+z) = 2^n\sqrt{\pi}m!z^{n-m}\cdot L_m^{n-m}(-2yz) \tag{95}$$

if $m \leq n$, and

$$\int_{-\infty}^{+\infty} \overline{\varphi_n(y)}y\varphi_m(y)dy = \sqrt{\frac{\hbar}{2mn}}[\sqrt{m+1}\delta_{n,m+1} + \sqrt{m}\cdot\delta_{n,m-1}] \tag{96}$$

where H_m and L_m^{n-m} are respectively Hermite and Laguerre polynomial, see [Grad] for the definition and for some of their properties. With these ingredients we get

$$x_{\gamma\mu}^{(1)} = (1-\delta_{p_\mu p_\gamma})(-1)^{p_\mu - p_\gamma}\frac{L_x e^{-y^2_{p_\mu p_\gamma}}}{2\pi_i(p_\mu - p_\gamma)}\mathcal{L}_{\gamma\mu} \tag{97}$$

$$x_{\gamma\mu}^{(2)} = \delta_{p_\mu p_\gamma}\{y_0^{(p_\gamma)}\delta_{n_\gamma n_\mu} + \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n_\mu+1}\delta_{n_\gamma,n_\mu+1} + \sqrt{n_\mu}\delta_{n_\gamma,n_\gamma-1}) \tag{98}$$

where

$$\mathcal{L}_{\gamma\mu} := \begin{cases} \sqrt{\frac{2^{n_\gamma}n_\mu!}{2^{n_\mu}n_\gamma!}}y_{p_\mu p_\gamma}^{n_\gamma - n_\mu}L_{n_\mu}^{n_\gamma - n_\mu}(2y^2_{p_\mu p_\gamma}) \text{ if } n_\mu \leq n_\gamma \\ \sqrt{\frac{2^{n_\mu}n_\gamma!}{2^{n_\gamma}n_\mu!}}(-y_{p_\mu p_\gamma})^{n_\mu - n_\gamma}L_{n_\gamma}^{n_\mu - n_\gamma}(2y^2_{p_\mu p_\gamma}) \text{ if } n_\gamma \leq n_\mu \end{cases} \tag{99}$$

and

$$y_{p_\mu p_\gamma} := \sqrt{\frac{m\omega}{4\hbar}}(y_0^{(p_\mu)} - y_0^{(p_\gamma)}) = \frac{\pi}{L_x}\sqrt{\frac{\hbar}{m\omega}}(p_\mu - p_\gamma) \tag{100}$$

Notice that wherever $p_\mu = p_\gamma$ formula (**??**) must be interpreted simply as $x_{\gamma\mu}^{(1)} = 0$.

These results are simpler if we restrict to the LLL. In this case we have $n_\gamma = n_\mu = 0$ and therefore, since $L_0^a(x) = 1$,

$$x_{\gamma\mu}^{(1)} = (1-\delta_{p_\mu p_\gamma})(-1)^{p_\mu - p_\gamma}L_x\frac{e^{-y^2_{p_\mu p_\gamma}}}{2\pi i(p_\mu - p_\gamma)} \tag{101}$$

$$x_{\gamma\mu}^{(2)} = y_0^{(p_\gamma)}\delta_{p_\mu p_\gamma} \tag{102}$$

To show how these results can be useful in the computation of the electron current we start noticing that, if ϱ is a state of the electron system then

$$\langle\vec{J}_\Lambda(t)\rangle_\varrho = \alpha_c\langle\frac{d}{dt}\vec{X}_\Lambda(t)\rangle_\varrho = \alpha_c\langle L(\vec{X}_\Lambda(t))\rangle_\varrho = \alpha_c Tr(\varrho L(\vec{X}_\Lambda(t))) \tag{103}$$

The vector $\langle \vec{J}_\Lambda(t)\rangle_\varrho$ will be computed in the next section for a particular class of states ϱ, and we will find the expressions for the conductivity tensor and its inverse, the resistivity matrix.

5 The conductivity tensor: some preliminary results

In this section we will use the different models discussed in Section 2 and their stochastic limits in order to obtain the associated conductivity tensors. In doing this we will also discuss different approximations. In particular we will show that fixing the model in the LLL appears as a too restrictive assumption because it does not allow to obtain a tensor σ which reproduces the experimental results. In particular the very first requirement for a plausibile σ is that, at least for certain values of the magnetic field, $\rho_{xy} \neq 0$ while $\rho_{yy} = 0$.

Here we will not go into the details for showing the "negative" results. We only mention the attempts which have not produced the desired form for σ, and then we give more details for the models that works!

First of all we remark that our attention has been focused on the second and third approximations since the first one leads to a master equation which is too difficult to be analyzed without further simplifying assumptions.

As we have already discussed the second approximation, which is naturally defined in the LLL, can be seen as a perturbative expansion of a model whose zeroth order in c_{-1} is trivial. The conclusion of our analysis is that also the first order in c_{-1} is trivial, as well as some modified version of the same model where the generator (??) is changed by adding some extra contributions properly chosen.

The third approximation is again trivial, when restricted to the LLL. This statement has been checked up to the second order in c_{-1} which is again a perturbation parameter as for the second approximation. Due to these results, and also following some recent suggestions widely discussed in the literature on the FQHE, we have finally moved to considering the easiest model, that is the one given the third approximation, not in the LLL. Let us compute the electric current in this case. We first need to find $\alpha L(\vec{X}_\Lambda)$, L being the generator defined in (??). Since $\vec{X}_\Lambda = \vec{X}_\Lambda^\dagger$, we have

$$L(\vec{X}_\Lambda) = L_1(\vec{X}_\Lambda) + h.c.,$$

where, as we find after a few computations,

$$L_1(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\alpha'\beta',\gamma} \delta_{\epsilon_{\alpha\beta},\epsilon_{\alpha'\beta'}} G_-^{\alpha\beta\alpha'\beta'} (\vec{x}_{\beta\gamma} a_\alpha^+ a_\gamma a_{\beta'}^+ a_{\alpha'} - \vec{x}_{\gamma\alpha} a_\gamma^+ a_\beta a_{\beta'}^+ a_{\alpha'}) \tag{104}$$

In the present paper we consider a situation of low temperature (the extension to arbitrary temperatures will be given elsewhere) and we compute the mean value of $L_1(\vec{X}_\Lambda)$ on a Fock state ψ_I:

$$\psi_I = a^+_{i_1} \dots a^+_{i_{N_I}} \psi_0, \qquad i_k \neq i_l, \forall k \neq l \tag{105}$$

where I is a set of possible quantum numbers ($I \subset (\mathbf{Z}, \mathbf{Z})$), N_I is the number of elements in I and ψ_0 is the vacuum vector. The order of the elements of I is important to fix uniquely the phase of ψ_I. Equation (**??**) must be replaced, in our case, by

$$\langle \psi_I, \vec{J}_\Lambda(t) \psi_I \rangle = \alpha_c \langle \psi_I, L(\vec{X}_\Lambda) \psi_I \rangle \tag{106}$$

Introducing now the characteristic function of the set I,

$$\chi_I(\alpha) = \begin{cases} 1 \text{ if } \alpha \in I \\ 0 \text{ if } \alpha \notin I, \end{cases} \tag{107}$$

we get

$$\langle a^\dagger_\gamma a_\alpha \psi_I, a^\dagger_{\beta'} a_{\alpha'} \psi_I \rangle = \delta_{\alpha\gamma} \delta_{\alpha'\beta'} \chi_I(\alpha) \chi_I(\alpha') + \delta_{\alpha\alpha'} \delta_{\gamma\beta'} \chi_I(\alpha)(1 - \chi_I(\gamma)). \tag{108}$$

Using this equality, together with

$$\delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha'\alpha'}} = \delta_{\varepsilon_\alpha, \varepsilon_\beta} \quad ; \qquad \delta_{\varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta'}} = \delta_{\varepsilon_\beta, \varepsilon'_\beta} \tag{109}$$

we find that the average current is proportional to

$$\langle L(\vec{X}_\Lambda) \rangle_{\psi_I} = \mathcal{L}_1(\vec{X}_\Lambda) + \mathcal{L}_2(\vec{X}_\Lambda) \tag{102}$$

where we isolate two contributions of different nature:

$$\mathcal{L}_1(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\alpha'} \delta_{\varepsilon_\alpha, \varepsilon_\beta} \{\chi_I(\alpha) - \chi_I(\beta)\} \chi_I(\alpha') (\vec{x}_{\alpha\beta} \overline{G_-^{\alpha\beta\alpha'\alpha'}} + \vec{x}_{\beta\alpha} G_-^{\alpha\beta\alpha'\alpha'}), \tag{110}$$

$$\mathcal{L}_2(\vec{X}_\Lambda) = e^2 \sum_{\alpha\beta\beta'} \delta_{\varepsilon_\beta, \varepsilon_{\beta'}} \{\vec{x}_{\beta\beta'} [G_-^{\alpha\beta\alpha\beta'} \chi_I(\alpha)(1 - \chi_I(\beta')) - \overline{G_-^{\beta\alpha\beta'\alpha}} \chi_I(\beta')(1 - \chi_I(\alpha))] -$$

$$- \vec{x}_{\beta'\beta} [G_-^{\beta\alpha\beta'\alpha} \chi_I(\beta')(1 - \chi_I(\alpha)) - \overline{G_-^{\alpha\beta\alpha\beta'}} \chi_I(\alpha)(1 - \chi_I(\beta'))]\}. \tag{111}$$

Remark. It is interesting to notice that if we replace $\delta_{\varepsilon_\alpha, \varepsilon_\beta}$ by $\delta_{\alpha,\beta}$, then we easily obtain $\langle L(X^{(1)}_\Lambda) \rangle_{\psi_I} = 0$, which is not the result we would like to obtain. This means that this approximation (taking $\alpha = \beta$ means to consider only one among the many contributions in the sums in (??), (??)!) is too strong and must be avoided.

Using now equations (??), (??) for $x^{(i)}_{\gamma\mu}$ we are able to obtain $\mathcal{L}_1(X^{(i)}_\Lambda)$ and $\mathcal{L}_2(X^{(i)}_\Lambda)$, $i = 1, 2$.

First of all we can show that, even if $\mathcal{L}_1(X^{(1)}_\Lambda)$ is not zero, nevertheless it does not depend on the electric field. Therefore

$$\frac{\partial}{\partial E}\mathcal{L}_1(X^{(1)}_\Lambda) = 0 \tag{112}$$

Secondly, the computation of $\mathcal{L}_2(X^{(1)}_\Lambda)$ gives rise to an interesting phenomenon: due to the definition of $X^{(1)}_{\gamma\mu}$ the sum in (??) is different from zero only if $p_\beta \neq p_{\beta'}$. Moreover all also must have $\varepsilon_\beta = \varepsilon_{\beta'}$, that is

$$n_\beta - n_{\beta'} = \frac{2\pi eE}{m\omega^2 L_x}\,(p_{\beta'} - p_\beta) \tag{113}$$

This equality can be achieved in two different ways:

1) if $\frac{2\pi eE}{m\omega^2 L_x}$ is not rational then (??) is satisfied only if $\beta = \beta'$. But this condition implies that $p_\beta = p_{\beta'}$, and we know already that whenever this condition holds, then $x^{(1)}_{\beta\beta'} = 0$.

2) If

$$\frac{2\pi eE}{m\omega^2 L_x} \in Q \tag{114}$$

we have two possibilities: the first one is

$$\beta = \beta'$$

which again, for the same reason, does not contribute to $\mathcal{L}_2(X^{(1)}_\Lambda)$. The second is

$$\frac{n_\beta - n_{\beta'}}{p_{\beta'} - p_\beta} = \frac{2\pi eE}{m\omega^2 L_x} \tag{115}$$

Therefore, we can state the following

Proposition. *In the context of Model #3, if the electric and the magnetic fields are such that*

$$\frac{2meE}{m\omega^2 L_x} \in \mathbf{R}\backslash\mathbf{Q}$$

then

$$\langle J^{(1)}_\Lambda(t)\rangle_{\psi_I}=0$$

and therefore $\sigma_{xy}=0$.

On the other hand, if condition (**??**) is satisfied, we can conclude that the sum $\sum_{\alpha\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_{\beta'}}(\dots)$ in (**??**) can be replaced by

$$\sum_{\alpha\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_{\beta'}}(\dots)=\sum_\alpha{\sum_{\beta\beta'}}'(\dots) \tag{116}$$

where $\sum_\alpha\sum'_{\beta\beta}$ means that the sum is extended to all the α and to those β and β' with $p_\beta\neq p_{\beta'}$ satisfying (**??**), which implies that $\varepsilon_\beta=\varepsilon_{\beta'}$.

Since, as it is easily seen, $g_{\alpha\beta}(k)\overline{g_{\alpha'\beta'}(\underline{k})}$ does not depend on $\vec{E}$, we find that

$$\frac{\partial}{\partial E}G_-^{\alpha\beta\alpha'\beta'}=-i\frac{he}{m\omega L_x}(p_\alpha-p_\beta)\Lambda_-^{\alpha\beta\alpha'\beta'} \tag{117}$$

where

$$\Lambda_-^{\alpha\beta\alpha'\beta'}=\int_{-\infty}^0 d\tau\tau\int d^2k g_{\alpha\beta}(\underline{k})\overline{g_{\alpha'\beta'}(\underline{k})}e^{i\tau(\omega(\underline{k})-\varepsilon_{\alpha\beta})} \tag{118}$$

so that, using also (**??**), we get

$$\frac{\partial}{\partial E}\mathcal{L}_2(X^{(1)}_\Lambda)=\frac{he}{m\omega L_x}\Theta_x \tag{119}$$

where

$$\Theta_x:=\sum_\alpha{\sum_{\beta\beta'}}'(p_\beta-p_\alpha)\tilde{x}^{(1)}_{\beta\beta'}\{\chi_I(\alpha)(1-\chi_I(\beta'))\cdot(\Lambda_-^{\alpha\beta\alpha\beta'}+\overline{\Lambda_-^{\alpha\beta\alpha\beta'}})$$
$$-\chi_I(\beta')(1-\chi_I(\alpha))(\Lambda_-^{\beta\alpha\beta'\alpha}+\overline{\Lambda_-^{\beta\alpha\beta'\alpha}})\} \tag{120}$$

and

$$\tilde{x}^{(1)}_{\beta\beta'}=ix^{(1)}_{\beta\beta'}\quad(\in\mathbf{R}) \tag{121}$$

Therefore we conclude that

$$\frac{\partial}{\partial E}\langle J^{(1)}_\Lambda(t)\rangle_{\psi_I}=\frac{\alpha_c he^3}{m\omega L_x}\Theta_x \tag{122}$$

Let us now compute the second component of the average current: $\langle\psi_I,L(X^{(2)}_\Lambda)\psi_0\rangle=\mathcal{L}_1(X^{(2)}_\Lambda)+\mathcal{L}_2(X^{(2)}_\Lambda)$.

The first contribution is easily shown, from (**??**) and (**??**), to be identically zero, since

$$\delta_{\varepsilon_\alpha,\varepsilon_\beta}\delta_{p_\alpha p_\beta}=\delta_{\alpha\beta} \tag{123}$$

On the contrary the second term, $\mathcal{L}_2(X_\Lambda^{(2)})$, is different from zero and it has a rather interesting expression: in fact, due to the factor δ_{p_μ,p_γ}, the only non trivial contributions in the sum $\sum_{\beta\beta'}\delta_{\varepsilon_\beta,\varepsilon_{\beta'}}$, in (**??**), are exactly those with $\beta=\beta'$. Taking all this into account, we find that

$$\mathcal{L}_2(X_\Lambda^{(2)}) = e^2\sum_{\alpha\beta}(y_0^{(p_\beta)}-y_0^{(p_\alpha)})\chi_I(\alpha)(1-\chi_I(\beta))(G_-^{\alpha\beta\alpha\beta}+\overline{G_-^{\alpha\beta\alpha\beta}}) \tag{124}$$

which is different from zero. Furthermore, using (**??**), we get

$$\frac{\partial}{\partial E}\mathcal{L}_2(X_\Lambda^{(2)}) = -2e^3\left(\frac{h}{m\omega L_x}\right)^2\Theta_y$$

were we have defined

$$\Theta_y = \sum_{\alpha,\beta}(p_\alpha-p_\beta)^2\chi_I(\alpha)(1-\chi_I(\beta))\ \mathrm{Im}\ (\Lambda_-^{\alpha\beta\alpha\beta}) \tag{125}$$

and $\Lambda_-^{\alpha\beta\alpha\beta}$ is given by (**??**). If we call now

$$j_{x,E} = \frac{\partial\langle J_\Lambda^{(1)}(t)\rangle_{\psi_I}}{\partial E} = \alpha_c\frac{\partial\langle L(X_\Lambda^{(1)})\rangle_{\psi_I}}{\partial E}$$

$$j_{y,E} = \frac{\partial\langle J_\Lambda^{(2)}(t)\rangle_{\psi_I}}{\partial E} = \alpha_c\frac{\partial\langle L(X_\Lambda^{(2)})\rangle_{\psi_I}}{\partial E}\ ,$$

we obtain the conductivity tensor (see [cha])

$$\sigma_{xx}=\sigma_{yy}=j_{y,E}, \qquad \sigma_{xy}=-\sigma_{yx}=j_{x,E} \tag{126}$$

and, for the resistivity tensor,

$$\rho_{xx}=\rho_{yy}=\frac{\sigma_{yy}}{\sigma_{yy}^2+\sigma_{xy}^2}, \qquad \rho_{xy}=-\rho_{yx}=\frac{\sigma_{xy}}{\sigma_{yy}^2+\sigma_{xy}^2} \tag{127}$$

After minor computations we conclude that

$$\rho_{xy}=\begin{cases} 0 \quad \text{if} \quad \frac{2\pi eE}{m\omega^2L_x}\notin\mathbf{Q} \\ \frac{m\omega L_x}{2e^3h\alpha_c}\frac{\Theta_x}{[\Theta_x^2+(\frac{h}{m\omega L_x})^2\Theta_y^2]} \quad \text{if} \quad \frac{2\pi eE}{m\omega^2L_x}\in\mathbf{Q}, \end{cases} \tag{128}$$

$$\rho_{xx}=\begin{cases} -(\frac{m\omega L_x}{h})^2\frac{1}{2\alpha_c e^3\Theta_y} \quad \text{if} \quad \frac{2\pi eE}{m\omega^2L_x}\notin\mathbf{Q} \\ -\frac{1}{2e^3\alpha_c}\frac{\Theta_y}{[\Theta_x^2+(\frac{h}{m\omega L_x})^2\Theta_y^2]} \quad \text{if} \quad \frac{2\pi eE}{m\omega^2L_x}\in\mathbf{Q}\ , \end{cases} \tag{129}$$

This result is close to the experimental data observed in the FQHE. In particular we want to stress that the behaviour of both the conductivity and the resistivity tensors are driven by the values of the magnetic field via a *fine tuning condition* ($\frac{2\pi e E}{m\omega^2 L_x} \in \mathbf{Q}$) which strongly reminds the sharp values of the filling factor (and therefore of the magnetic field) for which the plateaux are observed.

Another interesting point concerns the asymptotic values of ρ_{xx} and ρ_{xy} for large values of the magnetic field, which is the typical experimental situation. It is possible to check that, for large values of $B = |\vec{B}|$, ρ_{xy} increases linearly in B while ρ_{xx} is independent of the magnetic field. These results reflect once again the behaviour of ρ_{ij} observed experimentally.

On the other hand we must say that our model does not reproduces *all* the experimental data. We believe that these differences can follow from the simplified model we have analyzed in these notes. However, the mechanism which produces the fine tuning condition looks very promising, in our opinion, for a deeper understanding of the FQHE and deserve further analysis.

We want to conclude this section, and the paper, with some open problems and projects for the future:

1) Even if the mechanism described by the present model seems to be correct, the numerical results do not yet fit with the experimental results. So the next natural step is to look for a different physical hamiltonian which reproduces the plateaux and the minima in the correct way.

2) The second step should be to extend this analysis also to non zero temperature. The main difficulty in doing this is that formula (**??**) should be replaced by the analogous one for non–zero temperature states.

3) The interpretation of the background field, in particular its connections with the impurities and the phonons, is still to be clarified.

4) Also, does the energy of the system presents some cusps in presence of the same values of B which satisfy the fine tuning condition?

5) Does the stochastic limit procedure give information about the ground state of the system? Or, alternatively, for what value of the filling the crystal ground state ceases to be energetically favourite with respect to the Laughlin wave function?

6) Which is the origin of the fractional excitations observed in the two-dimensional electron gases? Is it possible to explain these anomalous

commutation relations as a result of the stochastic limit?

These are only few of the open points which remains to be understood for a complete understanding of the FQHE. We hope to be able to answer to some of these questions in the future.

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