On the Γ-equivariant form of the Berezin's quantization of the upper half plane

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Introduction

Let Γ be a Fuchsian subgroup of $PSL(2,\mathbb{R})$. In this paper we consider the Γ-equivariant form of the Berezin's quantization of the upper half plane $\mathbb{H}$ which will correspond to a deformation quantization of the (singular) space $\mathbb{H}/\Gamma$. Our main result is that the von Neumann algebra associated to the $\Gamma-$ equivariant form of the quantization is stably isomorphic to the von Neumann algebra associated with Γ. Moreover, the dimension of each algebra in the deformation quantization, as a left module over the group von Neumann algebra $\mathcal{L}(\Gamma)$, is a linear function of the deformation parameter ("Planck's constant").

Recall that the Muray-von Neumann construction, [Mvn] for the dimension of projective, left modules over type II von Neumann algebras with trivial center, allows all positive real numbers as possible value for the dimension. Consequently the above isomorphism is meaningful for all values of the deformation parameter.

This will be particularly interesting when Γ is the group $\Gamma = PSL(2,\mathbb{Z})$. We use the terminology (introduced in [KD], [FR], see also [DV]) of von Neumann algebras $\mathcal{L}(F_t), t > 1$ corresponding to free groups with a (possible) fractional "number t of generators" (even if the group itself may not make sense). In this case the von Neumann algebras associated with the equivariant form of the Berezin quantization will be free group von Neumann algebras, where the "number of generators" is a bijective function of the deformation parameter.

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The difference between the Berezin quantization of the upper half plane and its Γ-equivariant form is easy to establish. In the classical case the von Neumann algebras associated to the deformation are simply isomorphic to $B(H)$, the algebra of all bounded operators on a Hilbert space. In the equivariant case these algebras are type II_1 factors. This is a consequence of the formulae for the traces in this algebras: classically the trace is an integral over $\mathbb{H}$ of the restriction to the diagonal of the reproducing kernel, while in the Γ-equivariant case the trace is the integral over a fundamental domain of Γ in $\mathbb{H}$. This last fact is in a particular a generalization of the computation in [GHJ] of the type II_1 factor trace of a product of two Toeplitz operators having automorphic forms as symbols.

There exists a remarkable analogy, at least at the formal level, between Rieffel's ([Ri]) construction for the irrational rotation C$^*-$algebra and the construction in this paper. In Rieffel's construction, a deformation quantization for the torus $\mathbb{T}^2$ is realized (in a more involved manner), starting from the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. The von Neumann algebras in the deformation for $\mathbb{T}^2$ are all isomorphic to the hyperfinite II_1 factor, with the exception of the rational values for the parameter.

For any deformation quantization, with suitable properties, there exists an associated $2-$cocycle in the Connes's cyclic cohomology of a certain "smooth" subalgebra, for each parameter value. In this paper, we construct in a natural way, a dense family of subalgebras on which the 2-cocycles live. This algebras are endowed with a norm that is a continuous analog of the $\sigma(l^1, l^\infty)$ norm on finite matrices.

Surprisingly, the formulae defining the 2-cocycle are very similar to the formulae in the paper of Connes and Moscovici ([CM]) where cyclic cocycles are constructed from Alexander-Spanier cycles. We will also show that the two cocycles appearing in the deformation quantization for $\mathbb{H}/\Gamma$ are related in a natural to a canonical element (see [Gr], [Gh]) in the second bounded cohomology group $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$.

Our construction, when $\Gamma = PSL(2, \mathbb{Z})$, could be taught of as yet another definition for the von Neumann algebras corresponding to free groups with "fractional number of generators". A possible advantage of such an approach could be the fact that the "smooth structure" (see [Co]) may be used to define a family of C^*-algebras that could be a candidate for the C^* -algebras corresponding to free groups with "fractional number of generators".

It would be very interesting if one could establish a direct relation between Voiculescu's random matrix model for free group factors, and the "continuous matrix" model coming from Berezin's equivariant quantization. Also it would be very interesting if a direct connection could be established between the deformation quantization constructed in [Za] and the approach in this paper.

We prove that, for deformation quantization of algebras with the property that the associated cyclic 2-cocycles are bounded with respect to the uniform norms on the algebras, there exist a time-dependent, linear differential equation, whose associated evolution operator induces an isomorphism between the algebras associated to distinct values of the deformation parameter. This depends on a rather standard technique to prove vanishing of cohomology groups of von Neumann algebras by fixed point theorems.

The cocycles we are constructing would be bounded if a certain bounded function on $\mathbb{H}^2$ (defined by $z, w \to \arg\ (z - \overline{w})$) would be a Schur multiplier (see [Pi],[CS]) on the Hilbert spaces of analytic functions $H^2(\mathbb{H}, y^{r-2}\mathrm{d}x\mathrm{d}y)$.

The proof of the main result of this paper is based on an observation, related to the Plancherel formula for the universal cover $\widetilde{PSL(2,\mathbb{R})}$ of $PSL(2,\mathbb{R})$ established by L. Pukanzsky in [Pu] (see also [Sa]). The fact we are using is that the projective, unitary representations of $PSL(2,\mathbb{R})$ in the continuous series that extends the analytic discrete series of $PSL(2,\mathbb{R})$ also have square integrable coefficients over $PSL(2,\mathbb{R})$ as the representations in the discrete series do.

The computation of the isomorphism class of the algebras in the deformation is then completed by a generalization, to projective, unitary representations, of a method found in [GHJ] (and referred there to [AS],[Co]). This method computes the Murray-von Neumann dimension of the Hilbert space of a representation of a Lie group as left module (via restriction) over the group algebra of a lattice subgroup.

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0. Definitions and outline of the proofs

Recall that a von Neumann algebra is a selfadjoint subalgebra of $B(H)$, which is unital and closed in the weak operator topology. A type II_1 factor is a von Neumann algebra M with trivial center and such that there exists a weakly continuous, linear functional (called trace) $\tau : M \to \mathbb{C}$ with $\tau(xy) = \tau(yx)$ and so that 0 is the only positive element in the kernel of τ. We normalize τ by $\tau(1) = 1$. Let $\mathcal{L}(\Gamma)$ be the weak closure of the group algebra $\mathbb{C}(\Gamma)$, represented in $B(l^2(\Gamma))$, by left convolution operators. If Γ has nontrivial, infinite conjugacy classes, then $\mathcal{L}(\Gamma)$ is a type II_1 factor, the trace being simply the evaluation at the neutral element in Γ.

Such algebras are usually associated with a discrete group Γ with infinite (nontrivial) conjugacy classes. Let $\mathcal{L}(\Gamma)$ be the weak closure of the group algebra $\mathbb{C}(\Gamma)$, represented in $B(l^2(\Gamma))$, by left convolution operators.

The following construction goes back to the original paper ([MvN]) of Murray and von Neumann. Let M be a type II_1 factor with trace τ and let t be a positive real number. Denote by τ also, when no confusion arises, the tensor product trace $\tau\overline{\otimes}\mathrm{tr}_{B(H)}$ on $M\overline{\otimes}B(H)$. Let p be any selfadjoint idempotent in $M\overline{\otimes}B(H)$. Then the isomorphism class of the type II_1 factor $p(M\overline{\otimes}B(H))p$ is independent on the choice of p as long as p has trace t. This type II_1 factor is usually denoted by M_t. Clearly $\mathcal{F}(M) = \{t|\ M_t \cong M\}$ is a multiplicative subgroup of $\mathbb{R}_+/\{0\}$, referred to, by Murray and von Neumann as the fundamental group of M.

In the same paper referred to above, given a weakly continuous representation of M into some $B(K)$, the authors define a positive real number $\dim_M K$ which measures the dimension of K as a left Hilbert module over M. The dimension, in type II_1 factors, may take any positive real value. The original terminology for $\dim_M K$ was the coupling constant of M in K.

This number has all the formal features of a dimension theory, that is $\dim_M(K_1 \oplus K_2) = \dim\,_M(K_1) + \dim\,_M(K_2)$. The dimension number is normalized (when M is the von Neumann algebra $\mathcal{L}(\Gamma)$ of a group Γ) by the condition $\dim_{\mathcal{L}(\Gamma)} l^2(\Gamma) = 1$. More generally, for arbitrary M, let $L^2(M,\tau)$ be the Hilbert space corresponding to Gelfand-Naimark-Segal construction for the trace τ on M, that is $L^2(M,\tau)$ is the

Hilbert space completion of M as a vector space with respect to the scalar product $\langle a, b\rangle_\tau = \tau(b^*a), a, b \in M$. Then we have $\dim_M(L^2(M,\tau)) = 1$.

To obtain a Hilbert space of an arbitrary dimension t, $0 < t \leq 1$ one takes a projection e' in the commutant

$$M' = \{x \in B(L^2(M,\tau)) | [x, M] = 0\},$$

of trace $\tau_{M'}(e') = t$. Then $e'L^2(M,\tau)$ is a left Hilbert module over M of dimension t. Note that M' is also a type II_1 factor. This last statement is more transparent in the case $M = \mathcal{L}(\Gamma)$. In this case M' is isomorphic and antisomorphic to M. In fact, $\mathcal{L}(\Gamma)'$ is the von Neumann algebra $\mathcal{R}(\Gamma)$ generated by right convolutors on $l^2(\Gamma)$. If $t > 1$, to obtain a module over M of dimension t one has to replace M by $M \otimes M(n, \mathbb{C})$ where n is any integer bigger than t.

From the above construction it is easy deduced that, if K is a left Hilbert module over M (that is we have a unital embedding of M into $B(K)$), then the commutant $M' = \{x \in B(K) | [x, M] = 0\}$ is antisomorphic to the algebra M_t with $t = \dim_M K$.

Since for $M = \mathcal{L}(\Gamma)$, M is always antisomorphic to itself, in this case we may replace antisomorphic simply by isomorphic everywhere in the above statements.

Finally, one may construct a left Hilbert module over the von Neumann algebra of a discrete group Γ which is a lattice subgroup in a semisimple Lie group G, for every representation π of G which belongs to the discrete series representations of G. Let H_π be the Hilbert space of the representation and let d_π be the coefficient with which this representation enters in the Plancherel formula for G. Then the following result was proved in [GHJ] (see also [Co],[AS]):

Let M be the von Neumann algebra generated by the image of Γ by π in $B(H_\pi)$. If Γ is a lattice then M is isomorphic to $\mathcal{L}(\Gamma)$ and

$$\dim_M H_\pi = (\text{covol}\ \ \Gamma) d_\pi.$$

We now recall the definition of the Berezin quantization from [Be1], [Be2]. Although the setting could be more general, we will restrict ourselves in this paper to the case when the phase space is $\mathbb{H}$ with the geometry given by the noneuclidian metric $(\mathrm{Im}\ z)^{-2} \mathrm{d}z\mathrm{d}\overline{z}$.

The meaning of quantization was recalled in the introduction to [Be1] and it was "a construction, starting from the classical mechanics of a system, of a quantum system which had the classical system as its limit as $h \to 0$, where h is Planck's constant" (we quote from the above mentioned paper).

In the case when the phase space is $\mathbb{H}$, Berezin's construction of an algorithm for the quantum system is the construction of a family of multiplications $*_h$ indexed by h on a suitable vector subspace of functions on $\mathbb{H}$, which are associative, whenever this comparison makes sense.

The property that the quantum system has the classical system as a limit, when $h \to 0$, means that for suitable functions f, g on $\mathbb{H}$

$$\lim_{h\to 0} \frac{1}{h}[f *_h g - g *_h f] = \{f, g\},$$

where the Poisson bracket $\{\cdot, \cdot\}$ is computed according to the formula:

$$\{f, g\} = (\mathrm{Im} z)^{-2}[(\frac{\mathrm{d}}{\mathrm{d}z} f)(\frac{\mathrm{d}}{\mathrm{d}\overline{z}} g) - (\frac{\mathrm{d}}{\mathrm{d}z} g)(\frac{\mathrm{d}}{\mathrm{d}\overline{z}} f)].$$

The Berezin algorithm for the multiplication operation $*_h$ is realized by identifying any suitable functions on $\mathbb{H}$ with bounded linear operators on the Hilbert space of analytic functions

$$H_{1/h} = H^2(\mathbb{H}, (\mathrm{Im}\ z)^{-2+1/h} \mathrm{d}z\mathrm{d}\overline{z}).$$

The multiplication $*_h$ then corresponds, via this identification, to the composition operation for linear operators on $H_{1/h}$. There exists two possibilities to realize the correspondence between functions on $\mathbb{H}$ and linear operators on $H_{1/h}$. In both methods the functions on $\mathbb{H}$ are identified with a special type of symbol for linear operators.

Let $P_{1/h}$ be the orthogonal projection from $L^2(\mathbb{H}, (\mathrm{Im}\ z)^{-2+1/h} \mathrm{d}z\mathrm{d}\overline{z})$ onto $H^2(\mathbb{H}, (\mathrm{Im}\ z)^{-2+1/h} \mathrm{d}z\mathrm{d}\overline{z})$ and let $M^{1/h}$ be the multiplication operator on $L^2(\mathbb{H}, (\mathrm{Im}\ z)^{-2+1/h} \mathrm{d}z\mathrm{d}\overline{z})$ with the function f. Then the (not necessary bounded) Toeplitz operator with symbol f is $T_f^{1/h} = P_{1/h} M^{1/h} P_{1/h}$.

A function f on $\mathbb{H}$ is the covariant symbol of a linear operator A on $H_{1/h}$ if A is the Toeplitz operator $T_f^{1/h}$ on $H_{1/h}$ with symbol f. We will use in this situation the notation $\overset{\circ}{A} = f$ or $\overset{\circ}{A}(z, \overline{z}) = f(z, \overline{z})$.

A function f on $\mathbb{H}$ is the contravariant symbol of a linear operator A on $H_{1/h}$ if f is the restriction to the diagonal $z = w$ of a function $\tilde{f}$ on $\mathbb{H}^2$ which is analytic in the first variable and antianalytic in the second variable. The relation between A and $\tilde{f}$ is explained bellow: Let $e_z^{1/h}, z \in \mathbb{H}$ be the evaluation vectors in $H_{1/h}$, that is $\langle f, e_z^{1/h} \rangle = f(z), z \in \mathbb{H}$, for all f in $H_{1/h}$. Then $\tilde{f}$ is given by

$$\tilde{f}(z, \overline{w}) = \frac{\langle A e_w^{1/h}, e_z^{1/h} \rangle}{\langle e_w^{1/h}, e_z^{1/h} \rangle}, z, w \in \mathbb{H}.$$

We will use the notation $\tilde{f} = \hat{A}$ or $\tilde{f}(z, \overline{w}) = \hat{A}(z, \overline{w})$.

The main theorem in Berezin's papers [Be1], [Be2] is that by using the correspondence between functions on $\mathbb{H}$ and linear operators on $H_{1/h}$, given by any of this two symbols, one gets a quantization which has the required classical limit.

Moreover there exist a natural duality relation between the two type of symbols which is realized by using the pairing given by the operatorial trace on $B(H_{1/h})$: For suitable bounded linear operators A, B on $H_{1/h}$ (see e.g. [Co] for a rigorous treatment) one has

$$(0.1) \qquad \mathrm{tr}_{B(H_{1/h})}(AB) = \int_{\mathbb{H}} \hat{A}(z, \overline{z}) \overset{\circ}{B}(z, \overline{z}) (\mathrm{Im} z)^{-2} \mathrm{d}z \mathrm{d}\overline{z}.$$

Finally the multiplication operation $*_h$ is covariant with respect to the action of group $PSL(2, \mathbb{R})$on $\mathbb{H}$. Hence there exists a projective, unitary representation $\pi_{1/h}$ on the Hilbert space $H_{1/h}$ with the following property: If $f = \hat{A}$ is the contravariant symbol of an operator A on $B(H_{1/h})$ then, for any group element g in $PSL(2, \mathbb{R})$ the function on $\mathbb{H}$ defined by $z \to f(g^{-1}z)$ is the symbol of the operator $\pi_{1/h}(g) A \pi_{1/h}(g^{-1})$.

Let Γ be a discrete subgroup of $PSL(2, \mathbb{R})$ of finite covolume. The covariance property for the Berezin multiplication shows that this operation is an inner operation on a suitable space of functions on $\mathbb{H}$, that are $\Gamma-$ equivariant. From a borelian viewpoint, $\Gamma-$equivariant, measurable functions on $\mathbb{H}$ are identified with functions on F where F is any fundamental domain in $\mathbb{H}$ for the action of Γ on $\mathbb{H}$.

We will show in the third paragraph of this paper that there exists a suitable vector space $\mathcal{V}_{1/h}$ of $\Gamma-$ equivariant functions on $\mathbb{H}$, dense in $L^2(F)$ and so that

$\mathcal{V}_{1/h}$ is an involutive algebra with respect to the product given by $*_{1/h}$. The main theorem of this paper is the following

Theorem. *Let* Γ *be a discrete subgroup of* $PSL(2,\mathbb{R})$ *of finite covolume. Let* F *be a fundamental domain in* $\mathbb{H}$ *for the action of* Γ *on* $\mathbb{H}$. *For every* $h>0$, *there exists a dense vector subspace* $\mathcal{V}_{1/h}$ *of* $L^2(F)$ *which is closed under conjugation and under the product* $*_h$. *Here* $*_h$ *is the Berezin's product, if we identify functions on* F *with* Γ-*equivariant functions on* $\mathbb{H}$. *Let* τ *be the functional on this vector space defined by* $\tau(f)=\int\limits_F f(z)(Im\ \ z)^{-2}dzd\bar{z}$. *Then* τ *is a trace, that is* $\tau(f *_h g)=\tau(g *_h f)$, *for all suitable* f,g.

Let $\mathcal{A}_{1/h}$ *be the von Neumann algebra obtained by taking the weak closure of the vector space* $\mathcal{V}_{1/h}$ *in the Gelfand-Naimark-Segal representation associated with the trace* τ *on* $\mathcal{V}_{1/h}$. *Then* $\mathcal{A}_{1/h}$ *is isomorphic to the type* II_1 *factor* $(\mathcal{L}(\Gamma))_t$ *with* $t=$ *(covol* $\Gamma)(\frac{1/h-1}{\pi})$.

This result is particularly interesting when $\Gamma=PSL(2,\mathbb{Z})$. In this case the algebras in the deformation are isomorphic to the type II_1 factors $\mathcal{L}(F_t)$ corresponding to free groups with real number of generators. This factors where introduced in ([KD] and independently in [FR]) based on the random matrix techniques developed by Voiculescu (see also [DV] for an entropy theoretic viewpoint definition of this factors).

The interesting feature that appears is that the (real) "number of generators" t in $\mathcal{L}(F_t)$, for the algebras in the deformation, is a bijective function on the deformation parameter (the Planck' constant). Recall that it is still an open problem (hinted in [MvN] and first time explicitly mentioned by R. Kadison) if the isomorphism class of $\mathcal{L}(F_N)$ depends on N.

The explanation of this behavior when $\Gamma = PSL(2,\mathbb{Z})$ is that in this case $\mathcal{L}(PSL(2,\mathbb{Z})) = \mathcal{L}(F_{\frac{7}{6}})$ (by [KD]) and that the following formula holds: $\mathcal{L}(F_t)_r = \mathcal{L}(F_{(t-1)r^{-2}+1})$, for all $t > 1$, $r > 0$, ([KD],[FR]).

The proof of the main theorem follows from the following facts. By the co-variance property recalled above is easy deduced that $\mathcal{A}_{1/h}$ is isomorphic to the commutant $\{\pi_{1/h}(\Gamma)\}'$ of the image of Γ through $\pi_{1/h}$ in $B(H_{1/h})$. If the group cocycle corresponding to the projective representation $\pi_{1/h}$ vanishes by restriction to Γ then $\pi_{1/h}|_\Gamma$ may be perturbed (by scalars of modulus 1) to a representation of Γ.

We are then in a situation which is very similar to the theorem we recalled at the beginning of this paragraph. If $1/h = r$ is an integer then $\pi_{1/h}$ is a representation in the discrete series of $PSL(2,\mathbb{R})$ with coefficient $(r-1)/\pi$.

Hence, if $1/h$ is an integer, then the dimension of $H_{1/h}$ as a left Hilbert module over $\mathcal{L}(\Gamma)$ (via $\pi_{1/h}|_\Gamma$) is $(r-1)/\pi(\text{covol }\Gamma)$. Consequently, by what we recalled at the beginning of this paragraph, the algebra $\mathcal{A}_{1/h}$, which is the commutant of $\pi_{1/h}(\Gamma)$ in $B(H_{1/h})$, will be isomorphic to $\mathcal{L}(\Gamma)_t$ with $t = (r-1)/\pi(\text{covol }\Gamma)$.

If $1/h$ is now a positive real number, not necessary an integer, then the projective representation $\pi_{1/h}$ is no longer a representation, so the above argument does no longer apply. There exists still a striking similarity with the previous situation which may be read off from the Plancherel formula for the universal cover $\widetilde{PSL(2,\mathbb{R})}$ of $PSL(2,\mathbb{R})$.

The projective representations $\pi_{1/h}$ lift to actual unitary representations of $\widetilde{PSL(2,\mathbb{R})}$, the simply connected universal covering group of $PSL(2,\mathbb{R})$. The lifted representation now belong, as it was observed in Pukanszky article ([Pu], see also

[Sa]), to the continuous series of representations of $\widetilde{PSL(2,\mathbb{R})}$. (In particular this part of the continuous series of $\widetilde{PSL(2,\mathbb{R})}$ extends the discrete series of $PSL(2,\mathbb{R})$. The coefficient with which the representation $\pi_{1/h}$ intervene in the continuous series is given by the same algebraic formula as in the integer case for $PSL(2,\mathbb{R})$, that is $(1/h-1)/\pi$.

The above mentioned property for the representations $\pi_{1/h}$ may be better understood if we look directly to the computations involved in determining the coefficient of a representation in the Plancherel formula for the discrete series.

If $1/h$ is an integer then the representation $\pi_{1/h}$ has square summable coefficients which are verifying the generalized Schur orthogonality relations (see [Go],[HC]). This relations are (with dg Haar measure on $PSL(2,\mathbb{R})$):

$$\int_{PSL(2,\mathbb{R})} |\langle \pi_{1/h}(g)\zeta,\eta\rangle|^2 \mathrm{d}g = \frac{1/h-1}{\pi}||\zeta||^2||\eta||^2, \zeta,\eta \in H_{1/h}.$$

If $1/h$ is not an integer then one can still check this relations holds true for the projective representation $\pi_{1/h}$. Note that this doesn't depend on the possible choice of scalars of modulus 1 which would appear if we consider $\pi_{1/h}$ be induced from a representation of the universal cover.

This fact means that each of the projective, unitary representations $\pi_{1/h}$ for $PSL(2,\mathbb{R})$ is contained in a "skewed" form of the left regular representation of $PSL(2,\mathbb{R})$. One can deduce from here an analogue of the theorem from the book [GHJ] holds true for the representations $\pi_{1/h}$ and hence that $\mathcal{A}_{1/h} = \{\pi_{1/h}(\Gamma)\}'$ is isomorphic to $\mathcal{L}(\Gamma)_t$ with $t = (\text{covol } \Gamma)(\frac{1/h-1}{\pi})$.

Note that in the preceding setting, if f is a Γ-equivariant function then the Toeplitz operator with symbol f in $B(H_{1/h})$ commutes with $\pi_{1/h}(\Gamma)$ and hence it belongs (for suitable functions f) to $\mathcal{A}_{1/h}$.

The duality relation (0.1) between the covariant and contravariant symbol for operators in $\mathcal{A}_{1/h}$ still exists if one replaces the operatorial trace with the trace on the type II_1 factor $\mathcal{A}_{1/h}$. The relation takes now the following form:

$$(0.2) \qquad \tau(AB) = \int_F \hat{A}(z,\overline{z})\overset{\circ}{B}(z,\overline{z})(\mathrm{Im} z)^{-2}\mathrm{d}z\mathrm{d}\overline{z}.$$

This is now a generalization of the formula 3.3.e in [GHJ], computing the trace of a product of two Toeplitz operators having symbols automorphic forms. In fact the covariant symbols for operators in $\mathcal{A}_{1/h}$ may be regarded as a generalization of automorphic forms (in fact any pair of automorphic forms gives rise to such a symbol which could be eventually the symbol of an unbounded operator).

With the terminology we have just introduced our main result also shows that

Proposition. *Let π_r be the projective representations of $PSL(2,\mathbb{R})$on the Hilbert space $H_r = H^2(\mathbb{H}, (Im\ z)^{-2+r} dz d\overline{z})$ which are given by the same formula as the unitary representations in the discrete analytic series for $PSL(2,\mathbb{R})$ when r is an integer. Assume that Γ is a lattice subgroup of $PSL(2,\mathbb{R})$ so that the second group cohomology cocycle for the projective representation π_r of $PSL(2,\mathbb{R})$ vanishes by restriction to $H^2(\Gamma,\mathbb{T})$.*

Then von Neumann algebra M generated by $\pi_r(\Gamma)$ in $B(H_r)$ is isomorphic to $\mathcal{L}(\Gamma)$ and $\dim_M H_r = \frac{r-1}{\pi}(covol\,\Gamma)$.

In the remaining part of the paper we will be concerned with certain cohomology classes that are associated with a deformation of algebras. Formally to any deformation quantization one could associate a 2-Hochschild cohomology cocycle $a, b \to a *_r' b$ (which lives on a suitable subalgebra on which derivations are possible) defined by

$$a *_r' b = \frac{\mathrm{d}}{\mathrm{d}r}(a *_r b).$$

That this formally verifies the properties of a 2-Hochschild cocycle can be seen easy by taking the derivative in the deformation parameter of the relation expressing the associativity of the multiplication.

This 2-cocycle should measure, in a certain sense, the obstruction for the algebras in the deformation to be isomorphic. In particular if this element vanishes in the second cohomology group, then one could hope to find an eventually unbounded operator X_r for all r so that

$$X_r(a *_r b) - (X_r a) *_r b - a *_r (X_r b) = a *_r' b, \tag{0.3}$$

for suitable a, b.

If this operator could be made selfadjoint and if the evolution equation

$$\dot{y}(r) = X_r y(r),$$

would have a solution for a dense set of initial values then let the associated evolution operator be $U(s,t)$. Recall that $U(s,t)$ is defined by the condition that $U(s,t)y$ is the solution of the differential equation at point s with initial y condition at t. Then $U(s,t)$ would be an isomorphism of algebras. Indeed we would have formally that if $\dot{y}(r) = X_r y(r)$ and $\dot{z}(r) = X_r z(r)$, then

$$\frac{\mathrm{d}}{\mathrm{d}r}(y(r) *_r z(r)) = \dot{y}(r) *_r z(r) + y(r) *_r' z(r) + y(r) *_r \dot{z}(r).$$

The identity (0.3) would then show that the last expression can be further reduced to $X_r(y(r) *_r z(r))$. The unicity of the solutions of the differential equation shows that the evolution operator must then be an isomorphism of algebras.

Of course to make this formal argument work properly, a lot of conditions about domains of unbounded operators should be checked and this seems practically impossible.

A possible approach to overcome this difficulties in particular cases would be to use the quadratic forms instead of looking at bounded operators. This amounts to looking at

$$\frac{\mathrm{d}}{\mathrm{d}r}\langle (a *_r b, c\rangle_\tau = \frac{\mathrm{d}}{\mathrm{d}r}\tau(a *_r *_r c^*),$$

for convenient a, b, c rather then looking at $\frac{\mathrm{d}}{\mathrm{d}r}(a *_r b)$.

If we normalize the previous trilinear functional by discarding the terms which come from the "skewing effect" due to the fact that the trace of a product of two elements depends on the deformation parameter parameter, we will end up with a cyclic cohomology two cocycle ψ_r associated to the deformation.

We introduce a new norm $||\cdot||\lambda, r$ for each r on the algebras in the deformation and denote the set of all elements in $B(H_r)$ that are finite with respect to this norm by $B(\hat{H}_r)$. This norm is the analogue of the usual $\sigma(l^1, l^\infty)$ norm on finite matrices and is defined for general operators on H_r by considering their kernels to be "continuous matrices" (see paragraph 2 for the precise definition of the norm $||\cdot||\lambda, r$. The remarkable property of the norms $||\cdot||\lambda, r$ is

Proposition. *The algebras $B(\hat{H}_r)$ consisting of all elements in $B(H_r)$ that are finite with respect to this norm are involutive Banach algebras not only with respect* *to the product* $*_r$ *but also with respect to all the products* $*_s$ *for* $s \geq r$. *Moreover* $B(\hat{H}_r)$ *is a (dense) vector subspace of* $B(\hat{H}_s)$.

Proposition. *We consider the subalgebras* $\hat{\mathcal{A}}_r = \mathcal{A}_r \cap B(\hat{H}_r)$. *Then the derivatives involved in the definition of* ψ_r *make sense for all* a, b, c *in* $\hat{\mathcal{A}}_r$ *and* ψ_r *is bounded with respect to this norm. More precisely*

$$|\psi_r(a,b,c)| \leq const||a||_{\lambda,r}||b||_{L^2(\mathcal{A}_r)}||c||_{L^2(\mathcal{A}_r)}, a,b,c \in \hat{\mathcal{A}}_r.$$

In the last paragraph of this paper we will show that

Theorem. *A cyclic two cocycle* ψ *on an arbitrary type* II_1 *factor, which has the property that*

$$\psi(a,b,c) \leq ||a||_\infty ||b||_2 ||c||_2, a,b,c \in M,$$

vanishes in the second cyclic cohomology group $H^2_\lambda(M, \mathbb{C})$ *(see A. Connes' article [Co] for the definition of this groups). Moreover in this case the cocycle implementing* ψ *could be chosen so as to correspond to an antisymmetric bounded operator on both* $L^2(M, \tau)$ *and* M.

But then such a solution would lead to a linear differential equation with bounded linear operators which is known to have a well defined evolution operator.

Thus if we have a deformation quantization of algebras in which the cyclic cohomology cocycle verifies the more restrictive boundedness condition mentioned above, then the associated evolution operator induces an isomorphism between the algebras in the deformation, at different values of the parameter.

In the case of the cocycles ψ_r that arise in connection with the equivariant form of the Berezin quantization, it is not clear if one could prove that ψ_r is such a bounded linear functional as above.

The best we can do is to write down an explicit formula for $\psi_r(A,B,C)$ in terms of the symbols of the operators A,B,C. The formula for $\psi_r(A,B,C)$ is the same as the formula $\tau(ABC)$. The difference between the two formulas is that for $\psi_r(A,B,C)$ one has to juxtapose to the integrand defining $\tau(ABC)$ an Alexander Spanier cocycle θ which is a diagonally $\Gamma-$ equivariant function on $\mathbb{H}^3$. If

$$(0.4)\quad \phi(z,\overline{\zeta}) = i\arg((z-\overline{\zeta})/2i) = \ln((z-\overline{\zeta})/2i) - \overline{\ln((z-\overline{\zeta})/2i)},\quad z,\zeta \text{ in } \mathbb{H}$$

then θ has the expression

$$\theta(z,\eta,\zeta) = \phi(z,\overline{\zeta}) + \phi(\zeta,\overline{\eta}) + \phi(\eta,\overline{z}),\quad z,\eta,\zeta \text{ in } \mathbb{H}.$$

The cocycle ψ_r would be bounded if one could prove that the function ϕ is a bounded Schur multiplier on the space $B(H_r$ for r is in an interval.

If $\Gamma = PSL(2,\mathbb{Z})$ then is easy to see that θ is vanishing as a $\Gamma-$ equivariant cocycle (in the $\Gamma-$equivariant Alexander-Spanier cohomology) because

$$(0.5)\qquad \theta(z,\eta,\zeta) = \tilde{\phi}(z,\overline{\zeta}) + \tilde{\phi}(\zeta,\overline{\eta}) + \tilde{\phi}(\eta,\overline{z}),\quad z,\eta,\zeta \text{ in } \mathbb{H},$$

with a diagonally, $\Gamma-$ equivariant $\tilde{\phi}$. The formula for $\tilde{\phi}$ is

$$\tilde{\phi}(z,\overline{\zeta}) = \arg((z-\overline{\zeta})/2i) + \ \arg\ (\Delta(z)) - \overline{\ \arg\ (\Delta(\zeta))},\quad z,\zeta \text{ in } \mathbb{H}.$$

Here Δ is the unique automorphic form of order 12. The disadvantage for $\tilde{\phi}$ is that $\tilde{\phi}$ is not a bounded function although θ is. The fact that one can not find a bounded $\Gamma-$equivariant $\tilde{\phi}$ solving the above equation is related to the non-vanishing of $H^2_{\mathrm{bound}}(PSL(2,\mathbb{Z})$ (see [Ghys]).

1. Berezin quantization of the upper half plane

In this paragraph we recall some facts concerning the Berezin's quantization of the upper half plane ([Be.1], [Be.2], [Upm]). Berezin realizes the deformation quantization for the upper half plane (and in fact for more general symmetric domains) by using symbols for bounded operators acting on Hilbert spaces of analytic functions. As the bounded operators on Hilbert spaces of analytic functions are always given by reproducing kernels, this symbols will always exist for bounded operators.

We let H_r be the Hilbert space of square integrable analytic functions on, the upper half plane $\mathbb{H}$ with respect to the measure ν_r, which has density $(\mathrm{Im}\ \ z)^{r-2}$ with respect to the canonical Lebesgue measure $dzd\overline{z}$ on $\mathbb{H}$. H_r is nonzero for $r > 1$ ([Ba]).

The choice of the measure ν_r for the Hilbert spaces H_r is dictated by the fact that this Hilbert spaces are enacted with projective, unitary representations $(\pi_r)_{r>1}$ of $PSL(2, \mathbb{R})$. We recall first from ([Ba], see also [Sa], [Puk]) the construction of this representations. Also recall that ν_0 is a invariant measure on $\mathbb{H}$ under the action of $SL(2, \mathbb{R})$.

Definition 1.1. *([Ba]) Let H_r, $r > 1$ be the Hilbert space of all analytic functions with $||f||_r^2 < \infty$ where the Hilbert norm is defined by*

$$||f||_r^2 = \int_{\mathbb{H}} |f(z)|^2 (Im\ \ z)^{r-2} dzd\overline{z} = \int_{\mathbb{H}} |f(z)|^2 d\nu_r(z).$$

Assume that $G = PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transforms:

$$G \times \mathbb{H} \ni (\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ z) \to \frac{az+b}{cz+d} \in \mathbb{H}$$

and let: $j(g,z) = (cz+d)$. *Also choose (see [Ma, p. 113]) a normal branch of* $arg(j(g,z)) = arg(cz+d)$, *for all* $z \in \mathbb{H}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ *where the function arg takes its values in* $-\pi < t \leq \pi$.

Using this branch for $(cz+d)^r = exp(r\,ln(cz+d))$ *one defines*

$$(\pi_r(g)f)(z) = (cz+d)^{-r} f(g^{-1}z),\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,\ f \in H_r,\ z \in \mathbb{H}\ .$$

Then([Ba]) $\pi_r : PSL_2(\mathbb{R}) \to B(H_r)$ *is a projective, unitary representation of* $PSL_2(\mathbb{R})$ *with cocycle* $c_r(g_1,g_2) \in \{z|\ |z| = 1,\ z \in \mathbb{C}\}$ *defined by*

$$\pi_r(g_1,g_2) = c_r(g_1,g_2)\pi_r(g_1)\pi_r(g_2),\ g_1,g_2 \in G.$$

If $r = 2,3,...$ *then* π_r *is an actual representation of* $PSL_2(\mathbb{R})$ *and belongs to the discrete series of representations for* $PSL_2(\mathbb{R})$ *(see[La]).*

Recall that any Hilbert space of analytic functions has a naturally associated reproducing kernel ([Aro]). For H_r this has been computed already by [Ba] and we recall the formulae.

Theorem 1.2. *([Ba]). The reproducing kernel* $k_r(z,\zeta)$ *for* $\mathbb{H}$ *is given by the formula*

$$k_r(z,\zeta) = \frac{c_r}{((z-\overline{\zeta})/2i)^r} for \ \ all z,\zeta \in \mathbb{H}.$$

In particular the following functions on $\mathbb{H}$ *defined for all* $z \in \mathbb{H}$,

$$e_z^r(\zeta) = \frac{c_r}{((\zeta-\overline{z})/2i)^r},\ \zeta \in \mathbb{H}$$

belong to $H_r = H^2(\mathbb{H},\nu_r)$ *and* $\langle f, e_z^r\rangle_r = f(z),\ $ *for all* $f \in H_r,\ z \in \mathbb{H}$.

Corollary. *1.3 ([Ba]). The orthogonal projection P_r from $L^2(\mathbb{H}, \nu_r)$ onto $H_r = H^2(\mathbb{H}, \nu_r)$ is given by the formula*

$$(P_r f)(z) = \langle f, e_z^r \rangle = c_r \int_{\mathbb{H}} \frac{f(\zeta)}{((z - \overline{\zeta})/2i)^r} d\nu_r(\zeta)$$

for all f in $L^2(\mathbb{D}, \nu_r)$. In the terminology of vector valued integral, this is

$$P_r f = \int_{\mathbb{H}} f(\zeta) e_\zeta^r d\nu_r(\zeta).$$

The fact that the Hilbert spaces H_r have evaluation vectors $e_z^r,\ z \in \mathbb{H}$, shows that all bounded linear operators B on $H_r = H^2(\mathbb{H}, \nu_r)$ are given by integral kernels. Berezin defined the contravariant symbol for an operator B on H_r to be its normalized integral kernel $\hat{B}(z, \overline{\zeta})$, which is a function on $\mathbb{H}^2$.

Definition 1.4. *([Be1,2]). For B in $B(H_r)$ let the contravariant symbol $\hat{B} = \hat{B}(z, \overline{\zeta})$ $z, \zeta \in \mathbb{H}$ be the function on $\mathbb{H}^2$, analytic in z, antianalytic in ζ defined by:*

$$\hat{B}(z, \overline{\zeta}) = \frac{\langle B e_\zeta^r, e_z^r \rangle}{\langle e_\zeta^r, e_z^r \rangle},\ z, \zeta \in \mathbb{H}^2.$$

Then $\hat{B}$ completely determines B by the formula:

$$(Bf)(z) = \langle Bf, e_z^r \rangle = \langle B(\int f(\zeta) e_\zeta^r d\nu_r(\zeta), e_z^r \rangle$$

$$= \int f(\zeta) \langle B e_\zeta^r, e_z^r \rangle d\nu_r(\zeta) = c_r \int_{\mathbb{H}} \frac{\hat{B}(z, \overline{\zeta}) f(\zeta)}{((z - \overline{\zeta})/2i)^r} d\nu_r(\zeta), z \in \mathbb{H},\ f \in H^2(\mathbb{H}, \nu_r).$$

The following properties for $\hat{B}$ are obvious consequences of the definition. In particular they show that the above integral is absolutely convergent.

Proposition. *1.5 ([Be]). Let B be any bounded linear operator acting on $H_r = H^2(\mathbb{H}, \nu_r)$ and let $\hat{B} = \hat{B}(z, \overline{\zeta}),\ z, \zeta \in \mathbb{H}$ be its symbol as above. Denote by $||B||_{\infty,r}$ the uniform norm of B (as an element of $B(H_r)$). Then:*

a) For all ζ in $\mathbb{H}$, the function on $\mathbb{H}$ defined by $z \to c_r((z-\overline{\zeta})/2i)^{-r}\hat{B}(z,\overline{\zeta})$ belongs to $H^2(\mathbb{H}, \nu_r)$ and has L^2norm less than $c_r^{1/2}||B||_{\infty,r}(Im\ \zeta)^{-r/2}$.

b) For all z in $\mathbb{H}$, the function on $\mathbb{H}$ defined by $\zeta \to c_r((z-\overline{\zeta})/2i)^{-r}\hat{B}(z,\overline{\zeta})$ belongs to $H^2(\mathbb{H}, \nu_r)$ and has L^2norm less than $c_r^{1/2}||B||_{\infty,r}(Im\ \zeta)^{-r/2}$.

c) If B is identified with P_rBP_r as an operator acting on $L^2(\mathbb{H}, \nu_r)$ then the formula

$$(Bf)(z) = c_r \int_{\mathbb{H}} \hat{B}(z, \overline{\zeta}) f(\zeta) d\nu_r(\zeta)$$

holds for all f in $L^2(\mathbb{H}, \nu_r)$.

d) $|B(z, \overline{z}) \leq ||B||_{\infty,r}$, for all z in $\mathbb{H}$.

e) The symbol $\hat{B^*} = B^*(z, \overline{\zeta}), z, \overline{\zeta} \in \mathbb{H}$ of B^* (the adjoint of B) is given by the formula:

$$\hat{B^*}(z, \overline{\zeta}) = \overline{B(\zeta, \overline{z})}, z, \zeta \in \mathbb{H}.$$

Proof. a) follows from the fact that $c_r((z-\overline{\zeta})/2i)^{-r}\hat{B}(z,\overline{\zeta})$ is by definition $\langle Be^r_\zeta, e^r_z \rangle$. Thus for fixed ζ in H, this is $(Be^r_\zeta)(z)$. The L^2 norm of Be^r_ζ is, by the boundedness of B, less than

$$||B||_{\infty,r}||e^r_\zeta|| = c_r^{1/2}||B||_{\infty,r}(\text{Im}\ \ \zeta)^{-r/2}$$

as $\overline{B(z,\zeta)}$ is antianalytic in ζ.

$$||e^r_\zeta||^2 = \langle e^r_\zeta, e^r_\zeta \rangle = e^r_\zeta(\zeta) = \frac{c_r}{(\text{Im}\ \zeta)^r}.$$

This completes the proof of a) and b) is similar. The point e) is obvious while for point d) we observe that

$$|B(z,\overline{z})| = |\langle Be_z^r, e_z^r\rangle\langle e_z^r, e_z^r\rangle^{-1}|$$

$$\leq ||B||_{\infty,r}||e_z^r||^2||e_z^r||^{-2} = ||B||_{\infty,r}.$$

Point c) now follows from the fact that all f in $L^2(\mathbb{H}, d\nu_r)$ we have

$$c_r\int_{\mathbb{H}}\hat{B}(z,\overline{\zeta})f(\zeta)d\nu_r(\zeta) = \langle f, B(z,\cdot)\rangle_r = \langle P_r\ f, \overline{B(z,\cdot)}\rangle_r$$

for all z in $\mathbb{H}$, as $\overline{B(z,\cdot)}$ is analytic in ζ.

We now recall the definition of the Berezin product of the symbols $\hat{A},\ \hat{B}$, which are functions on $\mathbb{H}^2$. The product will depend on the quantization variable r, and it represents in fact the symbol of the composition (in $B(H_r)$) of the bounded operators on H_r that are represented by the symbols $\hat{A},\ \hat{B}$.

Definition. *([Be 1]). Let $A,\ B$ be two functions on $\mathbb{H}^2$, analytic in the first variable, antianalytic in the second and so that $((z-\overline{\zeta})/2i)^r A(z,\overline{\zeta})$, as a function of $\overline{\zeta}$ on H, keeping z in $\mathbb{H}$ fixed, belongs to $H^2(\mathbb{H},\nu_r)$, and so that $((z-\overline{\zeta})/2i)^r B(z,\overline{\zeta})$ as a function of z, on $\mathbb{H}$, keeping ζ in $\mathbb{H}$ fixed, belongs to $H^2(\mathbb{H},\nu_r)$. For $r>1$ the product of the two symbols A, B is defined by the formula:*

$$(A *_r B)(z,\overline{\zeta}) = c_r((z-\overline{\zeta})/2i)^r\int_{\mathbb{H}}((z-\overline{\eta})/2i)^{-r}A(z,\overline{\eta})((\eta-\overline{\zeta})/2i)^{-r}B(\eta,\overline{\zeta})d\nu_r(\eta).$$

for all z,ζ in $\mathbb{H}^2$.

The following formula is an easy consequence of the integral representation for operators acting on $H_r = H^2(\mathbb{H},\nu_r)$. It shows that the above product is the symbol of the composition of the operators in $B(H_r)$.

Proposition. *([Be]). If* $\hat{A} = \hat{A}(z,\zeta), \hat{B} = \hat{B}(z,\zeta), z,\zeta \in \mathbb{H}^2$ *are the symbols of two bounded linear operators* A *and* B*, respectively, acting on* H_r*, then* $\hat{A} *_r \hat{B} = (\hat{A} *_r \hat{B})(z,\overline{\zeta}), z,\zeta \in \mathbb{H}^2$ *is the symbol of the composition* AB *in* $B(H_r)$ *of the two operators* A *and* B*.*

Proof. We have to compute $\langle ABe^r_\zeta, e^r_z\rangle_r \langle e^r_\zeta, e^r_z\rangle_r^{-1}$ for z,ζ in $\mathbb{H}$, and show that this is equal to $(\hat{A} *_r \hat{B})(z,\overline{\zeta})$, if we know that $\hat{A}(z,\overline{\zeta}) = \langle Ae^r_\zeta, e^r_z\rangle_r \langle e^r_\zeta, e^r_z\rangle_r^{-1}$ and similarly for B. We have:

$$\langle ABe^r_\zeta, e^r_z\rangle_r = \langle A \int_{\mathbb{H}} (Be^r_\zeta)(\eta) e^r_\eta d\nu_r(\eta), e^r_z\rangle_r =$$

$$\int_{\mathbb{H}} (Be^r_\zeta)(\eta)\langle Ae^r_\eta, e^r_z\rangle d\nu_r(\eta) = \int_{\mathbb{H}} \langle Be^r_\zeta, e^r_\eta\rangle_r \langle Ae^r_\eta, e^r_z\rangle d\nu_r(\eta)$$

and hence for all z,ζ in $\mathbb{H}$

$$\langle ABe^r_\zeta, e^r_z\rangle_r \langle e^r_\zeta, e^r_z\rangle_r^{-1} = \langle e^r_\zeta, e^r_z\rangle_r^{-1} =$$

$$\langle e^r_\zeta, e^r_z\rangle_r^{-1} \int_{\mathbb{H}} \hat{A}(z,\overline{\eta})\langle e^r_\eta, e^r_z\rangle_r \hat{B}(\eta,\overline{\zeta})\langle e^r_\zeta, e^r_\eta\rangle d\nu_r(\eta) =$$

$$c_r((z-\overline{\zeta})/2i)^r \int_{\mathbb{H}} \hat{A}(z,\overline{\eta})((\eta-\overline{z})/2i)^{-r} \hat{B}(\eta,\overline{\zeta})((\eta-\overline{\zeta})/2i)^{-r} d\nu_r(\eta).$$

Finally, we recall the fact that together with the above contravariant symbols for operators acting on H_r, Berezin also introduced another type of symbols called covariant symbols for operators acting on H_r. Contrary to the contravariant symbols for a bounded operator, the covariant symbol does not always exists.

Definition. *([Be]). Let* A *be a bounded operator on* H_r *and let* f *be a bounded measurable function* f *on* $\mathbb{H}$*. Let* M^r_f *be the bounded multiplication operator on*

$L^2(\mathbb{H},\nu_r)$ with the function f and let $T_f^r = P_r M_f^r P_r$ in $B(H_r)$ be the Toeplitz operator with symbol f. Then f is called the covariant symbol of A and one uses (following [Be]) the notation $f = \mathring{A}$, if $A = T_f^r$. Clearly in this case the uniform norm of A is bounded by the essential norm of f.

The relation between the two symbols for a given bounded operator A on H_r is obtained as follows.

Proposition. *([Be]) If f is any bounded measurable function on $\mathbb{H}$, and $T_f^r = P_r M_f^r P_r$ is the corresponding Toeplitz operator with symbol f, acting on the Hilbert space H_r, then the (contravariant) symbol $\hat{A} = \hat{A}(z,\overline{\zeta})$ for the operator $A = T_f^r$ is*

$$\hat{A}(z,\overline{\zeta}) = \langle e_\zeta^r, e_z^r\rangle^{-1} \int_{\mathbb{H}} f(a)\langle e_a^r, e_z^r\rangle\langle e_\zeta^r, e_a^r\rangle d\nu_r(a) =$$

$$c_r^{-1}\int_{\mathbb{H}} f(a)\frac{\langle e_a^r, e_z^r\rangle\langle e_\zeta^r, e_a^r\rangle}{\langle e_\zeta^r, e_z^r\rangle\langle e_a^r, e_a^r\rangle}d\nu_0(a) = c_r\int_{\mathbb{H}} f(a)\frac{((z-\overline{\zeta})/2i)^r((a-\overline{a})/2i)^r}{((z-\overline{a})/2i)^r((a-\overline{\zeta})/2i)^r}d\nu_0(a).$$

In particular $\hat{A}(z,\overline{z}) = (B_r f)(z,\overline{z})$, where B_r is the operator on $L^2(\mathbb{H},\nu_0)$ with kernel the function on $\mathbb{H}^2$ defined by $(z,a) \to [\frac{\mathrm{Im}\ z\ \mathrm{Im}\ a}{|z-\overline{a}|^2}]^2 = k_r(z,a)$.

Denote $\rho(z,a)$ to be $\frac{\mathrm{Im}\ z\ \mathrm{Im}\ a}{|z-\overline{a}|^2}$. Then it is well known from hyperbolic geometry that ρ is the inverse of the hyperbolic cosine of the hyperbolic distance between z and a, for z, a in $\mathbb{H}^2$. In particular the kernel k_r is invariant under the diagonal action of the group $PSL(2,\mathbb{R})$ and hence by ([Se], see also [Ku]) the operator B_r is a function $B_r(\Delta)$ (in the sense of functional calculus for unbounded selfadjoint operators) of the invariant laplacian $\Delta = (\mathrm{Im}\ z)^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$ on $\mathbb{H}$.

The explicit formula for $B_r = B_r(\Delta)$ as function of the laplacian is determined also in [Be] as an infinite product of resolvants of Δ. As we will not need the explicit formula for B_r we will only recall the essential properties of B_r as stated in [Be].

Theorem. *([Be]). The operator $B_r$is a bounded, positive operator on $L^2(\mathbb{H}, \nu_0)$. Moreover $||B_r|| \leq 1$ and the operators B_r pairwise commute for $r > 1$. Also B_r converges strongly to 1 when $r \to \infty$.*

Finally we will recall the duality relation between the covariant and the contravariant symbols. The rigorous form of this statements may be found in the paper by Coburn [Co].

Theorem. *([Be]). Let A be a bounded linear operator on H_r of contravariant symbol $\hat{A} = \hat{A}(z, \overline{\zeta})$ $z, \overline{\zeta} \in \mathbb{H}$ and let B be another bounded linear operator of covariant symbol $f \in L^\infty(\mathbb{H}, \nu_0)$, $f(z) = \mathring{B}(z, \overline{z})$, $z \in \mathbb{H}$. If the operator AB is in the ideal of trace class operators, then*

$$tr_{B(H_r)}(AB) = \int_{\mathbb{H}} \hat{A}(z, \overline{z}) \mathring{B}(z, \overline{z}) d\nu_0(z).$$

The deformation quantization of the upper halfplane is now realized by Berezin, by observing that for any two functions f, g on the upper halfplane, which are so that there exists functions $\hat{A} = \hat{A}(z, \zeta)$, $\hat{B} = \hat{B}(z, \overline{\zeta})$, $z, \zeta \in \mathbb{H}$, analytic in z and antianalytic in ζ with

$$f(z) = \hat{A}(z, \overline{z});\ g(z) = \hat{B}(z, \overline{z}),\ z \in \mathbb{H},$$

one may define

$$(f *_r g)(z) = (\hat{A} *_r \hat{B})(z, \overline{z}),\ z \in \mathbb{H}.$$

This product is well defined for example as long as $\hat{A}, \hat{B}$ are symbols of bounded operators acting on the spaces H_r.

The main result in [Be] is that under suitable conditions on the functions f, g the limits $\lim\limits_{r\to\infty} f *_r g$ and $\lim\limits_{r\to\infty} r(f *_r g - f *_r g)$ exists and are equal to respectively fg and

$$(\mathrm{Im}\ z)^2 (\frac{\partial f}{\partial z}\frac{\partial g}{\partial \overline{z}} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}}).$$

The last limit is the $SL(2,\mathbb{R})$ invariant form of the Poisson bracket on $\mathbb{H}$.

Finally, all the above formulae hold true if one replaces $\mathbb{H}$ by $\mathbb{D}$ the unit disk. The form of the formulae will be the same except that we will have to replace the factor $(z-\overline{\zeta})/2i$ by $(1-z\overline{\zeta})$ (so that Im ζ will be replaced by $1-|\zeta|^2$).

In the case of the unit disk the group which replaces $SL(2,\mathbb{R})$ is $SU(1,1)$, which is the group of all matrices

$$\{\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\}$$

The modular factor $(cz+d)$ is now replaced by $\overline{b}z + \overline{a},\ z \in \mathbb{D}$.

2. Smooth algebras associated to the Berezin quantization

In this paragraph, by analogy with finite matrices, we construct a (weakly dense) subalgebra $\hat{B}(H_r)$ of $B(H_r)$, which is a Banach algebra with respect to a certain norm (the analog of the (l^1, l^∞) norm on finite matrices). We will show that the algebras $\hat{B}(H_r)$ are well behaved, globally, with respect to the Berezin product $*_s$ for all $s \geq r$. In particular the symbols corresponding to the operators in $\hat{B}(H_r)$ will form an algebra under all the operation $*_s$. Thus, operations, such as differentiation of $A *_s B$ will have a sense for A, B in $\hat{B}(H_r)$.

The norm which defines the algebras $\hat{B}(H_r)$ is the analog (modulo a weight) of the supreme (after lines) of the absolute sum of elements in all the rows of a given matrix.

We first start by a criteria (which is essentially contained in Aronszjan memorium ([Aro])) on the contravariant symbol of a bounded operator, for the operator to be positive.

Lemma 2.1. *Let A in $B(H^2(\mathbb{H}, \nu_r))$ be a positive, bounded operator on H_r of uniform norm $||A||_{\infty,r}$ and with contravariant symbol $\hat{A} = \hat{A}(z, \overline{\zeta})$ $z, \zeta \in \mathbb{H}$. Then there exists a constant $M > 0$ so that the following matrix inequality holds for all N in $\mathbb{N}$ and all $z_1, z_2, ...z_N$ in $\mathbb{H}$:*

$$0 \leq [\hat{A}(z_i, \overline{z_j})((z_i - \overline{z_j})/2i)^{-r}]_{i,j=1}^{N} \leq [M((z_i - \overline{z_j})/2i)^{-r}]_{i,j=1}^{N}$$

Moreover, given A, the best constant M for which the above inequality holds for all N, $z_1, z_2, ...z_N$ is the uniform norm $||A||_{\infty,r}$.

Conversely, let K be a kernel $k = k(z, \overline{\zeta})$ on $\mathbb{H}^2$, analytic in z, antianalytic in ζ for which there exists a positive constant M such that the above inequality holds with k replacing $\hat{A}$, for all N in $\mathbb{N}$ and all $z_1, z_2, ...z_N$ in $\mathbb{H}$.

Then k is the contravariant symbol of a positive bounded operator on $\mathbb{H}$ of uniform norm less than M.

Proof. We start with the direct part of our statement. The inequality we have to prove is equivalent to showing (with $M = ||A||_{\infty,r}$) that for all f in $L^2(\mathbb{H}, \nu_r)$

one has

$$0 \leq \int\int_{\mathbb{H}^2} \langle Ae^r_\zeta, e^r_z\rangle f(\zeta)\overline{f(z)}d\nu_r(z,\zeta) \leq M \int\int_{\mathbb{H}^2} \langle e^r_\zeta, e^r_z\rangle f(\zeta)\overline{f(z)}d\nu_r(z,\zeta).$$

We identify A with P_rAP_r as an operator acting boundedly on $L^2(\mathbb{H},\nu_r)$. By proposition 1.5c A has as reproducing kernel, the function on $\mathbb{H}^2$, $(z,\zeta) \to \langle Ae^r_\zeta, e^r_z\rangle$. Hence the above inequality is

$$0 \leq \langle Af, f\rangle \leq M\langle P_r f, f\rangle = M\langle P_r f, P_r f\rangle, f \text{ in } L^2(\mathbb{D},\nu_r).$$

Since $A = P_rAP_r$ and A is positive, of norm $M = ||A||_{\infty,r}$, this holds true. The converse is along this lines. This completes the proof.

We introduce the following notation for a square root of the inverse of the hyperbolic distance between points in the upper half plane.

Notation 2.2. *For z, ζ in $\mathbb{H}$ denote by $d(z,\overline{\zeta})$ the quantity:*

$$d(z,\overline{\zeta}) = \frac{(Im\ z)^{1/2}(Im\ \zeta)^{1/2}}{[(z-\overline{\zeta})/2i]},\ \textit{for all } z,\zeta \textit{ in } \mathbb{H}$$

or in the unit disk representation

$$d(z,\overline{\zeta}) = \frac{(1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}}{1-z\overline{\zeta}}, \textit{for all } z,\zeta \textit{ in } \mathbb{D}.$$

Let $D_h(z,\zeta)$ be the hyperbolic distance between points in $\mathbb{H}$ (respectively $\mathbb{D}$). Then

$$|d(z,\overline{\zeta})|^2 = \rho(z,\zeta) = cosh^{-1}(D_h(z,\zeta)),\ \textit{for all } z,\zeta \textit{ in } \mathbb{H}.$$

In the next lemma we will show that the growth of a symbol $\hat{A} = \hat{A}(z,\overline{\zeta})$, when z,ζ are approaching the boundary of $\mathbb{H}$, is of the same type (in absolute value) as the growth of $\rho(z,\zeta)^{-r/2}$ when z,ζ are approaching the boundary. Moreover the estimate depends only on the uniform norm of A.

Corollary 2.3. *Let A in $B(H_r)$ be a bounded operator and let $\hat{A} = \hat{A}(z, \overline{\zeta})$ $z, \zeta \in \mathbb{H}$ be its (contravariant) symbol. Then*

i) if A is positive then

$$|A(z, \overline{\zeta})||d(z, \zeta)|^r \leq (A(z, \overline{z}))^{1/2}(A(\zeta, \overline{\zeta}))^{1/2} \leq \sup_{w \in \mathbb{H}} A(w, \overline{w}), z, \zeta \text{ in } \mathbb{H}.$$

ii) If A is arbitrary in $B(H_r)$ then

$$|A(z, \overline{\zeta})(d(z, \zeta))^r| \leq 4||A||_{\infty, r} \text{ for all } z, \zeta \in \mathbb{H}.$$

Proof. i) If A is a positive element in $B(H_r)$, then by the preceding proposition, for all z, ζ in $\mathbb{H}$, the matrix

$$\begin{pmatrix} A(z, \overline{z})(\text{Im } z)^{-r} & A(z, \overline{\zeta})(z - \overline{\zeta})/2i)^{-r} \\ \overline{A(z, \overline{\zeta})(z - \overline{\zeta})/2i)^{-r}} & A(\zeta, \overline{\zeta})(\text{Im } \overline{\zeta})^{-r} \end{pmatrix}$$

is positive. (The expression for the bottom left corner is justified by Proposition 1.5.d.

It is obvious that if $(a_{i,j})_{i,j=1}^2$ is a positive matrix, then so is $(a_{i,j} \ \lambda_i \overline{\lambda_j})_{i,j=1}^2$ for all λ_1, λ_2 in $\mathbb{C}$. We let $\lambda_1 = (\text{Im } z)^{r/2}, \ \lambda_2 = (\text{Im } \zeta)^{r/2}$ and apply this statement to the above matrix. We obtain that the following matrix,

$$\begin{pmatrix} A(z, \overline{z}) & A(z, \overline{\zeta})(d(z, \overline{\zeta}))^r \\ \overline{A(z, \overline{\zeta})(d(z, \overline{\zeta}))^r} & A(\zeta, \overline{\zeta}) \end{pmatrix}$$

is positive for all z, ζ in $\mathbb{H}$. The condition that its determinant be positive proves condition i).

To prove ii) we observe that by i) we already know that for all positive elements A in $B(H_r)$ one has

$$|A(z,\overline{\zeta})(d(z,\zeta))^r| \leq \sup_{w\in\mathbb{H}} A(w,\overline{w}) \leq ||A||_{\infty,r}$$

(we use again Proposition 1.5.c). To get the general statement we use the fact that any element A in $B(H_r)$ has the expression ([Dix]).

$$A = [(\text{Re } A)_+ - (\text{Re } A)_-] + i[(\text{Im } A_+) - (\text{Im } A_-)]$$

where $(\text{Re } A)_{\underline{+}}, (\text{Im } A)_{\underline{+}}$ are positive of uniform norm less then the uniform norm $||A||_{\infty,r}$ of A.

The preceding proposition shows already that the symbol $\hat{A} = \hat{A}(z,\overline{\zeta})$ of an operator A in $B(H_r)$ will define, by the integral formula, a bounded operator on all the Hilbert spaces H_s, with $s \geq r$. Moreover the uniform norm doesn't increase.

Proposition 2.4. *Fix $r > 1$ and let $s \geq r$. Let $j_{s,r}$ be the map which assigns to an operator A on $B(H_r)$ with contravariant symbol $\hat{A} = \hat{A}(z,\overline{\zeta}),\ z,\overline{\zeta} \in \mathbb{H}$ the operator acting on H_s with the same contravariant symbol, that is*

$$(j_{s,r}(A)f)(z) = \int_{\mathbb{H}} \hat{A}(z,\overline{\zeta})\langle e^s_\zeta, e^s_z\rangle f(\zeta) d\nu_s(\zeta),\ f \text{ in } H_s.$$

Then $j_{s,r}$ takes its values in $B(H_s)$ and

$$||j_{s,r}(A)||_{\infty,s} \leq 4||A||_{\infty,r}$$

$$j_{s,r}(B(H_r)_+) \subseteq B(H_s)_+.$$

Moreover $j_{s,r}$ has a weakly dense image in $B(H_s)$. We convene to denote $j_{s,r}(A)$ also by A, this being justified by the fact that both operators have the same symbol.

Proof. As in the preceding proof, to show the estimate on the norm, it is sufficient to prove that $j_{s,r}(B(H_r)_+) \subseteq B(H_s)_-$ and that

$$||j_{s,r}(A)||_{\infty,s} \leq ||A||_{\infty,r} \text{ for } A \text{ in } B(H_r)_+.$$

If A belongs to $B(H_r)_+$, we have seen in Proposition 2.1, with $M = ||A||_{\infty,r}$, that for all N in $\mathbb{N}$, $z_1, z_2, ...z_N$ in $\mathbb{H}$ one has the following matrix inequality:

$$(2.1) \quad 0 \leq [A(z_i, \overline{z}_j)((z_i - \overline{z_j})/2i)^{-r}]_{i,j} \leq M[((z_i - \overline{z_j})/2i)^{-r}]_{i,j}^N.$$

From general matrix theory we know that if the matrix inequality $0 \leq (a_{i,j})_{i,j=1}^N \leq (b_{i,j})_{i,j=1}^N$ holds for some matrices $(a_{i,j})_{i,j=1}$, $(b_{i,j})_{i,j=1}$ in $M_N(\mathbb{C})$ then so does the matrix inequality

$$0 \leq (c_{i,j} \cdot a_{i,j})_{i,j=1}^N \leq (c_{i,j} \cdot b_{i,j})_{i,j=1}^N$$

for any other positive matrix $(c_{i,j})_{i,j=1}^N$ in $M_N(\mathbb{C})$.

On the other hand by ([Shapiro, Shields]), for $s \geq r$ the matrix

$$[\frac{1}{[(z_i - \overline{z_j})/2i]^{s-r}}]_{i,j=1}^N$$

is always positive for all N in $\mathbb{N}$, $z_1, z_2, ...z_N$ in $\mathbb{H}$. Hence the above remarks and formula (2.1) prove that for all N in $\mathbb{N}$, $z_1, z_2, ...z_N$ in $\mathbb{H}$, the following matrix inequality holds:

$$0 \leq [A(z_i, \overline{z}_j)((z_i - \overline{z_j})/2i)^{-s}]_{i,j}^N \leq ||A||_{\infty,r}[((z_i - \overline{z_j})/2i)^{-s}]_{i,j}^N.$$

The converse to Proposition 2.1 shows that $j_{s,r}(A)$ is a positive operator in $B(H_s)$ of uniform norm less than $||A||_{\infty,r}$.

It remains to check that $j_{s,r}(B(H_r)$ is weakly dense in $B(H_s)$. It is sufficient to check that $j_{s,r}(\mathcal{C}_2(H_r)$ is weakly dense in $B(H_s)$ (we use here the standard notation $\mathcal{C}_2(H)$ for the Hilbert- Schmidt operators on H). In the canonical identification of $\mathcal{C}_2(H_r)$ with $H_r \overline{\otimes} H_r$ (see [Sch],[B.S]) an operator A in $j_{s,r}(B(H_r))$ with symbol $\hat{A} = \hat{A}(z, \overline{\zeta})$ will correspond to the function on $\mathbb{H}^2$, defined by

$$(z, \overline{\zeta}) \to \hat{A}(z, \overline{\zeta}[(z - \overline{\zeta})/2i]^{-(s-r)}.$$

For fixed η_1, η_2 in $\mathbb{H}$ and $\epsilon > 0$, the function on $\mathbb{H}^2$,defined by

$$l_{\eta_1,\eta_2,\epsilon}(z, \overline{\zeta}) = (z - \overline{\zeta})/2i]^{(s-r)}[(z - \overline{\eta_1})/2i]^{-s}][(\eta_2 - \overline{\zeta})/2i]^{-s}] \ \exp(-\epsilon z) \exp(-\epsilon \overline{\zeta})$$

belongs to $H_r \overline{\otimes}\ \overline{H}_r$. As $\epsilon \to 0$ the function $(z, \overline{\zeta}) \to (1 - z\overline{\zeta})^{(s-r)} \phi_{\eta_1,\eta_2,\epsilon}(z, \overline{\zeta})$ converges weakly in $H_r \overline{\otimes}\ \overline{H}_r$ to the function on $\mathbb{H}^2$

$$(z, \overline{\zeta}) \to [(z - \overline{\eta_1})/2i]^{-s}][(\eta_2 - \overline{\zeta})/2i]^{-s}].$$

But this is the symbol of the 1 dimensional operator in $B(H_s)$ defined (on H_s) by

$$f \to e^r_{\eta_1} \langle f, e^r_{\eta_2} \rangle.$$

This proves the weak density of $j_{s,r}(\mathcal{C}_2(H_r)$ in $B(H_s)$ and this completes the proof.

We now define for A in $B(H_r)$, which by looking at its symbol, may be taught of as a continuous matrix, the analogue of the (l^∞, l^1) norm for finite matrices. This norm will make the inclusions $j_{s,r},\ s \geq r$ continuous and it will induce a new, involutive, Banach algebra structure on a subalgebra in $B(H_r)$.

Definition 2.5. *Let $|d(z, \overline{\zeta})|$ be the function on the hyperbolic distance between z and ζ in $\mathbb{H}$ defined in 2.2. For A in $B(H_r)$ with contravariant symbol $\hat{A} =$*

$\hat{A}(z,\overline{\zeta}),\ z,\overline{\zeta}\in\mathbb{H}$ define $||A||_{\lambda,r}$ to be the maximum of the following integrals:

$$\sup_{z\in\mathbb{H}} c_r \int_{\mathbb{H}} |A(z,\overline{\zeta})d(z,\overline{\zeta})^r| d\nu_0(\zeta) \ \textit{and}$$

$$\sup_{\zeta\in\mathbb{H}} c_r \int_{\mathbb{H}} |A(z,\overline{\zeta})d(z,\overline{\zeta})^r| d\nu_0(z).$$

Let $\hat{B}(H_r)$ be the vector space of all A in $B(H_r)$ for which the quantity $||A||_{\lambda,r}$ is finite. Clearly $||\ ||_{\lambda,r}$ is a norm on $\hat{B}(H_r)$.

We will now show that with respect to the norm $||\ ||_{\lambda,r}$, $\hat{B}(H_r)$ becomes an involutive Banach algebra. Moreover the image of $\hat{B}(H_r)$ in $B(H_s)$ for $s\geq r$ is closed under the products $*_s$ for $s\geq r$ and this will allow us to differentiate the product.

Lemma 2.6. *Let A,B be bounded linear operators in $B(H_r)$ with symbols $\hat{A}=\hat{A}(z,\overline{\zeta})$ and respectively $\hat{B}=\hat{B}(z,\overline{\zeta}),\ z,\overline{\zeta}\in\mathbb{H}$. If A,B belong to the algebra $\hat{B}(H_r)$ defined in in the preceding statement, then for all $s\geq r$, the product*

$$j_{s,r}(A) *_s j_{s,r}(B)$$

which we denote simply by $A*_sB$, belongs to $\hat{B}(H_r)$ (more precisely to $j_{s,r}(\hat{B}(H_r))$). Moreover

$$||A *_s B||_{\lambda,r} \leq 2^{s-r}(\frac{c_s}{c_r})||A||_{\lambda,r}||B||_{\lambda,r}.$$

Proof. It is sufficient to show that the integral:

$$c_r c_s \int_{\mathbb{H}} |\hat{A}(z,\overline{\eta})||(z-\overline{\eta})/2i|^{-s}|\hat{B}(\eta,\overline{\zeta})||(\eta-\overline{\zeta})/2i|^{-s} d\nu_s(\eta)|d(z,\overline{\eta})|^r d\nu_0(\zeta)$$

(and a similar integral corresponding the other term in the definition of $|| \, ||_{\lambda,r}$) are bounded by $2^{s-r}(\frac{c_s}{c_r})||A||_{\lambda,r}||B||_{\lambda,r}$.

Regrouping the factors in the integral it follows that we have to estimate:

$$c_r^2(\frac{c_s}{c_r})\int\int_{\mathbb{H}}|\hat{A}(z,\overline{\eta})||d(z,\overline{\eta})|^r|\hat{B}(\eta,\overline{\zeta})||d(\eta,\overline{\zeta})|^r(M(z,\eta,\zeta))^{s-r}d\nu_0(\eta,\zeta)$$

where M is the positive valued function on $\mathbb{H}^3$ defined by the formula

$$M(z,\eta,\zeta)=|\text{Im }\eta||(z-\overline{\zeta})/2i|(z-\overline{\eta})/2i|^{-1}|(\eta-\overline{\zeta})/2i|^{-1}$$

We will show in the next lemma that $M(z,\eta,\zeta)\leq 2$ for all z,η,ζ in $\mathbb{H}$. Hence the integral is bounded by

$$2^{s-r}(\frac{c_s}{c_r})c_r^2\int\int_{\mathbb{H}}|\hat{A}(z,\overline{\eta})||d(z,\overline{\eta})|^r|\hat{B}(\eta,\overline{\zeta})||d(\eta,\overline{\zeta})|^r d\nu_0(\eta,\zeta)\leq$$

$$\leq 2^{s-r}(\frac{c_s}{c_r})c_r\int_{\mathbb{H}}|\hat{A}(z,\overline{\eta})||d(z,\overline{\eta})|^r(c_r\int_{\mathbb{H}}|\hat{B}(\eta,\overline{\zeta})||d(\eta,\overline{\zeta})|^r d\nu_0(\zeta))d\nu_0(\eta)\leq$$

$$\leq 2^{s-r}(\frac{c_s}{c_r})||B||_{\lambda,r}c_r\int_{\mathbb{H}}|\hat{A}(z,\overline{\eta})||d(z,\overline{\eta})|^r d\nu_0(\eta)\leq 2^{s-r}(\frac{c_s}{c_r})||A||_{\lambda,r}||B||_{\lambda,r}.$$

This completes the proof of the lemma, subject to checking the following estimate.

Lemma. *2.6 Let $M=M(z,\eta,\zeta)$ be the function $\mathbb{H}^3$ defined by the formula*

$$M(z,\eta,\zeta)=\frac{(Im\ \eta)|(z-\overline{\zeta})/2i|}{|(z-\overline{\eta})/2i||(\eta-\overline{\zeta})/2i|},\ z,\eta,\zeta\in\mathbb{H}.$$

Then $0\leq M(z,\eta,\zeta)\leq 2$ for all z,η,ζ in $\mathbb{H}$.

Proof. We use the formula

$$\rho(z,\zeta)=\frac{\text{Im } z\text{ Im }\zeta}{|(z-\zeta)/2i|^2}=\cosh^{-1}(D_h(z,\zeta)),\ z\zeta\in\mathbb{H},$$

(recall that we used the notation $D_h(z,\zeta)$ for the hyperbolic distance between the points z, ζ in $\mathbb{H}$). Then

$$M(z,\eta,\zeta) = \rho(z,\eta)\rho(\eta,\zeta)\rho^{-1}(z,\zeta),\ \ z,\eta,\zeta$$

and hence M is invariant under the diagonal action of the group $PSL_2(\mathbb{R})$ on $\mathbb{H}^3$. Consequently $M(z,\eta,\zeta)$ is an invariant function in the hyperbolic geometry of $\mathbb{H}$. Hence we may replace M, to compute its supreme, by the corresponding form of M in the terminology of the hyperbolic geometry on the unit disk.

Hence we may assume that

$$M(z,\eta,\zeta) = \frac{(1-|\eta|^2)|1-z\overline{\zeta}|}{|1-z\overline{\eta}||1-\eta\overline{\zeta}|},\ \ z,\eta,\zeta \in \mathbb{D}.$$

Since M is invariant under the diagonal action of the group $SU(1,1)$ and since this group acts transitively on $\mathbb{D}$ it is sufficient to estimate this when $\eta = 0$.

Thus we want an upper estimate for

$$M(z,0,\zeta) = |1-z\overline{\zeta}| \text{ for } z,\ \zeta \in \mathbb{D}.$$

The number 2 is clearly the lowest upper bound. This completes the proof.

In the following lemma we prove that $\hat{B}(H_r)$ is continuously embedded $B(H_r)$. In fact we will show a stronger statement; any kernel function $k = k(z,\zeta)$ on $\mathbb{H}^2$ which has the property that the norm $||k||_{\lambda,r}$ is finite defines an integral operator on $L^2(\mathbb{H},\nu_r)$ of uniform norm less than $||k||_{\lambda,r}$.

This will be an easy consequence of the operator interpolation technique used by Berezin in [Be 2].

Proposition 2.7. *Let $k = k(z,\zeta)$ be a function on $\mathbb{H}^2$ such that the maximum of the following expressions is finite:*

$$\sup_z c_r \int_{\mathbb{H}} |k(z,\zeta)| \cdot |d(z,\overline{\zeta}|^r d\nu_0(\zeta)$$

$$\sup_\zeta c_r \int_{\mathbb{H}} |k(z,\zeta)| \cdot |d(z,\overline{\zeta}|^r d\nu_0(\zeta).$$

Denote the maximum of the above two expressions by $||k||_{\lambda,r}$. Then the integral operator k on $L^2(\mathbb{D},\nu_r)$ defined by the formula:

$$(kf)(z) = c_r \int_{\mathbb{H}} \frac{k(z,\zeta)}{|(z-\overline{\zeta})/2i|^r} f(\zeta) d\nu_r(\zeta),\ z \in \mathbb{H},\ f \in L^2(\mathbb{D},\nu_r)$$

is continuous and its norm is bounded by $||k||_{\lambda,r}$. In particular for all B in $B(H_r)$ we have that

$$||B||_{\infty,r} \leq ||B||_{\lambda,r}.$$

Proof. Proving that k is bounded on $L^2(\mathbb{D},\nu_r)$ is equivalent, by using the isometry

$$g \to g(z)(\text{Im } z)^{r/2}$$

from $L^2(\mathbb{D},\nu_r)$ onto $L^2(\mathbb{D},\nu_0)$, proving that the operator k_0 on $L^2(\mathbb{D},\nu_0)$ defined by

$$(k(g))(z) = \int_{\mathbb{H}} |k(z,\zeta)||d(z,\zeta)|^r g(\zeta) d\nu_0(\zeta),\ z \in \mathbb{H},\ g \in L^2(\mathbb{H},\nu_0)$$

is a bounded linear operator. Moreover the uniform norms of k and k_0 coincide.

Our hypothesis is that

$$\sup_{z\in\mathbb{H}} c_r \int_{\mathbb{H}} |k(z,\zeta)||d(z,\zeta)|^r d\nu_0(\zeta) \leq ||k||_{\lambda,r}$$

$$\sup_{\zeta\in\mathbb{H}} c_r \int_{\mathbb{H}} |k(z,\zeta)||d(z,\zeta)|^r d\nu_0(z) \leq ||k||_{\lambda,r}.$$

The interpolation technique used in [Be], Th 2.4, pp1131, shows that k_0 is bounded, of uniform norm less then $||k||_{\lambda,r}$.

We now summarize the properties that we obtained so far for the algebras $\hat{B}(H_r)$ and $B(H_r)$. The remarkable point about this is that the algebra $\hat{B}(H_r)$ is closed under all the products $(*_s)$ for $s \geq r$ and moreover with respect to the norm $||\ ||_{\lambda,r}$, $\hat{B}(H_r)$ is a banachique algebra with respect to the product $(*_s)$ and the norm $||\ ||_{\lambda,r}$. (That is $||A *_s B||_{\lambda,r} \leq$ const $||A||_{\lambda,r}||B||_{\lambda,r}$ for all A, B in$\hat{B}(H_r)$ with a constant depending only on $s \geq r$).

Corollary 2.8. *$\hat{B}(H_r)$ is an involutive Banach algebra with respect to the norm $||\ ||_{\lambda,r}$ from Definition 2.5 (and the product $(*_s)$). Moreover, with respect to this norm on $\hat{B}(H_r)$, and the uniform norm on $B(H_r)$, the inclusion of $\hat{B}(H_r)$ into $B(H_r)$ is continuous with weakly dense image.*

Moreover for all $s \geq r$, $\hat{B}(H_r)$ is mapped by $j_{s,r}$ continuously into $\hat{B}(H_s)$ and the image of $\hat{B}(H_r)$ in $\hat{B}(H_s)$ is closed under the multiplication $*_s$ and we have

$$||A *_s B||_{\lambda,r} \leq (\frac{c_s}{c_r})2^{s-r}||A||_{\lambda,r}||B||_{\lambda,r}$$

for all A, B in $\hat{B}(H_r)$, $s \geq r$.

Finally $\hat{B}(H_r)$ is weakly dense in $B(H_s)$ for all $s \geq r$.

Proof. The last statement will follow from the next lemma in which we show that $B(H_{r-2-\epsilon})$ is contained in $\hat{B}(H_r)$ for all $\epsilon > 0$. One uses here also the fact that $j_{s,r}(B(H_r))$ is weakly dense in $B(H_s)$ for all $s \geq l$. The only assertion that needs to be proved is that $\hat{B}(H_r)$ embeds continuously into $\hat{B}(H_r)$ for $s \geq r$.This follows from the fact that d takes only subunitary values (see definition 2.1).

It is easy to see that the uniform norm on $B(H_r)$ is not equivalent to the norm $|| \, ||_{\lambda,r}$, i.e. $\hat{B}(H_r)$, as expected, is a strictly smaller algebra than $B(H_r)$. (This may be verified by looking at the values of the two norms $|| \, ||_{\lambda,r}$ and $|| \, ||_{\infty,r}$ on one dimensional projections in $B(H_r)$.

However the two norms are equivalent on the image of $B(H_{r-2-\epsilon})_+$ in $\hat{B}(H_r)$ for all $\epsilon > 0$. We will sketch a proof of this (simple) fact although we are not going to use it.

First we note a corollary of Lemma 2.

Corollary 2.8. *If* $1 < r < s - 2 - \epsilon$, *and* ϵ *is strictly positive then* js, r *maps* $B(H_r)$ *into* $B(H_s)$ *and*

$$||js, r(A)||_{\lambda,s} \leq (const)(\frac{s-r}{2} - 1)^{-1}||A||_{\infty,r}$$

for all A *in* $B(H_r)$.

Proof. If A belongs to $B(H_r)$ then we have proved that

$$\sup_{z,\zeta \in \mathbb{H}^2} |A(z,\zeta)||d(z,\zeta)|^r \leq 4||A||_{\infty,r}$$

for all A in $B(H_r)$. Hence

$$c_r \int |A(z,\zeta)||d(z,\zeta)|^s d\nu_0(\zeta) \leq 4||A||_\infty \frac{c_r}{c_s} c_s \int |d(z,\zeta)|^{s-r} d\nu_0(\zeta) =$$

$$= 4||A||_{\infty,r} \int_{\mathbb{H}} [\frac{(\text{Im } z)(\text{Im } \zeta)}{|(z-\overline{\zeta})/2i|}]^{s-r} d\nu_0(\zeta) =$$

$$= \text{const } ||A||_{\infty,r} (\text{Im } z)^{s-r} \langle e_z^{\frac{s-r}{2}}, e_z^{\frac{s-r}{2}} \rangle = \text{const } ||A||_{\infty,r} (\frac{s-r}{2} - 1)^{-1}.$$

Corollary. *2.9 Let $r < s-2-\epsilon$ with ϵ strictly positive. Then for all A in $B(H_r)_+$, which is identified, by js, r, with an element in $B(H_r)$, we have*

$$||A||_{\lambda,s} \leq const(\frac{s-r}{2} - 1)^{-1} ||A||_{\infty,r}.$$

Proof. We proved as in the proof of the above corollary. Since A is positive in $B(H_r)$ we have in addition that

$$\sup_{z,\zeta\in\mathbb{H}} |A(z,\overline{\zeta})||d(z,\zeta)|^r \leq \sup_{z\in\mathbb{H}} |A(z,\overline{z})| \leq ||A||_{\infty,s}$$

(we use here proposition 1.5.d). The argument then follows as above.

Finally we mention the following interesting behavior of the weak topology on the unit ball in $\hat{B}(H_r)$ which is a corollary of the preceding discussion (although we are not going to make use of this fact, either.)

Lemma 2.10. *The unit ball of $\hat{B}(H_r)$ (with respect to the norm $|| \, ||_{\lambda,r}$) is compact with respect to the weak operator topology on $B(H_r)$.*

Proof. We already know that $\hat{B}(H_r)_1 \subseteq B(H_r)_c$ for some $c > 0$ (by Corollary 2.9). Let B_n be any sequence $\hat{B}(H_r)_1$. by the preceding assertion we may assume that B_n converges weakly to B for some B in $B(H_r)_c$.

Let

$$f_n(z,\zeta) = |B_n(z,\zeta)||d(z,\zeta)|^r, \ z,\zeta \in \mathbb{H},$$

$$f(z,\zeta) = |B(z,\zeta)||d(z,\zeta)|^r,\ z,\zeta \in \mathbb{H}.$$

We know that the positive functions $f_n(z,\zeta)$, $f(z,\zeta)$ on $\mathbb{H}^2$ are uniformly bounded by C on $\mathbb{H}^2$. Moreover f_n converges punctually to f.

We want to take to the limit the following inequality that holds for all the functions f_n:

$$\int_{\mathbb{H}} f_n(z,\zeta) d\nu_0(\zeta) \leq 1$$

for all z in $\mathbb{H}$.

For fixed z in $\mathbb{H}$, we use the Lebesgue dominated convergence theorem (as all functions are uniformly bounded by c) to deduce that for any subset E of $\mathbb{H}$ of finite ν_0-measure we have

$\int_E f(z,\zeta) d\nu_0(\zeta) \leq 1$ for all z in $B(H_r)$.

As f is positive we may now use Fatou lemma to deduce that

$$\sup_{z\in\mathbb{H}} \int_{\mathbb{H}} f(z,\zeta) d\nu_0(\zeta) \leq 1.$$

A similar computation holds for $\sup\limits_{\zeta\in\mathbb{H}} \int_{\mathbb{H}} f(z,\zeta) d\nu_0(\zeta)$. Hence B belongs to $\hat{B}(H_r)_1$.

3. The Berezin quantization for quotient space $\mathbb{H}/\Gamma$

In this paragraph we are analyzing the Γ-invariant form of the Berezin quantization. Let$(*_s)_{s\in(a,b)}$ be the product for Berezin symbols which was derived from the composition rule of linear operators acting on the Hilbert spaces $H^2(\mathbb{H}, (\text{Im } z)^{-2+s} \mathrm{d}z\mathrm{d}\overline{z})$. Let Γ be a Fuchsian subgroup of $PSL(2,\mathbb{R})$ which we allow to be of finite or infinite covolume. Since the Berezin quantization is constructed such that $PSL(2,\mathbb{R})$ acts as a group of symmetries, it follows that the symbols $k = k(z,\overline{\zeta}),\ z,\zeta \in \mathbb{H}$, that are invariant under the diagonal action of the group Γ (i.e. $k(\gamma z,\overline{\gamma\zeta}) = k(z,\overline{\zeta}),\ \gamma \in \Gamma, z,\zeta \in \mathbb{H}$) are closed under any of the products $*_s$.

By analogy with [Be] (see also [KL]) to obtain a deformation quantization for $\mathbb{H}/\Gamma$, we let the algebras in the corresponding deformation to be the vector space of Γ-invariant symbols and we let the multiplication be defined by the products $*_s$. For the integrals entering the formula of $*_s$ to be convergent we impose suitable condition on the growth of the symbols.

This algebras are then identified with the commutant of the (projective) representation of Γ on H_r that is obtained by restriction of the projective representation π_r of $PSL(2,\mathbb{R})$ to Γ.

Using a generalization of a theorem by [AS], [Co], [GHJ], in the form stated in the monograph [GHJ] we will prove that the algebras in the deformation quantization are type II_1 factors or respectively properly semifinite algebras (corresponding the case when Γ has finite or ,respectively, infinite covolume).

The above mentioned generalization of the theorem by [A.S.], [Co], [GHJ] to projective representations of $PSL(2,\mathbb{R})$, is used to determine the isomorphism class of the algebras in the deformation: they are all stably isomorphic to the von Neumann algebra $\mathcal{L}(\Gamma)$ associated to Γ. Moreover, the dimension of the Hilbert space H_r as a left module over the corresponding algebra in the deformation, tends to zero when the deformation parameter $r = 1/h$ tends to infinity.

Finally, the Berezin formula computing the trace for operators on $B(H_r)$ as the integral of the restriction of the symbol on $\mathbb{H}$, is now replaced by an integral over a fundamental domain. This generalizes a result in [GHJ], (formula 3.3.e, see also the manuscript notes [Jo]). In particular we will show that the $\Gamma-$invariant Berezin are a natural generalization of the notion of automorphic forms for the group Γ.

The following definition extends the formal dimension ([Go]) of a representation with square integrable coefficients to more general projective representations.

Definition 3.1. *Let G be a unimodular locally compact group with Haar measure dg. Let $\pi : G \to B(H)$ be a projective, unitary representation of G. Assume that π is (topologically) irreducible and also assume that π has square integrable coefficients: i.e. there exists at least one nonzero η in H so that*

$$\int_G |\langle \pi(g)\eta, \eta\rangle_H|^2 dg = d_\pi^{-1} ||\eta||^2,$$

for some strictly positive number d_π. Then d_π is independent on the choice of η. Moreover, the following equality holds true (the Schur orthogonality relations):

$$\int_G |\langle \pi(g)\zeta, \eta\rangle_H|^2 dg = d_\pi^{-1} ||\zeta||^2 ||\eta||^2 \text{ for all } \zeta, \eta \text{ in } H.$$

Before proving the lemma, we recall the following folklore lemma:

Lemma. *Let G be an unimodular locally compact group. Let $\{c(g,h)\}_{g,h\in G}$ be a family of modulus 1 complex numbers defining an element in $H^2(G,\mathbb{T})$, that is*

$$c(g,hk)c(h,k) = c(gh,k)c(g,h), \; g,h,k \text{ in } G.$$

For all h in G define an unitary operator R_h on $L^2(G)$ by the following relation:

$$(R_h f)(g) = c(g,h)^{-1} f(gh), \; f \in L^2(G), \; g,h \text{ in } G.$$

Then $(R_h)_{h\in G}$ *is a projective, unitary representation of* G *on* $L^2(G)$ *with cocycle* $\{c(g,h)\}_{g,h\in H}$. *Thus*

$$R_{hk} = c(h,k)R_hR_k, \ \ k,h \ in \ G.$$

Proof. We have $(R_{hk}f)(g) = c(g,hk)^{-1}f(ghk)$. On the other hand

$$(R_h(R_kf))(g) = c(g,h)^{-1}(R_kf)(gh) = c(g,h)^{-1}c(gh,k)^{-1}f(ghk).$$

Hence

$$(R_{hk}f)(g) = c(g,hk)^{-1}c(g,h)c(gh,k)(R_h(R_kf))(g) =$$

$$= c(g,h)(R_h(R_kf))(g), \quad \text{for all } f \in L^2(G), g,h \in G.$$

This completes the proof.

Let π be a projective, unitary representation of G. Let $c(g,h)$ be the complex number of modulus 1 defined by the relation

$$\pi(gh) = c(g,h)\pi(g)\pi(h), \quad \text{for } g,h \in G$$

Because of the above lemma, we may think to the projective, unitary representation π as subrepresentation of the "skewed", left regular representation of G on $L^2(G)$, with cocycle c.

Proof of Proposition 3.1. We follow the lines contained in the proof of the analogue result for the discrete series of representations of G, as it is found in the exposition in [Ro], chapter 17.

Keeping η fixed in H, we define the linear map T from the vector space $W = \{\zeta \in H | \int_G |\langle \pi(g)\zeta, \eta\rangle_H|^2 dg < \infty\}$ into $L^2(G)$ by $(T\zeta)(g) = \langle \pi(g)\zeta, \eta\rangle$. Then

$$(T\pi(h)\zeta)(g) = \langle \pi(g)\pi(h)\zeta, \eta\rangle =$$

$$c(g,h)^{-1}\langle \pi(g,h)\zeta, \eta\rangle = R_h(T\zeta)(g) \text{ for all } g \in G.$$

Thus $T\pi(h) = R_hT$ and T is an (eventually) unbounded, closed operator with domain W. Moreover the operators $\pi(h), h \in G$ map W into W.

The theorem 15.13 in [Ro] shows that T is a multiple of an isometry and in particular $W = H$. In particular we find that

$$\int_G |\langle \pi(g)\zeta, \eta\rangle|^2 dg < \infty, \quad \text{for all } \zeta \text{ in } H.$$

Similar arguments involving the, cocycle perturbed, left, regular representation of G on $L^2(G)$, will show that:

$$\int_G |\langle \pi(g)\zeta, \eta'\rangle|^2 dg < \infty, \text{ for all } \zeta, \eta' \text{ in } H$$

The rest of the argument is exactly as in the proof of Theorem 16.3 in [Ro]. This completes the proof.

For the projective representation $(\pi_r)_{r>1}$ Recall that the discrete series of representation $(\pi_n)_{n\in\mathbb{N}}$ of $PSL(2,\mathbb{R})$ have formal dimension $\frac{n-1}{\pi}$. We will prove a similar statement for the projective representation $(\pi_r)_{r>1}$. It will then follow that the projective, unitary representations $(\pi_r)_{r>1}$ have similar properties to square integrability with $d_{\pi_r} = \frac{r-1}{\pi}$

The explanation for this fact should be looked into the Plancherel formula for the universal cover $\widetilde{PSL(2,\mathbb{R})}$ of $PSL(2,\mathbb{R})$. L. Pukanszki [Pu] has shown that the representations $(\pi_r)_{r>1}$ are a summand in the continuous series in the Plancherel formula for $\widetilde{PSL(2,\mathbb{R})}$. The coefficient of the representation π_r in this formula is $d_{\pi_r} = \frac{r-1}{\pi}$.

The main result of this paragraph is the following:

Theorem 3.2. *Let Γ be a Fuchsian group in $PSL(2,\mathbb{R})$ of finite or infinite covolume. Let $r > 1$ and let $\mathcal{A}_r$ and $\hat{\mathcal{A}}_r$ be the vector space of all symbols $k = k(z,\overline{\zeta})$, $z,\overline{\zeta} \in \mathbb{H}$ analytic in z and antianalytic in ζ, that are Γ invariant (i.e. $k(z,\overline{\zeta}) = k(\gamma z, \overline{\gamma\zeta})$, $\gamma \in \Gamma$, $z, \zeta \in \mathbb{H}$) and which correspond to bounded operators in $B(H_r)$ or $\hat{B}(H_r)$, respectively. Then the vector spaces $\mathcal{A}_r$ and $\hat{\mathcal{A}}_r = \hat{B}(H_r) \cap (A_r)$ are closed under the product $*_r$.*

Let F be a fundamental domain for the action of Γ in $\mathbb{H}$ (see [Leh]). Then

i) $\mathcal{A}_r$ is a type II_1 factor or an infinite semifinite von Neumann corresponding respectively to the case when Γ is of finite or infinite covolume. Moreover $\mathcal{A}_r$ is isomorphic to the commutant $\{\pi_r(\Gamma)\}' \subseteq B(H_r)$ of the image of Γ in $B(H_r)$ via the representation π_r.

ii) In the finite covolume case the trace on $\mathcal{A}_r$ is given by the formula

$$\tau(k) = \frac{1}{area(F)} \int_F k(z,\overline{z}) d\nu_0(z), \ k \in \mathcal{A}_r.$$

When Γ has infinite covolume in $PSL(2,\mathbb{R})$, one defines a semifinite, faithful, normal trace on $\mathcal{A}_r$ by the formula

$$\tau(k) = \int_F k(z,z) d\nu_0(z).$$

iii) Assume that the group cocycle in the second cohomology group of $PSL(2,\mathbb{R})$, $H^2(PSL^2(\mathbb{R}),\mathbb{T})$ associated with the projective representation of $PSL(2,\mathbb{R})$ on H_r, vanishes by restriction to $H^2(\Gamma,\mathbb{T})$. Then $\mathcal{A}_r$ is isomorphic to $e(\mathcal{L}(\Gamma) \otimes B(H_r))e$ where e is any projection in $L(\Gamma) \otimes B(H_r)$ of trace $\tau(e) =$ covol $(\Gamma) d_{\pi_r} = \frac{r-1}{\pi}(covol(\Gamma))$. ∎

Remark. *Note that the condition in (iii) is always satisfied when* Γ *is not cocompact (see [Pa2]).*

The proof of the theorem will be splitted into more lemmas. In a slightly different form the first two lemmas may be found in the book of [Ro]. The first lemma shows that the representations π_r have the property that they move (modulo scalars) $\pi_r(g)$ maps the vector e_z^r into $e_{g^{-1}z}^r$ for all z in $\mathbb{H}$.

Recall from paragraph 1, that the representation π_r were defined by the formula

$$(\pi_r(g)f)(z) = (j(g,z))^{-r} f(g^{-1}z),$$

for all g in G, f in $H^2(\mathbb{H}, \nu_r)$ and z in $\mathbb{H}$. They are projective representations because the factor

$$(j(g,z))^r = \exp(r \ln(j(g,z)) = \exp(r \ln(cz+d))$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $PSL(2,\mathbb{R})$, z in $\mathbb{H}$, involves the choice of a branch for $\ln(cz+d)$.

Lemma 3.3. *([Ro]). Let* $r > 1$ *and let* $e_z^r \in H^2(\mathbb{H}, d\nu_r)$ *be the evaluation vector at* z*, i.e.* $\langle f, e_z^r \rangle_r = f(z)$ *for all* f *in* $H_r = H^2(\mathbb{H}, d\nu_r)$*. Then there exists* $\Theta_g \in \mathbb{C}$ *of modulus 1 so that*

$$\pi_r(g) e_z^r = \Theta_g (j(g^{-1}, z))^r e_{g^{-1}z}^r$$

for all g *in* G*,* $z \in \mathbb{H}$*.*

Proof. Indeed for any f in H_r we have

$$\langle f, \pi_r(g), e_z^r \rangle_{H_r} = \langle \pi_r(g^{-1}) f, \pi_r(g^{-1}) \pi_r(g) e_z^r \rangle_{H_r}$$

by the unitary of π_r. Since π_r is a projective representation of G, there exists a scalar $\overline{\Theta_g}$ of modulus 1 so that $\pi_r(g^{-1})\pi_r(g) = \overline{\Theta_g}\ Id_{B(H_r)}$. Hence

$$\langle f, \pi_r(g), e^r_z\rangle = \Theta_g\langle \pi_r(g^{-1})f, e^r_z\rangle = \Theta_g(\pi_r(g^{-1})f)(z) =$$

$$\Theta_g(j(g^{-1}, z))^{-r} f(g^{-1}z) = \Theta_g(j(g^{-1}, z))^r\langle f, e^r_{g^{-1}z}\rangle_r.$$

It is now easy to prove that an operator A in $B(H_r)$ commutes with $\pi_r(\Gamma)$ if and only if its symbol has the property that its symbols is Γ invariant. This will complete the proof of i) by showing that the Γ invariant symbols correspond to elements in the commutant of $\pi_r(\Gamma)$ in $B(H_r)$, and hence they are clearly closed under multiplication.

Lemma. *3.4 An operator A in $B(H_r)$ of Berezin symbol $\hat{A} = \hat{A}(z, \zeta), z, \zeta$ in $\mathbb{H}$ commutes with $\pi_r(\Gamma)$ if and only if $\hat{A}$ is Γ invariant under the diagonal action, that is*

$$\hat{A}(\gamma z, \overline{\gamma\zeta}) = \hat{A}(z, \zeta) \quad \textit{for all } z, \zeta \textit{ in } \mathbb{H}, \gamma \in \Gamma.$$

Proof. Assume that A commutes with $\pi_r(\Gamma)$. Then for all γ in Γ,

$$A(\gamma^{-1}z, \gamma^{-1}\zeta) = \frac{\langle Ae^r_{\gamma^{-1}\zeta}, e^r_{\gamma^{-1}z}\rangle}{\langle e^r_{\gamma^{-1}\zeta}, e^r_{\gamma^{-1}z}\rangle} =$$

$$\frac{\langle A(\Theta_{\gamma^{-1}}(j(\gamma, \zeta))^{-r})\pi_r(\gamma^{-1})e^r_\zeta, (\Theta_{\gamma^{-1}}(j(\gamma, z))^{-r})\pi_r(\gamma^{-1})e^r_z\rangle}{\langle(\Theta_{\gamma^{-1}}(j(\gamma, \zeta))^{-r})\pi_r(\gamma^{-1})e^r_\zeta, (\Theta_{\gamma^{-1}}(j(\gamma, z))^{-r})\pi_r(\gamma^{-1})e^r_z\rangle}$$

where we used the preceding lemma. The scalars are canceling themselves and hence we get

$$\hat{A}(\gamma^{-1}z, \overline{\gamma^{-1}\zeta}) = \frac{\langle A\pi_r(\gamma^{-1})e^r_\zeta, \pi_r(\gamma^{-1})e^r_z\rangle}{\langle \pi_r(\gamma^{-1})e^r_\zeta, \pi_r(\gamma^{-1})e^r_z\rangle}.$$

Since $\pi_r(\gamma^{-1})$ is unitary and $\pi_r(\gamma^{-1})$ commutes with A, we get $A(z,\overline{\zeta})$. The converse follows this lines too. This completes the proof of i). (We also use here the fact that $\hat{B}(H_r)$ is closed under multiplication and hence so is $\hat{\mathcal{A}}_r = \hat{B}(H_r) \cap \mathcal{A}_r$).

Proof of ii). To prove ii) we will first check that the formula for τ defines indeed a trace. Note that by point i) $k(\gamma^{-1}z, \overline{\gamma^{-1}z}) = k(z,\overline{z})$ for all z in $\mathbb{H}$, γ in Γ and hence the integral for τ doesn't depend on the choice of F.

To check that τ is a trace it is sufficient to check that $\tau(k^* *_r k) = \tau(k *_r k^*)$ for every $\Gamma-$ equivariant kernel $k = k(z,\overline{\zeta})$ on $\mathbb{H}^2$ which is so that the integrals in the formulae for $\tau(k^* *_r k)$ and $\tau(k *_r k^*)$ are absolutely convergent.

Assume that k is as above and that k is so that the integrals involved in $\tau(k^* *_r k)$ or $\tau(k *_r k^*)$ are convergent.

We have: $k^*(z,\overline{\zeta}) = \overline{k(\zeta,\overline{z})}$ for z, ζ in $\mathbb{H}$ and hence

$$(k^* *_r k)(z,\overline{\zeta}) = c_r[(z-\overline{\zeta})/2i]^r \int_{\mathbb{H}} \frac{k^*(z,\eta)}{[(z-\overline{\eta})/2i]^r} \frac{k(\eta,\overline{\zeta})}{[(\eta-\overline{\zeta})/2i]^r} d\nu_r(\eta)$$

so that

$$(k^* *_r k)(z,\overline{z}) = c_r \int_{\mathbb{H}} k^*(z,\overline{\eta}) k(\eta,\overline{\zeta}) |d(z,\overline{\eta})|^{2r} d\nu_0(\eta)$$

and hence

$$\tau(k^* *_r k) = (\mathrm{const}) c_r \int_F (\int_{\mathbb{H}} |k(\eta,\overline{z})|^2 |d(z,\overline{\eta})|^{2r} \mathrm{d}\nu_0(\eta)) \mathrm{d}\nu_0(z).$$

Similarly

$$\tau(k *_r k^*) = (\mathrm{const}) c_r \int_F (\int_{\mathbb{H}} |k(\eta,\overline{z})|^2 |d(z,\overline{\eta})|^{2r} \mathrm{d}\nu_0(\eta)) \mathrm{d}\nu_0(z).$$

The constant in front of the two integrals equals 1 if F has infinite covolume and equals (area $F)^{-1}$ if Γ has finite covolume (the hyperbolic) area is computed here relative to ν_0).

By renaming the variables in the integral for $\tau(k^* *_r k)$ we get:

$$\tau(k^* *_r k) = (\text{const})c_r \int_F (\int_{\mathbb{H}} |k(z,\overline{\eta})|^2 |d(\eta,z)|^{2r} d\nu_0(z)) d\nu_0(\eta).$$

The second expression for $\tau(k^* *_r k)$ is different from the one for $\tau(k *_r k^*)$ only in the choice of the domain of integration (as $|d(z,\overline{\eta})| = |d(\overline{\eta},z)|$ for all z,η). But in both integrals we are integrating over a fundamental domain for the diagonal action of Γ in $\mathbb{H}^2$, while the integrand is $\Gamma-$invariant under the same action of Γ. Since we are integrating positive functions, the two integrals must be equal.

We will now show that the functional τ is positive definite. Let k represent a positive operator in $\mathcal{A}_r \subseteq B(H_r)$. Then by Lemma 2.1., $k(z,\overline{z})$ is positive $z \in \mathbb{H}$. Hence if $\tau(k) = (\text{const}) \int_F k(z,\overline{z}) d\nu_0(z)$ is 0 then it follows that $k(z,\overline{z}) = 0$ for all z in F and thus for all z in $\mathbb{H}$ by the Γ-invariance.

But $k = k(z,\overline{\zeta})$ is analytic in z and antianalytic in ζ. Hence if k vanishes on the diagonal $z = \zeta,\ z \in \mathbb{H}$, then it must be identically zero. Thus we have shown that τ is definite, i.e. that if $\tau(k) = 0$ and k is the symbol of a positive element in $\mathcal{A}_r$ then $k = 0$.

In the case of a group Γ of finite covolume, τ is well defined on $\mathcal{A}_r$. Indeed let $||k||_{\infty,r}$ be the uniform norm of the operator on H_r defined by k. We then have

$$|\tau(k)| = |\frac{1}{\text{area}(F)} \int_F k(z,\overline{z}) d\nu_0(z)| \leq \sup_{z \in F} |k(z,\overline{z})| \leq ||k||_{\infty,r}.$$

We used for the last inequality Lemma 1.5.d.

In the case of a group Γ of infinite covolume we will have to check in addition that there exists sufficiently many positive elements in $\mathcal{A}_r$ so that the trace τ takes finite value on them, and so that 1 (the unit of $\mathcal{A}_r$) is a weak increasing limit of positive elements having finite trace.

To obtain such elements we let $k_f = k_f(z,\overline{z}),\ z,\overline{z} \in \mathbb{H}$ be the symbol of a Toeplitz operator $T_f^r = P_r M_f P_r$ on $B(H_r)$. The symbol f of the operator T_f^r is assumed to be a positive function f with compact support in the interior $\overset{\circ}{F}$ of F and then extended by Γ invariance to the whole upper half plane $\mathbb{H}$, (see also the next paragraph).

Since f is $\Gamma-$ equivariant, the operator T_f^r commutes with $\pi_r(\Gamma)$. Also T_f^r is clearly a positive operator. If we choose an increasing net of such functions f that converges point wise to the constant function 1 then it follows that $1_{\mathcal{A}_r}$ is a weak (increasing) limit of operators of the form T_f^r.

The functional τ evaluated on T_f^r gives:

$$\int_F k_f(z,\overline{z})d\nu_0(z) = \int_F c_r[(\mathrm{Im}\ z)^r \int \frac{f(a)}{|1-\overline{z}a|^{2r}} d\nu_0(a)]d\nu_0(z) =$$

$$= \mathrm{const} \int_F f(a)[\int_{\mathbb{H}} |d(z,\overline{a})|^{2r} d\nu_r(a)]d\nu_0(z).$$

But $\sup\limits_z \int_{\mathbb{H}} |d(z,\overline{a})|^{2r} d\nu_0(a) < \infty$ and hence $\tau(T_f^r)$ is finite. This completes the proof of ii).

We now turn to the proof of iii). We will follow the lines for the proof in the case of actual representations of $PSL(2,\mathbb{R})$ which is contained in the monograph [GHJ]. The only difference is that the representation $\pi_r$is not a subrepresentation in the left regular representation of G.Instead, for arbitrary real r, one considers π_r is a subrepresentation of the (projective) unitary representation $\tilde{\pi_r}$ of G into $B(L^2(\mathbb{H},\nu_r))$ and $\pi_r|_\Gamma$ is a subrepresentation in the regular representation of Γ into $B(l^2(\Gamma))$.

Proposition. *3.5 Let $\tilde{\pi_r} : G = PSL(2,\mathbb{R}) \to B(L^2(\mathbb{H},\nu_r))$ be the (projective) unitary representation of G onto $L^2(\mathbb{H},\nu_r)$ defined by the same formula as π_r:*

$$(\tilde{\pi_r}(g)f)(z) = (j(g,z))^{-r} f(g^{-1}z),\ \ f \in L^2(\mathbb{H},\nu_r),\ \ g \in G,\ \ z \in \mathbb{H}.$$

Assume that there exists complex numbers $c(\gamma)$ of modulus 1 so that $\gamma \to c(\gamma)\tilde{\pi_r}(\gamma) = \tilde{\pi}_r^0(\gamma)$ is a unitary representation of Γ on $L^2(\mathbb{H},\nu_r)$. Let F be a fundamental domain for Γ in $\mathbb{H}$. Let $(e_\gamma)_{\gamma\in\Gamma}$ be the canonical orthonormal basis for $l^2(\Gamma)$ and let $V_r : l^2(\Gamma) \otimes L^2(F,\nu_r) \to L^2(\mathbb{H},\nu_r)$ be defined by the formula

$$V_r(\sum_\gamma e_\gamma \otimes f_\gamma) = \sum_\gamma \tilde{\pi}_r^0(\gamma)(\gamma^{-1})(f_\gamma)$$

for all elements $\sum\limits_\gamma e_\gamma \otimes f_\gamma$ in $l^2(\Gamma) \otimes L^2(F,\nu_r)$. Note that the functions f_γ are identified with elements in $L^2(\mathbb{H},\nu_r)$ by defining them to be zero outside F.

Let $R_\gamma : \Gamma \to B(l^2(\Gamma))$ be the right regular representation of Γ (i.e. $R_\gamma e_h = e_h\gamma^{-1}$, for all $h,\gamma \in \Gamma$).

Then V_r is an unitary and

$$V_r^* \tilde{\pi_r}(\gamma) V_r = R_\gamma \overline{\otimes} Id_{B(L^2(F,d\nu_r))},\quad \textit{for all } \gamma \quad \textit{in } \Gamma.$$

Proof. We construct first an inverse U_r for V_r which is defined on $L^2(\mathbb{H},\nu_r)$ with values into $l^2(\Gamma) \otimes L^2(F,\nu_r)$ by

$$(3.1)\quad U_r f = \sum_\gamma e_\gamma \otimes f_\gamma,\ \ f_\gamma = \tilde{\pi}_r^0(\gamma)(f\chi_{\gamma^{-1}F}),\quad \text{for all}\quad f \quad \text{in } L^2(\mathbb{H},\nu_r).$$

We denote by χ_A the characteristic function of A in $\mathbb{H}$. Clearly $\tilde{\pi}_r^0(\gamma)(f\chi_{\gamma^{-1}F})$ has its support in F for all γ in Γ as

$$\tilde{\pi}_r^0(\gamma)(f\chi_{\gamma^{-1}F})(z) = c(\gamma)(j(\gamma,z))^{-r}(f\chi_{\gamma^{-1}F})(\gamma^{-1}z)$$

is nonzero only for z in F. Hence U_r is well defined. We check first that $V_rU_rf = f$ for all f in $L^2(F,\nu_r)$. Indeed if f_γ are as formula (3.1) then

$$V_r(\sum_\gamma e_\gamma \otimes f_\gamma) = \sum_\gamma \tilde{\pi}_r^0(\gamma^{-1})(f_\gamma) = \sum_\gamma \tilde{\pi}_r^0(\gamma)[\tilde{\pi}_r^0(\gamma^{-1})(f\chi_{\gamma^{-1}F})] =$$

$$= \sum_\gamma f\chi_{\gamma^{-1}F} = f$$

Clearly U_r is surjective as F is a fundamental domain for Γ in $\mathbb{H}$ so that $\mathbb{H}$ is covered by translates of F by Γ. Moreover, U_r is unitary because, given f in $L^2(F,\nu_r)$ and letting f_γ be defined by formula (3.1), then

$$||U_rf||^2 = ||\sum_\gamma e_\gamma \otimes f_\gamma||^2_{l^2(\Gamma)\otimes L^2(F,\nu_r)} = \sum_\gamma ||\tilde{\pi}_r^0(\gamma^{-1})(f\chi_{\gamma^{-1}F})||^2_{L^2(F,\nu_r)} =$$

$$= \sum_\gamma ||f\chi_{\gamma^{-1}F}||^2_{L^2(\mathbb{H},\nu_r)} = ||f||^2_{L^2(\mathbb{H},\nu_r)}.$$

We have so far checked that V_r is unitary from $l^2(\Gamma)\otimes L^2(F,\nu_r)$ onto $L^2(\mathbb{H},\nu_r)$ and that its inverse is given by the relation (3.1).

It remains to check that

$$U_r\tilde{\pi}_r^0(\gamma)V_r = R_\gamma \overline{\otimes} Id_{B(L^2(F,\nu_r))}, \text{for all} \gamma \in \Gamma.$$

Indeed for any element $\sum\limits_\gamma e_\gamma \otimes f_\gamma$ in $l^2(\Gamma)\otimes L^2(F,\nu_r)$ and any $\sigma \in \Gamma$, we have

$$\{U_r\tilde{\pi}_r^0(\sigma)V_r\}(\sum_\gamma e_\gamma \otimes f_\gamma) = U_r\tilde{\pi}_r^0(\sigma)(\sum_\gamma \tilde{\pi}_r^0(\gamma^{-1})f_\gamma) =$$

$$= \sum_{\gamma} U_r(\tilde{\pi}_r^0(\sigma\gamma^{-1})f_\gamma) = U_r(\sum_{\gamma} \tilde{\pi}_r^0(\gamma^{-1})(f_{\gamma\sigma}\chi_F)) =$$

$$\sum_{\delta} e_\delta \otimes (\tilde{\pi}_r^0(\delta)(\sum_{\gamma} \tilde{\pi}_r^0(\gamma^{-1})(f_{\gamma\sigma}\chi_F)\chi_{\delta^{-1}F}).$$

Note that in general for any α in Γ and g in $L^2(F, \nu_r)$ the support of the function $\tilde{\pi}_r^0(\alpha^{-1})(g\chi_F)$ is in $\chi_{\alpha^{-1}F}$. Hence, in the above sum, the only nonzero terms that may occur, are those for which $\delta = \gamma$. Consequently

$$(U_r\tilde{\pi}_r^0(\sigma)V_r)(\sum_{\gamma} e_\gamma \otimes f_\gamma) =$$

$$= \sum_{\delta} e_\delta \otimes (\tilde{\pi}_r^0(\delta)[\tilde{\pi}_r^0(\delta^{-1})(f_{\delta\sigma})] =$$

$$= \sum_{\delta} e_\delta \otimes f_{\delta\sigma} = \sum_{\delta} e_{\delta\sigma^{-1}} \otimes f_\delta =$$

$$= R_\sigma \overline{\otimes} \mathrm{Id}(\sum_{\delta} e_\delta \otimes f_\delta).$$

This completes the proof of the proposition.

Corollary 3.6. *Assume the hypothesis iii) from Theorem 3.2. Then $\{\tilde{\pi}_r(\Gamma)\}'$ is isomorphic to $e(\mathcal{L}(\Gamma) \otimes B(H))e$ for a projection e in $\mathcal{L}(\Gamma) \otimes B(H)$.*

Proof. We use the notations from Proposition 3.5. Note that P_r, the projection from $L^2(\mathbb{H}, \nu_r)$ on $H^2(\mathbb{H}, \nu_r)$ commutes with $\tilde{\pi}_r(g)$, for all $g \in G$ (as $\tilde{\pi}_r(g)$ maps analytic functions into analytic functions).

Hence, by [St., Zs.], $\{\pi_r(\Gamma)\}'$ is isomorphic to $P_r\{\tilde{\pi}_r(\Gamma)\}'P_r$. The condition iii) in Theorem 3.2 shows that proposition 3.5 applies and hence $\{\tilde{\pi}_r(\Gamma)\}'$ is isomorphic to $\mathcal{L}(\Gamma)\overline{\otimes}B(L^2(F, \nu_r))$ (since in general $\{\mathcal{R}(\Gamma)\}'$ is isomorphic to $\cong \mathcal{L}(\Gamma)$).

We use the notations in Proposition 3.5 and Corollary 3.6 and also assume the condition iii) in Theorem 3.2. Let $Q_r = V_r^* P_r V_r$ be the projection in $B(l^2(\Gamma) \otimes L^2(F, \nu_r)$ which corresponds to P_r through the identification of $L^2(\mathbb{H}, \nu_r)$ with $l^2(\Gamma) \otimes L^2(F, \nu_r)$, by the unitary U_r.

We have just proved that Q_r commutes with $R_\gamma \overline{\otimes} \mathrm{Id}_{B(L^2(F,\nu_r)}$ for all $\gamma \in \Gamma$ (as P_r commutes with $\tilde{\pi}_r(\gamma)$ for all $\gamma \in \Gamma$). Moreover we have just proved that

Proposition 3.7. *With the notations in proposition 3.5, we have that $\{\tilde{\pi}_r(\Gamma)\}'$ is isomorphic to*

$$Q_r(\mathcal{L}(\Gamma) \overline{\otimes} B(L^2(F, \nu_r)))Q_r \subseteq B(l^2(\Gamma) \overline{\otimes} B(L^2(F, \nu_r)).$$

Moreover Q_r belongs to the algebra $\mathcal{L}(\Gamma) \overline{\otimes} B(L^2(F, \nu_r))$, which is the commutant of

$$\{R_\gamma \overline{\otimes} Id_{B(L^2(F,\nu_r))}, \gamma \in \Gamma\}'.$$

To determine the isomorphism class of the algebra $\{\tilde{\pi}_r(\Gamma)\}'$ we will choose a trace τ_1 on $\mathcal{L}(\Gamma) \overline{\otimes} B(L^2(F, \nu_r))$. The trace τ_1 will be normalized by the condition that τ_1 takes the value 1 on minimal projections in $B(L^2(F, \nu_r))$. We will use the method explained in [GHJ].

Proposition 3.8. *We use the notations in Proposition 3.5. Let τ_1 be the trace τ_1 on $\mathcal{L}(\Gamma) \overline{\otimes} B(L^2(F, \nu_r))$, normalized by the condition that τ_1 takes the value 1 on minimal projections in $B(L^2(F, \nu_r))$. Let $\{\tilde{\epsilon}_n\}_{n \in \mathbb{N}}$ be an orthonormal basis in $L^2(F, \nu_r)$. Let $\{\delta_\gamma\}_{\gamma \in \Gamma}$ be the canonical orthonormal basis for $l^2(\Gamma)$ and let e be the neutral element in Γ. Then*

$$\tau_1(Q_r) = \sum_{n \in \mathbb{N}} ||Q_r(\delta_e \otimes \tilde{\epsilon}_n)||^2_{l^2(\Gamma) \overline{\otimes} L^2(F,\nu_r)}.$$

Proof. Let $\tau_{\mathcal{L}(\Gamma)}$ be the canonical normalized trace on $\mathcal{L}(\Gamma)$ and let

$$\mathrm{tr} = \mathrm{tr}_{B(L^2(F,\nu_r))}$$

be the semifinite trace on $B(L^2(F,\nu_r))$ taking value 1 on the one dimensional projections. Then $\tau_1 = \tau_{\mathcal{L}(\Gamma)}\overline{\otimes}\mathrm{tr}$. Moreover for any element $x\overline{\otimes}y$ in $\mathcal{L}(\Gamma)\overline{\otimes}B(L^2(F,\nu_r))$, with y of trace class, we have:

$$\tau_1(x\overline{\otimes}y) = \tau_{\mathcal{L}(\Gamma)}(x)\mathrm{tr}(y) =$$

$$= \tau_{\mathcal{L}(\Gamma)}(x)[\sum_{n\in\mathbb{N}}\langle y\tilde{\epsilon}_n, \tilde{\epsilon}_n\rangle_{L^2(F,\nu_r)}] =$$

$$= \sum_{n\in\mathbb{N}}\langle (x\overline{\otimes}y)(\delta_e\overline{\otimes}\tilde{\epsilon}_n), \delta_e\overline{\otimes}\tilde{\epsilon}_n\rangle_{l^2(\Gamma)\overline{\otimes}L^2(F,\nu_r)}.$$

Since Q_r is a weak limit of linear combinations of elements $x\overline{\otimes}y$ as above, this concludes the proof of the lemma.

Remark. *Let* $\{\epsilon_n\}_{n\in\mathbb{N}} \subseteq L^2(\mathbb{H},\nu_r)$ *be the image under* V_r *of the orthonormal system* $\{\delta_e\overline{\otimes}\tilde{\epsilon}_n \mid n \in \mathbb{N}\} \subseteq l^2(\Gamma)\overline{\otimes}L^2(F,\nu_r)$. *As*

$$\{(R_\gamma\overline{\otimes}Id_{B(L^2(F,\nu_r))})(\delta_e\overline{\otimes}\tilde{\epsilon}_n) \mid \gamma \in \Gamma, n \in \mathbb{N}\}$$

is an orthonormal basis for $l^2(\Gamma)\overline{\otimes}L^2(F,\nu_r)$ *it follows that*

$$\{\tilde{\pi}_r(\gamma)\epsilon_n \mid \gamma \in \Gamma, n \in \mathbb{N}\}$$

is an orthonormal basis for $L^2(\mathbb{H},\nu_r)$.

The properties of the orthonormal system $\{\epsilon_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{H},\nu_r)$ are summarized in the next proposition (whose proof has already been completed).

Proposition 3.9. *We use the above notations. Let Q_r be the projection $U_r P_r V_r$ in $\mathcal{L}(\Gamma)\overline{\otimes}B(L^2(F,\nu_r))$. Then $\{\pi_r(\Gamma)\}'$ is isomorphic to $Q_r(\mathcal{L}(\Gamma)\overline{\otimes}B(L^2(F,\nu_r)))Q_r$.*

Let τ_1 be the trace τ_1 on $\mathcal{L}(\Gamma)\overline{\otimes}B(L^2(F,\nu_r))$, normalized by the condition that τ_1 takes the value 1 on minimal projections in $B(L^2(F,\nu_r))$ and so that $\tau_1 = \tau_{\mathcal{L}(\Gamma)}$ if restricted to $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Gamma)\overline{\otimes}Id_{B(L^2(F,\nu_r))}$.

Then there exists an orthonormal system $\{\epsilon_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{H},\nu_r)$ so that

i). $\{\tilde{\pi}_r(\gamma)\epsilon_n \mid \gamma\in\Gamma, n\in\mathbb{N}\}$ is an orthonormal basis for $L^2(\mathbb{H},\nu_r)$.

ii). $\tau_1(Q_r) = \sum\limits_{n\in\mathbb{N}} ||P_r\epsilon_n||^2$.

We conclude now the proof of iii). in Theorem 3.2.

Proposition 3.10. *Assuming the hypothesis from Proposition 3.9, it follows that*

$$\sum_{n\in\mathbb{N}} ||P_r\epsilon_n||^2 = d_{\pi_r}\, covol(\Gamma) = \frac{r-1}{\pi} covol(\Gamma).$$

Proof. The proof is now exactly as in [GHJ]. Since our context is a bit different we will recall it anyway.

Let η be any unit vector in

$$H_r = H^2(\mathbb{H},\nu_r) = P_r(L^2(\mathbb{H},\nu_r)).$$

Then $\tilde{\pi}_r(g)\eta = \eta, g \in G$ and

$$||\tilde{\pi}_r(g)\eta||_{L^2(\mathbb{H},\nu_r)} = ||\eta||_{L^2(\mathbb{H},\nu_r)} = 1, \text{ for all } g \in G.$$

Let $\mathcal{F}$ be a fundamental domain for Γ acting on the right on G. Then covol(Γ) is $\int\limits_{\mathcal{F}} 1 \mathrm{d}g$ and this may be infinite. We have

$$\mathrm{covol}(\Gamma) = \int\limits_{\mathcal{F}} 1\mathrm{d}g = \int\limits_{\mathcal{F}} ||\tilde{\pi}_r(g)\eta||_{L^2(\mathbb{H},\nu_r)}\mathrm{d}g =$$

$$= \sum_{n\in\mathbb{N},\gamma\in\Gamma} \int_{\mathcal{F}} |\langle \tilde{\pi}_r(g)\eta, \tilde{\pi}_r(\gamma)\epsilon_n\rangle|^2 \mathrm{d}g =$$

$$= \sum_{n\in\mathbb{N}} \int_{G} |\langle \tilde{\pi}_r(g)\eta, \epsilon_n\rangle|^2 \mathrm{d}g =$$

$$= \sum_{n\in\mathbb{N}} \int_{G} |\langle \pi_r(g)\eta, P_r\epsilon_n\rangle|^2 \mathrm{d}g =$$

$$= \sum_{n\in\mathbb{N}} (d_{\pi_r})^{-1} ||\eta||^2 \cdot ||P_r\epsilon_n||^2 =$$

$$= \sum_{n\in\mathbb{N}} (d_{\pi_r})^{-1} ||P_r\epsilon_n||^2 = (d_{\pi_r})^{-1}\tau_1(Q_r).$$

Here we used Proposition 3.1 (the generalized form of Schur orthogonality relations for projective representations). This and Lemma 3.9 concludes the proof for Theorem 3.2, if one assumes that the orthogonality relations hold.

Note that we proved in fact a slightly more precise statement than the determination of the isomorphism class of $\{\pi_r(\Gamma)\}'$ in the case of $\mathcal{L}(\Gamma)$ being a factor (compare [JHG]).

Proposition. *Let Γ be a discrete fuchsian subgroup of $G = PSL(2,\mathbb{R})$ and assume that $\mathcal{L}(\Gamma)$ is a factor.*

Then the von Neumann algebra $\{\pi_r(\Gamma)\}''$ generated in $B(H_r)$ by $\pi_r(\Gamma)$ is isomorphic to $\mathcal{L}(\Gamma)$. Denote (following [J]) by $dim_M H$ the coupling constant ([MvN]) for a type II_1 factor M acting on a Hilbert space H. Then

$$dim_{\{\pi_r(\Gamma)\}''} H^2(\mathbb{H}, \nu_r) = dim_{\mathcal{L}(\Gamma)} H^2(\mathbb{H}, \nu_r) = covol(\Gamma) d_{\pi_r}, r > 1.$$

To complete the proof of our theorem we need to show that the projective unitary representations $(\pi_r)_{r>1}$ have all square integrable coefficients in the sense of Definition 3.1. We have

Proposition 3.11. *The projective unitary representations* $(\pi_r)_{r>1}$*have all finite formal dimension* $d_{\pi_r} = \frac{r-1}{\pi}$. *In particular*

$$\int\limits_{PSL(2,\mathbb{R})} |\langle \pi_r(g)\zeta, \eta\rangle_{H_r}|^2 dg = \frac{\pi}{r-1}||\zeta||^2 \cdot ||\eta||^2, \zeta, \eta \in H^2(\mathbb{H}, \nu_r).$$

Proof. As in [HGJ]it is sufficient to prove this statement for the case of projective representations $\pi_r : G = \mathrm{SU(1,1)} \to B(L^2(\mathbb{D}, \mu_r))$.

Recall that μ_r is the measure on the unit disk $\mathbb{D}$ with density

$$z \to (1 - |z|^2)^{r-2}$$

with respect to the Lebesgue measure on the disk. The representations π_r act on $H^2(\mathbb{D}, \mu_r)$ according to the formula

$$(\pi_r(g)f)(z) = (j(g,z))^{-r} f(g^{-1}z), f \in H^2(\mathbb{D}, \mu_r), g \in G, z \in \mathbb{D}.$$

The modular factor $j(g,z)$ is now given by the formula

$$j(g,z) = (\overline{b}z + \overline{a}), z \in \mathbb{D}$$

for

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1$$

an arbitrary element in SU(1,1). To define $j(g,z)^{-r}$ one chooses a normal branch for $t = \arg(\overline{b}z + \overline{a})$ with values in the interval $\pi < t \leq \pi$.

Because of definition 3.1 it is sufficient to prove the statement for a single non zero vector $\zeta = \eta$ and we choose this vector to be the constant function 1 on $\mathbb{D}$.

We use the method in ([Ro],chapter 20). Let

$$g = \begin{pmatrix} a & b \\ \bar{b} & \overline{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1,$$

be an arbitrary element in SU(1,1). Then

$$|\langle \pi_r(g)1, 1 \rangle| = |\int\limits_{\mathbb{D}} \frac{1}{(\bar{b}z + \overline{a})^{-r}} (1 - |z|^2)^{r-2} \mathrm{d}z \mathrm{d}\overline{z}| =$$

$$= |\int\limits_0^1 (1 - t^2)^{r-2} [\int\limits_0^{2\pi} (\overline{a} + \bar{b}te^{i\theta})^{-r} \mathrm{d}\theta] \mathrm{d}t| =$$

$$= |\int\limits_0^1 (1 - t^2)^{r-2} [\int\limits_{|w|=1} (iw)^{-1} (\overline{a} + \bar{b}w)^{-r} \mathrm{d}w] \mathrm{d}t|.$$

Since $|a| > |b|$, the function $w \to (iw)^{-1}(\overline{a} + \bar{b}w)^{-r}$ has its only pole in $\mathbb{D}$ at $w = 0$ and thus the above integral is

$$|\int\limits_0^1 (1 - t^2)^{r-2} \frac{2\pi}{(\overline{a})^r} \mathrm{d}t| = \frac{\pi}{(r-1)|a|^r}.$$

The proof now follows line by line the one in [Ro] and we end up with the equality

$$\int\limits_G |\langle \pi_r(g)1, 1 \rangle_{H_r}|^2 \mathrm{d}g = ((r-1)/\pi)^{-1} ||1||^2_{H_r}.$$

(See also [GHJ] for a disscution on the difference in constants that arises by working with SU(1,1) instead of $PSL(2, \mathbb{R})$). This finishes the remaining part of the proof of Theorem 3.2.

We end this paragraph by explaining why the kernels $k = k(z,\zeta) = k(\gamma z, \gamma\zeta), \gamma \in \Gamma$ on $\mathbb{D}^2$, analytic in z and antianalytic in ζ should be considered as a generalization of automorphic forms. We will discuss only the case of automorphic forms of integral, even weight for $\Gamma = PSL(2,\mathbb{Z})$.

Recall that an automorphic form of even, integral weight $2k$ for $\Gamma = PSL(2,\mathbb{Z})$ is an analytic function f on $\mathbb{H}$ so that

$$f(\gamma z) = j(\gamma, z)^{-2k} f(z), z \in \mathbb{H}, \gamma \in \Gamma$$

and

$$|f(z)| \leq \mathrm{const}(\mathrm{Im}\ z)^{-k}, z \in \mathbb{H}.$$

It was shown in [GHJ] (see also Jones manuscript notes) that if f, g are automorphic forms of same integral weight $2k$ then the linear multiplication operators M_f^n and respectively M_g^n, on H_n, with the functions f and g respectively are bounded, with values in H_{n+2k}. Moreover both M_f^n and M_g^n are intertwining operators for the representations π_n, π_{n+2k} restricted to Γ, that is

$$M_f^n \pi_n(\gamma) = \pi_{n+2k}(\gamma) M_f^n$$

and similarly for g.

Hence $(M_f^n)^* M_g^n$ belongs to $\mathcal{A}_n$ and $M_g^n (M_f^n)^*$ belongs to $\mathcal{A}_{n+2k}$. Moreover the value of the (unique) traces on $\mathcal{A}_n$ and $\mathcal{A}_{n+2k}$ on this elements is computed in [GHJ] and it is equal (modulo constants depending on n and k) to the Petterson scalar product $\langle f, g\rangle_{\mathrm{Pet}}$ (see [Ma]) which is defined by

$$\langle f, g\rangle_{\mathrm{Pet}} = \mathrm{const} \int\limits_F f(z)\overline{g(z)} \mathrm{Im}^{2k-2} \mathrm{d}\nu_0(z) = \int\limits_F f(z)\overline{g(z)} \mathrm{d}\nu_{2k}(z).$$

We note that this computation is now generalized by the trace formula in Theorem 3.2. This follows from the following:

Proposition 3.12. *Let f, g be automorphic forms of integral weight $2k$. Let M_f^n and respectively M_g^n be the linear, continuous multiplication operators with the functions f and g respectively, on H_n with values in H_{n+2k}. Then the Berezin symbol $k = k(z,\zeta) = k(\gamma z, \gamma\zeta), \gamma \in \Gamma$, for the operator $M_g^n(M_f^n)^*$ in $\mathcal{A}_{n+2k} \subseteq B(H_{2k+n})$ is given by the formula:*

$$k(z,\zeta) = \frac{c_n}{c_{n+2k}} \overline{f(\zeta)} g(z)((z-\zeta)/2i)^{2k}, z, \zeta \in \mathbb{H}.$$

Note that the factor $((z-\zeta)/2i)^{2k}$ makes this symbol Γ invariant. In particular (modulo a scalar), by Theorem 3.2, the trace of $M_g^n(M_f^n)^*$ in $\mathcal{A}_{n+2k}$ is $\int\limits_F f(z)\overline{g(z)} Im^{2k-2} d\nu_0(z)$.

Proof. It is obvious that for all $z \in \mathbb{H}$ one has:

$$(M_f^n)^* e_z^{n+2k} = \overline{f(z)} e_z^n.$$

Hence the symbol for $M_g^n(M_f^n)^*$ is

$$k(z,\zeta) = \frac{\langle (M_f^n)^* e_\zeta^{n+2k}, (M_g^n)^* e_z^{n+2k} \rangle}{\langle e_\zeta^{n+2k}, e_z^{n+2k} \rangle} =$$

$$\overline{f(\zeta)} g(z) \frac{\langle e_\zeta^n, e_z^n \rangle}{\langle e_\zeta^{n+2k}, e_z^{n+2k} \rangle}, z, \zeta \in \mathbb{H}.$$

Remark 3.13. . *This shows that the union of all symbols in $\mathcal{A}_r$ when r tends to infinity exhausts all possible pairs of automorphic functions.*

We have to take $r \to \infty$ above, since for fixed r, only the automorphic forms of weight $2k < r - 1$ may occur in symbols of bounded operators in $\mathcal{A}_r$by the above method.

4.The covariant symbol in invariant Berezin quantization

In this paragraph we will analyze the deformation quantization for $\mathbb{H}/\Gamma$ from the viewpoint of covariant symbols. We use the notation M_f^r for the multiplication operator on $H^2(\mathbb{H}, \nu_r)$ with the function f. Let P_r be the orthogonal projection from $L^2(\mathbb{H}, \nu_r)$ onto $H^2(\mathbb{H}, \nu_r)$. Recall that $A \in B(H^2(\mathbb{H}, \nu_r))$ admits a contravariant Berezin's symbol f in $L^\infty(\mathbb{H})$ if A is a Toeplitz operator on $H_r = H^2(\mathbb{H}, \nu_r)$ with symbol f, that is $A = T_f^r = P_r M_f^r P_r$.

The relation between the Berezin's contravariant and covariant symbols is a duality relation involving the trace on $B(H_r)$. We use the notation $\overset{\circ}{A}(z, \overline{z}) = f(z)$ for the contravariant symbol of A. Let B be any element in $B(H_r)$ of covariant symbol $\hat{B}(z, \overline{z})$. Assume that AB is a trace class operator. The duality relation between the two type of symbols is given by the following equality:

$$(4.0) \quad \mathrm{tr}_{B(H_r)}(AB) = \textstyle\int_{\mathbb{H}} \hat{B}(z, \overline{z}) \overset{\circ}{A}(z, \overline{z}) d\nu_0(z).$$

In this paragraph we will extend this relation to the case of Γ- invariant symbols. These symbols correspond to linear operators in $B(H_r)$ that commute with $\pi_r(\Gamma)$.Recall that we used in Proposition 3.5 the notation $\tilde{\pi}_r : G = PSL(2, \mathbb{R}) \to B(L^2(\mathbb{H}, \nu_r))$ for the projective, unitary representation of G on $L^2(\mathbb{H}, \nu_r)$ that is defined by the same algebraic formula as π_r.

If f is any bounded, measurable function on $\mathbb{H}$ then M_f^r commutes with $\tilde{\pi}_r(\Gamma)$. Moreover, P_r commutes with $\tilde{\pi}_r(\Gamma)$ (and in fact with $\tilde{\pi}_r(G)$ as $H^2(\mathbb{H}, \nu_r)$ is invariant by $\tilde{\pi}_r(G)$). Hence $T_f^r = P_r M_f^r P_r$ commutes with $\pi_r(\Gamma)$. We have thus proved that if $A = T_f^r$ and if f is a Γ-invariant and bounded function on $\mathbb{H}$ then A belongs to $\mathcal{A}_r = \{\pi_r(\Gamma)\}'$.

The duality relation (4.0) between the covariant and contravariant symbol will now be replaced with a new relation in which the trace $\mathrm{tr}_{B(H_r)}$ on $B(H_r)$ is replaced by a trace on the semifinite von Neumann algebra $\mathcal{A}_r$. This will correspond to the fact that in the formula (4.0) we will be rather integrating over F, a fundamental domain for the action of Γ on $\mathbb{H}$ rather then on $\mathbb{H}$ as in the classical setting.

In the last part of this paragraph we will introduce a third type of a symbol for operators on H_r. We will call this type of symbol an intermediate symbol for operators in $B(H_r)$ because it inherits properties from both the covariant and the contravariant symbol.

Recall that $B_r(\Delta)$ was the positive operator which assigns to a function f on $\mathbb{H}$ the restriction to the diagonal of the contravariant symbol of the associated Toeplitz operator T^r_f. The intermediate symbol for an operator A in $B(H_r)$ is the operator function $[B_r(\Delta)]^{1/2}$ applied to the the covariant symbol of A.

We start with a rigorous definition of the operator $B_r(\Delta)$ in the Γ-invariant case.

Proposition 4.1.. *Let $r > 1$and let F be a fundamental domain for the action of Γ on the upper half plane. Let f be any bounded function on $\mathbb{H}$ that is $\Gamma-$ invariant and let $A = T^r_f = P_r M^r_f P_r$ be the Toeplitz operator on $H^2(\mathbb{H}, \nu_r)$ with symbol f.*

Then A commutes with $\{\pi_r(\Gamma)\}$ (with the notations in the third paragraph, this is $A \in \mathcal{A}_r$). Let $\hat{A} = \hat{A}(z, \zeta)$ be the contravariant symbol of $A = T^r_f$. Let $K_r(z, \zeta)$ be the kernel function on $L^2(F)$ defined by

$$K_r(z, \eta) = c_r \sum_{\gamma \in \Gamma} |d(z, \gamma\eta)|^{2r},$$

where $|d(z, \eta)|^2 = (Im\ z)(Im\ \eta)|z - \overline{\eta}|^2, z, \eta \in \mathbb{H}$ is a function on the hyperbolic *distance between z and η in $\mathbb{H}$. Then*

$$\hat{A}(z,\overline{z}) = (B_r f)(z) =$$

$$= c_r \int_{\mathbb{H}} f(\eta)|d(z,\eta)|^{2r}\, d\nu_r(z) = \int_F K_r(z,\eta) f(\eta)\, d\nu_0(\eta).$$

Moreover the linear operator B_r defined above on $L^\infty(F)$ with values in $L^\infty(F)$ extends to a bounded, positive, contractive operator on $L^2(F,\nu_r)$. The operator B_r is injective and the operators $(B_r)_{r>1}$ are pairwise commuting. Moreover B_r tends strongly to 1 as r tends to infinity.

Proof. The kernel $K_r(z,w), z, w \in F$ is symmetric with positive values. Moreover

$$\int_F K_r(z,w)\mathrm{d}\nu_0(w) = c_r \int_{\mathbb{H}} |d(z,w)|^{2r}\mathrm{d}\nu_0(w) =$$

$$= c_r(\mathrm{Im} z)^r \int_{\mathbb{H}} \frac{1}{|(z-\overline{w})/2i|^{2r}}\mathrm{d}(w) = c_r^{-1}(\mathrm{Im} z)^r \int_{\mathbb{H}} \frac{c_r^2}{|(z-\overline{w})/2i|^{2r}}\mathrm{d}(w) =$$

$$= c_r^{-1}(\mathrm{Im} z)^r \langle e_z^r, e_z^r\rangle_{H_r} = c_r^{-1}(\mathrm{Im} z)^r e_z^r(z) =$$

$$= c_r^{-1}(\mathrm{Im} z)^r \frac{c_r}{((z-\overline{z})/2i)^r} = 1, \text{ for all } z \in F.$$

Hence the interpolation arguments in Theorem 2.4, page 1131, [Be] show that B_r extends to a contractive operator from $L^2(F,\nu_r)$ into $L^2(F,\nu_r)$ and it also extends to a bounded operator from $L^1(F,\nu_r)$ into $L^1(F,\nu_r)$.

That B_r is a positive injective operator follows from the Corollary on page 66 in Patterson paper ([Pa]) in Math. Proc. Cambr, (81). We could have proved the positivity of B_r by using the corresponding property of the similar operator on $L^2(\mathbb{H}, \nu_r)$.

The pairwise commutativity of the operators B_r follows, for example, from the above quoted paper ([Pa]). In fact the operators B_r are functions of the invariant laplacian on $\mathbb{H}/\Gamma$. This type of analysis was first considered by Selberg [Se]. This completes the proof.

We now prove the duality relation between the two type of Berezin's symbols in the $\Gamma-$ equivariant case. Since obviously (at least in the infinite covolume case) the trace formula can not hold true for all elements in $\mathcal{A}_r$, we will restrict our consideration to elements in $\hat{\mathcal{A}}_r = \hat{B}(H_r) \cap \mathcal{A}_r$.

Proposition 4.2. *Let $\mathcal{A}_r$ be the commutant of $\pi_r(\Gamma)$ in $B(H_r)$ and let $\hat{\mathcal{A}}_r = \hat{B}(H_r) \cap \mathcal{A}_r$. Let $r > 1$ and let A be any operator in $\hat{\mathcal{A}}_r$. Let $\hat{A}(z, \overline{\zeta})$, $z, \zeta \in \mathbb{H}$, be the contravariant symbol of the operator A. We choose a fundamental domain F for Γ in $\mathbb{H}$ and let f be $\Gamma-$ equivariant function on $\mathbb{H}$. Assume that f is in $L^1(F, \nu_0)$. Let $T^r_f = P_r M^r_f P_r$ be the Toeplitz operator on $H^2(\mathbb{H}, \nu_r)$ with symbol f. Denote by $\tau_{\mathcal{A}_r}$ the (semifinite) faithful trace on $\mathcal{A}_r$ that was constructed in Theorem 3.2. Then*

$$\tau_{\mathcal{A}_r}(AT^r_f) = (const) \int_F \hat{A}(z.\overline{z}) f(z) d\nu_o(z).$$

The value of the constant in front of the integral is (area $F)^{-1}$ in the finite covolume case and 1 otherwise. Moreover

$$|\tau_{\mathcal{A}_r}(AT^r_f)| \leq c_r ||A||_{\lambda,r} ||f||_{L^1(F,\nu_0)}.$$

Proof. The symbol for the operator AT_f^r is the iterated integral:

$$k(z,\overline{\zeta}) = c_r^2((z-\overline{\zeta})/2i)^r \int\limits_{\mathbb{H}}\int\limits_{\mathbb{H}} \frac{\hat{A}(z,\overline{\eta})f(a)}{[(z-\overline{\eta})/2i]^r[(\eta-\overline{a})/2i]^r[(a-\overline{\zeta})/2i]^r}\mathrm{d}\nu_r(a,\eta),$$

were we first integrate a and then η.

Consequently

$$\tau_{\mathcal{A}_r}(AT_f^r) = (\mathrm{const})c_r^2 \int\limits_{F,z}\int\limits_{\mathbb{H},\eta}\int\limits_{\mathbb{H},a} \frac{\hat{A}(z,\overline{\eta})f(a)}{[(z-\overline{\eta})/2i]^r[(\eta-\overline{a})/2i]^r[(a-\overline{z})/2i]^r}\mathrm{d}\nu_r(z,a,\eta),$$

were we first integrate a, then η and then z. We indicated above for each variable the domain of integration. By letting the measure be $\mathrm{d}\nu_0(z,a,\eta$ and by collecting all the densities into the integrand, the integrand itself becomes a $\Gamma-$invariant function in the variables z,a,η on $\mathbb{H}^3$. If the integral were to be absolute convergent then we could integrate on any fundamental domain of Γ acting on $\mathbb{H}^3$, e.g.we could integrate on $\mathbb{H}_z\times F_a\times\mathbb{H}_\eta$. In this case the integral would be (modulo a constant):

$$\int\limits_F f(a)\{c_r^2\int\int\limits_{\mathbb{H}} \frac{\hat{A}(z,\overline{\eta})}{[(z-\overline{\eta})/2i]^r[(\eta-\overline{a})/2i]^r[(a-\overline{z})/2i]^r}\mathrm{d}\nu_r(z,\eta)\}\mathrm{d}\nu_r(a) =$$

$$\int\limits_F f(a)\frac{\hat{A}(a,\overline{a})}{[(a-\overline{a})/2i]^r}\mathrm{d}\nu_r(a) = \int\limits_F f(a)\hat{A}(a,\overline{a})\mathrm{d}\nu_0(a).$$

To prove the absolute convergence of the integrand it is sufficient to check this on any fundamental domain. Thus it is sufficient to estimate:

$$(4.1)\quad \int\limits_F |f(a)|[c_r^2\int\int\limits_{\mathbb{H}} \frac{|\hat{A}(z,\overline{\eta})|}{|[(z-\overline{\eta})/2i]|^r|[(\eta-\overline{a})/2i]|^r|[(a-\overline{z})/2i]|^r}\mathrm{d}\nu_r(z,\eta)]\mathrm{d}\nu_r(a).$$

By Proposition 2.7, if A belongs to $\hat{B}(H_r)$, then the operator on $L^2(\mathbb{H}, \nu_r)$, with integral kernel $\frac{|\hat{A}(z,\overline{\eta})|}{|[(z-\overline{\eta})/2i]|^r}$ is bounded of uniform norm less than $||A||_{\lambda,r}$. Moreover the function on $\mathbb{H}$ defined by $z \to |[(a-\overline{z})/2i]|^{-r}$, belongs to $L^2(\mathbb{H}, \nu_r)$ and has norm less than

$$c_r \int\limits_{\mathbb{H}} |[(a-\overline{z})/2i]|^{-2r} \mathrm{d}\nu_r(z) = c_r^{-1} \langle e_a^r, e_a^r \rangle_{H_r} = (\mathrm{Im}\ a)^{-r}.$$

Hence the inner integral in the formula (4.1) is estimated by $||A||_{\lambda,r}(\mathrm{Im}\ a)^{-r}$ and thus the integral itself is bounded by

$$(\mathrm{const}) \int\limits_F |f(a)| \mathrm{d}\nu_0(a) = (\mathrm{const}) ||A||_{\lambda,r} \cdot ||f||_1.$$

This ends the proof.

In the next proposition we show that Toeplitz "operators" may be defined even if the symbol function is not bounded, but only in L^2. In this case the corresponding Toeplitz operator will be an element in $L^2(\mathcal{A}_r)$.

Lemma 4.3. *We identify $L^\infty(F)$ with a subspace of all bounded functions on $\mathbb{H}$, by extending them outside F by $\Gamma-$invariance. Let $r > 2$. Let S_r be the bounded linear operator on $L^\infty(F)$ with values in $\mathcal{A}_r$ defined by $S_r f = T_f^r = P_r M_f^r P_r$. Then S_r extends to a contractive linear map from $L^2(F)$ into $L^2(\mathcal{A}_r)$. Moreover*

$$\langle S_f^r, S_g^r \rangle_{L^2(\mathcal{A}_r)} = \langle B_r f, g \rangle_{L^2(F)}, \text{ for all } f, g \in L^2(F).$$

In particular, $\{S_f^r | f \in L^2(F) \cap L^\infty(F)\} \subseteq L^2(\mathcal{A}_r) \cap \mathcal{A}_r$.

Proof. We compute the term $\langle S_f^r, S_g^r\rangle_{L^2(\mathcal{A}_r)}$. We assume first that f is in $L^1(F) \cap L^\infty(F)$ and g is in $L^\infty(F)$. In this case the iterated integral defining $\langle S_f^r, S_g^r\rangle_{L^2(\mathcal{A}_r)}$ is (modulo a constant which is (area $F)^{-1}$ in the finite covolume case and 1 otherwise)

$$\int\limits_{F_z}((z-\overline{z}/)2i)^r \int\limits_{\mathbb{H}_\eta}\int\int\limits_{\mathbb{H}^2_{a,b}} \frac{f(a)\overline{g(b)}}{[(z-\overline{a})/2i]^r[(a-\overline{\eta})/2i]^r[(\eta-\overline{b})/2i]^r} \mathrm{d}\nu_r(a,b)\mathrm{d}\nu_r(\eta)\nu_0(z).$$

The above integral is an iterated integral: first we integrate a, b and then η and z.

If the integral is absolutely convergent, then we may integrate in any order the variables a, b, η, z. In this case, the integrand with respect to the measure $\mathrm{d}\nu_0(a, b, \eta, z)$ is $\Gamma-$ invariant. To evaluate the integral, we may henceforth change the domain of integration into $F_a \times \mathbb{H}_b \times \mathbb{H}_{\eta,z}$. Under the absolute convergence assumption the integral will be thus equal to

$$(4.2) \int\limits_{F_a}\int\limits_{\mathbb{H}_b} f(a)g(b)[\int\int\limits_{\mathbb{H}_{\eta,z}} \frac{\mathrm{d}\nu_r(\eta)}{[(a-\overline{\eta})/2i]^r[(\eta-\overline{b})/2i]^r} \frac{\mathrm{d}\nu_r(z)}{[(a-\overline{z})/2i]^r[(z-\overline{b})/2i]^r}]\mathrm{d}\nu_r(a,b) =$$

$$\int\limits_{F_a} f(a)(\int\limits_{\mathbb{H}_b} g(b)|[(a-\overline{b})/2i]|^{2r}\mathrm{d}\nu_r(b))\mathrm{d}\nu_r(a) = \langle f, B_r g\rangle_{L^2(F)}.$$

By Fubini's theorem, to show the absolute convergence of the integral, it is sufficient to check absolute convergence for the first integral in formula (4.2). The integral of the absolute value of the integrand is bounded by

$$(4.3) ||g||_\infty \int\limits_F |f(a)|\ M_r(a)\mathrm{d}\nu_0(a),$$

where

$$M_r(a) = \int\int\int_{\mathbb{H}^3} |d(a,\eta)d(\eta,b)d(b,z)d(z,a)|^r \mathrm{d}\nu_0(z,\eta,b).$$

Recall that we use the notation $d(z,\overline{\zeta}) = (\mathrm{Im}\ z)^{1/2}(\mathrm{Im}\ \zeta)^{1/2}[(z-\overline{\zeta})/2i]^{-1}$ and recall that $|d(z,\overline{\zeta})|$ is a function of the hyperbolic distance between z and ζ, $z,\zeta \in \mathbb{H}$.

It is easy to conclude that $M_r(a)$ is a $PSL(2,\mathbb{R})-$invariant function on $\mathbb{H}$. Since $PSL(2,\mathbb{R})$ acts transitively on $\mathbb{H}$ it is thus sufficient to check that the integral defining $M_r(a)$ is convergent for a single value of a. Also to check the finiteness of the integral defining $M_r(a)$ we may use the unit disk $\mathbb{D}$ formalism instead of the corresponding formalism for the upper half plane $\mathbb{H}$. We let $a = 0 \in \mathbb{D}$ and hence we have to estimate:

$$\int\int\int_{\mathbb{D}^3_{z,\eta,b}} (1-|z|^2)^{r/2}(1-|\eta|)^{r/2}(d(z,b))^r(d(\eta,b))^r \mathrm{d}\nu_0(b)\mathrm{d}\nu_0(z,b).$$

By integrating first in the parameter b and then in the parameters z,η and since the quantities

$$\sup_z \int_{\mathbb{D}} ((d(z,b))^{2r}\mathrm{d}\nu_0(b),\ \sup_\eta \int_{\mathbb{D}} (d(\eta,b))^{2r}\mathrm{d}\nu_0(b)$$

are finite, by using the Cauchy-Buniakowsky-Schwarz inequality, we obtain that the above integral is dominated by

$$(\mathrm{const})\int\int_{\mathbb{D}^2} (1-|z|^2)^{r/2}(1-|\eta|)^{r/2}\mathrm{d}\nu_0(z,\eta),$$

which is finite if $r > 2$.

Thus we have proved in particular that for f in $L^1(F) \cap L^\infty(F)$ the following equality holds:

$$\langle S^r f, S^r f\rangle_{L^2(\mathcal{A}_r)} = \langle B_r f, f\rangle_{L^2(F)}.$$

As B_r is bounded and contractive, this shows that

$$||S^r f||_{L^2(\mathcal{A}_r)} \leq ||f||_{L^2(F)}$$

Hence the above equality also extends to all $f \in L^2(F)$.

This concludes the proof of the lemma. Incidentally we have also proved the following:

Proposition 4.4. *Let $r > 2$. Let $S^r : L^2(F) \to L^2(\mathcal{A}_r)$ be defined as above by*

$$S^r f = T^r_f = P_r M^r_f P_r.$$

Then there exists a constant $C_r > 0$ so that for all g in $L^\infty(F)$ and for all f in $L^1(F, \nu_0)$ one has the inequality

$$|\tau_{\mathcal{A}_r}((S^r f)(S^r g))| \leq C_r ||f||_{L^1(F,\nu_0)} ||g||_\infty.$$

In fact it is easier to understand the map S^r by looking at its adjoint. In the next proposition we will identify $(S^r)^*$ with the restriction map R_r on $\mathcal{A}_r$, which associates to any kernel $k(z, \overline{\zeta})$ on $\mathbb{H}^2$ its restriction to the diagonal $z = \zeta$.

Proposition 4.5. *Let $r > 2$, let F be a fundamental domain for the action of* Γ *on $\mathbb{H}$. Let $R_r : \mathcal{A}_r \to L^\infty(F)$ be the map associating to an operator A in $\mathcal{A}_r$ with kernel $\hat{A}$, the function on F defined by: $z \to \hat{A}(z, \overline{z})$.*

Then R_r extends to a continuous linear map $R_r : L^2(\mathcal{A}_r, \tau) \to L^2(F)$. Moreover R_r is the adjoint of S^r and $R_r^* R_r = B_r$.

Proof. Indeed we have already checked that

$$\langle T_f^r, T_g^r \rangle_\tau = \langle B_r f, g \rangle_{L^2(F)} = \langle R_r(T_f^r), g \rangle_{L^2(F)},$$

for all $f, g \in L^2(F)$ or for all $f \in L^1(F, \nu_0)$ and all $g \in L^\infty(F)$.

This implies that

$$\langle T_f^r, S_r(g) \rangle_{L^2(\mathcal{A}_r} = \langle R_r(T_f^r), g \rangle_{L^2(F)}$$

for all g in $L^2(F)$ and all f in $L^\infty(F)$. Hence R_r is graph-contained (as an eventually unbounded operator) in the adjoint of S_r. Since S_r is a bounded linear operator and R_r is closable it follows that R_r extends to a bounded operator. This completes the proof.

We will now introduce a third type of symbol which is intermediate between Berezin's contravariant and covariant symbols. For $A \in B(H_r)$ we let $V_r A$ to be the the value of the operatorial inverse square root of B_r applied to the function obtained by restriction to the diagonal of the contravariant symbol $\hat{A}$ of A.

When using the intermediate symbol $V_r A$, the main simplification occurs in computations involving the scalar product: the scalar product $\langle A, B \rangle_{L^2(\mathcal{A}_r)}$ of two elements A, B is equal to the canonical scalar product in $L^2(F)$ of their intermediate symbols $V_r A, V_r B$.

We will use this property to define a different representation of the Berezin deformation. In this representation the trace of a product of two elements (viewed as symbols functions on $\mathbb{H}$) doesn't depend on the deformation parameter. The main properties for the symbol V_r are outlined in the following proposition.

Proposition 4.6. *Let $U_r : L^2(F) \to L^2(\mathcal{A}_r)$ be the unitary operator defined by*

$$U_r f = T^r_{B_r^{-1/2}} f, f \in L^2(F).$$

Note that U_r is first defined on a dense set and then extended by continuity.

For A in $\mathcal{A}_r$ with symbol $\hat{A} = \hat{A}(z, \overline{\zeta}), z, \zeta \in \mathbb{H}$, let $R_r A$ be the function on $\mathbb{H}$ defined by $(R_r A)(z) = \hat{A}(z, \overline{z})$. The inverse of U_r is the unitary $V_r : L^2(\mathcal{A}_r) \to L^2(F)$ defined by the formula:

$$V_r A = B_r^{-1/2}(R_r A), A \in L^2(\mathcal{A}_r).$$

Let $j_{s,r} : \mathcal{A}_r \to \mathcal{A}_s$ be the map that associates to A in $\mathcal{A}_r$ the element $j_{s,r}(A)$ in $\mathcal{A}_s$ having the same contravariant symbol as A.

Then the following diagram is commutative:

$$\begin{array}{ccc} L^2(\mathcal{A}_r) & \overset{j_{s,r}}{\longrightarrow} & L^2(\mathcal{A}_s) \\ \uparrow{\scriptstyle U_r} & & \uparrow{\scriptstyle U_s} \\ L^2(F) & \overset{B_r^{1/2} B_s^{-1/2}}{\longrightarrow} & L^2(F) \end{array}$$

Proof. We first define U_r on the set $\mathcal{S} = B_r^{1/2}(L^2(F) \cap L^\infty(F))$. By the injectivity and continuity of B_r, $\mathcal{S}$ is a dense subspace of $L^2(F)$. For every vector f in $B_r^{1/2}(L^2(F) \cap L^\infty(F))$ we have that

$$||U_r f||^2_{L^2(\mathcal{A}_r)} = \langle T^r_{B_r^{-1/2}} f, T^r_{B_r^{-1/2}} f \rangle_{L^2(\mathcal{A}_r)}.$$

By Proposition 4.4 this is

$$\langle B_r B_r^{-1/2} f, B_r^{-1/2} f \rangle_{L^2(F)} = ||f||_{L^2(F)}.$$

Thus U_r extends by continuity to an isometry on $L^2(F)$. To prove that U_r is in addition an unitary, it will be sufficient to check that V_r is a left inverse for U_r.

First we note that V_r is also a well defined isometry. Observe that if A is of the form T_f^r with f a $\Gamma-$equivariant function on $\mathbb{H}$, then V_r is well defined as $V_r(T_f^r) = B_r^{1/2} f$. Hence V_r is well defined on the following dense subset of $L^2(\mathcal{A}_r)$:

$$\{T_f^r |\ f \in L^2(F) \cap L^\infty(F)\}.$$

Moreover

$$||V_r A||^2_{L^2(F)} = \langle V_r A, V_r A\rangle_{L^2(F)} =$$

$$\langle B_r^{-1/2} R_r A, B_r^{-1/2} R_r A\rangle_{L^2(F)} = \langle B_r^{-1} B_r f, B_r f\rangle_{L^2(F)}$$

and by Proposition 4.4 this is $||T_f^r||_{L^2(\mathcal{A}_r)}$. Thus V_r also extends to an isometry on $L^2(\mathcal{A}_r)$.

The restriction to the diagonal of the contravariant symbol of the operator $U_r f$ is equal to the restriction to the diagonal of the symbol of $T^r_{B_r^{-1/2} f}$ which is

$$B_r(B_r^{-1/2}) f = B_r^{1/2} f, f \in L^2(F).$$

Hence the restriction to the diagonal of $U_r(V_r A)$ is equal to

$$B_r^{1/2}(V_r A) = B_r^{1/2} B_r^{-1/2} R_r A = R_r A$$

for all $A \in \mathcal{A}_r \cap L^2(\mathcal{A}_r)$. Thus for such A's, the restriction to the diagonal of the symbol of $U_r(V_r A)$ and the restriction to the diagonal of the symbol of A are equal. Hence $U_r(V_r A) = A$ for A in a dense set and hence

$$U_r V_r = \mathrm{Id}_{L^2(\mathcal{A}_r)}.$$

Similarly, for $f \in L^2(F) \cap L^\infty(F)$

$$V_r U_r f = V_r(T^r_{B_r^{-1/2} f}) = B_r^{-1/2} R_r T^r_{B_r^{-1/2} f} = B_r^{-1/2} B_r B_r^{1/2} f = f.$$

Thus U_r, V_r are unitaries, inverse one to the other.

To complete the proof of the proposition it remains to check the commutativity for the diagram. It is obvious that $j_{s,r}$ is bounded with respect to the L^2 norms on the corresponding spaces (since the absolute value of the function d entering the formulae for these norms takes only subunitary values). Also it will be proved bellow that the operator $B_r^{1/2} B_s^{-1/2}$ is bounded and contractive. Hence it will be sufficient to check the commutativity of the diagram, for vectors in a dense set.

For f in $L^2(F) \cap L^\infty(F)$, the restriction to the diagonal of $j_{s,r}(U_r f)$ is

$$R_s(T^r_{B_r^{-1/2} f}) = R_r(T^r_{B_r^{-1/2} f}) = B_r^{1/2} f.$$

On the other hand

$$R_s(U_s(B_s^{-1/2} B_r^{1/2} f)) = R_s T^s_{B_s^{-1/2} B_r^{1/2} B_s^{-1/2}} =$$

$$R_s T^s_{B_s^{-1} B_r^{1/2}} = B_s B_s^{-1} B_r^{1/2} f = B_r^{1/2} f.$$

Since any two elements in $\mathcal{A}_s$ whose symbols coincide on diagonal, are equal it follows that the diagram is commutative.

Corollary 4.7. *For $s \geq r > 1$, the vector space $j_{s,r}(\mathcal{A}_r)$ is weakly dense in $\mathcal{A}_s$. Hence, for $r > 3$, the algebra $\hat{\mathcal{A}}_r$ is weakly dense in $\mathcal{A}_r$.*

Proof. The statement is equivalent to showing that $j_{s,r}(L^2(\mathcal{A}_r))$ is dense in $L^2(\mathcal{A}_s)$. By the commutativity of the diagram in the previous proposition, this is equivalent to proving that $B_s^{-1/2}B_r^{1/2}$ has a dense image which is also equivalent (as $B_s^{-1/2}B_r^{1/2}$ is selfadjoint) to showing that this bounded operator has trivial kernel which will be proved by the arguments at the end of this paragraph.

To prove that the algebra $\hat{\mathcal{A}}_r$ is weakly dense in $\mathcal{A}_r$ we use the fact (already proved in the paragraph 2 that for all strictly positive ϵ, the algebra $\mathcal{A}_{r-2-\epsilon}$ is contained in $\hat{\mathcal{A}}_r$. This completes the proof of the corollary.

The diagram in Proposition 4.7 also shows that the operatorial absolute value $|j_{s,r}|$ in the polar decomposition of the inclusion map $j_{s,r} : \mathcal{A}_r \to \mathcal{A}_s$ is unitary equivalent (by the unitary V_r) to the operator $B_s^{-1/2}B_r^{1/2}$.

This will be useful in understanding the differentiation, with respect the deformation parameter, of the Berezin quantization.

Corollary 4.8. *Let $s \geq r > 1$ and let $|j_{s,r}|$ in $B(L^2(\mathcal{A}_r))$ be the operatorial absolute value in the polar decomposition of the inclusion map $j_{s,r} : \mathcal{A}_r \to \mathcal{A}_s$ (i.e. $|j_{s,r}| = ((j_{s,r})^* j_{s,r})^{1/2}$).*

Then the following diagram is commutative:

$$\begin{tikzcd} L^2(\mathcal{A}_r) \arrow[r, "|j_{s,r}|"] & L^2(\mathcal{A}_r) \\ L^2(F) \arrow[u, "U_r"] \arrow[r, "B_r^{1/2}B_s^{-1/2}"] & L^2(F) \arrow[u, "U_r"] \end{tikzcd}$$

Proof. By the previous corollary we have that

$$(j_{s,r})^* j_{s,r} = U_r(B_s^{-1/2}B_r^{1/2})U_s^* U_s(B_s^{-1/2}B_r^{1/2})U_r^* = U_r(B_s^{-1}B_r)U_r^*.$$

The assertion now follows from the fact that $B_s^{-1}B_r$ is a positive operator.

We will now show that the "coefficient" function $s \to \langle U_s,\ U_t g\rangle_t$ is differentiable at any point s for f, g in a dense subspace. Moreover its derivative will be equal to an expression involving the bilinear functional giving the derivative (in the deformation parameter) of the scalar product. This will allow us to simplify the expression of the derivative of the Berezin product when using intermediate symbols.

Corollary 4.10. *Let $t > 1$ and let f, g be two arbitrary vectors in $L^2(F)$ so that the function α given by the formula*

$$\alpha(s) = \langle U_s f,\ U_t g\rangle_{L^2(\mathcal{A}_s)}$$

is defined in a neighborhood of t and is differentiable at $s = t$. We also assume that the function $\psi(s)$ defined by

$$\psi(s) = \langle U_s f, U_t g\rangle_{L^2(\mathcal{A}_t)}$$

is also defined in a neighborhood of t.

Then the function $\psi(s)$ is also differentiable at t and

$$\psi'(t) = \frac{d}{ds}\langle U_s f, U_t g\rangle_{L^2(\mathcal{A}_t)} = -1/2\alpha'(t).$$

Proof. The proof is essentially contained in the following observation:

Observation. *Let D be a selfadjoint (eventually) unbounded operator acting on a Hilbert space H. Let $a_p = a_p(D)$ be positive, selfadjoint,injective (eventually unbounded) operators, that are functions on D (in the sense of the borelian functional calculus), for p in an interval $(t-\epsilon, t+\epsilon)$. We assume that $a_t(D) = Id_H$ and that the map $p \to a_p(D)\eta$ is continuously strongly differentiable in p in a neighborhood of p, for η in a dense subset of H.*

Let f, g be two vectors in H so that the function $\alpha(p) = \langle a_p(D)f, g\rangle_H$ is defined in a neighborhood of t and it is differentiable at $p = t$. We assume also that the function $\beta(p) = \langle a_p^{-1/2} f, g\rangle$ is defined in a neighborhood of t.

Then β is differentiable $p = t$ and $\beta'(t) = -1/2\alpha'(t)$.

Proof. With no loss of generality we may assume that D has multiplicity 1. The setting is then the following: we are given a measure μ on a subset $\sigma = \sigma(D)$ and functions $a_p = a_p(x), x \in \sigma$, positive on σ and non vanishing at any atom of μ. The operator D is then the operator of multiplication with the independent variable on $H = L^2(d, \mu)$.

The vectors f, g are now simply two functions in $L^2(d, \mu)$. By hypothesis, for $\mu-$ almost all $x \in \sigma$, the application $p \to a_p(x)$ is differentiable (and finite) in a neighborhood of t.

Also by hypothesis, the functions on σ, in the variable p, are defined in a neighborhood of t by

$$\alpha(p) = \int_\sigma a_p(x) f(x)\overline{g(x)} \mathrm{d}\mu(x)$$

and

$$\beta(p) = \int_\sigma (a_p(x))^{-1/2} f(x)\overline{g(x)} \mathrm{d}\mu(x)$$

are the integrals are absolutely convergent. Finally, the hypothesis also gives that $\alpha(p)$ is differentiable at $p = t$ and that a_t is the constant function 1.

The conclusion of the observation then simply follows from Lebesgue theorem of differentiability under the integral sign.

Proof of the Corollary. We have $\langle U_s f, U_t g\rangle_t = \langle a_{t,s} f, g\rangle_{L^2(F)} = \langle a_{s,t}^{-1} f, g\rangle_{L^2(F)}$.

Moreover $\langle a_{s,t}^2 f, g\rangle = \langle |j_{s,t}|^2 U_t f, U_t g\rangle = \langle U_t f, U_t g\rangle_s$ (we use here the following property of $j_{s,t}$:

$$\langle j_{s,t}^* j_{s,t} A, B\rangle_t = \langle A, B\rangle_s \quad \text{for all } A, B \text{ in } L^2(\mathcal{A}_r)).$$

Thus if the derivative $\frac{d}{ds}\langle U_t f, U_t g\rangle_s|_{s=t}$ exists then this will imply that the derivative

$$\frac{d}{ds}\langle a_{st}^2 f, g\rangle_s|_{s=t} \quad \text{exists.}$$

The proof now follows from the observation.

Finally to describe the multiplication of Berezin symbols of operators in the expression given by the intermediate symbols and to be able to define the differentiation of this multiplication with respect to the deformation parameter we will have to find a dense set of vectors that are well behaved under this type of operations. This is realized in the following lemma.

Lemma 4.10. *Let $r > 1$ be fixed and let (a, b) be an interval with $r < a < b$. Then there exists a weakly dense set $\mathcal{E}$ in $L^\infty(F)$ so that $\mathcal{E} \cap L^2(F)$ is dense in $L^2(F)$ and so that for all $s \in (a, b)$ and all $f \in \mathcal{E}$, $U_s f = T^s_{B_s^{-1/2}}$ belongs to $\mathcal{A}_s \cap L^2(\mathcal{A}_s)$ and the function on (a, b) defined by*

$$s \to ||U_s f||_{\infty,r}$$

is locally bounded.

Proof. We will let $\mathcal{E}$ be the vector space $B_r^{3/2}(L^\infty(F))$. It is clear that $\mathcal{E} \cap L^2(F)$ is dense in $L^2(F)$ as this follows from the fact that

$$B_r^{3/2}(L^\infty(F) \cap L^2(F)) \subseteq B_r^{3/2}(L^\infty(F)) \cap L^2(F) \subseteq \mathcal{E} \cap L^2(F).$$

The first inclusion follows from the fact that $B_r^{3/2}$ maps $L^2(F)$ into $L^2(F)$ and thus $B_r^{3/2}(L^\infty(F) \cap L^2(F)) \subseteq L^2(F)$. On the other hand, $L^\infty(F) \cap L^2(F)$ is dense in $L^2(F)$. Since $B_r^{3/2}$ is continuous, it follows that $B_r^{3/2}(L^\infty(F) \cap L^2(F))$ is dense in $B_r^{3/2}(L^2(F))$. Since $B_r^{3/2}$ has a dense range (as it is a selfadjoint operator with zero kernel) it follows that also $B_r^{3/2}(L^\infty(F) \cap L^2(F))$ is dense in $L^2(F)$.

We want to prove that, for any s and any $f \in \mathcal{E} \cap L^2(F)$ there exists g (depending on s and f) so that T_g^r is bounded and so that

$$U_s f = T^s_{B_s^{-1/2} f} = j_{s,r}(T_g^r) = j_{s,r}(U_r(B_r^{1/2} g)).$$

Since $(U_s)^* j_{s,r} U_r = B_s^{-1/2} B_r^{1/2}$ (by Corollary 4.7), the above equality is equivalent to

$$f = B_r^{1/2} B_s^{-1/2} B_r^{1/2} g = B_s^{-1/2} B_r g$$

and hence this is equivalent to

$$g = B_s^{1/2} B_r^{-1} f.$$

We need to prove that, with this g, T_g^r is bounded. As f is an element in $B_r^{3/2}(L^\infty(F))$ and thus $f = B_r^{3/2}(\theta)$ for some θ in $L^\infty(F)$. Hence

$$g = B_s^{1/2} B_r^{-1} f = B_s^{1/2} B_r^{1/2} \theta, \theta \in L^\infty(F).$$

Thus, to complete the proof of the statement, it will be sufficient to show that the bounded operator $B_s^{1/2} B_r^{1/2}$ maps $L^\infty(F)$ into $L^\infty(F)$ and that the function $s \to ||U_s \theta||_{\infty,r}$ is locally bounded for all θ in $L^\infty(F)$.

It will be sufficient to show that the operator $B_s^{1/2}B_r^{1/2}$ in $B(L^2(F))$, is given, by analogy with B_r, by an integral, $PSL(2,\mathbb{R})-$ invariant, kernel function on $\mathbb{H}$ ([Ku],[Sel]).

More precisely,it suffices to find a kernel function $L = L_{s,r}(z,\overline{w})$ on $\mathbb{H}^2$ such that

$$B_s^{1/2}B_r^{1/2}f(z) = \int\limits_F L_{s,r}(z,\overline{w})f(w)\mathrm{d}\nu_0(w), z \in \mathbb{H}.$$

Moreover it is required that there exists a kernel function $l_{s,r} = l_{s,r}(z,\overline{w}), z, w \in \mathbb{H}$ on $\mathbb{H}^2$. The following properties should hold true for $l_{s,r}$:

a).$l_{s,r}$ is $PSL(2,\mathbb{R})-$, diagonally invariant, that is

$$l_{s,r}(gz,\overline{gw}) = l_{s,r}(z,\overline{w}), z, w \in \mathbb{H}, g \in PSL(2,\mathbb{R}).$$

b).$l_{s,r}(z,\overline{w}) = \sum\limits_{\gamma} l_{s,r}(\gamma z,\overline{w}), z, w \in \mathbb{H}$.

c). The expression

$$M(s) = \sup_{\zeta\in\mathbb{H}} \int\limits_{\mathbb{H}} |l_{s,r}(z,\overline{w})|\mathrm{d}\nu_0$$

is finite and moreover the function $M(s)$ is locally finite.

Assume we have find such an $l_{s,r}$ having the properties a), b), c). Then, for any $\theta \in B_r^{3/2}(L^\infty(F) \cap L^2(F))$ we will have that

$$B_s^{1/2}B_r^{1/2}f(z) = \int\limits_{\mathbb{H}} l_{s,r}(z,\overline{w})\theta(w)\mathrm{d}\nu_0(w), z \in \mathbb{H}.$$

Hence $s \to \sup||B_r^{1/2}B_s^{1/2}\theta||_\infty$ is locally bounded.

To prove that the operator $B_r^{1/2}B_s^{1/2}$ has the required property, we will use the technique of the Selberg transform (which in fact, as mentioned in [Ve], is a particular case of a more general transform).

Assume that B_r, B_s are given by the functions h_r, h_s as a function of the invariant laplacian. Then $(h_r h_s)^{1/2}$ is the function corresponding to the operator $B_r^{1/2}B_s^{1/2}$.

Let $\tilde{h}_r(t), \tilde{h}_s(t)$ be the functions $h_r(t^2+1/4)$, $h_s(t^2+1/4)$. Let $\phi_{r,s}(t) = (h_r h_s)^{1/2}(t^2+1/4)$.

By Selberg Theorem (see also [Za]), $(h_r h_s)^{1/2}$ will be represented by a kernel $k_{r,s}$ as we wanted if ϕ_{rs} has an holomorphic continuation in the strip $|\mathrm{Im} t| < 1/2$ and $\phi_{rs}(t)$ is of rapid decay in this strip.

On the other hand, Berezin formula for B_r as function of the laplacian shows that

$$\tilde{h}_r(t) = \prod_{n=1}^{\infty}[1+(t^2+1/4)[(1/r+n)(1/r+(n-1)]^{-1}]^{-1}.$$

This expression shows that ϕ_{rs} has the required property.

5. A cyclic 2-cocycle associated to a deformation quantization

In the paragraph 3 we have constructed a family of semifinite von Neumann algebras $\mathcal{A}_r \subseteq B(H_r)$ which are a deformation quantization, in the sense of Berezin, for $\mathbb{H}/\Gamma$. In this section we are constructing a cyclic 2-cocycle which is defined on a weakly dense subalgebra of $\mathcal{A}_r$, for each r. This 2-cocycle is associated in a canonical way with the deformation. It is likely that an abstract setting for this construction should be found in the general machinery developed in [R.N,B.T].

In the particular case of the deformation for $\mathbb{H}/\Gamma$, the cyclic 2-cocycle we obtain this way is very similar in form with the cyclic cocycles that are constructed in the paper by Connes and Moscovici ([CM]). We will prove that that the cyclic 2-cocycle of the equivariant deformation may be obtained from a $\Gamma-$invariant Alexander-Spanier two cocycle on $\mathbb{H}$ by a procedure very similar to the constructions in chapter 4 in the above mentioned paper.

The relation between the cyclic 2-cocycle and the deformation becomes more transparent if one uses the intermediate type of Berezin's symbols $A \to U_r A$ that we have introduced in the preceding paragraph. This happens due to the fact that this type of symbols have the advantage that the trace of a product of two symbols is constant in the deformation parameter. We first introduce a rather formal abstract setting in which this type of cocycles may be constructed. In the last part of this paragraph we will perform more precise computations for the Berezin's, $\Gamma-$ invariant quantization.

Definition 5.1. *Let $(\mathcal{B}_s)_{s\in(a,b)}$ be a family of semifinite von Neumann algebras. Denote by $(*_s)_{s\in(a,b)}$ the corresponding products operations on these algebras. We will call $(\mathcal{B}_s, *_s)_{s\in(a,b)}$ a "nice deformation" if the following properties hold true:*

i. *For $r \leq s$, $\mathcal{B}_r$ is contained as a vector subspace in $\mathcal{B}_s$. Let $j_{s,r}$ be the corresponding inclusion map. We assume that $\mathcal{B}_r$ is weakly dense in $\mathcal{B}_s$ and that $j_{s,r}$ is weakly and normic continuous. Moreover, for all $s \geq r$, $j_{s,r}$ preserves the involution and the unit.*

ii. *There exists a linear functional τ on a subalgebra of the union $\bigcup_{s\in(a,b)} \mathcal{B}_s$, so that for each s, τ is a semifinite, normal faithful trace on $\mathcal{B}_s$. In addition the maps $j_{s,r}$ are trace preserving.*

iii. *Let $L^2(\mathcal{B}_s, \tau)$ be the Hilbert of the Gelfand-Naimark-Segal construction for τ on $\mathcal{B}_s$. We assume that $j_{s,r}$ extends to a contractive linear map from $L^2(\mathcal{B}_r, \tau)$ into*

$L^2(\mathcal{B}_s, \tau)$. Moreover, $j_{s,r}$ maps the positive part of $\mathcal{B}_r$ into the positive part of $\mathcal{B}_s$.(In fact we are not going to make use of this last property for $j_{s,r}$.)

iv. There exist a selfadjoint vector subspace $\mathcal{D} \subseteq \bigcap_{s\in(a,b)} \mathcal{B}_s$ which is closed under all the multiplication operations $(*_s)_{s\in(a,b)}$. Also, the space $\mathcal{D}$ be dense in the Hilbert spaces $L^2(\mathcal{B}_s, \tau)$, for all s.

Remark 5.2. *All of the above properties hold true for the family of algebras $(\mathcal{A}_s)_{s\in(a,b)}$ in the (equivariant) Berezin deformation quantization of $\mathbb{H}/\Gamma$ that was constructed in Theorem 3.2. In this case, one could let $\mathcal{D}$ be any of the algebras $\mathcal{A}_r$ for $r < a-2$, or one might take $\mathcal{D} = \hat{\mathcal{A}}_a$.*

For a "nice" deformation" as above, by requiring some additional properties, we construct a cyclic 2-cocycle, which measures, to a certain extent, the obstruction on the algebras in the deformation to be mapped isomorphically one onto the other by a family of isomorphisms depending smoothly on the deformation parameter.

Definition 5.3. *Let $(\mathcal{B}_s)_{s\in(a,b)}$ be a "nice deformation" as in the Definition 5.1. We will call $(\mathcal{B}_s)_{s\in(a,b)}$ a "nice differentiable deformation" if in addition there exists weakly dense, selfadjoint subalgebras $\hat{\mathcal{B}}_s \subseteq \mathcal{B}_s$, for all s, with the following properties:*

i). The algebras $(\hat{\mathcal{B}}_s)_{s\in(a,b)}$ are all unital with the same unit as $\mathcal{B}_s$. Moreover $L^2(\mathcal{B}_s, \tau) \cap \hat{\mathcal{B}}_s$ is weakly dense in $\mathcal{B}_s$. Also, we let $|| \; ||_{\lambda,s}$ be a Banach algebra norm on the algebras $\hat{\mathcal{B}}_s$ for all s. The unit balls for the norms $|| \; ||_{\lambda,s}$ are are weakly compact. Moreover $||B^*||_{\lambda,s} = ||B||_{\lambda,s}$ for all s.

ii). For $s \geq r$, the inclusions $j_{s,r}$, map $\hat{\mathcal{B}}_r$ continuously into $\hat{\mathcal{B}}_s$ with respect to the norms $|| \; ||_{\lambda,r}$ and $|| \; ||_{\lambda,s}$, correspondingly. Moreover $\hat{\mathcal{B}}_r$ is closed under the *products* $(*)_s$ *for all* $s \geq r$. *There exist positive constants* $c_{r,s}$ *so that the function* $s \to c_{r,s}$ *is locally bounded for all* r *and so that* $||A *_s B||_{\lambda,r} \leq c_{r,s}||A||_{\lambda,r}||B||_{\lambda,r}$, *for all* A, B *in* $\hat{\mathcal{B}}_r$.

iii). The space $\mathcal{D}$ *is contained in* $\bigcap_s(\hat{\mathcal{B}}_s \cap L^2(\mathcal{B}_s, \tau))$.

iv). For all $s \in (a, b)$ *The following functionals are well defined on* $\mathcal{D}$.

a). $\mu_t(c, (a, b)) = \frac{\mathrm{d}}{\mathrm{d}s}\tau(c *_t (a *_s b))|_{s=t}$.

b). $\phi_t(a, b) = \frac{\mathrm{d}}{\mathrm{d}s}\tau(a *_s b)|_{s=t}$.

c). $\theta_t(a, b, c) = \frac{\mathrm{d}}{\mathrm{d}s}\tau(a *_s b *_s c)|_{s=t}$.

Again we note that all of the above conditions will hold true for the deformation quantization from Theorem 3.2, with $\mathcal{B}_s = \mathcal{A}_s$ and $\hat{\mathcal{B}}_s = \hat{\mathcal{A}}_s = \mathcal{A}_s \cap \hat{B}(H_s)$, for all $s \in (a, b)$. We also let $\mathcal{D}$ be $\mathcal{A}_r \cap L^2(\mathcal{A}_r)$ for some $r < a-2$ or we let $\mathcal{D} = \hat{\mathcal{A}}_a \cap L^2(\mathcal{A}_a)$.

The above cocycles have the formal properties listed in the following proposition.

Proposition 5.4. *Let* $\mathcal{D} \subseteq \hat{\mathcal{B}}_s \subseteq \mathcal{B}_s$, $s \in (a, b)$ *be a "nice differentiable deformation" as in Definition 5.3. Let* t *be fixed in* (a, b). *Define on* $\mathcal{D}$ *the following additional cocycles:*

$$\alpha_t(a, b, c) = \phi_t(a *_t b, c) + \phi_t(b *_t c, a) + \phi_t(c *_t a, b),$$

$$\psi_t(a, b, c) = \theta_t(a, b, c) - (1/2)\alpha_t(a, b, c),$$

for $a, b, c \in \mathcal{D}$. *Then the following properties hold true:*

i). The linear functionals $\psi_t, \theta_t, \alpha_t$ *are cyclic, that is* $\psi_t(a, b, c) = \psi_t(b, c, a)$ *and similarly for* θ_t, α_t. *Moreover* ϕ_t *is antisymmetric, that is* $\phi_t(a, b) = -\phi_t(b, a)$, *for all* $a, b, c \in \mathcal{D}$.

ii). For all $a, b, c \in \mathcal{D}$ *one has* $\theta_t(a, b, c) = \mu_t(c, (a, b)) + \phi_t(a *_t b, c)$.

iii. ψ_t *belongs to* $Z^2_\lambda(\mathcal{D},\mathbb{C})$*, that is* ψ_t *is a cyclic two cocycle in the sense of Connes' cyclic cohomology (Co]):*

$$\psi_t(a,b,c)=\psi_t(b,c,a),$$

$$\psi_t(a*_s b,c,d)-\psi_t(a,b*_s c,d)+\psi_t(a,b,c*_s d)-\psi_t(d*_s a,b,c)=0, a,b,c,d\in\mathcal{D}.$$

iv. μ_t *is a Hochschild 2-cocycle, that is* μ *verifies the second property listed for* ψ_t *above.*

v. The following equality holds true $\overline{\psi_t(a,b,c)}=\psi_t(b^*,a^*,c^*)$ *for all* $a,b,c\in\mathcal{D}$*. Moreover if* $1\in\mathcal{D}$*, (which corresponds to the case when all the algebras* $\mathcal{B}_s$ *are finite) then* $\psi_t(1,b,c)=0$ *for all* b,c *in* $\mathcal{D}$*.*

Proof. The statement in i). follows from the fact that τ is a trace. The assertion ii). is a consequence of the product rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(a*_s b*_s c)|_{s=t}=\frac{\mathrm{d}}{\mathrm{d}s}\tau(c*_s(a*_s b))|_{s=t}+\frac{\mathrm{d}}{\mathrm{d}s}\tau(c*_s(a*_s b))|_{s=t}=$$

$$\phi_t(c,a*_s b)+\mu_t(c,(a,b)), a,b,c\in\mathcal{D}.$$

In particular this implies that the following formula, relating ψ_t,θ_t,ϕ_t, holds true:

(5.1) $\psi_t(a,b,c)=\theta(a,b,c)-1/2\phi_t(a,b,c)=$

$$\mu_t(c,(a,b))+1/2[-\phi_t(c*_s a,b)+\phi_t(c,a*_s b)-\phi_t(b*_s c,a)].$$

Note that formula (5.1) shows that i). and iv). imply iii).

The property iv). follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(d*_s((a*_s b)*_s c))|_{s=t}=\frac{\mathrm{d}}{\mathrm{d}s}\tau(d*_s(a*_s(b*_s c))|_{s=t},$$

by using the product rule for differentiation.

Finally property v) is a consequence of the following two equalities:

$$\theta_t(a,b,c) = \overline{\theta_t(b^*,a^*,c^*)}$$

and $\phi_t(a,b) = \overline{\phi_t(b^*,a^*)}$, which hold for all $a,b,c \in \mathcal{D}$. These properties follow both from corresponding properties of the trace. This ends the proof.

We will check that the model described in Theorem 3.2 has the properties in Definition 5.3. We will obtain bounds for the cocycles by estimating the absolute for the integrals representing the cocycles.

Proposition 5.5. *Let $a > 1$ and let $\mathcal{D}$ be the vector space $L^2(\mathcal{A}_a) \cap \hat{\mathcal{A}}_a$. Then $\mathcal{D}$ is weakly dense in $\mathcal{A}_s$, for all s in (a,b) and the conditions in Definition 5.3 hold true for $\mathcal{D}$. In particular $\mathcal{D}$ is closed under all the multiplication the operations $(*_s)_{s\in(a,b)}$. For any t in (a,b) and $r < a$, the cocycles μ_t, θ_t, ϕ_t are defined on $\mathcal{D}$. Moreover there exists a constant c depending on r so that for all $A,B,C \in \hat{\mathcal{A}}_r \cap L^2(\mathcal{A}_t)$ we have:*

$$|\phi_t(A,B)| \leq c||A||_{2,t}||B||_{2,r},$$

$$|\phi_t(A *_s B, C)| \leq c||C||_{\lambda,r}||A||_{2,t}||B||_{2,t},$$

$$|\mu_t(C,(A,B))| \leq c(||C||_{\lambda,r}||A||_{2,t}||B||_{2,t} + ||A||_{\lambda,r}||B||_{2,t}||C||_{2,t} + ||B||_{\lambda,r}||C||_{2,t}||A||_{2,t}). \blacksquare$$

Proof. We only have the check the inequalities. By Lebesgue theorem on differentiation under the integral sign, the derivatives involved in ϕ_t, μ_t will exist as soon as the absolute value of the derivatives of the integrands have finite integral.

For A,B,C in $\hat{\mathcal{A}}_r \cap L^2(\mathcal{A}_t)$, we let $A_t(z,\overline{\eta}) = A(z,\overline{\eta})[(z-\overline{\eta})/2i]^{-t}, z,\eta \in \mathbb{H}$ and we use a similar notation for B and C. We deduce the following expressions for

$\mu_t(C,(A,B))$ and $\phi_t(A*_s B, C)$ (were by const we denote (area $F)^{-1}$ or 1 according to the case when Γ has finite or infinite covolume in $PSL(2,\mathbb{R})$):

(5.2) $\mu_t(C,(A,B)) = \frac{\mathrm{d}}{\mathrm{d}s}\tau(C *_s (A *_s B))|_{s=t} = \frac{c_t'}{c_t}\tau(A *_s B *_s C)+$

$$c_t^2(\mathrm{const})\int\limits_{F_z}\int\limits_{\mathbb{H}_\eta}\int\limits_{\mathbb{H}_\zeta} A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{z})m(z,\eta,\zeta)\mathrm{d}\nu_t(z,\eta,\zeta),$$

with

(5.3) $m(z,\eta,\zeta) = \ln[(\eta-\overline{\eta})/2i] + \ln[(z-\overline{\zeta})/2i] - \ln[(z-\overline{\eta})/2i] - \ln[(\eta-\overline{\zeta})/2i], z,\eta,\zeta \in \mathbb{H}.$

Similarly,

(5.4) $\phi_t(A *_s B, C) = \frac{c_t'}{c_t}\tau(A *_s B *_s C)+$

$$c_t^2(\mathrm{const})\int\limits_{F_z}\int\limits_{\mathbb{H}_\eta}\int\limits_{\mathbb{H}_\zeta} A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{z})\ln|d(z,\overline{\zeta})|^2\mathrm{d}\nu_t(z,\eta,\zeta).$$

We will carry over only the estimate for μ_t, since the other one is similar. We recall that we used the notation $d(z,\overline{\zeta}) = (\mathrm{Im}\ z)^{1/2}(\mathrm{Im}\ \overline{\zeta})^{1/2}[(z-\overline{\zeta})/2i]^{-1}, z,\zeta \in \mathbb{H}$. Also recall that the absolute value $|d(z,\overline{\zeta})|^2$ depends only on the hyperbolic distance between z,ζ in $\mathbb{H}$. Let $l(z,\zeta)$ be the function on $\mathbb{H}^2$ defined by

(5.5)$l(z,\zeta) = \ln(\mathrm{Im}\ z)^{1/2} + \ln(\mathrm{Im}\ \overline{\zeta})^{1/2} - \ln[(z-\overline{\zeta})/2i]^{-1}, z,\zeta \in \mathbb{H}.$

We obviously have:

(5.6) $m(z,\eta,\zeta) = l(z,\eta) + l(\eta,\zeta) - l(z,\zeta), z,\eta,\zeta \in \mathbb{H}.$

Consequently to show that the absolute value of the integral in the formula (5.2) is convergent, it is sufficient to estimate the following integral (and two other similar ones).

$$\int_{F_z}\int_{\mathbb{D}_\eta}\int_{\mathbb{D}_\zeta} |A_t(z,\overline{\eta})||B_t(\eta,\overline{\zeta})||C_t(\zeta,\overline{z})||l(\eta,\zeta)|\mathrm{d}\nu_t(z,\eta,\zeta).$$

For fixed z in F, denote $f_z(\eta) = |A_t(z,\overline{\eta})|$, $g_z(\zeta) = |B_t(\eta,\overline{z})|$ for $\eta \in \mathbb{H}$. By Proposition 1.5.a, the functions f_z, g_z belong to $L^2(\mathbb{H},\nu_r)$ for all z in F. Moreover

$$\int_F ||f_z||^2_{L^2(\mathbb{H},\nu_r)} = \int_F\int_{\mathbb{H}} |A_t(z,\overline{\eta})|^2\mathrm{d}\nu_t(z,\eta) = ||A||^2_{L^2(\mathcal{A}_t)}.$$

On the other hand if B belongs to $\mathcal{A}_r$, then

$$\sup_{z\in\mathbb{H}} c_r \int_{\mathbb{H}} |B(z,\overline{\zeta})||d(z,\overline{\zeta})|^r\mathrm{d}\nu_0(\zeta) \leq ||B||_{\lambda,r}.$$

Since, if $r < t$, there exists a constant $c(r)$ such that

$$x^t \ln x \leq c(r)x^r, 0 \leq x \leq 1$$

and since $|l(z,\zeta)| \leq |\ln|d(z,\overline{\zeta})||$ for all $z,\zeta \in \mathbb{H}$, it follows that there exists a constant $c(r)$ so that:

$$\sup_{z\in\mathbb{H}} \int_{\mathbb{H}} |B(\eta,\overline{\zeta})||l(\eta,\overline{\zeta})|^t\mathrm{d}\nu_0(\zeta) \leq c(r)||B||_{\lambda,r}.$$

By Proposition 2.7, it follows that the kernel

$$K_B(\eta,\overline{\zeta}) = |\frac{B(\eta,\overline{\zeta})l(\eta,\overline{\zeta})}{[(\eta-\overline{\zeta})/2i]^t}|, \eta,\zeta \in \mathbb{H},$$

defines a bounded operator on $L^2(\mathbb{H},\nu_r)$ of operatorial uniform norm less than $c(r)||B||_{\lambda,r}$.

The integral in formula (5.6) is clearly equal to

$$\int\limits_{F_z}\int\limits_{\mathbb{D}_\eta}\int\limits_{\mathbb{D}_\zeta} K_B(\eta,\overline{\zeta})f_z(\eta),g_z(\zeta)\mathrm{d}\nu_t(\eta,\zeta)\mathrm{d}\nu_t(z).$$

By the above arguments this is less than

$$\int\limits_{F_z} ||K_B||_{B(L^2(\mathbb{H},\nu_r))}||f_z||_{H_r}||g_z||_{H_r}\mathrm{d}\nu_t(z) \leq$$

$$c(r)||B||_{\lambda,r}[\int\limits_{F} ||f_z||^2_{H_r}\mathrm{d}\nu_t(z)]^{1/2}[\int\limits_{F} ||g_z||^2_{H_r}\mathrm{d}\nu_t(z)]^{1/2} =$$

$$c(r)||B||_{\lambda,r}||A||_{2,t}||B||_{2,t}.$$

This is an upper bound for one of the three terms that appear in the expression of μ_t. The other two terms, listed in the statement of the proposition, are, by similar arguments, upper bounds for the other two integrals. This completes the proof.

We will now determine the expression for the cyclic 2-cocycle ψ_t, $t\in(a,b)$ that is associated to the deformation $(\mathcal{A}_s)_{s\in(a,b)}$. The expression for ψ_t will be very similar to the one that appears in the construction in the paper by Connes and Moscovici. The formula for $\psi_t(A,B,C)$ is obtained by superposition in the integral formula for the trace $\tau(A*_s B*_s C)$ of a $\Gamma-$ invariant Alexander-Spanier cocycle θ.

Recall that by using the notation $A_t(z,\overline{\eta})=A(z,\overline{\eta})[(z-\overline{\eta})/2i]^{-t}, z,\eta\in\mathbb{H}$ and similarly for B,C, the formula for $\tau(A*_s B*_s C)$ is

$$\tau(A*_s B*_s C)=c_t^2(\mathrm{const})\int\limits_{F_z}\int\limits_{\mathbb{H}_\eta}\int\limits_{\mathbb{H}_\zeta} A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{z})\mathrm{d}\nu_t(z,\eta,\zeta).$$

The cocycle θ is a bounded measurable function on $\mathbb{H}^3$ and this may be used to get better estimates for ψ_t. We find such an estimate in the next statement.

Proposition 5.7. *Let $1 < r < t$ and let A, B, C be arbitrary elements in $\hat{\mathcal{A}}_r \cap L^2(\mathcal{A}_r)$. Let $\phi(z,\eta) = i\,arg[(z-\overline{\eta})/2i] = \ln[(z-\overline{\eta})/2i] - \ln[(\eta-\overline{z})/2i]$ for $z,\eta \in \mathbb{H}$. Let*

$$\theta(z,\eta,\zeta) = 1/2[\phi(z,\zeta)+\phi(\zeta,z)+\phi(\eta,\zeta)], z,\eta,\zeta \in \mathbb{H}.$$

Clearly θ is a $\Gamma-$ invariant, bounded function on $\mathbb{H}$. We use the notation $A_t(z,\overline{\eta}) = A(z,\overline{\eta})[(z-\overline{\eta})/2i]^{-t}, z,\eta \in \mathbb{H}$ and similarly for B, C. Then

$$\psi_t(A,B,C) = 1/2(\frac{c_t'}{c_t})\tau(A *_s B *_s C)+$$

$$c_t^2(const)\int\limits_{F_z}\int\limits_{\mathbb{H}_\eta}\int\limits_{\mathbb{H}_\zeta}\theta(z,\eta,\zeta)A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{z})\,d\nu_t(z,\eta,\zeta).$$

Moreover the following estimates holds for ψ_t.

$$|\psi_t(A,B,C) \leq const[||A||_{\lambda,t}||B||_{2,t}||C||_{2,t}], \text{ for all } A,B,C \in \hat{\mathcal{A}}_r \cap L^2(\mathcal{A}_r).$$

Proof. To deduce the expression for $\psi_t(A,B,C)$ we use the formulae 5.2 and 5.4. Because

$$\psi_t(A,B,C) = \mu_t(C,(A,B)) + 1/2[-\phi_t(C *_s A, B) + \phi_t(C, A *_s B) - \phi_t(B *_s C, A)]$$

we obtain that

$$\psi_t(A,B,C) = 1/2(\frac{c_t'}{c_t})\tau(A *_s B *_s C)+$$

$$c_t^2(\text{const})\int\limits_{F_z}\int\limits_{\mathbb{H}_\eta}\int\limits_{\mathbb{H}_\zeta}\gamma(z,\eta,\zeta)A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{z})\mathrm{d}\nu_t(z,\eta,\zeta),$$

where γ has the following expression:

$$\gamma(z,\eta,\zeta) = m(z,\eta,\zeta) + 1/2[-\ln|d(\eta,\overline{\zeta})|^2 + \ln|d(z,\overline{\zeta})|^2 - \ln|d(\eta,\overline{z})|^2].$$

Since

$$\ln|d(z,\overline{\zeta})|^2 = \ln(\text{Im } z) + \ln(\text{Im } \zeta) - \ln[(z-\overline{\zeta})/2i] - \ln[(\zeta-\overline{z})/2i],$$

and

$$m(z,\eta,\zeta) = \ln(\text{Im } \eta) + \ln[(z-\overline{\zeta})/2i] - \ln[(z-\overline{\zeta})/2i] - \ln[(\eta-\overline{\zeta})/2i], \text{for all } \ z,\eta,\zeta \in \mathbb{H},$$

one obtains that $\gamma = \theta$.

The estimate for ψ_t is now obtained by the same procedure as the one used for η_t in the preceding paragraph, with the only difference that computations are now easier by the fact that the function is bounded.

Indeed we have to estimate the following integral:

$$\int\limits_{F_z}\int\limits_{\mathbb{D}_\eta}\int\limits_{\mathbb{D}_\zeta} |A_t(z,\overline{\eta})||B_t(\eta,\overline{\zeta})||C_t(\zeta,\overline{z})| \mathrm{d}\nu_t(z,\eta,\zeta)$$

for $A,B,C \in \hat{\mathcal{A}}_r \cap L^2(\mathcal{A}_r)$. One denotes for a fixed z in F, $f_z(\eta) = |A_t(z,\overline{\eta})|$, $g_z(\zeta) = |B_t(\eta,\overline{\zeta})|$ for $\eta,\zeta \in \mathbb{H}$. Since B is in $\hat{\mathcal{A}}_t$ it follows that the kernel on $\mathbb{H}^2$ defined by $\zeta,\eta \to |B_t(\eta,\overline{\zeta})|$ defines a bounded operator $L^2(\mathbb{H},\nu_r)$ of norm less than $||B||_{\lambda,t}$. But then the above integral is (modulo a universal constant) less than

$$||B||_{\lambda,t} \int\limits_{F_z} ||f_z||_t ||g_z||_t \mathrm{d}\nu_t(z) = ||B||_{\lambda,t}||A||_{2,t}||B||_{2,t}.$$

This completes the proof.

In the last part of this paragraph we will the use the intermediate symbols U_r on the algebras $\mathcal{A}_r$ for yet another approach to the construction of the cocycle ψ_t. We will prove that there exists a dense domain $\mathcal{E} \subseteq L^2(F)$, which is closed under all the multiplication operations in all the algebras $\mathcal{A}_s$. Also, we will prove that on $\mathcal{E}$ the following formula holds true:

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(U_s f *_s U_s g *_s U_s h)|_{s=t} = \psi_t(U_t f, U_t G, U_t h).$$

This formula explains more clearly the reason for which ψ_t is a cyclic cocycle.

This is because if we transfer the product operation on $\mathcal{D}$ by

$$f \circ_t g = U_t^*(U_t F *_s U_t g), f, g \in \mathcal{E}$$

and define a trace by

$$\tau(f) = \mathrm{const} \int\limits_F f \mathrm{d}\nu_0, f \in \mathcal{E}$$

and $\tilde{\psi}(f, g, h) = \psi_t(U_t f, U_t g, U_t h)$, then the above formula will show that:

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(f \circ_s g \circ_s h) = \tilde{\psi}(f, g, h), f, g \in \mathcal{D}.$$

The reason for which $\tilde{\psi}_t$ is a cyclic two cocycle is now easy to deduce because $\tau(f \circ_t g)$ is a constant (depending on f, g only). We will use for the next statement the formalism introduced in Definition 5.1 and 5.3.

Proposition 5.8. *With the formalism in Definitions 5.1 and 5.3 let $\mathcal{D} \subseteq \hat{B}_s\mathcal{B}_s$, $s \in (a, b)$ be a "nice differentiable deformation". Let t be an arbitrary in (a, b). We assume in addition that there exists a Hilbert space H and unitaries $U_s \to L^2(\mathcal{B}_s, \tau)$ with the following additional property:*

Whenever f, g be vectors in H so that if $U_s f, U_s g$ belong to $\mathcal{D}$ for s in a small neighborhood of t, then

$$\frac{d}{ds}[U_s f, U_s g]_{L^2(\mathcal{B}_t,\tau)} = (-1/2)\phi(U_t f, (U_t g)^*).$$

Then if f, g, h in H have the property that $U_s, U_s g, U_s h$ belong to $\mathcal{D}$ for s in a small neighborhood of t, then

$$\frac{d}{ds}\tau(U_s f *_s U_s g *_s U_S h)|_{s=t}$$

exists and is equal to $\psi_t(U_t f, U_t g, U_t h)$.

6. Bounded cohomology and the cyclic 2-cocycle of the Berezin's deformation quantization

In this section we prove some facts about the cyclic 2-cocycle that was constructed in the previous section for a deformation quantization of algebras. Recall that ψ_t was a cyclic 2-cocycle defined on a dense $*$-subalgebra $\hat{\mathcal{A}}_t$ of the deformation quantization $\mathcal{A}_t$ for $\mathbb{H}/\Gamma$ constructed in paragraph 3. We will show that the cohomology class of ψ_t in the second cyclic cohomology group $H^2(\hat{\mathcal{A}}_t, \mathbb{C})$ ([Co]) is closely related to a canonical element in the bounded cohomology of the group Γ.

In the last part of this paragraph we will show that a deformation in which the 2-cyclic cocycle is bounded with respect to the uniform norms from the von Neumann algebras will have the property that the algebras in the deformation are isomorphic. Indeed in this case, by the next paragraph, the cyclic 2-cocycle vanishes in the cyclic cohomology group of the von Neumann algebra. We then prove that there exists a linear, nonautonomuous differential equation, with bounded linear operators, whose evolution operator implements the isomorphism.

We first recall the integral formulae and the estimates that we found in the last paragraph for the cyclic 2-cocycle ψ_t associated to the deformation quantization of $\mathbb{H}/\Gamma$ that we introduced in paragraph 3. Let $\phi(z,\overline{\zeta}) = i\arg((z-\overline{\zeta})/2i) = \ln((z-\overline{\zeta})/2i) - \overline{\ln((z-\overline{\zeta})/2i)}$, for z,ζ in $\mathbb{H}$ and let

$$\theta(z,\eta,\zeta) = \phi(z,\overline{\zeta}) + \phi(\zeta,\overline{\eta}) + \phi(\eta,\overline{z}), \quad z,\eta,\zeta \text{ in } \mathbb{H}.$$

Then θ is a Γ-invariant continuous function on $\mathbb{H}^3$ which is an Alexander-Spanier cocycle. Let $t > 1$ and let A, B, C be elements in $\hat{\mathcal{A}}_t$. Let $\hat{A} = \hat{A}(z,\eta)$ be the Berezin's, contravariant symbol of A. We use the notation $A_t(z,\overline{\eta}) = \hat{A}(z,\eta)((z-\overline{\eta})/2i)^{-t}$, $z,\eta \in \mathbb{H}$, and similarly for B and C. Then the formula for ψ_t is

$$(6.1) \quad \psi_t(A,B,C) = 1/2(\frac{c_r'}{c_r})\tau(A *_t B *_t C) +$$

$$\int\limits_{F_z} \int\limits_{\times} \int\limits_{\mathbb{H}^2_{\eta,\zeta}} \theta(z,\eta,\zeta) A_t(z,\overline{\eta}) B_t(\eta,\overline{\zeta}) C_t(\zeta,\overline{z}) d\nu_t(z,\eta,\zeta).$$

The formula for $\psi_t(A,B,C)$ should be compared with the similar formula for the trace $\tau(A *_t B *_t C)$ which is

$$\tau(A *_t B *_t C) = \int\limits_{F_z} \int\limits_{\times} \int\limits_{\mathbb{H}^2_{\eta,\zeta}} A_t(z,\overline{\eta}) B_t(\eta,\overline{\zeta}) C_t(\zeta,\overline{z}) d\nu_t(z,\eta,\zeta).$$

Note that the integral formula for ψ_t like the formula for $\tau(A *_t B *_t C)$ is an iterated integral. This integral converges absolutely if A belongs to $\hat{\mathcal{A}}_t$. In fact for the absolute convergence of the integral it is sufficient that

$$\sup_z \int\limits_{\mathbb{H}} |A(z,\overline{\zeta})| \, |d(z,\overline{\zeta})|^r d\nu_0(\zeta)) \leq ||A||_{\lambda,t} < \infty.$$

Also recall that we found the following estimate:

$$(6.2)\quad |\psi_t(A,B,C)| \leq \mathrm{const}||A||_{\lambda,t}||B||_{L^2(\mathcal{A}_t)}||C||_{L^2(\mathcal{A}_t)}$$

The relation between the 2-cocycle ψ_t with the deformation quantization is more transparent from the viewpoint of the the intermediate symbols $V_r A = B_r^{-1/2}(A(z,\overline{z}))$, A in $\mathcal{A}_r$ introduced at the end of paragraph 4. Recall that V_r maps $L^2(\mathcal{A}_r)$ isometrically onto $L^2(F)$. If U_r is the inverse for V_r then we found that

$$\psi_t(U_t f, U_t g, U_t h) = \frac{d}{ds}\tau(U_s f *_s U_s g *_s U_s h)|_{s=t}$$

if f, g, h run through a dense subset of $L^2(F)$.

We will single out some obstructions for the cocycle ψ_t to be trivial in the cyclic cohomology group $H^2_\lambda(\hat{\mathcal{A}}_t, \mathbb{C})$. The condition that ψ_t vanishes in this cohomology group is equivalent to the existence of a bilinear form on $\mathcal{A}_t$ so that

(6.3) $\chi_(A,B) = -\chi_t(B,A)$,

(6.4) $\psi_t(A,B,C) = \chi_t(B *_s A, C) - \chi_t(B, A *_s C) + \chi_t(C *_s B, A)$,

for all A, B, C in $\hat{\mathcal{A}}_t$. Because of the antisymmetry for χ_t, the relation (6.4) is equivalent to:

(6.5) $\psi_t(A,B,C) = \chi_t(B *_s A, C) + \chi_t(A *_s C, B) + \chi_t(C *_s B, A)$.

A natural candidate for a bilinear form χ_t is given by the formula

$$\chi_t(A,B) = \int\limits_{F_z}\int\limits_{\mathbb{H}} A_t(z,\overline{\eta})B_t(\eta,\overline{z})d(z,\overline{\eta},\eta,\overline{z})\mathrm{d}\nu_t(z,\eta),$$

for a suitable $\Gamma-$equivariant function d on $\mathbb{H}^2$.

In fact we will rather use this formula to construct a solution for a perturbed problem with respect to the equation in (6.5). This is contained in the following statement.

Proposition 6.1. *Let $d = d(z, \overline{\eta}, \eta, \overline{z})$ be a $\Gamma-$equivariant function on $\mathbb{H}^2$, having purely imaginary values and with the following properties :*

$$d(z, \overline{\eta}, \eta, \overline{z}) + d(\eta, \overline{\zeta}, \zeta, \overline{\eta}) + d(\zeta, \overline{z}, z, \overline{\zeta}) = \theta(z, \eta, \zeta),$$

$$d(z, \overline{\eta}, \eta, \overline{z}) = -d(\eta, \overline{z}, z, \overline{\eta}),$$

for all $z, \eta, \zeta \in \mathbb{H}$. Let A, B be in $L^2(\mathcal{A}_r)$, with Berezin's contravariant symbols $\hat{A}, \hat{B}$. We use the notation $A_t(z, \overline{\eta})$ for $\hat{A}(z, \overline{\eta})[(z - \overline{\eta})/2i]^{-r}$ and similarly for B. Let $\chi_t(A, B)$ be defined by

$$\chi_t(A, B) = \int\limits_{F_z} \int\limits_{\mathbb{H}} A_t(z, \overline{\eta}) B_t(\eta, \overline{z}) d(z, \overline{\eta}, \eta, \overline{z}) d\nu_t(z, \eta).$$

Then χ_t is an antisymmetric bilinear form ($\chi_t(A, B)$=-$\chi_t(B, A)$). The form domain for χ_t is the linear space of all A, B for which the integrals in the definition of χ_t are absolutely convergent. Moreover the following equality holds true for all operators A, B, C in $L^2(\mathcal{A}_t)$ whose symbols are so that the integrals involved in the formula are absolute convergent:

$$(6.6)\ \psi_t(A, B, C) - 1/2(\tfrac{c_r'}{c_r})\tau(A *_t B *_t C) = \chi_t(B *_s A, C) + \chi_t(A *_s C, B) + \chi_t(C *_s B, A).$$

Proof. Indeed, if the integrals are absolutely convergent, then we have that:

$$\chi_t(A *_s B, C) = \int\limits_{F_z} \int\limits_{\mathbb{H}_\eta} \int\limits_{\mathbb{H}_\zeta} d(z, \overline{\zeta}, \zeta, \overline{z}) A_t(z, \overline{\eta}) B_t(\eta, \overline{\zeta}) C_t(\zeta, \overline{z}) \mathrm{d}\nu_t(z, \eta, \zeta),$$

and similarly for the other two terms.

Because the absolute value of the integrands has finite integral and the integrands are $\Gamma-$ invariant functions on $\mathbb{H}^3$, by Fubini's theorem, we may choose any domain of integration, as long it is a fundamental domain for Γ in $\mathbb{H}^3$. Then (6.6) reduces to the first property for the function d (by using also formula (6.1)). This completes the proof.

In general it is difficult to check that the domain were the above identity holds is sufficiently rich. In fact, in the case of groups Γ of finite covolume, the identity vector $1 \in \mathcal{A}_t \subseteq L^2(\mathcal{A}_t)$ makes the integrals involved in the formulae divergent.

We will find a condition on the group Γ for which a function d with the properties in Proposition 6.1 exists. To do this we need to recall the construction of canonical a group 2-cocycle in the second cohomology group $H^2(PSL(2,\mathbb{R}),\mathbb{Z})$.

Definition 6.2. *Let $N(\gamma_1,\gamma_2)$, $\gamma_1,\gamma_2 \in \Gamma$ be the group cocycle in the second cohomology group $H^2(PSL(2,\mathbb{R}),\mathbb{Z})$, defined by the formula:*

$$(2\pi)N(g_1,g_2) = arg\ j(g_1g_2,z) - arg\ j(g_1,g_2z) - arg\ j(g_2,z),$$

for all $g_1,g_2 \in PSL(2,\mathbb{R})$ and for all $z \in \mathbb{H}$.

Then N is a non-trivial element in $H^2(PSL(2,\mathbb{Z})$. The only possible values for N are -1 , 0 or 1. (see e.g the book of Maass ([Ma], pp. 113). Denote by N_Γ the restriction of N to $\Gamma\times\Gamma$. Then N_Γ vanishes in $H^2(\Gamma,\mathbb{Z})$ if Γ is not cocompact (see e.g [Pat]).

The reasons for which the 2-cocycle N_Γ is a coboundary in $H^2(\Gamma,\mathbb{Z})$ when Γ has finite covolume are more transparent in the case when $\Gamma = PSL(2,\mathbb{Z})$. In this case (because the commutator subgroup of $PSL(2,\mathbb{Z})$ is cyclic of finite order) there exists at most one $\mathbb{Z}-$ valued cocycle $c = c(\gamma), \gamma \in \Gamma$ so that

$$N_\Gamma(\gamma_1, \gamma_2) = c(\gamma_1\gamma_2) - c(\gamma_1) - c(\gamma_2), \gamma_1, \gamma_2 \in \Gamma.$$

Also, when $\Gamma = PSL(2,\mathbb{Z})$ it is easy to determine the cycle c. A possible formula for c is

$$c(\gamma) = \ln(\Delta(\gamma z)) - \ln(\Delta(z)), \gamma \in \Gamma, z \in \mathbb{H}.$$

The explicit formula for c in terms of the generators for Γ has been determined already by Rademacher in [Ra]. (Recall that Δ is the unique modular form for Γ of order 12 and that $\ln \Delta$ is defined in $\mathbb{H}$.)

In the next proposition we find a sufficient criteria on a discrete, Fuchsian subgroup Γ of $PSL(2,\mathbb{R})$ so that there exists a bounded function d for Γ with the properties in Proposition 6.1. The fact that d is bounded implies that the bilinear form χ_t constructed in that proposition is bounded. As we will see later the criteria on the group Γ can not hold true unless Γ has infinite covolume.

Proposition 6.3. *Let Γ be a Fuchsian group such that N_Γ vanishes not only in $H^2(\Gamma,\mathbb{Z})$ but also in the bounded cohomology group $H^2_{\text{bound}}(\Gamma,\mathbb{Z})$ (that is there exist a bounded cochain $c : \Gamma \to \mathbb{Z}$ so that $N_\Gamma(\gamma_1,\gamma_2) = c(\gamma_1\gamma_2) - c(\gamma_1) - c(\gamma_2), \gamma_1, \gamma_2 \in \Gamma$.)*

Then there exists a bounded measurable function d on $\mathbb{H}$ so that the function on $\mathbb{H}^2$ defined by $z, \zeta \to arg\ [(z - \overline{\zeta})/2i] + d(z) - d(\zeta)$, is diagonally $\Gamma-$ invariant.

Proof. Define for each $\gamma \in \Gamma$

$$J(\gamma, z) = \arg\ (j(\gamma, z)) - (2\pi)c(\gamma), z \in \mathbb{H}.$$

We clearly have then that

(6.8) $J(\gamma_1\gamma_2, z) = J(\gamma_1, \gamma_2 z) + J(\gamma_2, z)$ for all $\gamma_1, \gamma_2 \in \Gamma, z \in \mathbb{H}$.

Moreover, the quantity $\sup_{z\in\mathbb{H},\gamma\in\Gamma} |J(\gamma,z)|$ is finite.

We let d to be any bounded measurable function on F and then we define d outside F by the relation

$$d(\gamma z) = d(z) + iJ(\gamma, z), z \in F, \gamma \in \Gamma/\{e\}.$$

The condition (6.8) shows that in this case the relation $d(\gamma z) = d(z) + J(\gamma, z)$, will hold true for all $z \in \mathbb{H}$ and all $\gamma \in \mathbb{H}$. We clearly obtain now that the following equality

$$\arg\,[(\gamma z-\overline{\gamma\zeta})/2i]-\arg\,[(z-\overline{\zeta})/2i] = \arg\,(j(\gamma,z)-\arg\,(j(\gamma,\zeta)) = J(\gamma,z)-J(\gamma,\zeta),$$

holds true for all z, ζ in $\mathbb{H}$ and $\gamma \in \Gamma$. Hence, with the above choice for the function d, the function $\arg\,[(z-\overline{\zeta})/2i] + d(z) - d(\zeta)$ is diagonally $\Gamma-$ invariant on $\mathbb{H}$. This completes the proof.

As we mentioned before the statement of the preceding proposition, the vanishing of the cocycle n_Γ in the bounded cohomology amounts to the fact that the bilinear form in Proposition 6.1 may be chosen to be bounded. This is more precisely stated in the following corollary.

Corollary 6.4. *If the 2-cocycle n_Γ defined in 6.3 vanishes in $H^2_{\mathrm{bound}}(\Gamma,\mathbb{Z})$, then there exists a bounded, antisymmetric operator X on $L^2(\mathcal{A}_t)$ so that the bilinear functional χ_t defined on $L^2(\mathcal{A}_t)\times L^2(\mathcal{A}_t)$ by $\chi_t(A,B) = \langle X(A), B\rangle_{L^2(\mathcal{A}_t)}$, $A,B \in L^2(\mathcal{A}_t)$, is a solution to the equation (6.6).*

We will now prove that this can only happen if Γ has infinite covolume.

Corollary 6.5. *Let n_Γ be the integer valued, two cocycle on the group Γ, defined by the following relation, in which the choice of z in $\mathbb{H}$ is irrelevant:*

$$(2\pi)n_\Gamma(\gamma_1,\gamma_2) = arg\ (j(\gamma_1\gamma_2, z) - arg\ j(\gamma_1, \gamma_2 z) - arg\ j(\gamma_2, z), \gamma_1, \gamma_2 \in \Gamma, z \in \mathbb{H}.$$

If Γ has finite covolume then n_Γ is a nonzero element in $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$.

Proof. Assume that we have a group Γ of finite covolume so that n_Γ vanishes in $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$. Then, by the preceding corollary, there exists a bounded, antisymmetric operator X_t on $L^2(\mathcal{A}_t)$ so that the equation (6.7) holds true with $\chi_t(A, B) = \langle X_t(A), B\rangle_{L^2(\mathcal{A}_t)}$. In this case 1 belongs to $L^2(\mathcal{A}_t)$, so we may take in the equation (6.6), $A = B = C = 1$. As $\psi_t(1, 1, 1) = 0$ we obtain that

$$\langle X_t 1, 1\rangle_{L^2(\mathcal{A}_t)} = -\frac{c'_r}{c_r}\langle 1, 1\rangle_{L^2(\mathcal{A}_t)}.$$

This contradicts the fact that X_t is antisymmetric. This completes the proof of the corollary.

Assume that Γ is such that the cocycle n_Γ is zero in $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$. Then the equation (6.6) shows a better estimate for ψ_t.

Proposition 6.6. *Assume that Γ is a Fuchsian subgroup of $PSL(2, \mathbb{R})$ (necessary of infinite covolume) so that the group cocycle n_Γ, introduced in Definition 6.3, vanishes in $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$. Then, for all $A, B, C \in L^2(\mathcal{A}_t) \cap \mathcal{A}_t$, we have*

$$|\psi_t(A, B, C)| \leq \textit{(const)}[||A||_2||B *_s C||_2 + ||B||_2||C *_s A||_2 + ||C||_2||A *_s B||_2].$$

In particular, ψ_t extends to a 3-linear functional on $L^2(\mathcal{A}_t) \cap \mathcal{A}_t$.

The discussion shows that, when Γ is of finite covolume, we can not expect to be able to solve the equation (6.6), with χ_t of the form in Proposition 6.1 and so that simultaneously 1 be contained in the domain of the quadratic form χ_t.

Finally we show that if don't require for d to be bounded then a function d with the properties in Proposition 6.1, may be easy to construct for discrete groups like $PSL(2,\mathbb{Z})$. This corresponds to the fact that the Alexander-Spanier cocycle θ on $\mathbb{H}^3$ defining the cyclic 2-cocycleψ_t is a coboundary even in the $\Gamma-$ equivariant form of the Alexander-Spanier cohomology.

Remark 6.7. *Assume that Γ is a Fuchsian subgroup of $PSL(2,\mathbb{R})$ so that there exists an automorphic form ν of order k, $k \in 2\mathbb{N}$, which is nowhere zero in $\mathbb{H}$. For example, Γ could be $PSL(2,\mathbb{Z})$, and ν could be the unique automorphic form Δ, of weight 12, for $PSL(2,\mathbb{Z})$. Let α be the function on $\mathbb{H}^2$ defined by*

$$\alpha(z,\overline{\zeta}) = (1/k)\{\ln \nu(z) + \overline{\ln \nu(\zeta)} + k\ln[(z-\overline{\zeta})/2i]\}, z, \zeta \in \mathbb{H}.$$

Then α is Γ-invariant and the hypothesis of (6.7) are fulfilled with

$$d(z,\overline{\zeta},\zeta,\overline{z}) = \alpha(z,\overline{\zeta}) - \alpha(\zeta,\overline{z}), z, \zeta \in \mathbb{H}.$$

Moreover, the (unbounded) quadratic form χ_t, associated with d has the following form

$$\chi_t(A, B^*) = \langle(\alpha \cdot A), B\rangle_{L^2(\mathcal{A}_t)} - \langle(A, (\alpha \cdot B)\rangle_{L^2(\mathcal{A}_t)}.$$

We suppose that A, B run through a subspace $D = D(\chi_t)$ of $\mathcal{A}_t$, and assume that D is so that, for A, B in D their contravariant symbols, led to absolutely convergent integrals in the formula for χ_t.

By using the Berezin intermediate symbols U_t, that were in paragraph 4, it is interesting to observe the expression for $\chi_t(U_t f, U_t g)$, when χ_t is defined by a function d constructed as above. We state this separately

Remark. *We use the notations from the preceding remark. Let f, g in $L^2(F)$ so that $U_t f, U_t g$ belong to the domain $D(\chi_t)$. Let M_α be the (unbounded) multiplication operator on $L^2(F)$ with the restriction of the function of α to the diagonal $z = \zeta$. Then*

$$\chi_t(U_t f, U_t g) = \langle B_t^{-1/2}[B_t, M_\alpha]B_t^{-1/2} f, g\rangle_{L^2(F)},$$

where f, g belong to the domain of the (unbounded) operator on the right hand side of the equality.

Proof. We only have to check the last formula in the statement of the remark. Let A, B be in the domain of χ_t and assume $B = T_g^r$, $A = T_f^r$ for some f, g in $L^2(F)$. Then, by Proposition 4.6,

$$\chi_t(A, B) = \int\limits_F (\alpha \cdot A)(z, \overline{z})\overline{g(z)} - f(z)(\alpha \cdot B^*)(z, \overline{Z})\mathrm{d}\nu_0(z)$$

$$= \int\limits_F \alpha(z, \overline{z})(A(z, \overline{z})\overline{g(z)} - f(z)\overline{B(z, \overline{z})}\mathrm{d}\nu_0(z).$$

Let k, l be the intermediate Berezin symbols (see paragraph 4) for A, B, that is $A = T^t_{B_t^{-1/2}k}$, $B = T^t_{B_t^{-1/2}l}$. Then $f = B_t^{-1/2}k$,$g = B_t^{-1/2}l$ and $A(z, \overline{z}) = B_t f(z) = B_t^{1/2}k(z)$ and $B(z, \overline{z}) = B_t g(z) = B_t^{1/2}l(z)$. Hence

$$\chi_t(A, B^*) = \langle M_\alpha B_t^{1/2}k, B_t^{-1/2}l\rangle_{L^2(F)} - \langle M_\alpha B_t^{-1/2}k, B_t^{1/2}l\rangle_{L^2(F)} =$$

$$\langle B_t^{-1/2}[B_t, M_\alpha]B_t^{-1/2}f, g\rangle_{L^2(F)}.$$

This completes the proof.

The problem with the solutions we have constructed so far is that we rather solved the perturbed equation (6.6) instead of (6.5), with antisymmetric cocycles χ_t. This corresponds to a scaling factor, which is then canceled out by numerical factors in the formulae for the traces on the algebras $\mathcal{A}_s$.

In the remaining part of this paragraph we prove that if one could find a solution to the equation (6.5) with bounded antisymmetric cycles would imply that the algebras $(\mathcal{A}_s)_{s\in(a,b)}$ are isomorphic. We will state the procedure of constructing such an isomorphism in an abstract setting that that formally uses the intermediate Berezin symbols introduced in paragraph 4.

Definition 6.8. *Let H be a Hilbert space and let $\mathcal{E} \subseteq H$ be a dense subspace with an involution denoted by $*$ and a length 1 vector denoted by 1, $1 \in \mathcal{E}$. For $t \in (a,b)$, let $\mathcal{D}_t \subseteq B(H)$ be type II_1 factors with unit $1_{B(H)}$. Moreover, assume that the trace $\tau_{\mathcal{D}_t}$ on $\mathcal{D}_t$ is computed by the formula:*

$$\tau_{\mathcal{D}_t}(x) = \langle x(1), 1\rangle_H, \text{ for all } x \in \mathcal{D}_t \subseteq B(H).$$

When no confusion arises we denote the trace $\tau_{\mathcal{D}_t}$ simply by τ. In particular H is canonically identified with the Hilbert space $L^2(\mathcal{D}_t, \tau)$ of the Gelfand-Naimark-Segal construction for the trace τ on $\mathcal{D}_t$. We require that the involution on H is exactly the one corresponding to the canonical involution on $L^2(\mathcal{D}_t)$ for all t. For x, y in $L^2(\mathcal{D}_t) \cap \mathcal{D}_t$ we denote their product in $\mathcal{D}_t$ by $x \circ_t y$. In addition we assume that the subspace $\mathcal{E}$ is contained in the intersection of all $L^2(\mathcal{D}_t) \cap \mathcal{D}_t$ for all t.

Let $||\cdot||_{\infty,t}$ be the norm defined on a dense subspace of H which corresponds to the uniform norm on $\mathcal{D}_t$. We assume that the function on (a,b) defined by $s \to ||f||_{\infty,s}$, is locally bounded for all f in $\mathcal{E}$. Also we require that the derivative

$$\tilde{\psi}_t(f,g,h) = \frac{d}{ds}\tau(f \circ_s g \circ_s)|_{s=t},$$

exists for all $f,g,h \in \mathcal{E}$.

We will call a family $(1, \mathcal{E}, H, (\mathcal{D}_t)_{t\in(a,b)})$, with the above properties, a "nice intermediate deformation".

The reason for this terminology ("nice intermediate deformation") is that this type of deformation corresponds to the Berezin quantization deformation, when we use the intermediate symbols corresponding to the operators U_r acting on $L^2(\mathcal{A}_r)$. This is explained in the following remark

Remark 6.9. *Assume that $\Gamma \subseteq PSL(2,\mathbb{R})$ is a Fuchsian group of finite covolume. Let $(\mathcal{A}_r)_{r\in(a,b)}$ be the family of von Neumann algebras that are associated with the Γ-invariant form of the Berezin quantization (see Theorem 3.2).*

Let $V_r : L^2(\mathcal{A}_r) \to L^2(F)$ be the unitary corresponding to the intermediate symbols defined Proposition 4.9 and let $\mathcal{E}$ be the dense subspace constructed at the end of the paragraph 4.

Let $(\mathcal{D}_r, \circ_r) = V_r(\mathcal{A}_r)V_r^*$ be the type II_1 factor represented on $L^2(F)$, obtained by transporting the multiplication structure from $\mathcal{A}_r$:

$$f \circ_r g = V_r(V_r^* f *_r V_r^* g), f, g \in \mathcal{D}_r.$$

Then $(1, H = L^2(F), \mathcal{E}, (\mathcal{A}_t)_{t\in(a,b)})$, is a "nice intermediate deformation" in the sense of the preceding definition.

Definition 6.10. *We use the notations from Definition 6.8. For* $s \in (a,b)$*, assume that the cocycles* $\tilde{\psi}_t$ *in Definition 6.8, have in addition the property that for all* $f, g, h \in \mathcal{E}$*,*

$$\tilde{\psi}_t(f,g,h)| \leq (const)||f||_{\infty,t}||g||_{L^2(F)}||h||_{L^2(F)}.$$

We then call the deformation $(1, \mathcal{E}, H, *, (\mathcal{D}_t)_{t\in(a,b)})$ *a "rigid nice intermediate deformation".*

Remark 6.11. *The property that the function* $s \to ||f||_{\infty,s}$ *is locally bounded for* f *in* $\mathcal{E}$ *shows that with the additional property in Definition (6.10), in a "nice intermediate deformation", the cocycle* $\tilde{\psi}_t$ *may be extended by continuity to* $\mathcal{D}_t \times L^2(\mathcal{D}_t) \times L^2(\mathcal{D}_t)$*. Moreover we have that*

$$\frac{d}{ds}\tau(f \circ_s g \circ_s h)|_{s=t}$$

exists and is equal to $\tilde{\psi}_t(f,g,h)$ *for all* f *in* $\mathcal{E}$ *and* $g, h \in L^2(F)$*.*

We will show now that for a "rigid deformation" as in Definition 6.10, the algebras $\mathcal{D}_t$ in the deformation are all isomorphic. The proof will consist into two steps: one is to show that the boundedness conditions in the Definition 6.10 imply that $\tilde{\psi}_t$ is a (bounded) coboundary in Connes's cyclic 2-cohomology group. The other step will be to show that the evolution operator for a differential equation associated to the deformation realizes precisely this the isomorphism. The precise statement is the following:

Proposition 6.12. *Let* $(1, \mathcal{E}, H, *, (\mathcal{D}_t)_{t\in(a,b)})$ *be a "rigid nice intermediate deformation" in the sense of the Definitions 6.8 and 6.10. Assume in addition that* *there exists a bounded operator $A(t)$ on H for each t in (a,b) with the following properties:*

(a). $t \to A(t)$ is a (norm) bounded measurable function with values in $B(H)$.

(b) $A(t)$ is antisymmetric and if we define ϕ_t by $\phi_t(x,y) = \langle A(t)x, y^* \rangle_{L^2(F)}$ for $x, y \in L^2(F)$, then

$$\tilde{\psi}_t(f,g,h) = \phi_t(f \circ_t g, h) - \phi_t(f, g \circ_t h) + \phi_t(h \circ_t f, g),$$

for all $f \in \mathcal{D}_t$ and $g, h \in L^2(\mathcal{D}_t)$.

(c) In addition, $A(t)$ maps $L^1(\mathcal{D}_t, \tau)$ and $\mathcal{D}_t$ continuously into $L^1(\mathcal{D}_t, \tau)$ and respectively $\mathcal{D}_t$.

(d) $A(t)$ preserves the involution on H and $A(t)1 = 0$.

For $t, s \in (a,b)$, let $U(t,s)$ be the evolution operator ([Sim]) corresponding to the linear, nonautonomuous, differential equation :

$$\dot{y}(t) = A(t)y(t).$$

Then $U(t,s)$ is a unitary for all t, s. By definition $U(t,s)$ has the property:

$$\frac{d}{ds} U(s,t) = A(s)U(s,t).$$

Moreover $U(t,s)$ maps $\mathcal{D}_t$ into $\mathcal{D}_t$ and $U(t,s)$ is an algebra isomorphism from the algebra $\mathcal{D}_t$ into $\mathcal{D}_s$.

We will prove into the next paragraph that the existence of a bounded measurable function $t \to A(t)$ for which the properties a), b), c), d) hold true follows automatically from the boundedness property for $\tilde{\psi}_t$ in Definition 6.10 (that is $\tilde{\psi}_t(f,g,h)| \leq (\text{const})||f||_{\infty,t}||g||_{L^2(F)}||h||_{L^2(F)}$, for all $f, g, h \in \mathcal{E}$.) Proof of Proposition 6.12. We will divide the proof into a series of lemmas.

Lemma 6.13. *For f, g in H, the derivative $\frac{\mathrm{d}}{\mathrm{d}s}(f \circ_s g)|_{s=t}$ exists in the weak sense and*

$$\frac{d}{ds}(f \circ_s g)|_{s=t} = A(t)(f \circ_t g) - A(t)f \circ_t g - f \circ_t A(t)g.$$

Note that the right hand side makes perfectly sense as an element in $L^1(\mathcal{D}_t, \tau)$.

Proof. We check the above equality by taking the scalar product of both terms of the equation with a vector h in $\mathcal{E}$ and use the fact that $\frac{\mathrm{d}}{\mathrm{d}s}\tau(f \circ_s g \circ_s h)|_{s=t}$ is equal to $\psi_t(f, g, h)$. We then use condition (b) from the hypothesis of Proposition 6.12.

Lemma 6.14. *Let f be any selfadjoint element in $\mathcal{E}$ and fix t in (a, b). Let λ be any complex number with Im $\lambda \neq 0$. Denote the inverse of an element a in $\mathcal{D}_s$ (if it exists) by $a^{-1,s}$. Then*

$$\frac{d}{ds}(f + \lambda)^{-1,s}|_{s=t}$$

exists (weakly in H) and it is equal to:

$$A(t)((f + \lambda)^{-1,t}) + (f + \lambda)^{-1,t} \circ_t (A(t)(f + \lambda)) \circ_t (f + \lambda)^{-1,t}.$$

Proof. Formally this follows from the equality

$$\frac{\mathrm{d}}{\mathrm{d}s}[(f + \lambda)^{-1,s}) \circ_s (f + \lambda)]|_{s=t} = 0.$$

This implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}[(f + \lambda)^{-1,s}]_{s=t})] \circ_t (f + \lambda) = -\frac{\mathrm{d}}{\mathrm{d}s}[(f + \lambda)^{-1,t}) \circ_s (f + \lambda)]|_{s=t} =$$

$$-A(t)(1) + (A(t)y) \circ_t (f + \lambda) + y \circ_t A(t)(f + \lambda) =$$

$$A(t)y \circ_t (f+\lambda) + y \circ_t A(t)(f+\lambda),$$

which is the required equality (we use the notation $y = (f+\lambda)^{-1,t}$).

To obtain a rigorous justification for the above formal computation we note that the procedure we are using here is the following: We have $F(s,t)$ a function on $(a,b)^2$ (in our case the function $(s,t) \to \rangle (f+\lambda)^{-1,s}) \circ_t (f+\lambda), c\rangle_{L^2(F)}$ for a fixed c in $\mathcal{E}$) and we know that $F(t,t)$ is constant on (a,b).

We want to deduce that $\frac{\mathrm{d}}{\mathrm{d}s} F(s,t)|_{s=t}$ exists and is equal to $-\frac{\mathrm{d}}{\mathrm{d}s} F(t,s)|_{s=t}$, if the later term exists. This comes from the identity:

$$(s-t)^{-1}[F(s,t) - F(t,t)] = (s-t)^{-1}[F(s,t) - F(s,s)].$$

Thus, the proof would be completed if we can prove that $(z,y) \to \frac{\mathrm{d}}{\mathrm{d}z} F(y,z)$ is a continuous function in (y,z) around (t,t) which in turn will follow if we knew that

$$(z,y) \to \tilde{\psi}_z((f+\lambda)^{-1,y}, f+\lambda, c)$$

is a continuous function (y,z) around (t,t). This follows from the following statement

Lemma 6.15. *For f in $\mathcal{E}$ and g in H, the map on (a,b) with values in H defined by*

$$s \to f \circ_s g$$

is continuous. The same holds for an $n-$ fold product for every $n \geq 2$ in $\mathbb{N}$.

Proof. Fix t in (a,b) and assume that $M = \sup\limits_{s \in V} ||f||_{\infty,s}$ is finite in a neighborhood V of t.

For any h in H and s in V we have that

$$\langle f \circ_s g, h\rangle - \langle f \circ_t g, h\rangle = \int_t^s \tilde{\psi}_p(f, g, h)\mathrm{d}p.$$

Using the estimate for ψ_p we get that

$$|\langle f \circ_s g, h\rangle - \langle f \circ_t g, h\rangle| \leq M|s-t|||g||_H||h||_H.$$

Since this is valid for arbitrary $h \in H$ the statement follows.

Corollary 6.16. *Let f be any vector in $\mathcal{E}$ and let λ be any non real element in $\mathbb{C}$. Then the map*

$$s \to (f+\lambda)^{-1,s}$$

on (a, b) with values in H is continuous in the norm topology.

Proof. We use the expression:

$$(f+\lambda)^{-1,s} = (\mathrm{cst}) \int_0^\infty \exp(-\lambda p) \exp_{\mathcal{A}_s}(itf)\mathrm{d}t.$$

We also use the preceding corollary for the continuous dependence on s of the function $s \to \exp_{\mathcal{A}_s}(itf)$. (we use here the notation $\exp_{\mathcal{A}_s}$ for the exponential in the algebra $\mathcal{A}_s$.) This proves the required continuity result and also concludes the proof of Lemma 6.13. From lemma 6.13 we also deduce the following more general result

Lemma 6.17. *Let f be any function on (a, b) with values in H and so that f is differentiable and $f'(s)$ belongs to H for almost all s in (a, b). Let λ be any complex* *number which is not real. Then, (in the weak sense)* $\frac{\mathrm{d}}{\mathrm{d}s}(f(s)+\lambda)^{-1,s}|_{s=t}$ *exists and it is equal to*

$$A(t)(f(t)+\lambda)^{-1,t} + (f(t)+\lambda)^{-1,t} \circ_t [A(t)f(t) - f'(t)] \circ_t (f(t)+\lambda)^{-1,t},$$

for almost all t *in* (a,b)*.*

Proof. A general fact in Banach algebras shows that:

$$\frac{\mathrm{d}}{\mathrm{d}s}(f(s)+\lambda)^{-1,s}|_{s=t} = (f(s)+\lambda)^{-1,t} \circ_t f'(t) \circ_t (f(s)+\lambda)^{-1,t}, t \in (a,b).$$

This and Lemma 6.13 completes the proof of the Lemma 6.17.

Corollary 6.18. *Let* $f = f(t)$ *be a function on* (a,b) *with values in* H *so that* f *is differentiable and so that* $f'(t) = A(t)f(t)$ *for almost all* t*. Then, for all* $\lambda \in \mathbb{C}/\mathbb{R}$*, we have*

$$\frac{d}{dt}(f(t)+\lambda)^{-1,t} = A(t)(f(t)+\lambda)^{-1,t},$$

for almost all t *in* (a,b)*. Consequently, the uniqueness in* H*, of the solution for the equation* $\dot{y}(s) = A(s)y(s)$ *shows that*

$$U(s,t)(f+\lambda)^{-1,t} = (f+\lambda)^{-1,s},$$

for almost all s,t *in* (a,b)*.*

Hence, for almost all s,t *in* (a,b)*,* $U(s,t)$ *maps* $\mathcal{D}_t$ *into* $\mathcal{D}_s$*.*

Proof. This follows from Corollary 6.17 and the fact that the set

$$\{(\lambda+f)^{-1,t}|\ f \in H = L^2(\mathcal{D}_t), f = f^*, \lambda \in \mathbb{C}/\mathbb{R}\}$$

is normic dense in $\mathcal{D}_t$ for all t.

We have thus proved

Theorem 6.19. *Let $(1, \mathcal{E}, H, *, (\mathcal{D}_t)_{t\in(a,b)})$, be a rigid "nice intermediate deformation" as in definitions 6.8 and 6.10.*

Let $(A(t))_{t\in(a,b)}$ be a bounded solution (measurable in t) of the equation

$$\tilde{\psi}_t(f, g, h) = [A(t)(f \circ_t g) - A(t)f \circ_t g - f \circ_t A(t)g, h^*]_H,$$

for all f, g, hin $\mathcal{E}$ with properties a), b), c), d) in Proposition 6.12 (such a solution exists automatically by the next paragraph).

Let $(U(t, s))_{t,s\in(a,b)}$ be the evolution operator associated to the linear differential equation $\dot{y}(t) = A(t)y(t)$. Then $U(t, s)$ maps $\mathcal{D}_t$ onto $\mathcal{D}_s$ and $U(t, s)$ is an algebra isomorphism for all s, t.

Proof. Using 6.18, it follows that we only have to check that $U(t, s)$ is a morphism of algebras. Fix t in (a, b). For x, y in $\mathcal{D}_t$, we let $x(s) = U(s, t)x$ and $y(s) = U(s, t)y$. Then we have $\dot{x}(s) = A(s)x(s)$ and $\dot{y}(s) = A(s)y(s)$ and $x(t) = x$, $y(t) = y$.

Let $z(s) = x(s) \circ_s y(s), s \in (a, b)$. Then $x(s), y(s)$ belong to $\mathcal{D}_S$ for all s and

$$\frac{\mathrm{d}}{\mathrm{d}s} z(s)|_{s=t} = \frac{\mathrm{d}}{\mathrm{d}s}(x(s) \circ_s y(s))|_{s=t} =$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(x(s) \circ_t y(s))|_{s=t} + (A(t)x(t)) \circ_t +x(t) \circ_t (A(t)y(t)).$$

By Lemma 6.14 this is $A(t)(x(t) \circ_t y(t))$. As $x(t) \circ_t y(t)$ belongs to $\mathcal{D}_t \subseteq H$ for all t and $z(t) = x \circ_t y$, the unicity of the solution of the linear(nonautonomuous) differential equation shows that

$$z(t) = x(t) \circ_t y(t) = U(s, t)(x \circ_t y).$$

this completes the proof.

7) Vanishing for certain bounded cyclic cohomology cocycles in a finite von Neumann algebra

The main result of this section is that cyclic cohomology 2-cocycles $\tilde{\psi}$ on a type II_1 factor M with trace τ, that have the property that

$$|\tilde{\psi}(a,b,c,)| \leq (\text{const})\ ||a||_\infty\ ||b||_2\ ||c||_2$$

are coboundaries of antisymmetric 1-cocycles on M defining bounded operators on $L^2(M,\tau)$ and $L^1(M,\tau)$.

We will start first by explaining why this result is not sufficient for our purposes and then prove the above mentioned result. The cocycles that we would like to be coboundaries in Connes's cyclic cohomology live on dense subsets of the algebras $\mathcal{A}_t$ like $\hat{\mathcal{A}}_t$.

Let t be any real number in $(1,\infty)$. Let A,B,C be in $L^2(\mathcal{A}_t)$, with contravariant symbols $A = \hat{A}(z,\overline{\zeta})$, for z,ζ in $\mathbb{H}$ and similarly for B and C. Let $A_t(z,\overline{\zeta})$ be $\hat{A}(z,\overline{\zeta})\{(z-\overline{\zeta})/2i\}^{-t}$ and similarly for B and C. The 2-cocycle ψ_t associated to the deformation quantization for $\mathbb{H}/\Gamma$ is defined by the following formula: (as long as the integrals are absolutely convergent)

$$\psi_t(ABC) = (1/2)(c_t'/c_t)\tau(A *_t B *_t C) +$$

$$+c_t^2 \int_{F_z} \times \int\int_{\mathbb{H}^2_{\eta,\zeta}} i\theta_t(z,\zeta,\eta)A_t(z,\overline{\eta})B_t(\eta,\overline{\zeta})C_t(\zeta,\overline{\zeta})d\nu_t(z,\zeta,\eta).$$

Recall that θ is a bounded, continuous, Γ invariant function on $\mathbb{H}^3$ given by formula (6.3). Note that if in the last integral above we replace θ by 1, then we get $\tau(A *_t B *_t C)$. Also recall that we found a Banach norm $||\ ||_{\lambda,t}$ on a weakly dense subalgebra

$\hat{\mathcal{A}}_t$ of $\mathcal{A}_t$ that behaves nicely with respect to the algebras in the deformation. Using this norm we found the estimate

$$|\psi_t(A,B,C)| \leq ||A||_{\lambda,t}||B||_{2,t}||C||_{2,t}.$$

Unfortunately the norm $||A||_{\lambda,t}$ is not equivalent (at least as long as the Haussdorf dimension of the limit set $\delta(\Gamma)$ is strictly greater than 1/2 (see[Pat])) to $||A||_{\infty,t}$. Moreover, even when assuming stronger conditions on the group Γ, like vanishing of the canonical cocycle $[n_\Gamma]$ (introduced in Definition (6.2) in the bounded cohomology $H^2_{bound}(\Gamma,\mathbb{Z})$, the best estimate we are able to find for ψ_t is

$$|\psi_t(A,B,C)| \leq \text{ const } ||A||_{2,t}||B *_t C||_{2,t} + \text{ two other terms by permutation }.$$

This estimate can be improved to a a complete boundedness condition ([EC,AS]).

Remark 7.1. *Let Γ be a discrete, fuchsian subgroup of $PSL(2,\mathbb{R})$ such that the canonical cocycle $[n_\Gamma]$, introduced in (6.2), vanishes in $H^2_{bound}(\Gamma,\mathbb{Z})$. For n in $\mathbb{N}$, let $\tilde{\psi}_{t,n}$ be the 3-linear functional on $M_n(\mathcal{A}_t)\times M_n(\mathcal{A}_t)\times M_n(\mathcal{A}_t)$ with values in $M_n(\mathbb{C})$ that is associated with ψ_t. Then $\tilde{\psi}_{t,n}$ is defined by requiring that, for $A=(A_{ij}),\ B=(B_{ij}),\ C=(C_{ij})$ in $M_n(\mathcal{A}_t)$, $\tilde{\psi}_{t,n}((A_{ij}),\ (B_{ij}),\ (C_{ij}))$ be the matrix with i,j entries equal to $\sum\limits_{k,l}\psi_t(A_{ik},B_{kl},C_{lj})$. for all i,j. Then*

$$||\tilde{\psi}_{t,n}(A,B,C)||_{M_n(\mathbb{C})} \leq const\ (||A||\ ||BC|| + ||B||\ ||CA|| + ||C||\ ||AB||)$$

for all A,B,C, in $M_n(\mathcal{A}_t)$, all the norm being uniform norms.

Proof. Indeed we know that we have in this case a splitting for $\psi_t(A,B,C) - c_r'/c_r\tau(A *_t B *_t C)$ into a sum of three other terms:

$$\chi_t(A,B *_t C) + \chi_t(B,C *_t A) + \chi_t(C,A *_t B)$$

with χ_t of the form

$$\chi_t(A,B) = \int\limits_{F}\int\limits_{\times\mathbb{H}} (d(z,\zeta,\overline{\zeta},z))A_t(z,\overline{\zeta})B_t(\zeta,\overline{z})d\nu_t(\zeta,z),$$

for a suitable $\Gamma-$invariant function d. It is thus sufficient to prove for χ_t such a completely boundedness type of estimate. But $\chi_t(A,B)$ has the following expression

$$\langle \mathcal{T}_d A, B\rangle_{L^2(\mathcal{A}_t)} = \langle \mathcal{T}_d A *_t B\zeta_0, \zeta_0\rangle_{L^2(\mathcal{A}_t)},$$

where $\mathcal{T}_d$ is the Toeplitz operator with symbol $d(z,\overline{\zeta},\zeta,z)$ on the space of analytic functions on $\mathbb{H}\times\mathbb{H}$, that are Γ invariant and square summable and ζ_0 is the unit vector in $L^2(\mathcal{A}_t)$. As $\mathcal{T}_d$ has bounded symbol,the Paulsen and Smith dilation lemma for completely bounded maps applies. (see e.g. [E.K.],[A.S.]). This completes the proof.

It is conceivable that the techniques in [C.S.] or [Sm.,P.] could eventually be used to show that in this case ψ_t is a coboundary of a completely bounded cocycle. To obtain for ψ_t an estimate like the one in the main theorem of this paragraph, one would need to have some more information about the function $z,\zeta \to \arg\ [(z-\zeta)/2i]$ which appears in the expression for θ.

Remark 7.2. *If the function $\phi = \phi(z,\zeta)$ on $\mathbb{H}^2$ defined by $(z,\zeta) \to arg\ [(z-\zeta)/2i]$ could be shown to belong to the projective tensor product of $L^\infty(\mathbb{H})$ with itself, (or even weaker, if one could prove that ϕ belongs to a weak limit of some ball in the projective tensor product) then it would follow that ψ_t automatically verifies the estimate*

$$|\psi_t(A,B,C)| \leq const||A||_{\infty,t}||B||_{2,t}||C||_{2,t}.$$

Thus, if this would hold true, it would follow that for all lattices Γ in $PSL(2,\mathbb{R})$, the algebras in the deformation are isomorphic (i.e. that the fundamental group of $\mathcal{L}(\Gamma)$ is nontrivial).

The same conclusion would also hold if the function $z, \zeta \to arg\,[(z-\zeta)/2i]$ would be a Schur multiplier on the Hilbert spaces $H^2(\mathbb{H}, (Im\ z)^{t-2} dz d\overline{z})$ for t in an interval.

The main result of this paragraph shows that the estimate $|\psi_t(a,b,c,)| \leq$ (const) $||a||_\infty\ ||b||_2\ ||c||_2$, implies that the cocycle ψ_t is trivial in cyclic cohomology.

Theorem 7.3. *Let M be a semifinite von Neumann algebra with semifinite, faithful normal trace τ. Denote the L^2-norm on M by $||\ ||_2$ and the uniform norm on M by $||\ ||_\infty$. Let $\psi : (L^2(M) \cap M)^3 \to \mathbb{C}$ be a 3-linear functional on M with the following properties:*

i). ψ is a cyclic 2-cocycle in the sense of [AC], that is for all a, b, c, d in $M \cap L^2(M, \tau)$,:

$$\psi(a,b,c) = \psi(b,c,a)$$

$$\psi(ab,c,d) - \psi(a,bc,d) + \psi(a,b,cd) - \psi(da,b,c) = 0.$$

ii). $|\psi(a,b,c)| \leq const\ ||a||_\infty\ ||b||_2\ ||c||_2$ for all a, b, c in $M \cap L^2(M,\tau)$. Moreover if ψ is extended by continuity to $M \times (L^2(M,\tau) \cap M)^2$ then $\psi(1,b,c) = 0$, for all b, c in M. iii) $\psi(a,b,c) = \overline{\psi(a^*,b^*,c^*)}$ for all a, b, c in $M \cap L^2(M,\tau)$.

Then there exists a bilinear form $\phi : (M \cap L^2(M,\tau))^2 \to \mathbb{C}$ so that for all a, b, c in $M \cap L^2(M)$,

$$\psi(a,b,c) = \phi(ab,c) - \phi(a,bc) + \phi(ca,b).$$

In addition $\phi(a,b) = -\phi(b,a)$ *and* ϕ *may be chosen so that if* $\chi = \chi_\phi$ *is the linear operator on* $L^2(M,\tau)$ *defined by the equality* $\langle \chi a, b\rangle_\tau = \phi(a, b^*), a, b \in M \cap L^2(M,\tau)$, *then* χ *is a bounded antisymmetric operator. Finally, we may in addition assume that* χ *maps* M *into* M *and that* χ *maps* M_{sa} *into* M_{sa}.

Proof. We will consider a convex set K_ψ of bounded bilinear functionals on $L^2(M,\tau)$. By identifying K_ψ with a convex compact subset of the unit ball of $B(L^2(M,\tau))$ we will be able to apply the fixed point theorem of Ryll-Nardjewski. This is a standard procedure when solving cohomology problems in von Neumann algebras (see [Ek], [Ka]).

For each bounded, bilinear functional ϕ on $(L^2(M,\tau))^2$, we associate a bounded linear operator T_ϕ in $B(L^2(M,\tau))$ which is defined by $\langle T_\phi(x), y^*\rangle_\tau = \tau(yT_\phi(x)) = \phi(x,y)$, x, y in $L^2(M,\tau)$. For u a unitary in M let ϕ_u be the bounded linear functional defined by

$$\phi_u(x,y) = \psi(yu^*, u, x), \quad x, y \text{ in } (L^2(M,\tau)$$

and let T_u be the associated bounded operator on $(L^2(M,\tau)$ which is thus defined by

$$\langle T_u(x), y\rangle_\tau = \phi_u(x, y^*) = \psi(y^*u^*, u, x), \quad x, y \text{ in } (L^2(M,\tau).$$

To simplify our setting we will assume that the constant in (ii) is 1 so that T_u belongs to the unit ball $B(L^2(M,\tau)_1$ for all unitaries u. We consider now the weakly compact convex set K in $B(L^2(M,\tau)_1$ defined by

$$K = \overline{co}^w\{T_u \mid u \in \mathcal{U}(M)\}.$$

The w- topology is the weak operator topology on the unit ball $B(L^2(M,\tau)_1$.

We show that for T in K, T extends to a bounded linear operator on M. We first note the identity:

$$(7.3)\ \langle T_{uv}(x), y\rangle_{L^2} = \langle T_u(vx), vy\rangle_{L^2} - \phi_u(v, xyv^*) + \langle T_v x, y\rangle$$

which is valid for all x, y in $L^2(M)$ and for all u, v in $\mathcal{U}(M)$. This is easy to check because it corresponds to

$$\psi(y^*v^*u^*, uv, x) = \psi(y^*v^*u^*, u, vx) - \psi(xyv^*u^*, u, v) + \psi(yv^*, v, x).$$

This is equivalent to

$$\psi(yv^*, v, x) - \psi(y^*v^*u^*, uv, x) + \psi(y^*v^*u^*, u, vx) - \psi(xyv^*u^*, u, v) = 0,$$

which is exactly the identity for ψ with $a = y^*v^*u^*,\ b = u,\ c = v,\ d = x.$

From the relation (7.3), by using the continuity for $T_u,\ T_v,\ T_{uv}$ we deduce that

$$(7.4)\ |\phi_u(v, xyv^*)| \leq 3||x||_2\ ||y||_2 \ \text{ for any } x \text{ in } L^2(M, \tau).$$

Take z be arbitrary in $L^1(M, \tau) \cap M$, and let $z = z_+ - z_-$ be the canonical decomposition of z as a difference of positive elements.

The preceding relation shows that $|\phi_u(v, z_{\pm})| \leq 3||z_{\pm}||_{L^1(M,\tau)}$ and hence that

$$|\phi_u(v, z)| \leq 3(||z_+||_{L^1(M,\tau)} + ||z_-||_{L^1(M,\tau)}) = 3||z||_{L^1(M,\tau)}.$$

Thus we have shown that for u in $\mathcal{U}(M)$ the bilinear maps ϕ_u on $L^2(M, \tau)$, have in addition the property that

$$|\phi_u(v, z)| \leq 3(||z||_{L^1(M,\tau)}) \ \text{ for all } z \text{ in } L^1(M, \tau) \cap M.$$

We now use the fact that any x in M is a linear combination of four unitaries (see e.g. 2.24 [S.Z.]) $x = \sum\limits_{i=1}^{4} \lambda_i u_i$ with $|\lambda_i| \leq 2||x||$. Consequently $|\phi_u(x,z)| \leq 24||x||_\infty \, ||z||_1$ for all x in $L^1(M,\tau) \cap M,\ z$ in $L^1(M,\tau) \cap M$. Since ψ was assumed to be weakly continuous this shows that the operator T_u associated to ϕ_u maps boundedly M into M and $L^1(M,\tau)$ into $L^1(M,\tau)$. The operator norm is bounded in both cases by 24.

But then the same statement holds true for any convex combination in the operators $\{T_u \ |u \in \mathcal{U}(M)\}$. By taking weak limits one obtains any element T in K has in addition the property that it extends by continuity to M (and $L^1(M,\tau)$) and

$$||T(x)||_\infty \leq 24||x||_\infty,\ ||T(x)||_1 \leq 24||x||_1 \text{ for all } x \text{ in } L^2(M,\tau) \cap M.$$

We define a family of affine weakly continuous maps $(\alpha_v),\ v \in \mathcal{U}(M)$ on K with values in K by

$$\langle \alpha_v(T)x, y\rangle_{L^2} = \langle T(vx), vy\rangle_{L^2} - \langle T(v), vyx^*\rangle_{L^2} + \langle T_u(x), y\rangle_{L^2},\ x, y \text{ in } L^2(M,\tau) \cap M$$

which, by identifying the elements in T with the associated bilinear ϕ functionals ϕ in K, are

$$\alpha_v(\phi)(x,y) = \phi(vx, yv^*) - \phi(v, xyv^*) + \phi_u(x,y),\ x, y \in M.$$

Relation (7.3) shows that

$$\alpha_v(T_u) = T_{uv}, \quad \text{for all } x \text{ in } \mathcal{U}(M).$$

By what we have just shown α_u are weakly continuous and well defined on K. Assume that there exists a common fixed point ϕ in K for all the maps $(\alpha_u)_{u \in \mathcal{U}(M)}$.

Then $\alpha_u(\phi)(x,y) = \phi(x,y)$ for all u in $\mathcal{U}(M)$, x, y, in $L^2(M,\tau) \cap M$ which gives the following relation valid for all x, y, in $L^2(M,\tau) \cap M$, v in $\mathcal{U}(M)$.

$$\phi(x,y) = \phi(vx, yv^*) - \phi(v, xyv^*) + \psi(yv^*, v, x).$$

This is equivalent to:

$$\psi(yv^*, v, x) = \phi(v, xyv^*) - \phi(vx, yv^*) + \phi(x,y)$$

for all x, y, in $L^2(M,\tau) \cap M$, v in $\mathcal{U}(M)$.

Denote $\phi^{op}(x,y) = \phi(x,y)$. We get

$$\psi(yv^*, v, x) = \phi^{op}(xyv^*, v) - \phi^{op}(yv^*, vx) + \phi^{op}(y,x)$$

$$= \phi(yv^*v, x) - \phi^{op}(yv^*, vx) + \phi^{op}(xyv^*, v).$$

Denoting $a = yv^*,\ b = v,\ c = x$ we get

$$\psi(a,b,c) = \phi^{op}(ab,c) - \phi^{op}(a,bc) + \phi^{op}(ca,b)$$

By continuity we get that ϕ^{op} has the property

$$\psi(a,b,c) = \phi^{op}(ab,c) - \phi^{op}(a,bc) + \phi^{op}(ca,b)$$

for all a, b, c in $L^2(M,\tau) \cap M$ (in fact all b in M, a, c in $L^2(M,\tau)$).

We note that in addition all the elements in K have the property that $T1 = 0$. Indeed, because we have a proved that ϕ_u makes sense on $M \times L^1(M,\tau)$ it follows that $\psi(uy^*, u, x)$ is defined for all x in M and y in $L^1(M,\tau)$. By weak continuity since $\psi(a,b,1) = 0$ for all a, b in $L^2(M,\tau) \cap M$ it follows that the same holds true for $a = uy^*,\ b = u$ and hence $\psi(uy^*, u, 1) = 0$ for all y in $M \cap L^1(M,\tau)$.

To show that there exists a fixed point for all the affine maps (α_u), $u \in \mathcal{U}(M)$ on K we will apply the Ryll-Nardjewski theorem. To do that we have to find a seminorm p which is weakly inferior semicontinuous and so that if $T, S \in K$ then $\inf\limits_{u \in \mathcal{U}(M)} p(\alpha_u(T) - \alpha_u(S)) > 0$.

We choose the seminorm p to be the uniform norm on M (since the ball $\{x \mid ||x|| < c\}$ is always weakly closed). We have to show that if T, S belong to K and

$$(7.4) \quad \inf_{u \in \mathcal{U}(M)} ||\alpha_u(T) - \alpha_u(S)||_{B(L^2(M,\tau))} = 0$$

then $R = S$. Denote $R = T - S$. Then $\alpha_u(R)$ is given by the formula

$$\langle \alpha_u(R)x, y\rangle_{L^2(M,\tau)} = \langle u^* R(ux), y\rangle_{L^2(M,\tau)} - \langle R(u), uyx^*\rangle$$

for all x, y in $L^2(M, \tau)$.

By (7.4) it follows that for any $\epsilon > 0$ there exists u in u in $\mathcal{U}(M)$ with

$$||u^* R(ux) - u^* R(u)x||_{L^2(M,\tau)} \leq \epsilon \; ||x||_2, \; x \text{ in } L^2(M, \tau).$$

Equivalently this means that for any ϵ there exists u with

$$||R(ux) - R(u)x||_{L^2} \leq \epsilon ||x||_{L^2}, \quad \text{for all } x \text{ in } M \cap L^2(M, \tau)$$

and consequently

$$||R(x) - (R(u)u^*)x||_{2, L^2(M,\tau)} \leq \epsilon ||x||_2$$

As $R(u)$ belongs to M (since we have shown this for all T_u), this shows that R belongs to the uniform norm closure in $\mathcal{L}(M, M)$ of the maps $\{L_a \mid a \in M\}$ where $L_a(x) = ax$ for x in M. But this uniform norm closure is $\{L_a \mid a \in M\}$ itself and hence R is of the form L_a for same a in M. As $R(1) = 0$ it follows that $R = 0$. Thus the Ryll-Nardjewski applies.

The relation $\psi(a,b,c) = \overline{\psi(b^*,a^*,c^*)}$ gives that

$$\psi(a,b,c) = \overline{\psi(b^*,a^*,c^*)} = \overline{\phi(b^*a^*,c^*)} - \overline{\phi(b^*,a^*c^*)} + \overline{\phi(c^*b^*,a^*)}.$$

Consequently, by using also the properties of ψ and writing $\psi(a,b,c) = \overline{\psi(b^*,a^*,c^*)} = 0$ we get

$$[\phi(ab,c) - \overline{\phi(b^*,a^*c^*)}] \cdot [\phi(a,bc) - \overline{\phi(c^*b^*,a^*)}] + [\phi(ca,b) - \overline{\phi(b^*,a^*c^*)}] = 0,$$

for all a,b,c in $M \cap L^2(M,\tau)$. By using the property $\phi(x,y) = -\phi(y,x)$, we get that

$$[\phi(ab,c) + \overline{\phi(c^*,b^*a^*)}] - [\phi(a,bc) + \overline{\phi(c^*b^*,a^*)} + [\phi(ca,b) - \overline{\phi(b^*,a^*c^*)}] = 0.$$

Now using the expression $\phi = \frac{\phi_1+\phi_2}{2}$ where $\phi_1(x,y) = \phi(x,y) + \overline{\phi(y^*,x^*)}$, it follows that there is no loss of generality when assuming the equation $\psi(a,b,c) = \phi(ab,c) - \phi(a,bc) + \phi(ca,b)$ to suppose that

$$\phi = \phi_2 = \phi(x,y) - \overline{\phi(y^*,x^*)}.$$

Hence $\phi(x,y) + \overline{\phi(y^*,x^*)} = 0,\ x,y$ in $L^2(M,\tau)$ which is the required condition for ϕ to be antisymmetric.

The only property that hasn't been checked is that $\chi(M_{sa}) \subseteq M_{sa}$ which is equivalent to

$$\chi(x^*) = \chi(x^*) \text{ for all } x \text{ in } M \cap L^2(M,\tau) \text{ or}$$

$$\langle \chi(x^*)y, z\rangle = \langle y, \chi(x)z\rangle \text{ or}$$

$$\langle \chi(x^*), zy^* \rangle = \langle yz^*, \chi(x) \rangle.$$

Denoting $b = zy^*$, this means that we want that

$$\langle \chi(x^*), b \rangle = \langle b^*, \chi(x^*) \rangle.$$

By the antisymmetry for χ this is

$$\langle \chi(x^* b), b \rangle = -\langle \chi(b^*), x \rangle$$

which is the same as $\phi(x^*, b^*) = -\phi(b^*, x^*)$.

This completes the proof of our theorem.

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