An Invariant for Subfactors in the von Neumann Algebra of a Free Group

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Dedicated to the Anniversary of Professor Richard V. Kadison

ABSTRACT In this paper we are considering a new invariant for subfactors in the von Neumann algebra $\mathcal{L}(F_k)$ of a free group. This invariant is obtained by computing the Connes' χ invariant for the enveloping von Neumann algebra of the iteration of the Jones' basic construction for the given inclusion. In the case of the subfactors considered in [Po2], [Ra1] this invariant is easily computed as a relative χ invariant, in the form considered in [Kaw].

As an application we show that, contrary to the case of finite group actions (or more general G-kernels) on the hyperfinite II_1 factor (settled in [Co1], [Oc2], [Jo3]), there exist non outer conjugate, injective homomorphisms (i.e two $\mathbb{Z}_2$-kernels) from $\mathbb{Z}_2$ into $\text{Out}(\mathcal{L}(F_k))$, with non-trivial obstruction to lifting to an action on $\mathcal{L}(F_k)$. Moreover, the algebraic invariants ([Co2]) do not distinguish between these two $\mathbb{Z}_2$-kernels. Also, there exists two non-outer conjugate, outer actions of $\mathbb{Z}_2$ on $\mathcal{L}(F_k)\otimes R$ that are neither almost inner or centrally trivial.

The aim of the present paper is to propose a new invariant for subfactors in the von Neumann algebra of a free group.

The invariant. *Given $A \subseteq B$ an inclusion of type II_1 factors, of finite index, we let B_∞ be the enveloping algebra for the tower of algebras in the (iterated) Jones's basic construction ([Jo1]) for $A \subseteq B$. Then*

$$\chi(B_\infty) \subseteq \mathit{Out}\ (B_\infty)$$

is a conjugacy invariant for $A \subseteq B$.

Here, for a type II_1 factor M, $\chi(M)$ denotes the Connes' invariant for M ([Co4]). The $\chi(M)$ invariant was introduced by Connes ([Co4]) in connection with

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a breakthrough construction that showed the existence of a type II_1 factor not anti-isomorphic to itself (thus giving a negative answer to Sakai's problem on the existence of type II_1 factors not coming from discrete groups).

We will show that for the subfactors in free group algebras, considered in [Ra1] (or for those considered in [Po2]), this invariant may be computed and it coincides with the relative form of the invariant χ ([Kaw]) for an associated pair of hyperfinite factors.

Using the full strength of this invariant (which comes, as in [Co3], by considering $\chi(B_\infty)$ as a subgroup in $\mathrm{Out}(B_\infty) = \mathrm{Aut}(B_\infty)/\mathrm{Int}(B_\infty)$, rather than as an abstract abelian group (see also [Jo2], [JS],[Su], [St])), we obtain, as an application, that the classification theory for finite group actions (and for G-kernels) on type II_1 factors, such as $\mathcal{L}(F_k)$(the von Neumann algebra of a free group F_k), is different from the corresponding classification theory on the hyperfinite II_1 factor ([Co1], [Co2],[Oc2],[Jo3]).

Theorem. *There exists two injective homomorphisms from $\mathbb{Z}_2$into $Out(\mathcal{L}(F_k))$, for any $k \in \mathbb{Z}, k \geq 2$, non-liftable to an action of $\mathbb{Z}_2$on $\mathcal{L}(F_k)$(i.e two $\mathbb{Z}_2$-kernels on $\mathcal{L}(F_k)$with non-trivial obstruction) that are not outer conjugate (in Out) and for which all the algebraic invariants ([Co1],[Jo3]) coincide.*

Explicitly, this means that there exist two (non-inner) automorphisms $\alpha_i, i = 1, 2$ on $\mathcal{L}(F_k)$, having non conjugate images (of order two) in Out $(\mathcal{L}(F_k))$. Moreover, both automorphisms are not liftable to actions of $\mathbb{Z}_2$ on $\mathcal{L}(F_k)$. The last statement means precisely that $\alpha_i^2 = Ad\ g_i$ and $\alpha_i(g_i) = -g_i$, where $g_i = g_i^+ - g_i^-$, with $g_i^\pm$ projections of trace 1/2 in $\mathcal{L}(F_k)$ (see [Co2],[Jo3]).

Note also that for actions of infinite cyclic groups on the von Neumann algebra of a free group with infinitely many generators, by [Ph], the actions obtained by multiplying the generators by complex numbers of modulus 1 are distinguished (modulo outer conjugacy) by the topology induced on $\mathbb{Z}$ from Aut $\mathcal{L}(F_\infty)$.

For free groups with finitely many generators or for finite cyclic groups the above arguments will not work. For example all $\mathbb{Z}_2$ actions on $\mathcal{L}(F_2)$ that are obtained by multiplying the generators with ± 1 or by switching the generators, are conjugate via elements in Aut $\mathcal{L}(F_2)$, which belong to the image of the $O(2)$ action on $\mathcal{L}(F_2)$, considered in [Vo2].

As a corollary we obtain that for the classification theory for subfactors in free group factors the usual invariants from the hyperfinite subfactors theory are not sufficient. That is, we may produce a pair of inclusions in $\mathcal{L}(F_N)$ of finite index, for which all the usual invariants are identical, but which, by looking at the invariant we have just introduced, are distinct.

Corollary. *There exist two non-conjugate, finite depth subfactors in $\mathcal{L}(F_k)$ having the same higher relative commutant invariants, and hence the same paragroups ([Oc1], [Jo1], [Po1]).*

The main technical ingredients used in proving this results are contained in the following statements:

Corollary (4). *Let $\mathcal{C} = (A \supseteq B \supseteq C; A \supseteq C \supseteq D)$ be an extremal commuting square ([Po1]) of finite dimensional algebras, which is λ-Markov ([Po1,We]), (i.e. there exist a λ-Markov trace, in the sense of Jones ([Jo1]) for $C \subseteq A$, which restricts to a λ-Markov trace for $D \subseteq B$). By ([Ra1]) the inclusion*

$$(\mathcal{L}(F_k) \otimes D) *_D C \subseteq (\mathcal{L}(F_k) \otimes B) *_B A$$

is a finite index inclusion of type II_1 factors (that are isomorphic to $\mathcal{L}(F_N)$ for some $N > 1$).

Let $\mathcal{A}_\infty$ be the enveloping von Neumann algebra in the tower of algebras in the iterated Jones's basic construction of the above inclusion. Let $B_\infty \subseteq A_\infty$ be the enveloping von Neumann algebras in the tower of algebras in the iterated Jones's basic construction for the inclusions $D \subseteq B$ and respectively $C \subseteq A$. Let $\chi(B_\infty, A_\infty)$ be the relative χ invariant ([Kaw]) for the inclusion of hyperfinite factors $B_\infty \subseteq A_\infty$. Then

$$\chi(\mathcal{A}_\infty) = \chi((\mathcal{L}(F_k) \otimes B_\infty) *_{B_\infty} A_\infty) \cong \chi(B_\infty, A_\infty).$$

Theorem (21). *Let $R_1 \subseteq R_0$ be the inclusion of type II_1 factors corresponding to the cross product by the action of $\mathbb{Z}_4$ corresponding to a $\mathbb{Z}_2$ kernel that can't be lifted to an action of $\mathbb{Z}_2$. We assume that the spectral projections, corresponding to the selfadjoint unitary realizing the obstruction to lifting, have trace $1/2$. Let $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$. Let N be the subfactor constructed in [Jo2], having the χ invariant isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

With the above notation, N is not isomorphic to $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$.

Theorem (22). *Let N be the factor constructed in [Jo2]. Then there exist subalgebras $A \subseteq B$ in N so that A is isomorphic to $\mathcal{L}(F_k)$, B is the cross product $A \rtimes_\theta \mathbb{Z}_4$, where θ is a $\mathbb{Z}_2$-kernel on A with non-trivial obstruction (to lifting). Moreover N is the enveloping algebra for the inclusion $A \subseteq B$.*

In a forthcoming paper we will carry out the analysis above to more general, finite abelian groups. It is plausible that an invariant of the type considered in this note could be used to settle Kadison's problem in [Ka] (see also Sakai's book, [Sa], Problem 4.4.44) on the isomorphism class of the algebras associated to free groups.

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0. Definitions

For u a unitary in M we denote by $\mathrm{Ad}_M u$ (or when M is clear from the context we use simply the notation $\mathrm{Ad}u$) the inner automorphism defined by $\mathrm{Ad}_M u(x) = uxu^*$. We let $\overline{\mathrm{Int}\ (M)}$ be the closure of the group of inner automorphisms of M, $\mathrm{Int}\ (M)$, in the pointwise, weak convergence topology on M. Let $\mathrm{Ct}\ (M)$ be the set of all automorphisms α of M that have the property that $||\alpha(x_n) - x_n||_2 \longrightarrow 0$, whenever $x_n, n \in \mathbb{Z}$ is a central sequence in M (i.e a bounded sequence in M with $||[x_n, y]||_2 \longrightarrow 0$ for any $y \in M$. Connes' invariant ([Co3]) is defined as follows:

$$\chi(M) = (\mathrm{Ct}\ (M) \cap \overline{\mathrm{Int}\ (M)})/\mathrm{Int}\ (M).$$

If M is a McDuff factor (i. e. $M \cong M \otimes R$, where R is the hyperfinite II_1 factor) then $\chi(M)$ is the center of the the group $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Int}(M)$. Let $N \subseteq M$ be an inclusion of finite, separable von Neumann algebras. We recall the definitions from [Kaw]. We let $\mathrm{Int}(N, M)$ be the set of all inner automorphisms of M that are implemented by unitaries in N. Let $\mathrm{Ct}(N, M)$ is the set of all automorphisms of M, leaving N invariant and acting asymptotically trivial on central sequences for M that are contained in N.

Let $N \subseteq M$ be an inclusion of separable, finite von Neumann algebras and let $\chi(N,M)$ be the relative χ invariant ([Kaw]) for the inclusion $N \subseteq M$:

$$\chi(N,M) = (\mathrm{Ct}(N,M) \cap \overline{\mathrm{Int}\ (N,M)})/\mathrm{Int}\ (N,M),$$

0. Outline of the proofs

The idea of our construction is as follows. Let N, M be finite von Neumann algebras so that the centers of N and M have trivial intersection. We will first use the technique, borrowed from Sakai's proof on the non-existence of non-trivial, central sequences in a free group factor, to show that for any amalgamated free products (of the type of those first considered in [Po2]) of the form $\mathcal{A} = \mathcal{L}(F_k) \otimes N) *_N M$, the central sequences in $\mathcal{A}$ are asymptotically contained in M.

By using this and the results from Jones's paper [Jo4], we are able to prove that the χ invariant $\chi(\mathcal{A})$ may be identified with the relative form of the χ invariant $\chi(N,M)$ considered in [Kaw]. Also in this correspondence, the obstruction to lifting for various elements in the χ groups are preserved.

Let $\mathcal{C} = (D \subseteq B \subseteq A; D \subseteq C \subseteq A)$ be an extremal commuting square with a λ Markov trace. Let $\mathcal{B}_\infty \subseteq \mathcal{A}_\infty$ be the index λ subfactor associated with the commuting square $\mathcal{C}$.

In [Ra1] we proved that

$$N_{\mathcal{C}} = (\mathcal{L}(F_k) \otimes D) *_D C \subseteq M_{\mathcal{C}} = (\mathcal{L}(F_k) \otimes B) *_B A,$$

is an index λ inclusion of free group factors, so that the enveloping von Neumann algebra of the iteration of the Jones's basic construction for the inclusion $N_{\mathcal{C}} \subseteq M_{\mathcal{C}}$ is

$$(\mathcal{L}(F_k) \otimes \mathcal{B}_\infty) *_{\mathcal{B}_\infty} \mathcal{A}_\infty.$$

By combining this with the result we presented above, it means that the χ invariant for the inclusion $N_{\mathcal{C}} \subseteq M_{\mathcal{C}}$ is

$$\chi(\mathcal{B}_\infty, \mathcal{A}_\infty).$$

We want to prove that this invariant is fine enough to distinguish between subfactors in free group factors, which have the same invariants coming from the higher relative commutants (i. e. the same paragroup [Oc1]).

Let θ be an outer $\mathbb{Z}_2$ kernel on the hyperfinite II_1 factor that can not be lifted to an action of $\mathbb{Z}_2$ on R. By the methods developed by Connes in [Co2] this means that $\theta^2 = \mathrm{Ad}_R g$ for some selfadjoint unitary g in R and that $\theta(g) = -g$. We will also assume (which is always possible by [Co2]) that the spectral projections of g have trace 1/2.

We will consider the commuting square $\mathcal{C}_\theta$, which by iteration of the Jones's basic construction yields an inclusion of the form $R \subseteq R \rtimes_\theta \mathbb{Z}_4$. Then it is easy to see that the inclusion $N_{\mathcal{C}_\theta} \subseteq M_{\mathcal{C}_\theta}$ associated with this subfactor is equivalent to an inclusion of the form $\mathcal{L}(F_k) \subseteq \mathcal{L}(F_k) \rtimes_{\theta'} \mathbb{Z}_4$, where θ' is a $\mathbb{Z}_2$ kernel on $\mathcal{L}(F_k)$ with exactly the same properties as θ. Moreover, the enveloping algebra for the inclusion $N_{\mathcal{C}_\theta} \subseteq M_{\mathcal{C}_\theta}$ is then isomorphic to

$$\mathcal{A} = (\mathcal{L}(F_k) \otimes R) *_R (R \rtimes_\theta \mathbb{Z}_4).$$

An explicit computation of the relative invariant $\chi(R, R \rtimes_\theta \mathbb{Z}_4)$ allows to prove that $\chi(\mathcal{A})$ is, as an abstract group, isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, for exactly one copy of $\mathbb{Z}_2$ in $\chi(\mathcal{A})$, which we will denote by ϕ, there is no obstruction to lifting it to a morphism of $\mathbb{Z}_2$ into $\mathrm{Aut}(\mathcal{A})$.

We analyze the factor $\mathcal{B} = \mathcal{A} \rtimes_\phi \mathbb{Z}_2$ and show that the dual action $\hat{\phi}$ is decomposed as $\phi = s_1 s_2$, where s_1 is in $\overline{\mathrm{Int}(\mathcal{B})}$, while s_2 is in $\mathrm{Ct}(\mathcal{B})$. Moreover, for a selfadjoint unitary h in $\mathcal{B}$ we have that

$$(*) \qquad s_i^2 = \mathrm{Ad}_{\mathcal{B}} h, \qquad s_i(h) = -h, i = 1, 2.$$

Now we consider the subfactor N introduced in [Jo2] by analogy with ([Co5]). Then N has also the property that that $\chi(N)$ is, as an abstract group, isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, for exactly one copy of $\mathbb{Z}_2$ in $\chi(N)$, which we will denote by γ, there is no obstruction to lifting it to morphism of $\mathbb{Z}_2$ into $\mathrm{Aut}(N)$.

We consider (following [Co4]) the factor $N \rtimes_\gamma \mathbb{Z}_2$ and show that here the dual action $\tilde{\gamma}$ on the cross product is also decomposed as $\tilde{\gamma} = s_1 s_2$, where s_1 belongs to $\overline{\mathrm{Int}(N \rtimes_\gamma \mathbb{Z}_2)}$, while s_2 belongs to $\mathrm{Ct}(N \rtimes_\gamma \mathbb{Z}_2)$. This time there exist two selfadjoint unitaries f_1, f_2 in $N \rtimes_\gamma \mathbb{Z}_2$ so that

$$(**) \qquad t_i^2 = \mathrm{Ad}_{[N \rtimes_\gamma \mathbb{Z}_2]} f_i, \qquad t_i(f_j) = (-1)^{i-j+1} f_j, \qquad i, j = 1, 2.$$

The relations $(*)$ and $(**)$, by the theory of invariants for finite group actions on type II_1 factors ([Jo3]), prevents one to obtain t_1, t_2, by a unitary cocycle perturbation, from s_1, s_2.

Since the algebras $\mathcal{B}$, $N \rtimes_\gamma \mathbb{Z}_2$ are canonically associated with $\mathcal{A}$ and N, as are s_1, s_2 and, respectively t_1, t_2, this shows that the factors N, $\mathcal{A}$ can't be isomorphic.

Moreover, by using Voiculescu's random matrix picture, the factor N is the enveloping von Neumann algebra in the iteration of the Jones's basic construction of an inclusion of the form $\mathcal{L}(F_k) \subseteq \mathcal{L}(F_k) \rtimes_{\theta''} \mathbb{Z}_4$, where θ'' is a $\mathbb{Z}_2$ kernel on $\mathcal{L}(F_k)$ with similar properties as θ'.

The fact that the algebras $\mathcal{A}$ and N are not isomorphic shows exactly that θ' and θ'' are not cocycle conjugate (although the two kernels have exactly the same algebraic invariants considered in [Jo3]).

1. Computations of the invariant for some subfactors of free group factors

The following lemma is essentially contained in Lemma 4.3.3 in [Sa] (which proves the absence of non-trivial central sequences in free group factors).

Lemma 1. *Let $N \subseteq M$ be a pair of separable, finite von Neumann algebras. Assume that N, M have centers with trivial intersection. Let $\mathcal{A}$ be the von Neumann algebra reduced free product*

$$\mathcal{A} = (\mathcal{L}(F_k) \otimes N) *_N M,$$

which by [Po2] is a type II_1 factor.Let $g_i, i = 1, 2$ are two of the k generators in F_k and let $||x||^2 = \tau(x^*x)$, $x \in \mathcal{A}$ where τ is the trace on $\mathcal{A}$. Let $E_N^{\mathcal{A}}$ be the normal, faithful, conditional expectation from $\mathcal{A}$ onto N.

Then $\mathcal{A}$ has the property that

$$||E_N^{\mathcal{A}}(y) - y||_2 \leq 14 \max_{i=1,2} ||[g_i^{\pm 1}, y]||_2.$$

In particular all central sequences in $\mathcal{A}$ are asymptotically contained in N.

Proof. We will follow closely the proof of Lemma 4.3.3 in the book of Sakai. For the sake of completeness we will repeat the argument by using our notations.

According to Chapter 3 in [Po2] any element x in the amalgamated free product $(\mathcal{L}(F_k)\otimes N) *_N M$ is a sum of elements of the form

$$m_0 w_1 m_1 w_2 ... w_N m_l, l \in \mathbb{N}, w_i \in F_k \backslash \{e\}, i = 1, 2, ..., l,$$

where $m_1, m_2, ..., m_{l-1}$ are elements in $M \ominus N$, while m_0, m_l are either 1 or in N. Moreover one of the terms in the sum may be in N. The different choices (for the number of indices or for the elements in F_k) will give orthogonal terms with respect to the canonical trace ([Po2], Prop. 3.1) on $(\mathcal{L}(F_k)\otimes N) *_N M$. Hence the above sum is an orthogonal sum in the Hilbert space associated with $(\mathcal{L}(F_k)\otimes N) *_N M$. Let $\mathcal{S}$ be the index set for the above sum (that is for different elements in $\mathcal{S}$ we get orthogonal elements). Denote by 1 the element corresponding to N in $\mathcal{S}$. Note that multiplying elements in $\mathcal{S}$ with elements in F_k, to the left or to the right, is a well defined operation on $\mathcal{S}$.

Fix an element y in $(\mathcal{L}(F_k)\otimes N) *_N M$. For s in $\mathcal{S}$ denote by y_s the component of y belonging to the subspace indexed by s. For a subset F in $\mathcal{S}$, let $\nu(F)$ be the sum of the squares of the Hilbert norms of the components of y that belong to F (e.g. $\sum_{s\in F} ||y_s||_\tau^2$). In particular $||\nu(\mathcal{S}\backslash 1)||_\tau^2$ is equal, according to the above definition to $C^2 = ||y - E_N(y)||^2$.

We let F be the subset of all elements in the index set $\mathcal{S}$ that correspond to elements that end in a nontrivial power of g_1. Then we will have that $\mathcal{S} = \cup g_1^{-1} F g_1 \cup F \cup \{1\}$ and moreover the sets F, $g_2^{-1} F g_2$, $g_2 F g_2^{-1}$ are disjoint in $\mathcal{S}\backslash 1$.

As in Lemma 4.3.3 in ([Sa]), fix

$$\epsilon^2 > \max_{i=1,2} \{ \sum_{s\in\mathcal{S}} ||(y_{g_i^{\pm 1} s g_i^{\mp 1}} - y_s)||_\tau^2 \}$$

$$= \max_{i=1,2} \{ ||y - g_i^{\pm 1} y g_i^{\mp 1}||_\tau^2 \}.$$

Then

$$|\nu(F) - \nu(g_i F g_i^{-1})| = |\nu(F)^{1/2} - \nu(g_i F g_i^{-1})^{1/2}||\nu(F)^{1/2} + \nu(g_i F g_i^{-1})^{1/2}|.$$

The second factor on the left hand side of the previous equality is clearly less than $2C$.

For the first factor, we note that

$$\epsilon > [\sum_{s\in\mathcal{S}} ||(y_{g_i s g_i^{-1}} - y_s)||_\tau^2]^{1/2}$$

$$= ||\sum_{s\in F}(y_{g_i s g_i^{-1}} - y_s)||_\tau^2$$

$$\geq |\ ||\sum_{s\in F} y_{g_i s g_i^{-1}}||_\tau - ||\sum_{s\in F} y_s||_\tau|$$

$$= |(\sum_{s\in F} ||y_{g_i s g_i^{-1}}||_\tau^2)^{1/2} - (\sum_{s\in F} ||y_s||_\tau^2)^{1/2}| = |\nu(g_i F g_i^{-1})^{1/2} - \nu(F)^{1/2}|.$$

Similarly

$$\epsilon > |\nu(g_i^{-1} F g_i)^{1/2} - \nu(F)^{1/2}|.$$

Thus,

$$|\nu(F) - \nu(g_i F g_i^{-1})| < 2C\epsilon.$$

Hence $\nu(g_i F g_i^{-1}) < \nu(F) + 2C\epsilon$. Consequently,

$$C^2 \leq \nu(g_i F g_i^{-1}) + \nu(F) \leq 2(\nu(F) + s\epsilon),$$

and hence $\nu(F) > \frac{C^2}{2} - C\epsilon$.

Since, $g_2 F g_2^{-1}$, $g_2^{-1} F g_2$ and F are disjoint subsets of $\mathcal{S}\backslash\{1\}$ we get that

$$C^2 \geq \nu(g_2^{-1} F g_2) + \nu(g_2 F g_2^{-1}) + \nu(F) > C^2 - 7C\epsilon.$$

This, by ([Sa]), completes the proof.

In the next proposition we use the methods in ([Jo5]) to prove that the Connes's χ invariant may be computed for certain amalgamated, free products of von Neumann algebras. In particular the invariant will be non-zero for some of this algebras.

Proposition 2. *Let $N \subseteq M$ be a pair of separable, hyperfinite von Neumann algebras. Assume that the centers of N, M have trivial intersection. Let*

$$\mathcal{A} = (\mathcal{L}(F_k) \otimes N) *_N M.$$

By ([Po2]) the von Neumann algebra $\mathcal{A}$ is a type II_1 factor. Then, the Connes's invariant $\chi(\mathcal{A})$ is determined by the initial inclusion $N \subseteq M$.

The χ invariant may be explicitly computed as $\chi((\mathcal{L}(F_k) \otimes N) *_N M) \cong \chi(N, M)$. Moreover in this identification, the corresponding elements in $\chi(\mathcal{A})$ and $\chi(N, M)$ have the same obstructions to lifting from the quotient.

Note that if the inclusion $N \subseteq M$ is stable (i. e. isomorphic to $N \otimes R \subseteq M \otimes R$), then $\mathcal{A}$ is a McDuff factor (i. e. $\mathcal{A} \cong \mathcal{A} \otimes R$).

The proof will is a clear consequence of the construction in the following proposition:

Proposition 3. *Let $N \subseteq M$ be a pair of separable, hyperfinite von Neumann algebras. Assume that the centers of N, M have trivial intersection. Let $\mathcal{A} = (\mathcal{L}(F_k) \otimes N) *_N M$. Let $\Phi : Aut(N, M) \to Aut(\mathcal{A})$ be the application which to every automorphism α of M, leaving N invariant, associates the automorphism*

$$\Phi(\alpha) = (Id_{\mathcal{L}(F_k)} \otimes \alpha_N) *_N \alpha,$$

(the construction of $\Phi(\alpha)$ is based on Proposition 3.4 in [Po2]).

Then Φ has the following properties

$$\Phi(Int(N, M)) \subseteq Int(\mathcal{A}), \quad \Phi(Ct(N, M) \subseteq \ Ct(\mathcal{A})$$

and Φ is continuous in the strong, pointwise convergence topology. Consequently Φ induces a map $\hat{\Phi}$ from $\chi(M, N)$ into $\chi(\mathcal{A})$. Moreover, $\hat{\Phi}$ is a surjective isomorphism.

Proof. Let u be any unitary in M, leaving N invariant. Since N commutes in $\mathcal{A}$ with the image of $\mathcal{L}(F_k)$ it follows that $\Phi(\mathrm{Ad}_M(u)) = \mathrm{Ad}_{\mathcal{A}}(u)$. Hence, $\Phi(\mathrm{Int}(N, M)) \subseteq \mathrm{Int}(\mathcal{A})$.

Moreover, because of the previous Lemma, any central sequence in $\mathcal{A}$ is asymptotically contained in N and hence $\Phi(\alpha)$ belongs to $\mathrm{Ct}(\mathcal{A})$ if and only if α belongs to $\mathrm{Ct}(M, N)$. Moreover, it is obvious that Φ is continuous in the strongly, pointwise convergence topology. Hence Φ induces indeed a map $\hat{\Phi}$ from

$$(\mathrm{Ct}(N, M) \cap \overline{\mathrm{Int}\ (N, M)})/\mathrm{Int}\ (N, M) \ \text{ into } \ \frac{\mathrm{Ct}(\mathcal{A}) \cap \overline{\mathrm{Int}(\mathcal{A})}}{\mathrm{Int}(\mathcal{A})}.$$

We check first the surjectivity for $\hat{\Phi}$. Let α belong to $\overline{\mathrm{Int}(\mathcal{A})}$. We recall the following lemma due to Jones

Lemma. *(Lemma 3, [Jo5]) Suppose $A \subseteq M$ is a subalgebra (of a finite von Neumann algebra M with a faithful state) with the property that any central sequence $\{x_n\}$ satisfies $||E_A(x_n) - x_n||_2 \to 0$ (E_A is the conditional expectation onto A). Then if $\alpha \in \overline{Int(M)}$, there exists a unitary $w \in M$ and a sequence z_n of unitaries in A such that $\alpha = Ad(w) \lim_{n\to\infty} Ad(z_n)$.*

By this lemma (and by using the preceding Lemma 1), we find that there exist unitaries w in $\mathcal{A}$ and u_n in N so that

$$\alpha = \mathrm{Ad}_{\mathcal{A}} w \lim_{n\to\infty} \mathrm{Ad}_{\mathcal{A}}(u_n).$$

But then, by the continuity of Φ, it follows that

$$\alpha = \mathrm{Ad}_{\mathcal{A}} w \lim_{n\to\infty} \Phi(\mathrm{Ad}_N(u_n)).$$

Thus Φ maps $\mathrm{Int}(M, N)$ onto $\overline{\mathrm{Int}(\mathcal{A})}$ (modulo $\mathrm{Int}(\mathcal{A})$). Assume that α also belongs to $\mathrm{Ct}(\mathcal{A})$. As $\mathrm{Ad}_{\mathcal{A}}(w)$ always belongs to $\mathrm{Ct}(\mathcal{A})$, it follows that $\lim_{n\to\infty} \mathrm{Ad}_{\mathcal{A}}(u_n)$ belongs to $\mathrm{Ct}(\mathcal{A})$ and hence, by Lemma 1, it follows that $\lim_{n\to\infty} \mathrm{Ad}_N(u_n)$ is in $\mathrm{Ct}(N, M)$.

It remains to show that $\hat{\phi}$ is injective. Thus assume that for some α in $\mathrm{Ct}(N, M)$ we have that $\Phi(\alpha) = Ad_{\mathcal{A}}(w)$, for some unitary in $\mathcal{A}$.

Since by construction $\Phi(\alpha)$ acts identically on N, it follows that w belongs to the relative commutant of $\mathcal{L}(F_k)$ and thus (by [Po2], Theorem 4.1) that $w \in N$, i.e that $\alpha = \mathrm{Ad}_M(w)$.

Corollary 4. *Let $\mathcal{C} = (A \supseteq B \supseteq C; A \supseteq C \supseteq D)$ be an extremal commuting square ([Po1]) of finite dimensional algebras, which is λ-Markov ([Po1,We]), (i.e. there exists a λ-Markov trace, in the sense of Jones ([Jo1]) for $C \subseteq A$, which restricts to a λ-Markov trace for $D \subseteq B$). By ([Ra1]) the inclusion*

$$(\mathcal{L}(F_k) \otimes D) *_D C \subseteq (\mathcal{L}(F_k) \otimes B) *_B A$$

is a finite index inclusion of type II_1 factors (that are isomorphic to $\mathcal{L}(F_N)$ for some $N > 1$).

Let $\mathcal{A}_\infty$ be the enveloping von Neumann algebra in the tower of algebras in the iterated Jones's basic construction of the above inclusion. Let $B_\infty \subseteq A_\infty$ be the enveloping von Neumann algebras in the tower of algebras in the iterated Jones's basic construction for the inclusions $D \subseteq B$ and respectively $C \subseteq A$. Note (by [Ra1]) that $\mathcal{A}_\infty$ is isomorphic to $(\mathcal{L}(F_k) \otimes B_\infty) *_{B_\infty} A_\infty$. Then

$$\chi(\mathcal{A}_\infty) = \chi((\mathcal{L}(F_k) \otimes B_\infty) *_{B_\infty} A_\infty) \cong \chi(B_\infty, A_\infty).$$

2. Computation of $\chi(R, R \rtimes_\theta \mathbb{Z}_4)$ for a $\mathbb{Z}_2$ kernel θ in Out with nontrivial obstruction to lifting to Aut

Let R be the hyperfinite, type II_1 factor and let θ be an action of $\mathbb{Z}_4$ into R. We will assume that θ defines a $\mathbb{Z}_2$-kernel, i.e that θ defines a homomorphism from $\mathbb{Z}_2$ into $\mathrm{Out}(R)$. Also we will assume that this $\mathbb{Z}_2$-kernel can not be lifted to a morphism from $\mathbb{Z}_2$ into $\mathrm{Aut}(R)$. By ([Co1]), the fact that θ can't be lifted to a morphism into $\mathrm{Aut}(R)$ is (uniquely) determined by the following obstruction:

Let g be the selfadjoint unitary (i.e $g^2 = 1$) defined by $\theta^2 = \mathrm{Ad}_R g$. The fact that such an unitary exists is warranted by the fact that θ defines an action of $\mathbb{Z}_2$ into Out. It is easy to observe (see [Co1]) that there exists an order 2 root of the unity α so that $\theta(g) = \alpha g$. An obstruction to lifting will exists if and only if $\alpha = -1$, that is if $\theta(g) = -g$.

We will also assume that θ has the property that the spectral projections of the selfadjoint unitary g have trace 1/2 (in general, again by [Co1], the value of the traces of the spectral projections of g is an invariant for conjugacy in $\mathrm{Aut}(R)$).

In this paragraph we will compute the relative invariant χ for an inclusion of hyperfinite, type II_1 factors having the form

$$R \subseteq R \rtimes_\theta \mathbb{Z}_4,$$

with θ as above.

In particular we will obtain that $\chi(R \rtimes_\theta \mathbb{Z}_4, R)$ is, (an abstract group), isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, exactly one of the three copies of $\mathbb{Z}_2$ has no obstruction to lifting from the quotient. This is to be compared with the case of a $\mathbb{Z}_2$ kernel with no obstruction to lifting. In this case, (by [Kaw]), for an order 2 outer action α on R, one has that $\chi(R, R \rtimes_\alpha \mathbb{Z}_2)$ is also $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We will then apply this result for the commuting square ([Po1], [GHJ], [Oc1]) generating the inclusion $R \subseteq R \rtimes_\theta \mathbb{Z}_4$. By considering the inclusion of free group factors associated with the commuting square as in Corollary 4, we will deduce that the enveloping algebra $\mathcal{A}$ for this inclusion has also the property that

$$\chi(\mathcal{A}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

as abstract groups. Moreover, as a subgroup of $\mathrm{Out}(\mathcal{A})$ exactly one of the two copies of $\mathbb{Z}_2$ has no obstruction to lifting to a morphism of $\mathbb{Z}_2$ into the automorphism group of $\mathcal{A}$.

Because we will need a very concrete knowledge of the structure of the inclusion $\chi(\mathcal{A}) \subseteq \mathrm{Out}(\mathcal{A})$, we will introduce a special model (similar the model in [Jo4]) for the (finite depth) subfactor $R \subseteq R \rtimes_\theta \mathbb{Z}_4$.

This model will also be used in the last paragraph of this paper, when realizing the Jones's example ([Jo5]) as an enveloping algebra.

The realization of this model is outlined in the following lemma. Some parts of the arguments are probably well known in the literature of subfactors. We include them here for the sake of completeness.

Lemma 5. *Let $(U_k)_{k\in\mathbb{Z}}$ is a family of unitaries with four point spectrum so that each spectral projection for U_k has trace $1/4$ and let $g = g_+ - g_-$ be a selfadjoint unitary whose spectral projections $g_\pm$ have trace 1/2. We assume the following relations:*

$$U_k^4 = 1,\ k \in \mathbb{Z}, \quad U_k g U_k^* = -g, \quad \textit{if } k = 0, -1, \quad U_k g U_k^* = g,\ \textit{if } k \in \mathbb{Z}/\{0, -1\}$$

and

$$U_k U_{k+1} U_k^* = \sqrt{-1} U_{k+1}, \quad k \in \mathbb{Z}.$$

The trace on this algebra generated by these unitaries is specified by requiring that each non-trivial monomial in the U_k's and g have zero trace).

Let (see [Jo3])

$$R_{-1} = \{gU_0^2, U_1, U_2, ...\}'' \subseteq \{g, U_0, U_1, ..., \}'' = R_0.$$

Then $R_{-1} \subseteq R_0$ is an inclusion of type II_1 factors, of index 4 and the relative commutant is generated by the (order 2) unitary g.

In addition, U_0 is normalizing R_{-1} (i. e. $Ad_{R_{-1}}(U_0)$ is a (non-inner) automorphism of R_{-1}). If $\theta = Ad_{R_{-1}}(U_0)$, then $\theta^4 = 1$ and R_0 is the cross product $R_{-1} \rtimes_\theta \mathbb{Z}_4$. Clearly

$$\theta^2 = Ad_{R_{-1}}(U_0^2) = Ad_{R_{-1}}(U_0^2 g)$$

and $h = U_0^2 g$ is a selfadjoint unitary in R_{-1} (with spectral projections of trace $1/2$) with $\theta(h) = -h$. Hence, θ defines a $\mathbb{Z}_2$ kernel into $Out(R_{-1})$ with nontrivial obstruction to lifting.

Moreover the k-th step in the iterated basic construction for this construction is:

$$R_k = \{gU_{-1}^2 ... U_{-k}^2, U_{-k}, U_{-(k-1)}, ...\}'', \quad k \geq 1.$$

Proof. Clearly θ as defined in the statement normalizes R_{-1} by the properties of the family $(U_k)_{k\in\mathbb{Z}}$ and g. Moreover g commutes with R_{-1} (and generates the relative commutant of R_{-1} in R_0) so that $\mathrm{Ad}_{R_{-1}}(U_0^2) = \mathrm{Ad}_{R_{-1}}(U_0^2 g)$. Hence, at least algebraically, $\mathrm{Ad}_{R_{-1}}(U_0)$ implements the cross product $R_{-1} \rtimes_\theta \mathbb{Z}_4$. The properties of the trace, on the family of all $(U_k)_{k\in\mathbb{Z}}$ and g, prove that this is valid when one considers also the topology on the algebraic cross product.

Also, since $\mathrm{Ad}_{R_0}(U_{-1})$ acts identically on $U_0^2 g$ and on $U_1, U_2, ...$ while $\mathrm{Ad}_{R_0}(U_{-1})$ multiplies U_0 by $\sqrt{-1}$, it follows that $\mathrm{Ad}_{R_0}(U_{-1})$ implements the dual action of $\mathbb{Z}_4$ on the cross product $R_{-1} \rtimes_\theta \mathbb{Z}_4$.

Hence the next step in the iterated basic construction of the inclusion $R_{-1} \subseteq R_0$ is

$$R_1 = \{U_{-1}, g, U_0, U_1, ...\}'' = \{gU_{-1}^2, U_{-1}, U_0, U_1, ...\}''.$$

The next steps in the basic construction are obtained by adjoining consecutively the unitaries $U_{-2}, U_{-3}, ...$

By using the above description for R_1 the notation becomes homogeneous, in the sense that gU_{-1}^2 is a selfadjoint unitary that anti-commutes with U_{-1} and with U_{-2}. Thus adjoining the unitary U_{-2} to the algebra R_1 we obtain the next step in the basic construction.

To construct a family of unitaries with the properties in the statement one has simply look for the unitaries implementing the cross products in the iterated basic steps of an inclusion of the form $R \subseteq R \rtimes_\beta \mathbb{Z}_4$, where β is an outer kernel of $\mathbb{Z}_2$ (with obstruction to lifting: $\beta^2 = \mathrm{Ad} v$, v a selfadjoint unitary with spectral projections of trace $1/2$). The unitary g of order 2 comes from the first relative commutant.

Lemma 6. *With the above notations, a downward tunnel ([Po1]) for $R_{-1} \subseteq R_0$ is defined as follows*

$$R_{-2} = \{gU_0^2U_1^2, U_2, U_3, ...\},$$

$$R_{-3} = \{gU_0^2U_1^2U_2^2, U_3, U_4, ...\}$$

$$R_{-k} = \{gU_0^2U_1^2...U_{k-1}^2, U_k, U_{k+1}, ...\}.$$

Proof. This in fact clear from the previous step as one can show directly, by the same method, that R_{-1} is the first basic construction step for the inclusion $R_{-3} \subseteq R_{-2}$.

Lemma 7. *With the above notations, a commuting square $\mathcal{C}_\theta = (D \subseteq B \subseteq A;\quad D \subseteq C \subseteq A)$, generating, by iteration of the basic construction for the inclusions $D \subseteq B$ and $C \subseteq A$ the inclusion $R_{-1} \subseteq R_0$ is obtained (by using the notations in Lemma 6) as follows:*

$$D = \{g\}'' \subseteq C = \{gU_{-1}^2, U_{-1}\}''; B = \{gU_0^2, U_0\}'' \subseteq A = \{gU_0^2U_{-1}^2, U_{-1}, U_0\}''.$$

Moreover $U_0 \in B$ normalizes A and $Ad_C U_0^2 = Ad_C g$. If we iterate the basic construction for the inclusions $D \subseteq B$ and $C \subseteq A$ we get the algebras B_k, A_k that are computed explicitly as follows:

$$B_k = \{gU_0^2U_1^2...U_k^2, U_k, U_{k-1}, ..., U_0\}'' \subseteq A_k = \{gU_{-1}^2U_0^2...U_k^2, U_{-1}, U_{k-1}, ..., U_0\}''.$$

Hence $B_k = R'_{-k-2} \cap R_0 \subseteq A_k = R'_{-k-2} \cap R_1$. In particular the inclusion $B_\infty \subseteq A_\infty$ is

$$\{g, U_0, U_1, ...\}'' \subseteq \{gU_{-1}^2, U_{-1}, U_0, U_1, ...\}'',$$

that is $R_0 \subseteq R_1$.

Proof. We let

$$D = R'_{-1} \cap R_0 \subseteq C = R'_{-1} \cap R_1,$$

$$B_0 = B = R'_{-2} \cap R_0 \subseteq A_0 = A = R'_{-2} \cap R_1,$$

and in general we have

$$B_k = R'_{k+2} \cap R_0 \subseteq A_k = R'_{k+2} \cap R_1.$$

Recall that

$$R_1 = \{gU_{-1}^2, U_{-1}, U_0, U_1, ...\}'',$$

$$R_0 = \{g, U_0, U1, ...\}'',$$

$$R_{-1} = \{gU_0^2, U_1, U_2, ..\}'',$$

$$R_{-2} = \{gU_0^2 U_1^2, U_2, U_3, ...\}''.$$

$$R_{-3} = \{gU_0^2 U_1^2 U_3^2, U_4, ...\}''.$$

An easy computation (by using the properties of the family $(U_k)_{k\in\mathbb{Z}}$ and g gives the result. This also shows that the commuting square $(D \subseteq B \subseteq A; D \subseteq C \subseteq A)$ verifies the condition to be an extremal commuting square as considered in [Po1], and thus that we may use the theorem in [Ra1]. This ends the proof.

In the next lemma we show that if the commuting square $\mathcal{C}$ corresponds to the commuting square $\mathcal{C}_\theta$, then the inclusion obtained by the taking the amalgamated free products will also be associated to a cross product.

Lemma 8. *Let $\mathcal{C}_\theta = (D \subseteq B \subseteq A; D \subseteq C \subseteq A)$ be defined as in the above lemma. The the free group factor $(\mathcal{L}(F_k) \otimes B) *_B A$ is isomorphic to the cross product of $(\mathcal{L}(F_k) \otimes D) *_D C$ by the $\mathbb{Z}_4$ action on $(\mathcal{L}(F_k) \otimes D) *_D C$ induced by $\theta = Ad_{[(\mathcal{L}(F_k)\otimes D)*_D C]} U_0$. Note that this action is in fact an injective $\mathbb{Z}_2$-kernel on $(\mathcal{L}(F_k) \otimes D) *_D C$ with obstruction -1 to lifting (to Aut)), as $\theta^2 = Ad\ g$ and g is a selfadjoint unitary in $(\mathcal{L}(F_k) \otimes D) *_D C$ whose spectral projections have trace $1/2$.*

Moreover, by preceding lemma 3, the enveloping algebra for the above inclusion is $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0 \cong (\mathcal{L}(F_k) \otimes R_0) *_{R_0} R_1$.

Proof. Denote for simplicity $Q = \mathcal{L}(F_k)$. We observe that the unitary U_0 in B has the property that $U_0 C U_0^*$ is C. Since $[U_0, Q] = 0$ it follows that that U_0 belongs to the normalizer $\mathcal{N}_{[(Q\otimes B)*_B A]}((Q \otimes D) *_D C)$ of $(Q \otimes D) *_D C$ in $(Q \otimes B) *_B A$. Moreover, $\mathrm{Ad}_C(U_0^2) = \mathrm{Ad}_C(g)$ as

$$[\mathrm{Ad}_C(U_0^2)](U_{-1}) = -U_{-1} = [\mathrm{Ad}_C(g)](U_{-1}),$$

$$\mathrm{Ad}_C(U_0^2)(g) = g = \mathrm{Ad}_C(g)(g).$$

Moreover, as $[g, Q] = 0$. it follows that the following equality also holds true:

$$\mathrm{Ad}_{[(Q\otimes D)*_D C]} U_0^2 = \mathrm{Ad}_{[(Q\otimes D)*_D C)]} g.$$

Finally, it is clear that

$$\{U_0, (Q \otimes D) *_D C\}'' = (Q \otimes B) *_B A.$$

Moreover, any monomial in $(Q \otimes B) *_B A$ may be written by using only one power of U_0. To finish the proof that θ implements the cross product we have to check that any monomial m involving a non-zero power of U_0 will have zero trace with respect to the canonical trace on $(Q \otimes B) *_B A$.

Say

$$m = q_0 c_0 q_1 c_1 ... q_i (c_i U_0) q_{i+1} c_{i+1} ... q_n c_n q_{n+1},$$

where q_i is in Q, q_i is not a multiple of the identity and c_i is in C.

Then by the definition of the trace on $(Q \otimes B) *_B A$, the trace of m is a a sum of products of the form

$$\tau(c_1')\tau(c_2')...\tau(c_{k-1}')\tau(c_k' U_0)\tau(c_{k+1}')....\tau(c_p'),$$

which is zero.

This completes the proof.

In the next lemma we compute the relative invariant $\chi(R_0, R_{-1})$.

Lemma 9. *(compare with[Oc1], [Kaw], [Lo], [Ch], [Ko]) Let θ a $\mathbb{Z}_4$ action on a copy R_{-1} of the hyperfinite II_1 factor so that θ induces a $\mathbb{Z}_2$-kernel on R_{-1}, with obstruction -1 ([Co1]). Let $R_0 = R_{-1} \rtimes_\theta \mathbb{Z}_4$. Then $\chi(R_{-1}, R_0) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Moreover, for exactly one of the two copies of $\mathbb{Z}_2$ in $\chi(R_{-1}, R_0)$, there exists no obstruction to lifting from the quotient.

Proof. In fact we will identify precisely also the generators of $\chi(R_{-1}, R_0) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By using the terminology used in the above lemmas, we have

$$R_{-1} = \{gU_0^2, U_1, U_2, ...\}'' \subseteq \{g, U_0, U_1, ..., \}'' = R_0.$$

Let $g = g_+ - g_-$, with $g_\pm$ projections of trace $1/2$. Then, for all complex numbers α, β of absolute value 1, the following automorphisms $\mathrm{Ad}_{R_0}(U_{-1}^i)$,$\mathrm{Ad}_{R_0}(U_0^j)$, $\mathrm{Ad}_{R_0}(\alpha g_+ + \beta g_-)$ belong to $\mathrm{Ct}(R_{-1}, R_0)$. Moreover $\mathrm{Ad}_{R_0}(U_{-1}^2) = \mathrm{Ad}_{R_0}(g)$ and hence

(1) $$\mathrm{Ad}_{R_0}(U_{-1}^2)\mathrm{Ad}_{R_0}(U_0^2) = \mathrm{Ad}_{R_0}(gU_0^2) \in \mathrm{Int}(R_{-1}, R_0).$$

Moreover

(2) $$[\mathrm{Ad}_{R_0}(U_{-1}), \mathrm{Ad}_{R_0}(U_0)] = 0$$

and

(3) $$\mathrm{Ad}_{R_0}(U_{-1}U_0)^2 = \mathrm{Ad}_{R_0}(gU_0^2),$$

while

(4) $$\mathrm{Ad}_{R_0}(U_{-1}U_0)(gU_0^2) = -gU_0^2,$$

(since $\mathrm{Ad}_{R_0}(U_{-1}U_0)(g) = g$).

The proof of Lemma 9 is split into the steps contained in the statements (10-13) bellow.

We will first determine the asymptotic inclusion ([Oc2],[Co3]) for the inclusion $R_{-1} \subseteq R_0$.

Lemma 10. *([Oc1]) With the notations in [Oc2], [Co2], [MD] and by considering the factors $R_{-1} \subseteq R_0$ from the previous lemma, we have that*

$$(R_{-1})^\omega \cap R_0' \subseteq (R_0)^\omega \cap R_0',$$

is an inclusion of type II_1 factors of index 2.

Let $\alpha = Ad_{R_0}(U_{-1})$. Then $(R_{-1})^\omega \cap R_0'$ is the fixed point of the (order 2) action of $(\alpha)^\omega = ((Ad_{R_0}(U_{-1}))^\omega$, restricted to $(R_0)^\omega \cap R_0'$.

In particular, by [GHJ], the only non-trivial automorphism of $(R_0)^\omega \cap R_0'$ which is the identity on $(R_{-1})^\omega \cap R_0'$ is $(\alpha)^\omega$.

Proof (of the lemma). We clearly have that $(\mathrm{Ad}_{R_0}(U_{-1}))^2 = \mathrm{Ad}_{R_0}(g)$ and since $\mathrm{Ad}_{R_0}(g)$ is in $\mathrm{Ct}(R_0)$ as g is in R_0 it follows (by [Co2]) that $(\alpha)^\omega = ((\mathrm{Ad}_{R_0}(U_{-1}))^2)^\omega$ acts as the identity on $(R_0)^\omega \cap R_0'$. Moreover, since R_{-1} is the fixed point algebra of the automorphism $\mathrm{Ad}_{R_0}(U_{-1})$, it follows that $(R_{-1})^\omega \cap R_0'$ is the fixed point algebra in $(R_0)^\omega \cap R_0'$ of α^ω. Clearly $((\mathrm{Ad}_{R_0}(U_{-1}))^\omega$ is non-trivial (since otherwise, by [Co2]) it would follow that $\mathrm{Ad}_{R_0}(U_{-1})$ belongs to $\mathrm{Int}(R_0)$, which is not the case).

This ends the proof of the lemma.

Corollary 11. *The automorphisms in* $Ct(R_0)$ *are of the form*

$$(Ad_{R_0}(U_{-1}))^j Ad_{R_0}(u),$$

for all u *in the normalizer* $\mathcal{N}_{R_0}(R_{-1})$.

Proof. Let $N \subseteq M$ be an inclusion of hyperfinite, type II_1 factors. Consider the map Φ which associates to any α in $\mathrm{Ct}(M, N)$ the corresponding automorphism α^ω in

$$\mathcal{S} = \{\beta \in \mathrm{Aut}(M^\omega \cap M') |\ \beta|_{N^\omega \cap M'} = \mathrm{Id}_{N^\omega \cap M'}\}.$$

We claim that the kernel of Φ consists of

$$\mathcal{T} = \{\mathrm{Ad}_M(u) \mid u \in \mathcal{N}_M(N)\}.$$

Indeed, (by [Co2]), any element in $\mathcal{T}$ is in the kernel of Φ. Also, by the definition of $\mathrm{Ct}(M, N)$ we have that Φ takes its values into $\mathcal{S}$.

Moreover if α^ω is in the kernel of Φ, then (by Connes's theorem asserting that centrally trivial automorphisms in the hyperfinite II_1 factor are the same as inner automorphisms), it follows that α must be of the form $\mathrm{Ad}_M(u)$ for some unitary u in M. As α is leaving N invariant, it follows that u is in $\mathcal{N}_M(N)$.

The proof is now completed by noticing that in the case $N = R_{-1}$ and $M = R_0$, by the preceding lemma, the only nontrivial element in $\mathcal{S}$ is $(\mathrm{Ad}_{R_0}(U_{-1}))^\omega$.

This completes the proof.

In the next lemma we will determine the normalizer $\mathcal{N}_{R_0}(R_{-1})$. Note that

$$\{\mathrm{Ad}_{R_0}(U_0^j)|\ j = 1, 2, 3\}$$

and

$$\{\mathrm{Ad}_{R_0}(\alpha_0 g_+ + \alpha_1 g_-)|\alpha_0, \alpha_1 \in \mathbb{T}\},$$

are subsets of $\mathcal{N}_{R_0}(R_{-1})$.

To see that this are all of the generators of $\mathcal{N}_{R_0}(R_{-1})$, we first note that any element u in $\mathcal{N}_{R_0}(R_{-1})$ has the property that $ugu^* = \pm g$ (as $\{g\}'' = R'_{-1} \cap R_0$). Thus, eventually by replacing u by uU_0, we may assume that u has the property that $ugu^* = g$. With this we obtain:

Lemma 12. *Modulo Int*(R_0)*, the group* $\mathcal{N}_{R_0}(R_{-1})$ *is generated by the (commuting) elements*

$$\{Ad_{R_0}(U_0^j)|\ j=1,2,3\}$$

and

$$\{Ad_{R_0}(\alpha_0 g_+ + \alpha_1 g_-)|\ \alpha_0,\alpha_1\in\mathbb{T}\}.$$

Proof. Let u in $\mathcal{N}_{R_0}(R_{-1})$ have the property that $ugu^* = g$ (conform remark above). By ([Po3]), we may describe $R_{-1}\subseteq R_0$ as the following inclusion

$$x\longrightarrow \begin{pmatrix} x & 0\\ 0 & \theta(x)\end{pmatrix}\in M_2(\mathbb{C})\otimes R_{-1}\cong R_0,\ \ x\in R_{-1},$$

where θ is an outer automorphism of R_{-1} with the property that $\theta^2 = \mathrm{Ad}_{R_{-1}}(h)$ for some selfadjoint unitary h, with $\theta(h) = -h$ and whose spectral projections have trace 1/2.

In this description g generates the diagonal algebra in $M_2(\mathbb{C})$. Let u be a unitary in R_0, with $ugu^* = g$:

$$u=\begin{pmatrix} u_0 & 0\\ 0 & u_1\end{pmatrix}, u_0,u_1\in\mathcal{U}(R_{-1}).$$

The condition that u normalizes R_{-1} may be rewritten as:

$$\mathrm{Ad}_{R_{-1}}u_1(\theta(x)) = \theta(\mathrm{Ad}_{R_{-1}}u_1(x)),\ \ x\in R_{-1}.$$

Since θ is an outer automorphism, this implies in turn that

$$u_1 = \alpha\theta(u_0),\ \text{ for }\ \text{some } \alpha\in\mathbb{T}.$$

This means that

$$u=\begin{pmatrix} u_0 & 0\\ 0 & \theta(u_0)\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & \alpha\end{pmatrix} = \tilde{u_0}g^{1,\alpha}.$$

But $\tilde{u_0}$ belongs to R_{-1} (in the above identification).

This completes the proof.

It remains to determine which elements in the picture described above are a system of representatives modulo Int(R_0,R_{-1}) for $\chi(R_0,R_{-1})$.

Corollary 13. *The group $\chi(R_1, R_0)$ consists of the following elements (representatives modulo Int(R_0, R_{-1})):*

$$Ad_{R_0}(U_0^2), Ad_{R_0}(U_0U_{-1}), Ad_{R_0}(U_0U_{-1}^3), Id.$$

Note that in the above set we may always replace $Ad_{R_0}(U_0^2)$ by $Ad_{R_0}(U_{-1}^2)$.

Proof. To do this, we will use Loi's result ([Lo], see also [Ko]) exact sequence

$$\overline{\mathrm{Int}(R_0, R_{-1})} \longrightarrow \mathrm{Aut}(R_0, R_{-1}) \longrightarrow \mathcal{G}.$$

Here $\mathcal{G}$ is the set of all automorphisms of R_0, fixing a downward tunnel and trace preserving and that are leaving invariant the set of increasing relative commutants in the tunnel. Moreover, the elements in $\mathcal{G}$ preserve the Jones projections in the tunnel. The image of an element α from $\mathrm{Aut}(R_0, R_{-1})$ into $\mathcal{G}$ is obtained by correcting, (by inner automorphisms), in stages, the automorphism α, so that it preserves each Jones projection in the downward tunnel (see [Lo] for more details).

Let γ be the automorphism in $\mathrm{Aut}(R_0, R_{-1})$ defined by $\gamma(g) = -g$, $\gamma(U_i) = U_i$. It is easily seen that the following formulae hold true:

$$\mathrm{Ad}_{R_0}(U_0) = \gamma \lim_{n\to\infty} \mathrm{Ad}_{R_0}(U_2^* U_4^* ... U_{2n}^*),$$

and similarly

$$\mathrm{Ad}_{R_0}(U_{-1}) = \gamma \lim_{n\to\infty} \mathrm{Ad}_{R_0}(U_1^* U_3^* ... U_{2n-1}^*).$$

It is easy to see that the terms involving a limit in the above formula, are exactly the necessary correction in $\overline{\mathrm{Int}(R_0, R_{-1})}$ to get from $\mathrm{Ad}_{R_0}(U_{-1})$ and $\mathrm{Ad}_{R_0}(U_0)$ the element γ in $\mathcal{G}$. As γ is non-trivial (and of order 2) it follows that

$$(\mathrm{Ad}_{R_0}(U_{-1}))^i (\mathrm{Ad}_{R_0}(U_0))^j$$

belongs to $\overline{\mathrm{Int}(R_0, R_{-1})}$ if and only if $i + j$ is even.

Loi's description of $\mathcal{G}$ for our particular inclusion $R_{-1} \subseteq R_0$, shows that for this inclusion, we have that $\mathcal{G} = \mathbb{T} \rtimes \mathbb{Z}_2$. Moreover the set

$$\{\mathrm{Ad}_{R_0}(g^{1,\alpha}) \mid \alpha \in \mathbb{T}\}$$

corresponds to the copy of the of $\mathbb{T}$ in $\mathbb{G}$, while γ corresponds to the generator of $\mathbb{Z}_2$.

The fact $\mathrm{Ad}_{R_0}(g^{1,\alpha})$ doesn't belong to $\overline{\mathrm{Int}(R_0, R_{-1})}$ could in fact be checked directly by using the matrix model for the inclusion for $R_{-1} \subseteq R_0$ as in Lemma 12.

We use the notation in Lemma 12. Assume, to get a contradiction, that there are unitaries u_n in R_{-1} so that

$$\mathrm{Ad}_{R_0}\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} = \lim_{n\to\infty} \mathrm{Ad}_{R_0}\begin{pmatrix} u_n & 0 \\ 0 & \theta(u_n) \end{pmatrix}.$$

This would imply that

$$\lim_{n\to\infty} \mathrm{Ad}_{R_0}(u_n) = \mathrm{Id}_{R_0},$$

while in the same time the other relation gives that

$$\lim_{n\to\infty} \mathrm{Ad}_{R_0}(\theta(u_n)) = \alpha \mathrm{Id}_{R_0}.$$

Hence, as θ is a $\mathbb{Z}_2$ kernel, by ([Co1]), this implies that $\alpha^2 = 1$. Hence for $\mathrm{Ad}_{R_0}(g^{1,\alpha})$ to belong to $\overline{\mathrm{Int}(R_0, R_{-1})}$, the only possibility is that $g^{1,\alpha}$ be equal to g.

By using the formulae (1), (2), (3), (4), this leaves for

$$\overline{\mathrm{Int}(R_{-1}, R_0)} \cap \mathrm{Ct}(R_{R_{-1}}, R_0)$$

(modulo $\mathrm{Int}(R_{-1}, R_0)$) only the possibilities announced in the statement. Note that

$$\mathrm{Ad}_{R_0}(U_{-1}^2) = \mathrm{Ad}_{R_0}(g) = \mathrm{Ad}_{R_0}(U_0^2).$$

This ends the proof. The steps outlined in the statements 10-13 complete the proof of Lemma 9.

Moreover, the proof shows to us some more precise information about the structure of the group $\chi(R_{-1}, R_0) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We collect this information in the next statement.

Corollary 14. *Let $\chi(R_{-1}, R_0)$ be the relative χ invariant for the inclusion $R_{-1} \subseteq R_0$. Then, as an abstract group we have that $\chi(R_{-1}, R_0) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, we have*

$$(Ad_{R_0}(U_{-1}^2))^2 = Id_{R_0};\ (Ad_{R_0}(U_0U_{-1}))^2 = Ad_{R_0}(U_0^2 g) \in Int(R_{-1}, R_0).$$

In addition,

$$(Ad_{R_0}(U_0U_{-1}))(U_0^2 g) = -U_0^2 g \in R_{-1}.$$

Hence for all copies of $\mathbb{Z}_2$ in $\chi(R_{-1}, R_0) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ there is an obstruction to lifting from $\frac{\mathrm{Aut}(R_{-1},R_0)}{\mathrm{Int}(R_{-1},R_0)}$ to $Aut(R_{-1}, R_0)$, except for the one generated by $Ad_{R_0}(U_{-1}^2)$.

We translate this into the language of $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$. We get

Corollary 15. *Let $R_1 \subseteq R_0$ be the inclusion of type II_1 factors corresponding to the cross product by the action of $\mathbb{Z}_4$ corresponding to a $\mathbb{Z}_2$ kernel that can't be lifted to an action of $\mathbb{Z}_2$. We assume that the spectral projections corresponding to the selfadjoint unitary realizing the obstruction to lifting have trace $1/2$ Let $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$. Let $\chi(\mathcal{A}) \subseteq Out(\mathcal{A})$ be the Connes's χ invariant for $\mathcal{A}$. We use the model for $R_{-1} \subseteq R_0$ outlined in Lemma 5. Then, as an abstract group $\chi(\mathcal{A})$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, the elements of $\chi(\mathcal{A})$ are (modulo $Int(\mathcal{A})$)*

$$Ad_{\mathcal{A}}(U_0^2), Ad_{\mathcal{A}}(U_0U_{-1}), Ad_{\mathcal{A}}(U_0U_{-1}^3), Id_{\mathcal{A}}.$$

In this set, all elements have order 2, but only for the element $Ad_{\mathcal{A}}(U_0^2)$ there exists no obstruction to lifting from $Out(\mathcal{A})$ to $Aut(\mathcal{A})$. Thus $Ad_{\mathcal{A}}(U_0^2)$ gives an (outer) action θ on $\mathcal{A}$.

In what follows, we will be interested in analyzing the factor $\mathcal{A} \rtimes_\theta \mathbb{Z}_2$. Note that because θ gives the only copy of $\mathbb{Z}_2$ in $\chi(\mathcal{A})$, with no obstruction to lifting, $\mathcal{A} \rtimes_\theta \mathbb{Z}_2$ is an invariant for the factor $\mathcal{A}$.

We note the following

Remark 16. *Let $N \subseteq M$ be finite, separable, von Neumann algebras whose centers have trivial intersection. Let G be a finite group acting on M that leaves N invariant. Let Q be a type II_1 factor. Then G induces an action on $\mathcal{C} = (Q \otimes N) *_N M$ and moreover $\mathcal{C} \rtimes G$ is isomorphic to $(Q \otimes (N \rtimes G)) *_{(N \rtimes G)} *(M \rtimes G)$.*

Proof. Let $M \rtimes G$ be generated by M and the unitaries $(u_g)_{g \in G}$. Clearly then $N \rtimes G$ is generated by N and the unitaries $(u_g)_{g \in G}$ (which implement the cross products). Then the unitaries $(u_g)_{g\in G}$ commute with Q in $(Q\otimes(N\rtimes G))*_{(N\rtimes G)}*(M\rtimes G)$. Clearly then (at least algebraically) the unitaries $(u_g)_{g\in G}$ are normalizing $\mathcal{A}$ in $(Q\otimes(N\rtimes G))*_{(N\rtimes G)}*(M\rtimes G)$. They also implement the corresponding action on $\mathcal{A}$. In addition any monomial in $(Q\otimes(N\rtimes G))*_{(N\rtimes G)}*(M\rtimes G)$ may be written as a sum of monomials containing just one of the u_g (because Q commutes with all of the unitaries $(u_g)_{g\in G}$. Moreover the trace on $(Q\otimes(N\rtimes G))*_{(N\rtimes G)}*(M\rtimes G)$ is so that any monomial containing a single unitary u_g different from the identity, has zero trace (because of the similar property for the trace on $M\rtimes G$).

This completes the proof.

In the next proposition we analyze the structure of the cross product $\mathcal{A}\rtimes_\theta\mathbb{Z}_2$ (with the notations in Lemma 15.

Proposition 17. *We use the notations in Lemma 15. Let θ be $Ad_{R_0}(U_{-1}^2)$. Then θ defines an action of $\mathbb{Z}_2$ on R_0, leaving R_{-1} invariant. Moreover, θ acts identically on R_{-1}. We may identify*

$$R_0\rtimes_\theta\mathbb{Z}_2=\{U_{-1}^2,g,U_0,U_1,...\}'',$$

$$R_{-1}\rtimes_\theta\mathbb{Z}_2=\{U_{-1}^2,gU_0^2,U_1,...\}'',$$

as subalgebras of R_1. The center of $R_0\rtimes_\theta\mathbb{Z}_2$ is generated by U_{-1}^2g, while the center of $R_{-1}\rtimes\mathbb{Z}_2$ is generated by U_{-1}^2. In particular $\mathcal{B}=\mathcal{A}\rtimes_{[\mathrm{Ad}_{\mathcal{A}}(U_{-1}^2)]}\mathbb{Z}_2$ is identified with

$$(\mathcal{L}(F_k)\otimes(R_{-1}\rtimes_\theta\mathbb{Z}_2))*_{R_{-1}\rtimes_\theta\mathbb{Z}_2}(R_0\rtimes_\theta\mathbb{Z}_2)$$

is a type II_1 factor.

Proof. Clearly the trace condition on the family of all unitaries $(U_k)_{k\in\mathbb{Z}}$ and g implies the identification of the cross products as in the statement.

Clearly U_{-1}^2 commutes with R_{-1} which is the same as saying that θ acts trivially on R_{-1}. On the other hand U_{-1}^2g commutes with g (obvious) but also with U_0 and obviously with all others U_k, $k\geq 1$. This completes the proof.

Lemma 18. *We use the above notations. Let s be the dual action of $\mathbb{Z}_2$ on the cross product $R_0\rtimes_\theta\mathbb{Z}_2$. Then s acts as $s(U_{-1}^2)=-U_{-1}^2$ and s acts identically on all the other generators for R_0, in the list of generators in the previous proposition.*

Then $sAd_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)$ *belongs to* $\overline{Int(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)}$ *while* $Ad_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)$ *belongs to* $Ct(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)$*.*

Proof. Observe that $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}U_{-1}$ acts identically on $R_{-1}\rtimes_\theta\mathbb{Z}_2$ so that $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1})$ belongs to $\mathrm{Ct}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)$.

On the other hand $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_0)$ belongs to $\mathrm{Ct}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)$.

Indeed, recall that $\mathrm{Ad}U_0$ belongs to $\mathrm{Ct}(R_0, R_{-1})$ (from Corollary 11) The typical central sequence for $R_0\rtimes_\theta\mathbb{Z}_2$ that is asymptotically in $R_{-1}\rtimes_\theta\mathbb{Z}_2$, but is not asymptotically in R_{-1} is $\{U_{-1}^2 g U_1^2 U_3^2 ... U_{2n+1}^2\}_{n\in\mathbb{Z}}$. But $\mathrm{Ad}U_0$ acts identically on this sequence. A similar argument as in Lemma 10 shows that is sufficient to check that $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_0)$ belongs to $\mathrm{Ct}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)$. We have that

$$\mathrm{Ad}_{R_0}(U_{-1}U_0) = \lim_{n\to\infty} \mathrm{Ad}_{R_0}(U_0U_1....U_n) \in \overline{\mathrm{Int}(R_{-1}, R_0)}.$$

On the other hand, this will not hold true in $R_0\rtimes_\theta\mathbb{Z}_2$ (that is, it is not true that $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)$ belongs to $\overline{\mathrm{Int}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)}$ because $[\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)](U_{-1}^2) = -U_{-1}^2$ and U_{-1}^2 is in the center of $R_0\rtimes_\theta\mathbb{Z}_2$. Clearly the term which corrects this obstruction is the automorphism s which has exactly the property that $s(U_{-1}^2) = -U_{-1}^2$, while accting identically on all other generators. Thus $s\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)$ belongs to $\overline{\mathrm{Int}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)}$; more precisely

$$s\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0) = \lim_{n\to\infty} \mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_0U_1....U_n).$$

The same argument shows that $\mathrm{Ad}_{R_0\rtimes_\theta\mathbb{Z}_2}(U_{-1}U_0)$ doesn't belong to $\mathrm{Int}(R_0\rtimes_\theta\mathbb{Z}_2, R_{-1}\rtimes_\theta\mathbb{Z}_2)$.

Proposition 19. *Let* $R_1\subseteq R_0$ *be the inclusion of type* II_1 *factors corresponding to the cross product by the action of* $\mathbb{Z}_4$ *corresponding to a* $\mathbb{Z}_2$ *kernel that can't be lifted to an action of* $\mathbb{Z}_2$*. We assume that the spectral projections corresponding to the selfadjoint unitary realizing the obstruction to lifting have trace* $1/2$*. Let* $\mathcal{A} = (\mathcal{L}(F_k)\otimes R_{-1}) *_{R_{-1}} R_0$*. Let* $\chi(\mathcal{A})\subseteq Out(\mathcal{A})$ *be the Connes'* χ *invariant for* $\mathcal{A}$*. We use the model for* $R_{-1}\subseteq R_0$ *outlined in Lemma 5.*

Let $\mathcal{B} = \mathcal{A}\rtimes\mathbb{Z}_2$*, where the* $\mathbb{Z}_2$*-action on* $\mathcal{A}$ *is induced by* $\phi = Ad_{\mathcal{A}}\ U_{-1}^2 \in \chi(\mathcal{A})$ *(this action is an invariant for* $\mathcal{A}$ *as* ϕ *represents the only copy of* $\mathbb{Z}_2$ *in* $\chi(\mathcal{A}) =$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ liftable to a morphism of $\mathbb{Z}_2$ to $Aut(\mathcal{A})$). Let s be the dual action on the cross product. Let $Q_{-1} = \{R_{-1}, U_{-1}^2\}''$, $Q_0 = \{R_0, U_{-1}^2\}''$.

Then $\mathcal{B}$ is isomorphic to

$$(\mathcal{L}(F_k) \otimes Q_{-1}) *_{Q_{-1}} Q_0.$$

Moreover, we have the following decomposition for s :

$$s = (sAd\, U_0 U_{-1})(Ad\, U_0^* U_{-1}^*) = s_1 s_2,$$

where $s_1 = sAd\, U_0 U_{-1} \in \overline{Int\ (\mathcal{B})}$, $s_2 = Ad\, U_0^* U_{-1}^* \in Ct(\mathcal{B})$. In addition $s_i^2 = Ad\, h$, with $h = U_0^2 g \in \mathcal{A}$ and $s_i(h) = -h$.

Proof. The proof of this statement follows, by using Lemma 18, from the isomorphism we constructed in Proposition 2 from $\chi(M, N)$ onto $\chi((\mathcal{L}(F_k) \otimes N) *_N M)$.

This ends the proof.

3. The non-isomorphism of the two algebras having the invariant χ equal to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

The decomposition, obtained in Proposition 19, for the dual action of $\mathbb{Z}_2$ into an almost inner part and a centrally trivial part on the algebra $\mathcal{B} = \mathcal{A} \rtimes \mathbb{Z}_2$ prevents $\mathcal{A}$ from being isomorphic to Jones' example ([Jo2], in analogy with the construction in [Co3]). Note that in both of this examples the Connes' χ invariant is equal, as an abstract group, to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover, in both cases, exactly one of the copies of $\mathbb{Z}_2$, has no obstruction to lifting to an action of $\mathbb{Z}_2$.

As in the Connes's original proof (see also [Jo2]), the χ invariant carries additional information, coming from its position into Out. This additional information is revealed by taking the cross product with the $\mathbb{Z}_2$ action.

We recall the Jones' example (constructed in analogy with [Co4]). One considers $N = (\mathcal{L}(F_N) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$ for a suitable action γ.

The $\mathbb{Z}_2$ action is defined as follows. Let α be a $\mathbb{Z}_2$ kernel on $\mathcal{L}(F_k)$ with $\alpha^2 = Ad_{\mathcal{L}(F_k)}(e)$, $\alpha(e) = -e$ trace 1/2 in $\mathcal{L}(F_k)$. Similarly, $R = R_0 = \{g, U_0, U_1, ...\}''$, while $\beta = \mathrm{Ad}_{R_0}(U_{-1}U_0)$, $\beta^2 = \mathrm{Ad}_{R_0}(g)$,, $\beta(g) = -g$. The $\mathbb{Z}_2$ action γ on $M = \mathcal{L}(F_N) \otimes R_0$ is constructed as follows

$$\gamma = (\mathrm{Ad}_{[\mathcal{L}(F_N) \otimes R_0]} W)(\alpha \otimes \beta),$$

where W is any square root of $e \otimes g$ in the fixed point algebra of $\alpha \otimes \beta$ in the algebra $\mathcal{L}(F_N) \otimes R_0$.

Then, by the Connes's exact sequence in ([Co4]), we have that $\chi(N) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where, for exactly one of the two copies of $\mathbb{Z}_2$ in $\chi(N) \subseteq Out\ (N)$, there exists no obstruction for lifting to an action on N. By [Co4] (see also [Jo2]) this copy (denoted by $\hat{\gamma}$) corresponds to the dual $\mathbb{Z}_2$ action for $N = (\mathcal{L}(F_N) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$. By Takesaki duality ([Ta]) we get:

Proposition 20. *With the above notations, the dual action $\tilde{\gamma}$ on the cross product $N \rtimes_{\hat{\gamma}} \mathbb{Z}_2$, has a decomposition of the form: $\tilde{\gamma} = t_1 t_2 Ad\, W$, where $t_1 \in Ct\ (N \rtimes_{\hat{\gamma}} \mathbb{Z}_2)$, $t_2 \in \overline{Int\ (N \rtimes_{\hat{\gamma}} \mathbb{Z}_2)}$. Moreover there exist selfadjoint unitaries $f^i = f^i_+ - e^i_-$, with $f^i_\pm$ projections of trace 1/2, so that $t_i^2 = Ad\ f^i$, $t_i(f^i) = -f^i$, $t_i(f^j) = f^j, i \neq j$.*

Proof. Indeed this decomposition comes from the fact that in $\chi(N)$ the copy of $\mathbb{Z}_2$ corresponding to an action (by Connes's exact sequence) is exactly the dual action of $\mathbb{Z}_2$ on the cross product in $(\mathcal{L}(F_N) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$. By Takesaki duality, $N \rtimes_{\hat{\gamma}} \mathbb{Z}_2$ is isomorphic to

$$(\mathcal{L}(F_N) \otimes R_0) \otimes B(l^2(\mathbb{Z}_2)),$$

with the dual action on the cross product being unitary equivalent to $\gamma \otimes \lambda_{\mathbb{Z}_2}$ ($\lambda_{\mathbb{Z}_2}$ is the left regular representation of $\mathbb{Z}_2$ on $l^2(\mathbb{Z}_2)$.

This ends the proof.

The information about the decomposition of the dual action of $\mathbb{Z}_2$ into centrally trivial part and a part in the closure of inner automorphism is now sufficient to distinguish in between the two algebras.

Theorem 21. *Let $R_1 \subseteq R_0$ be the inclusion of type II_1 factors corresponding to the cross product by the action of $\mathbb{Z}_4$ corresponding to a $\mathbb{Z}_2$ kernel that can't be lifted to an action of $\mathbb{Z}_2$. We assume that the spectral projections, corresponding to the selfadjoint unitary realizing the obstruction to lifting, have trace $1/2$. Let $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$. Let N be the subfactor constructed in [Jo2] which has the χ invariant isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

With the above notation, N is not isomorphic to $\mathcal{A} = (\mathcal{L}(F_k) \otimes R_{-1}) *_{R_{-1}} R_0$.

Proof. Assume the contrary. Then $\mathcal{B} = \mathcal{A} \rtimes \mathbb{Z}_2$ and $N \rtimes_\gamma \mathbb{Z}_2 = (\mathcal{L}(F_N) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$ would be isomorphic (since in each of the two cases the $\mathbb{Z}_2$ action is canonical as the $\mathbb{Z}_2$ action is determined in each of the two cases by the χ invariant). Moreover, under that assumption it would follow that the corresponding dual actions on the cross products by $\mathbb{Z}_2$ would have the same image in Out, and thus by [Co1], we may assume that they are equal.

Since in $N = (\mathcal{L}(F_N) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$, the decomposition of an automorphism as the product of a centrally trivial automorphism and an almost inner automorphism, is unique (modulo inner automorphisms), it would follow that the decomposition $s = s_1 s_2$ (regarded as an action of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$) is outer conjugate to the similar action described by t_1, t_2 in the preceding lemma. But this is impossible because the two actions have different characteristic invariants (see [Jo3]). This is because $t_1(f^2) = f^2, (t_i)^2 = Ad\ f^i$, while, with the notations in Proposition 19, $s_i(h) = -h, s_i^2 = Ad\ h$ (and also $[s_1, s_2] = 0$).

For the sake of the completeness we include an ad-hoc argument for this last assertion (instead of referring to [Jo3]).

So assume to get a contradiction, that in the group Out, the following equality holds true: $s_1 s_2 {=} t_1 t_2$. Hence, for some unitary W, we have that

$$t_1 t_2 \mathrm{Ad} W = s_1 s_2$$

and hence there exists a unitary w so that

$$t_1 \mathrm{Ad} w = s_1, \quad \mathrm{Ad} w^* t_2 Ad W = s_2.$$

Hence

$$s_1 = t_1 \mathrm{Ad} w_1, \quad s_2 = \mathrm{Ad} w_2^* t_2$$

for some unitaries w_1, w_2.

Moreover we have

$$t_i^2 = \mathrm{Ad} f^i, t_i(f^i) = -f^i, t_j(f^i) = f^i, \qquad i, j = 1, 2, \quad i \neq j.$$

Moreover we know that s_1, s_2 commute.

Then

$$s_1^2 = t_1^2 \mathrm{Ad} w_1 t_1 \mathrm{Ad} w_1 = t_1^2 \mathrm{Ad}(t_1^{-1}(w_1) w_1) = \mathrm{Ad}(f^1 t_1^{-1}(w_1) w_1),$$

$$s_2^2 = \mathrm{Ad} w_2^* t_2 \mathrm{Ad} w_2^* t_2 = \mathrm{Ad}(w_2^* t_2(w_2^*) f^2).$$

Note that

$$s_2(w_2^* t_2(w_2^*) f^2) = w_2^*(t_2(w_2^a st) t_2^2(w_2^*) t_2(f^2)) w_2$$

(5)
$$= w_2^*(t_2(w_2^*) f^2 w_2^* f^2(-f^2)) w_2$$

$$= -w_2^* t_2(w_2^*) f^2.$$

We want to exploit the commutativity hypothesis (i. e. that $[s_1, s_2] = 0$).

We thus have

$$t_1 \mathrm{Ad} w_1 \mathrm{Ad} w_2^* t_2 = \mathrm{Ad} w_2^* t_2 t_1 \mathrm{Ad} w_1,$$

and hence

$$\{\mathrm{Ad}[t_1(w_1 w_2^*)]\} t_1 t_2 = \{\mathrm{Ad}[w_2^* t_2 t_1(w_1)]\} t_1 t_2.$$

Hence

$$\mathrm{Ad}[t_1(w_1 w_2^*)] = \mathrm{Ad}[w_2^* t_2 t_1(w_1)],$$

and therefore there exists a complex number μ of absolute value 1 so that

$$t_1 t_2(w_1) = \mu w_2 t_1(w_1 w_2^*),$$

or equivalently

(6)
$$t_1(w_1 w_2^*) = \mu w_2^* t_1 t_2(w_1).$$

We want to determine the following quantity:

$$s_1(w_2^* t_2(w_2^*) f^2) = t_1(\mathrm{Ad} w_1(w_2^* t_2(w_2^*) f^2)) =$$

$$t_1(w_1 w_2^* t_2(w_2^*) f^2 w_1^*) = t_1(w_1 w_2^*) t_1 t_2(w_2^*) f^2 t_1(w_2^*).$$

We would like (to get a contradiction) that this last expression is $w_2^* t_2(w_2^*) f^2$, i.e. we want to check that

(8)
$$t_1(w_1 w_2^*) t_1 t_2(w_2^*) f^2 t_1(w_1^*) = w_2^* t_2(w_2^*) f^2,$$

which is equivalent to

$$[w_2 t_1(w_1 w_2^*)] t_1(t_2(w_2^*)) f^2 t_1(w_2^*) = t_2(w_2^*) f^2.$$

By using (6) we get the equivalent form to be checked

$$\overline{\mu} t_1 t_2(w_1) t_1 t_2(w_2^*) f^2 t_1(w_1^*) = t_2(w_2^*) f^2.$$

Applying t_2 both sides, we have to check successively

$$\overline{\mu} t_1 t_2^2(w_1 w_2^*) t_2(f^2) t_1 t_2(w_1^*) = t_2^2(w_2^*) t_2(f^2),$$

$$\overline{\mu} t_1(f^2 w_1 w_2^* f^2)(-f^2) t_1 t_2(w_1^*) = f^2 w_2^* f^2(-f^2),$$

$$\overline{\mu} f^2 t_1(w_1 w_2^*) t_1 t_2(w_1^*) = f^2 w_2^*.$$

But $t_1(w_1 w_2^*) = \mu w_2^* t_1 t_2(w_1^*)$ implies $t_1(w_1 w_2) t_1 t_2(w_1^*) = \mu w_2^*$, so that the relation to be checked is equivalent to

$$f^2 w_2^* = f^2 w_2^*.$$

Thus if the relation $s_1 s_2 = t_1 t_2$ holds in Out then both relations (5) and (8) have to hold true in the same time. But this is a clear contradiction. This completes the proof, but we want to point that what we have checked in the last part of the proof is the following.

What we have checked (by an ad-hoc argument) is a part of [Jo3]). We have proved that for a cocycle perturbation of t_1, t_2 written as $s_1 = t_1 \mathrm{Ad} w_1; s_2 = \mathrm{Ad} w_2^* t_2$, we necessary have that $s_1^2 = \mathrm{Ad} g_1$, $s_2^2 = \mathrm{Ad} g_2$ for some selfadjoint unitaries g_1, g_2. Moreover $s_i(g_j) = -g_j$ if $i = j$ and $s_i(g_j) = g_j$ if $i \neq j$.

Because in fact $s_i(g_j) = -g_j$ for all i, j, it follows that there is no cocycle perturbation from s_1, s_2 to t_1, t_2. Thus the factors N and $\mathcal{B}$ can not be isomorphic.

This completes the proof.

4. The factor constructed in [Jo2], with χ invariant $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, is the enveloping algebra of an inclusion of free group factors

In this section we use the tools provided by Voiculescu's random matrix model ([Vo1]) for free group algebras to prove that the example considered in [Jo2] of a type II_1 factor N having the χ invariant equal to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ may be identified with the enveloping algebra in the iterated basic construction $A \subseteq B$ of free group factors.

We will also show that this B is identified with $A \rtimes_\theta \mathbb{Z}_4$, where θ is a $\mathbb{Z}_2$ kernel on A with non-trivial obstruction to lifting. The spectral projections of the unitary implementing the obstruction have trace $1/2$.

In lemma 8 we showed that the subfactor $\mathcal{A}$ considered in Theorem 21 is the enveloping algebra of an inclusion of free group factors which also may be identified with the cross product by $\mathbb{Z}_4$ with an action coming from a $\mathbb{Z}_2$-kernel with the same properties as θ.

It will then follow, by the non-isomorphism result in Theorem 21 that we found in this way, two outer $\mathbb{Z}_2$ kernels on a free group factor which are not conjugate. In this way we accomplish the proof of the results announced in the introduction.

Thus it remains to prove the following result:

Theorem 22. *Let N be the factor constructed in [Jo2]. Then there exist subalgebras $A \subseteq B$ in N so that A is isomorphic to $\mathcal{L}(F_k)$, B is the cross product $A \rtimes_\theta \mathbb{Z}_4$, where θ is a $\mathbb{Z}_2$-kernel on A with non-trivial obstruction (to lifting). Moreover N is the enveloping algebra for the inclusion $A \subseteq B$.*

Proof. We will do this into two steps. Step I consists of showing that N is the term at the infinity (in the iterated o Jones' basic construction) of an inclusion of the form $A \subseteq B = A \rtimes_\theta \mathbb{Z}_4$.

We recall construction of N from above.

The type II_1 factor N is $(\mathcal{L}(F_k) \otimes R_0) \rtimes_\gamma \mathbb{Z}_2$, for the $\mathbb{Z}_2$ action γ defined as follows. Let α be a $\mathbb{Z}_2$ kernel on $\mathcal{L}(F_k)$ with $\alpha^2 = Ad_{\mathcal{L}(F_k)}(e)$, $\alpha(e) = -e$ where e is a selfadjoint unitary whose spectral projections $e_\pm$ have trace $1/2$ in $\mathcal{L}(F_k)$.

For the β action we use the following model: $R = R_0 = \{g, U_0, U_1, ...\}''$, while $\beta = \mathrm{Ad}_{R_0}(U_{-1}U_0)$, $\beta^2 = \mathrm{Ad}_{R_0}(g)$, $\beta(g) = -g$. The $\mathbb{Z}_2$ action γ on $M = \mathcal{L}(F_N) \otimes R_0$ is constructed as follows

$$\gamma = (\mathrm{Ad}_{\mathcal{L}(F_N)\otimes R_0} W)(\alpha \otimes \beta),$$

where W is any square root of $e \otimes g$ in the fixed point algebra of $\alpha \otimes \beta$ in $\mathcal{L}(F_k) \otimes R_0$.

We need a more precise description for the action α. For that purpose we use the terminology introduced in ([Dy], [Ra1]) of free group factors with fractional number of generators: we let $k = 5/2$ and hence $\mathcal{L}(F_k) = \mathcal{L}(F_{5/2})$ has the free generators: X, Y, e with the following properties: X, Y are semicircular, while e is a selfadjoint unitary whose spectral projections $e_\pm$ have trace $1/2$. We let α be defined by the requirement:

$$\alpha(X) = [\mathrm{Ad}_{\mathcal{L}(F_{5/2})}(e)](Y), \alpha(Y) = X, \alpha(e) = -e.$$

Assume that $g = g_+ - g_-$. Then one may choose W to be equal to

$$W = (e_+g_+ + e_-g_-) + \sqrt{-1}(e_+g_- + e_+g_-).$$

Hence, $W^2 = eg$ as $W^2 = e_+g_- + e_+g_- - (e_+g_+ + e_-g_-)$ and

$$(\alpha \otimes 1)(W) = (e_+g_- + e_+g_-) + \sqrt{-1}(e_+g_+ + e_-g_-) = \sqrt{-1}W^*,$$

and

$$(1 \otimes \beta)(W) = \sqrt{-1}W^*.$$

In particular $(\alpha \otimes \beta)(W) = W$.

In $N = M \rtimes_\gamma \mathbb{Z}_2$, we let p be the selfadjoint unitary implementing the cross product. Thus the spectral projection of p in N have trace $1/2$.

We then have that

$$\mathrm{Ad}p(g) = -g; \mathrm{Ad}p(e) = -e,$$

since $[W, e] = 0, [W, g] = 0$. Moreover:

$$\mathrm{Ad}p(x) = \mathrm{Ad}W(\alpha(x)), \text{ for all } x \in \{X, Y\}'',$$

$$\mathrm{Ad}p(y) = \mathrm{Ad}W(\beta(y)), \text{ for all } y \in \{U_0, U_1, ...\}''.$$

We want to see the action of $\mathrm{Ad}U_0$ on this elements. We note that

$$pU_0p = W\alpha(U_0)W^* = \sqrt{-1}WU_0W^*.$$

But $U_0gU_0^* = -g$ and hence $U_0g_\pm U_0^* = g_\mp$ and consequently

$$U_0WU_0^* = \sqrt{-1}W^*, \text{ hence } U_0W^*U_0^* = -\sqrt{-1}W.$$

Thus

$$pU_0p = \sqrt{-1}WU_0W^* = \sqrt{-1}W(-\sqrt{-1})WU_0 = W^2U_0.$$

Thus

$$U_0 p U_0^* = pW^2 = peg.$$

Thus we may describe the action of U_0 on the algebra

$$A = \{X, Y, e, p, g\}'',$$

as follows:

$$\mathrm{Ad}U_0(X) = X,\ \mathrm{Ad}U_0(Y),\ \mathrm{Ad}U_0(e) = e,$$

$$\mathrm{Ad}U_0(g) = -g,\ \mathrm{Ad}U_0(p) = peg.$$

This implies that $(\mathrm{Ad}U_0)^2$ acts identically on $\{X, Y, e, g\}$, while

$$(\mathrm{Ad}U_0)^2(p) = \mathrm{Ad}U_0(peg) = (peg)e(-g) = -p.$$

Thus

$$(\mathrm{Ad}_A U_0)^2 = \mathrm{Ad}_A g,$$

(since $\mathrm{Ad}_A g(p) = -p$, while $\mathrm{Ad}_A g$ acts identically on X, Y, e, g).

Moreover, since $\mathrm{Ad}U_0(g) = -g$ it follows that, at least algebraically, the algebra $B = (A \cup \{U_0\})''$ is identified with the cross product $A \rtimes_{\mathrm{Ad}_A U_0} \mathbb{Z}_4$ with U_0 implementing the cross product. Moreover $\mathrm{Ad}_A U_0$ implements on A an $\mathbb{Z}_2$-outer kernel with obstruction to lifting.

The only condition that remains to be checked is that any monomial in the variables $\{X, Y, e, g, p, U_0\}$ may be written by using only one occurrence of U_0 to some power. We will also have to check that any such monomial has zero trace if the power of U_0 is not a multiple of 4.

Because we already know that p implements a cross product any such monomial will be of the form pm or m where m belongs to $\mathcal{L}(F_{5/2}) \otimes R_0$. By the construction of R_0, U_0 may be used at most once in m. Also by the definition of the trace on the cross product the trace of a monomial of the form mp is zero. For a monomial of the type m we are back in the case of R_0 where the statement holds.

Moreover it is clear that the unitaries $U_1, U_2, \ldots$ will implement the consecutive terms in the iterated basic construction of $A \subseteq A \rtimes_{\mathrm{Ad}_A U_0} \mathbb{Z}_4$. This completes the proof of the first step.

We collect what we found so far in a form of a Corollary.

Corollary 23. *Let N be the example constructed in [Jo2] of a type II_1 factor having $\chi(N) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then N is the term at the infinity of the iteration of Jones's basic construction in an inclusion of the form $A \subseteq A \rtimes_\theta \mathbb{Z}_4$, where θ is a $\mathbb{Z}_2$ outer kernel on A (that is $\theta^2 = Ad_A g$, g a selfadjoint unitary in A with $\theta(g) = -g$ and so that his spectral projections have trace $1/2$).*

By using the notation we introduced above in the first step of the proof, we have that $A = \{X, Y, e, p, g\}$ and we are given a trace on this algebra with respect to which X, Y, e are free, X, Y are semicircular, e, g, p are selfadjoint unitaries whose spectral projections $e_\pm, g_\pm, p_\pm$ have trace $1/2$.

Let $W = (e_+g_+ + e_-g_-) + \sqrt{-1}(e_+g_- + e_+g_-)$, so that $W^2 = eg$. Then g commutes with $\{X, Y, e\}$ and $Ad_A p(e) = -e$, $Ad_A p(g) = -g$. Moreover

$$[Ad_A p](X) = [Ad_A W](Ad_A e(Y))$$

$$= [g_+ Ad_A(e_+ - \sqrt{-1}e_-)(Y)] + [g_- Ad(e_+ + \sqrt{-1}e_-)(Y)]$$

and

$$[Ad_A p](Y) = [Ad_A W](X) = [g_+ Ad(e_+ + \sqrt{-1}e_-)(X)] + [g_- Ad_A(e_+ - \sqrt{-1}e_-)(X)].$$

Moreover the trace on the algebra A is described as follows:

If any monomial in X, Y, e, p, g contains p then the trace is zero. If the monomial doesn't contain p then the trace is the same as the trace on $\{X, Y, e\}'' \otimes \{g\}''$.

We claim that in this conditions, the algebra A is a a free group factor (eventually with a fractional number of generators, as in [Dy],[Ra1]).

Proof. This will also complete the proof of step II of our lemma. The proof is based on the Voiculescu's random matrix model introduced in [Vo1].

The proof of this is again in two steps. We will give an explicit random matrix model for the algebra A and then by using this model we will show that A is a free group factor.

The model is as follows: note that by ([Jo1]) the selfadjoint unitaries $\{e, p, g\}$ generate a copy of $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, with the minimal projections of trace $1/4$. Take $M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \subseteq M_4(\mathbb{C})$ and assume that we are using a system of matrix units $(e_{ij})_{i,j=1,\dots,4}$ so that the first copy is generated by e_{12} while the second is generated by e_{34}. Let D be any W^*-algebra with finite trace τ, containing the free elements a, b, c, d, f where a, c, e, f are semicircular, while b, d are circular.

We let in $M_4(\mathbb{C}) \subseteq M_4(\mathbb{C}) \otimes D$:

$$e_+ = e_{22} + e_{33};\ g_+ = e_{11} + e_{33};\ p = \frac{1}{2}(\text{Re }(e_{12}) + \text{Re }(e_{34})).$$

With this choice e, p, g verify our conditions. We let

$$X = a \otimes e_{11} + c \otimes e_{22} + e \otimes e_{33} + f \otimes e_{44} + 2\text{Re }(b \otimes e_{14}) + 2\text{Re }(d \otimes e_{23}),$$

or by using matrix notation

$$X = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & d^* & e & 0 \\ b^* & 0 & 0 & f \end{pmatrix}.$$

Clearly, we have that X is free with e given in matrix notation as

$$e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and X commutes with g given in matrix notation as

$$g_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We recall that $W = (e_+g_+ + e_-g_-) + \sqrt{-1}(e_+g_- + e_+g_-)$. We have to find that $Y = \text{Ad}_A(pW)(X)$. Now $e_-g_- = e_{44}$, $e_+g_+ = e_{33}$, $e_-g_+ = e_{11}$, $e_+g_- = e_{22}$. Hence p, W are given in matrix notation as

$$p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$W = \begin{pmatrix} \sqrt{-1} & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence the matrix expression for Y is

$$Y = \begin{pmatrix} c & 0 & 0 & (-\sqrt{-1})d \\ 0 & a & \sqrt{-1}b & 0 \\ 0 & (-\sqrt{-1})b^* & f & 0 \\ \sqrt{-1}d & 0 & 0 & e \end{pmatrix}.$$

Since a, b, c, d, e, f is a free family it follows that X, Y, e are free with respect to the unique (normalized) trace on $M_4(\mathbb{C}) \otimes D$. Moreover, X, Y, e are (probability) independent (with respect to the trace) to g. Finally if we have any monomial pm containing a p (which necessary occurs only once). Then m commutes with g, so it has the form

$$m = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix}.$$

When we hit this operator with p we get something with zero on the diagonal and thus of null trace. This completes the matrix model for the algebra A.

To show that A is a free group factor (it is clearly a factor) we reduce A by the projection $e_{22} + e_{33}$. The reduced algebra is then generated by e_{23} and $c \otimes e_{22} + 2\mathrm{Re}\ (d \otimes e_{23}) + e \otimes e_{33}$ and a similar element involving a, b, f. By [Vo1], this is $M_2(\mathbb{C}) \otimes \mathcal{L}(F_2)$ which if we further reduce it by e_{33} is a free group algebra.

The last part of the argument can me made more precise by determining (as in [Dy], [Ra1]) the exact number of generators to show that A is $\mathcal{L}(F_{11/4})$. If from the beginning we add to the generators of $\mathcal{L}(F_{5/2})$ a number of generators on which α acts trivially, then we could get any number of generators for A (bigger than 11/4).

This completes the proof.

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