A type III factor with core isomorphic to the free group von Neumann algebra of a free group, tensor B(H)

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In this paper we obtain a type III_λ factor by using the free product construction from [Vo1,Vo2] and show that its core ([Co]) is $L(F_\infty) \otimes B(H)$. We will prove that

$$M_2(\mathbf{C}) * L^\infty([0,1],\nu)$$

is a type III_λ factor if $M_2(\mathbf{C})$ is endowed with a nontracial state. Moreover we will show that the core ([Co]) of this type III_λ factor (when tensorized by $B(H)$) is $L(F_\infty) \otimes B(H)$ and we will give an explicit model for the associated (trace scaling) action of Z on the core (cf. [Co], [Ta]). Here $B(H)$ is the space of all linear bounded operators on a separable, infinite dimensional Hilbert space H.

Recall from [Vo1], that a family $(A_i)_{i\in I}$ of subalgebras in a von Neumann algebra M with state ϕ, is free with respect to ϕ if $\phi(a_1 a_2 ... a_k) = 0$ whenever

$$\phi(a_i) = 0, a_i \in A_{j_i}, i = 1, 2, ...k, j_1 \neq j_2, ...j_{k-1} \neq j_k.$$

Reciprocally given a family $(A_i, \phi_i), i \in I$ of von Neumann algebras with faithful normal states ϕ_i , one may construct (see[Vo1]) the (reduced) free product von Neumann algebra $*A_i$, which contains $A_i, i \in I$ and has a faithful normal state ϕ so that $\phi|_{A_i} = \phi_i$ and so that the algebras $(A_i)_{i\in I}$ are free with respect to ϕ.

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The aim of this paper is to show the following result.

Theorem *Let* $E = M_2(\mathbf{C}) * \mathbf{L}^\infty([0,1]),\nu)$ *be endowed with the free product state* ϕ *where* $M_2(\mathbf{C})$ *is endowed with the state* ϕ_0 *which is subject to the condition*

$$\phi_0(e_{11})/\phi_0(e_{22}) = \lambda \in (0,1) \quad \textit{and} \quad \phi(e_{12}) = \phi(e_{21}) = 0,$$

while $L^\infty([0,1],\nu)$ *has the state given by Lebesgue measure on [0,1]. With these hypothesis,* $M_2(\mathbf{C}) * L^\infty([0,1],\nu)$ *is a type* III_λ *factor and its core is isomorphic to* $L(F_\infty) \otimes B(H)$.

In the proof of the theorem we will also obtain a model for the core of $E \otimes B(H)$ and for the corresponding (dual) action on the core, of the modular group of the weight $\phi \otimes tr$ (tr is the canonical semifinite trace on $B(H)$). This model will be a submodel of the one parameter action of $R_+/\{0\}$ on $L(F_\infty) \otimes B(H)$,that we have constructed in [Ra].

The model *Model for the core of (* $M_2(\mathbf{C}) * L^\infty([0,1],\nu)) \otimes B(H)$ *and of the corresponding dual action on the core for the modular group of automorphism for the weight* $\phi \otimes tr$*:*

Let A_0 be the subalgebra in the algebraic free product

$$L^\infty(\ R) * (\ C[X] *\ C[Y])$$

generated by $\{pXp, pYp, p |\ p\ finite\ projection\ in\ L^\infty(\ R)\}$ where $L^\infty(\ R)$ is endowed with the Lebesgue measure.

Let τ be the unique trace on A_0 defined by the requirement that the restriction τ_p to the algebra generated in $p\ Ap$ by $pXp, pYp, pL^\infty(\ R)$ is subject to the following conditions:

(i) The three algebras generated respectively by pXp, pYp, $pL^\infty(\ R)$ are free with respect to τ_p

(ii) $\tau(p)^{-1/2}pXp, \tau(p)^{-1/2}pYp$ are semicircular (with respect τ_p)(see [Vo1] for the definition of a semicircular element).

Such a construction is possible because of the Theorem 1 in [Ra].

Assume that pXp, pYp are selfadjoint and let A be the weak completion of A_0 in the G.N.S. representation for τ. Then (cf. [Ra]), A is a type II_∞ factor isomorphic to $L(F_\infty) \otimes B(H)$ and the trace τ extends to a semifinite normal trace on A (which we also denote by τ).

Recall (by [Ra]) that in this case, there exists a one parameter group of automorphism $(\alpha_t)_{t \in\ R_+ \setminus \{0\}}$ on A, scaling trace by t, for each $t \in\ R_+ \setminus \{0\}$, which is induced by $d_t * M_t$ on $L^\infty(\ R) * (\ C[X] *\ [Y])$ where d_t is dilation by t on $L^\infty(\ R)$, while $M_t(X) = t^{-1/2}X; M_t(Y) = t^{-1/2}Y$, $t > 0$.

Let B the von Neumann subalgebra of A generated by

$$q_n = \chi_{[\lambda^{n-1},\lambda^n]}, n \in\ Z,$$

the characteristic functions of the intervals $[\lambda^{n-1}, \lambda^n]$ and by the following subsets of A:

$$\tilde{X} = \{q_n X q_m | n, m \in\ Z, |n-m| \le 1\},$$

$$\tilde{Y} = \{q_n Y q_n | n \in Z\}.$$

Clearly B is invariant under $\{\alpha_{\lambda^n}\}_{n\in Z}$ and by Lemma 3 in [Ra], B is isomorphic to $L(F_\infty) \otimes B(H)$. Let $\beta_n = \alpha_{\lambda^n}|\ B$.

Let $D = B \times_\beta Z$ be the cross product of B by the action Z given by β. Then by [Co], D is a type III_λ factor. Let $u \in D$ be the unitary implementing the cross product. Moreover let ψ be the normal semifinite faithful weight on D obtained as the composition expectation from D onto B.

We will prove that B, with the action of Z given by $(\beta_n)_{n\in Z}$ is isomorphic to the core of $E \otimes B(H)$, with the dual action (on the core) for the modular group of automorphisms of the weight $\phi \otimes tr$ on $E \otimes B(H)$. Our main result will be a consequence of the following statement.

Proposition. *Let E be the von Neumann algebra free product $M_2(\mathbf{C}) * L^\infty([0,1],\nu)$, with the free product state $\phi = \phi_0 * \nu$, where $M_2(\mathbf{C}) = (e_{ij})_{i,j=1}^2$ is endowed with the normalized state ϕ_0 with $\phi(e_{11})/\phi(e_{22}) = \lambda$ and $\phi(e_{12}) = \phi(e_{21}) = 0$. Then, with the above notation E is isomorphic to $(q_o + q_1)\ D(q_0 + q_1)$.*

Moreover the state ϕ on E is (via this identification) the (normalized) restriction of ψ to $(q_o + q_1)\ D(q_0 + q_1)$.

(Here $D = B \times_\beta Z$, where B is the von Neumann subalgebra in A generated by $\tilde{X} = \{q_n X q_m | n, m \in Z, |n - m| \leq 1\}$, $\tilde{Y} = \{q_n Y q_n | n \in Z\}$ and the characteristic functions $q_n = \chi_{[\lambda^{n-1},\lambda^n]}, n \in Z$, $q_n \in L^\infty(R) \subseteq A$. Moreover $\beta_n = \alpha_{\lambda^n}, n \in Z$.)

Recall from above that the von Neumann algebra A is a type II_∞ factor isomorphic to $L(F_\infty) \otimes B(H)$ and A is generated by

$$\{pXp, pYp, p|\ p\ finite\ projection\ in\ L^\infty(R)\}.$$

Here $\alpha_t, t > 0$ acts as dilation by t on $L^\infty(R)$ and multiplies X, Y by $t^{-1/2}$. The trace on A is subject to the above conditions (i), (ii) and it is scaled by the automorphisms $\alpha_t, t > 0$.

This proposition will be a consequence of the following two lemmas.

Lemma 1. *With A, B, D, ψ, τ and u as before let*

$$e_{11} = q_1 u = u q_0; e_{11} = q_0; e_{22} = q_1$$

Let $a = x + y$, where

$$x = (q_0 + q_1) X (q_0 + q_1) - q_0 X q_0$$

$$y = q_0 Y q_0.$$

Then $M_2(\mathbf{C}) = (e_{ij})_{i,j=1}^2$ is free with respect to

$$\psi_1 = (\psi(q_0 + q_1))^{-1} \psi|_{(q_o+q_1)\ D(q_0+q_1)},$$

to the semicircular element a, in the algebra $(q_0 + q_1)\ D(q_0 + q_1)$ with unit $q_0 + q_1$.

Proof. We have to check freeness, which means that the value of ψ_1 on certain monomials in a, u, e_{11}, e_{22} is null. Since by definition , ψ_1 vanishes the monomials containing a different number of u 's and u^* 's, we have only to check this if the number of occurrences for u is equal to the one for u^*.

Let $p_n = q_n + q_{n+1} = \chi_{[\lambda^{n-1},\lambda^{n+1}]}$.

Using the fact that u implements β_1 on D it follows that we only have to check $\psi_1(m) = 0$ if

$$m = p_0 f_1 q_{i_1} f_2 q_{i_2} f_3 ... q_{i_n} f_{n+1} p_0$$

where the following conditions are fulfilled:

(a) i_{j+1} is either i_j or $i_j \pm 1$.

(b) Card $\{s | i_j = s, j = 1, 2, ..., n\}$ is even for every s.

(c) f_k is a product

$$f_1^k a_1^k ... f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k, \; n_k \geq 1$$

where f_s^k, $s = 1, 2, 3...n_k$, is an element of null value under the state ψ_1 in the algebra generated by $\alpha_j(a)$ while a_s^k is an element of null trace in the algebra generated by q_j, q_{j+1}. Here j is an integer which is completely determined, for each k. If $i_k \neq i_{k+1}$ then j is the minimum of the i_k and i_{k+1}. If $i_k = i_{k+1}$ then j is either i_k if $i_{k-1} \leq i_k$ or either $i_k - 1$ if $i_{k-1} > i_k$.

To see that those are all the monomials of null state that may appear in the algebra generated by $M_2(\mathbf{C})$ and a it is sufficient to note that any string

$$f_1 e_{21} f_2 e_{21} ... f_p e_{21} f_{p+1} e_{12} f_{p+2} e_{12} ... e_{12} f_{2p+1} =$$

$$= f_1 (q_1 u) f_2 q_1 u ... f_p q_1 u f_{p+1} (u^* q_1) ... (u^* q_1) f_{2p+1},$$

after cancelation, is equal to

$$f_1 (q_1 u) f_2 ... q_1 u f_p q_1 \beta_1 (f_{p+1}) q_1 f_{p+2} (u^* q_1) ... (u^* q_1) f_{2p+1} =$$

$$= f_1 (q_1 u) f_2 ... q_1 \beta_1 (f_p) q_2 \beta_2 (f_{p+1}) q_2 \alpha_1 (f_{p+2}) q_1 ... (u^* q_1) f_{2p+1} =$$

$$= f_1 q_1 \alpha_1 (f_2) q_2 ... \beta_{p-1} (f_p) q_p \beta_p (f_{p+1}) q_p \beta_{p-1} (f_{p+2}) ... q_1 f_{2p+1}$$

and similarly for a string in which each $q_1 u$ is replaced by $u^* q_1$ and conversely.

Here the f_i 's are products of the form $f_1^i a_1^i f_2^i a_2^i ... f_n^i$ where f_j^i are elements of null trace in the algebra generated by $a = (q_0 + q_1) a (q_0 + q_1)$, while a_j^i are elements of null trace in the algebra generated by q_0, q_1.

The monomials in the algebra generated by $M_2(\mathbf{C})$ and a that are to be checked for having zero value under ψ_1 are obtained by replacing certain f_j by other strings of this form, or by putting together such strings.

To show that the value of $\psi_1(m)$ is zero we will use the following observation which is a consequence of Lemma 3.1 in [Vo2]. This observation will be used to replace the elements $f_1, ..., f_{n+1}$ in the monomial m by elements of null trace.

Observation. *Let B be a W^*-algebra with trace τ, let X be a semicircular element and p a nontrivial projection that is free with X. Then any element of null trace in the algebra (with unit p) generated by pXp is a sum of monomials which are products either of elements of null trace in the algebra generated by pXp or either of the form $p - \tau(p)$, but no such monomial is $p - \tau(p)$ itself.*

Proof. Indeed if x is such an element then $pxp = x$, and moreover any other such monomial, which is different from $p - \tau(p)$, when multiplied with p, preserves the property of having null trace.

On the other hand

$$\tau(p(p - \tau(p))) = 1 - \tau(p) \neq 0.$$

This ends the proof of the observation.

To conclude the proof of Lemma 1 we let p a projection which is greater than the supremum of all the projections $\{q_i | i \in I_m\}$ that are involved in m.

We may then assume by construction that we are given a finite family of semicircular elements z^j so that $z^j = pz^jp$ and so that (modulo a multiplicative constant) $\alpha_j(a) = (q_j + q_{j+1})z^j(q_j + q_{j+1})$ for j in I_m.

Using the above observation we may express $f_k = f_1^k a_1^k ... f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k$ as a sum of products of null trace in the algebras generated by $\{q_j\}$ and $\{z_j\}$ (adjacent elements are allways in different algebras).

(Note that $(q_j + q_{j+1})(q_j - \tau(q_j)(\tau(q_j + q_{j+1}))^{-1}(q_j + q_{j+1})$ has always null trace).

Again the above observation shows that each of these monomials must contain at least on term in z^j. Since consecutive f^i involve different elements in the set $\{z^j\}$ it follows that $\psi_1(m) = 0$.

This ends the proof of Lemma 1.

Lemma 2 *With B, D as before we have that $(q_0 + q_1)B(q_0 + q_1)$ coincides with the von Neumann subalgebra $C \subseteq (q_o + q_1)\ D(q_0 + q_1)$ (with unit $q_0 + q_1$) that is generated by the monomials with an equal number of e_{12} 's and e_{21} 's.*

Proof. We have to show that the subalgebra $(q_0 + q_1)B(q_0 + q_1)$ coincides with the subalgebra $C \subseteq (q_0 + q_1)(B \times_\beta\ Z)(q_0 + q_1) = (q_0 + q_1)\{B, u\}''(q_0 + q_1)$ that is generated by monomials in a and $(e_{ij})_{i,j=1}^2$ containing an equal number of e_{12} 's and e_{21} 's.

Clearly C is invariant under the action of R (or T) on D given by the modular group of ψ which acts by $\sigma_t^\psi(u) = \lambda^{it}u, \sigma_t^\psi|_{\ B} = Id_{\ B}$ so that $C \subseteq (q_0+q_1)\ D^{\ R}(q_0+q_1) = (q_0 + q_1)B(q_0 + q_1)$.

Hence we have to only prove the reverse inclusion. But due to the specific form of the generators in B, we obtain that B is generated by elements of the form

$$m = f^1 q_{i_1} f^2 q_{i_2} ... f^n q_{i_n} f^{n+1}$$

where the conditions on $i_1, ..., i_n$ are

$$a)\ i_{j+1} \in \{i_j, i_j - 1, i_j + 1\},\ j = 1, 2, ..., n,\ i_0, i_n \in \{0, 1\}$$

$$b)\ \ card\ \{j | i_j = s\}\ is\ even\ ,$$

while f is one of the elements

$$\alpha_s(q_0 X q_0);\ \alpha_s(q_0 X q_1); \alpha(q_1 X q_0)\ or\ \alpha_s(q_1 Y q_1),$$

where s is either i_j or i_{j+1} if $i_j \neq i_{j+1}$. If $i_j = i_{j+1}$, then either $s = i_j$ and $f^j = \alpha_s(q_0 X q_0)$ or either $s = i_{j-1}$ and $f^j = \alpha_s(q_1 Y q_1)$.

The assumptions we made are sufficient to show that in such a monomial we have some symbols corresponding to $\alpha_s(a)$ which are then necessary followed by symbols corresponding to $\alpha_{s+1}(a)$ (or to $\alpha_{s-1}(a)$). Moreover in m this sets of symbols are always separated by one of the projections q_p ($p \in \{s, s \pm 1\}$).

If we replace in m any such q_p by $q_1 u$ (or respectively by $u^* q_1$) and we replace the symbols from $\alpha_s(a)$ by the corresponding symbols in a we get the same m, but this time expressed as an element in the subalgebra of C, generated by monomials with equal occurrence number of e_{12} 's and e_{21} 's. This ends the proof of Lemma 2.

To conclude the proof, we note the following observation:

Remark. *Let* $B, D = B \times_\beta Z$ *and* $u, \{q_i\}_{i \in Z}$ *be as before . Then* $(q_0+q_1)D(q_0+q_1)$ *coincides with the algebra generated by* $(q_0+q_1)B(q_0+q_1)$ *and* $e_{12} = q_1 u = u q_0$.

Proof. With q_0, q_1 as before we have to show that $q_0(u^*)^n b q_0 = q_0(u^*)^n q_n b q_0$ is contained in the algebra generated by $(q_0+q_1)B(q_0+q_1)$. Assume $n > 1$; we may express $q_n b q_0$ as

$$q_n b_n q_{n-1} b_{n-1} q_{n-2} ... q_1 b_1 q_0.$$

Then

$$q_0(u^*)^n b q_0 = q_0(u^*)^n q_n b_n q_{n-1} ... q_1 b_1 q_0 =$$
$$= q_0 u^* \alpha_{n-1}(b_n) q_0 u^* \alpha_{n-2}(b_{n-1}) q_0 ... q_0 u^* b_1 q_0$$

which is an element in the algebra generated by $(q_0+q_1)B(q_0+q_1)$ and $u q_0$.

Proof of the theorem

Clearly the subalgebra generated by e_{12} and all the elements in

$$M_2(\mathbf{C}) * L^\infty([0,1], \nu)$$

with equal occurrence number of e_{12} 's and e_{21} 's coincides with the algebra itself. Thus $M_2(\mathbf{C}) * L^\infty([0,1], \nu)$ with the free product state ϕ is identified with $(q_0+q_1)D(q_0+q_1)$ with the restriction of ψ (which is generated by $u q_0 = q_1 u$, and a). In particular the modular group of ϕ is $\sigma_t^\phi(e_{ij}) = \lambda^{it} e_{ij}$ and σ_t^ϕ is the identity on $L^\infty([0,1], \nu)$.

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