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ON THE FRONTIER OPERATOR

by

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1. INTRODUCTION

It is well known that there are many equivalent ways of introducing a topology on a set : by Kuratowski closure operator, be the class of open sets (closed sets), by neighborhoods. We shall study a new method to define a topology, namely by the frontier operator, the relation between the closure operator and the frontier operator and also the characterization of connected sets, continuous functions using the frontier operator.

2. DEFINITIONS

Let X be a set and let 2^X be the family of all subsets of X. Let us denote the relative complement of A with respect to X by $C_X A$ whenever A is a set. Let Ø be the void set.

Let (X, τ) be a topological space. Let $\mathscr{F}$ be the family of τ-closed sets. The τ-closure operator $k : 2^X \mapsto 2^X$ is defined by $kA = \cap \{Z : Z \in \mathscr{F}$ and $A \subseteq Z\}$; the τ-interior operator $i : 2^X \mapsto 2^X$ is defined by $iA = \cup \{Z : Z \in \tau$ and $Z \subseteq A\}$; the τ-frontier operator $f : 2^X \mapsto 2^X$ is defined by $fA = kA \cap kC_X A = C_{kA}(iA)$, whenever $A \in 2^X$.

Let $k : 2^X \mapsto 2^X$ be an operator such that the following statements hold for all $A, B \in 2^X$;

$$k_1 \cdot k\text{Ø} = \text{Ø}.$$

$$k_2 \cdot A \subseteq kA.$$

$$k_3 \cdot kkA = kA.$$

$$k_4 \cdot (kA) \cup kB = k(A \cup B).$$

Then the set $\tau_k = \{Z : Z \subseteq X$ and $kC_X Z = C_X Z\}$ is a topology on X and kA is the τ_k-closure of A (see [1], [2], [4]).

Let $i : 2^X \mapsto 2^X$ be an operator such that the following statements hold for all $A, B \in 2^X$:

$$I_1 \cdot iX = X.$$

$$I_2 \cdot iA \subseteq A.$$

$$I_3 \cdot iiA = iA.$$

$$I_4 \cdot (iA) \cap iB = i(A \cap B).$$

Then the set $\tau_i = \{Z : Z \subseteq X \text{ and } iZ = Z\}$ is a topology on X, and iA is the τ_i-interior of A (see [1], [2]).

A topological space (X, τ) is called connected if X is not the union of two non-void separated sets A, B (that is : if k is the τ-closure operator, then $kA \cap B$, $kB \cap A$ are both void).

Let (X, τ), $(Y, \mathcal{U})$ be two topological spaces. A function $f : X \mapsto Y$ is called continuous if and only if the inverse of each $\mathcal{U}$-open set is τ-open.

3. THE FRONTIER OPERATOR

We shall now give eight sets of independent axioms for the frontier operator and construct the associated topology. Such a set is given also in [4] but it differs from ours.

L e m m a 1. *Let (X, τ) be a topological space and let F_2 be the frontier operator. The fololowing statements hold whenever* $A, B \subseteq X$ (see [1]).

1). $F_2F_2A \subseteq A \cup F_2A$.

2). $C_AF_2A \subseteq C_XF_2C_XF_2A$.

3). $(A \cup F_2A) \cup (B \cup F_2B) = A \cup B \cup F_2(A \cup B)$.

4. $C_AF_2A \subset C_BF_2B = C_MF_2N$ if $M = A \cap B$.

5). $F_2\emptyset = \emptyset$.

6). $F_2X = \emptyset$.

7).$F_2C_XA = F_2A$.

L e m m a 2. *Let $f : 2^X \mapsto 2^X$ be an operator such that the following statements hold for $A, B \subseteq X$*

$$B_1 \cdot ffA \subseteq A \cup fA.$$

$$B_2 \cdot (A \cup fA) \cup (B \cup fB) = A \cup B \cup f(A \cup B).$$

$$B_3 \cdot f\emptyset = \emptyset.$$

Then the operator $ClA = A \cup fA$ for $A \in 2^X$ *is a closure operator and the associated topology is* $\tau_k = \{A : A \subseteq X$ *and* $fC_XA \subseteq X_XA\}$.

P r o o f. From B_3, $Cl\emptyset = \emptyset$. By virtue of axiom B_2 it follows that $ClA \cup ClB = Cl(A \cup B)$. By definition $A \subseteq ClA$. Since $ClClA = Cl(A \cup fA) = ClA \cup ClfA) \cup ffA$, by axiom B_1 it follows that $ClClA = ClA$. Hence Cl is a closure operator and the associated topology is $\{A : A \subseteq X$ and $ClC_XA = C_XA\} = \tau_k$.

L e m m a 3. *Let* $f : 2^X \mapsto 2^X$ *be an operator such that the following statements hold for* $A, B \subseteq X$.

$$B_1' \cdot C_AfA \subseteq C_XfC_XfA.$$

$$B_2' \cdot C_Afa \cap C_BfB = C_MfM \text{ if } M = A \cap B.$$

$$B_3' \cdot fX = \emptyset.$$

Then the operator $Im : 2^X \mapsto 2^X$ *defined by* $InA = C_AfA$ *for* $A \in 2^X$ *is an interior operator associated topology is* $\tau_i = \{A : A \subseteq X$ and $fA \subseteq C_XA\}$.

P r o o f. By definition $InA \subseteq A$. By axiom B_2' it follows that $InA \cap InB = In(A \cap B)$. By B_3', $InX = X$. Since $InInA = In(A \cap C_XfA) = InA \cap InC_XfA = (A \cap C_XfA) \cap C_XfC_XfA$, by virtute of axiom B_1', $InInA = InA$. Hence In is an interior operator and the associated topology is $\{A : A \subseteq X$ and $InA = A\} = \tau_i$.

L e m m a 4. *If* $f : 2^X \mapsto 2^X$ *is an operator such that*: $B_4 \cdot fC_XA = fA$ holds for $A \in 2^X$, *then the following statements are true, for* $i \in \{1, 2, 3\}$:

$$E_i \cdot f \text{ satisfies the axiom } B_i \Leftrightarrow f \text{ satisfies axiom } B_i'.$$

P r o o f. Whenever $A \in 2^X, ffA \subseteq A \cup fA$ is equivalent to $C_XA \cap C_XfA \subseteq CffA \Leftrightarrow C_XA \cap C_XfC_XA \subseteq C_XfC_XfC_XA \Leftrightarrow B \cap C_XfB \subseteq C_XfC_XfB$, whenever $B \subseteq X$ (that is $B = C_XA$ for $A \in 2^X$) and E_1 follows.

Whenever $A, B \in 2^X$, $(A \cup fA) \cup B \cup fB = A \cup B \cup f(A \cup B) \Leftrightarrow (C_XA \cap C_XfC_XA) \cap (C_XB \cap C_XfC_XB) = C_XA \cap C_XB \cap C_Xf(C_XA \cap C_XB) \Leftrightarrow (N \cap C_XfN) \cap (P \cap C_XfP) = M \cap C_XfM$ if $M = N \cap P$ and $N, P \in 2^X$, $N = C_XA$; $P = C_XB$. Hence follows E_2.

E_3 follows from $f\emptyset = fX$.

P r o p o s i t i o n 1. *Let* $f : 2^X \mapsto 2^X$ *be an operator which satisfies one of the following sets of axioms* :

$$S_1 = \{B_1, B_2, B_3, B_4\}; \qquad S_2 = \{B_1, B_2, B_3', B_4\};$$

$$S_3 = \{B_1, B_2', B_3, B_4\}; \qquad S_4 = \{B_1, B_2', B_3', B_4\};$$

$$S_5 = \{B_1', B_2, B_3, B_4\}; \qquad S_6 = \{B_1', B_2, B_3', B_4\};$$

$$S_7 = \{B_1', B_2', B_3, B_4\}; \qquad S_8 = \{B_1', B_2', B_3', B_4\}.$$

Using the notations from Lemma 2 and Lemma 3, the following statements are true :

$F_1 \cdot Cl$ is a closure operator and the associated topology is τ_k.

$F_2 \cdot In$ is an interior operator and the associated topology is τ_i.

$F_3 \cdot \tau_f = \tau_i$. Let $\tau = \tau_k = \tau_i$.

$F_4 \cdot$ The τ-frontier operator is f.

P r o o f. By virtue of Lemma 4, if f satisfies one of the following sets of axioms : $S_1, S_2, \ldots, S_8$, then it satisfies the axioms $B_1, B_1', B_2, B_2', B_3, B_3', B_4$. By virtue of Lemmas 1 and 2, F_1 and F_2 follow.

Whenever $A \in 2_X$, $A \in \tau_k \Leftrightarrow FA = fC_XA \subseteq C_XA \Leftrightarrow A \in \tau_i$. Hence $\tau_k = \tau_i = \tau$.

Whenever $N \in 2^X$, $fN = ClN \cap C_XInN$. Because Cl is the τ-closure operator and In is the τ-interior operator it follows that f is the τ-frontier operator.

P r o p o s i t i o n 2. *The sets of axioms $S_1, S_2, \ldots, S_8$ are composed of independent axioms.*

P r o o f. We shall give a proof by exemples.

E x e m p l e 1. Let $X = \{1, 2\}$, $F : 2^X \mapsto 2^X$ defined by $FA = \{1\}$ for $A \subseteq X$. Then F satisfies $B_1, B_1', B_2, B_2', B_4$ but not B_3, B_3' because $F\emptyset = FX = \{1\}$. Hence the axioms B_3, B_3' are independent of the others.

E x a m p l e 2. Let $X = \{1, 2, 3\}$, $f : 2^X \mapsto 2^X$ defined by $f\{2, 3\} = f\{1\} = \{1, 2\}$, $f\emptyset = fX = \emptyset$, $f\{1, 2\} = f\{3\} = f\{2\} = f\{1, 3\} = X$. Then F satisfies the axioms $B_2, B_2', B_3, B_3', B_4$ but not B_1, B_1' because $ff\{1\} = X$ and $\{1\} \cup f\{1\} = \{1, 2\}$. Hence the axioms B_1', B_1 are independent of the others.

E x a m p l e 3. Let $X = \{1, 2, 3\}$ and $f : 2^X \mapsto 2^X$ defined by $f\emptyset = fX = f\{1\} = f\{2, 3\} = \emptyset$; $f\{2\} = f\{1, 3\} = \{2\}$; $f\{3\} = f\{1,2\} = \{3\} \cdot f$ satisfies $B_1, B_1', B_3, B_3', B_4$ but not B_2, B_2' because $\{1\} \cup \{3\} \cup f\{1\} \cup f\{3\} \neq \{1\} \cup \{3\} \cup f\{1, 3\}$. Hence, the axioms B_2, B_2' are independent of the others.

E x a m p l e 4. Let $X = \{1, 2\}$ and $f : 2^X \mapsto 2^X$ defined by $f\{1\} = \{1\}$ $f\{2\} = \{2\}$, $f\emptyset = fX = \emptyset$. f satisfies $B_1, B_1', B_2, B_2', B_3, B_3'$, but not B_4. Hence B_4 is an independent axiom.

4. CONNECTED SETS AND CONTINUOUS FUNCTIONS

We shall study now some problems concerning relativization, connected sets and continuous functions.

(X, τ) being a topological space, we shall denote with Cl_τ, In_τ and Fr_τ the τ-closure, τ-interior and τ-frontier operators.

L e m m a 5. *Let (X, τ) be a topological space $(Y, \mathcal{U})$ a subspace of (X, τ) and $(Z, \mathcal{V})$ a subspace of $(Y, \mathcal{U})$. Then whenever $A \subseteq Y$, the following statements hold :*

$$R_1 \cdot Fr_{\mathcal{U}}A = Cl_\tau A \cap Cl_\tau C_Y A \cap Y.$$

$$R_2 \cdot Fr_{\mathcal{V}}(A \cap Z) \subseteq Fi_{\mathcal{U}}A.$$

Proof. By the definition of relativization R_1 follows. Let $T = Y \cap C_X Z(Y = Z \cup T;\ Z \cap T = \emptyset)$. $Fr_{\mathscr{U}} A = Cl_\tau A \cap Cl_\tau C_Y A \cap Y = (Cl_\tau A \cap (Cl_\tau C_Z A) \cap Z) \cup (Cl_\tau A \cap Cl_\tau C_Z A \cap T) \cup (Cl_\tau \cap Cl_\tau (C_Y A \cap T) \cap Y)$. Hence $Fr_{\mathscr{V}}(A \cup Z) = Cl_\tau(A \cup Z) \cap Cl_\tau C_Z A \cap Z \subseteq Fr_{\mathscr{U}} A$.

Using Lemma 5 we shall prove the following known statement: Let (X, τ) be a topological space and $(Y, \mathscr{U})$ a topological connected subspace of (X, τ). Then the τ-closure of Y is also a connected set.

Proof. Let $M = Cl_\tau Y$. Let $\mathscr{V}$ be the relativization of τ to M. Whenever A is a subset of M such that $Fr_{\mathscr{V}} A = \emptyset$, by Lemma 5 it follows that $Fr_{\mathscr{U}}(A \cap Y) = \emptyset$ and since Y is connected, $A \cap Y = \emptyset$ or $A \cap Y = Y$.

We suppose that $A \cap Y = Y \Leftrightarrow Y \subseteq A$. Since $Cl_{\mathscr{V}} Y = Cl_\tau Y \cap Cl_\tau Y = Cl_\tau Y$ and $Cl_{\mathscr{V}} A = A$ and $Y \subseteq A$ it follows that $Cl_{\mathscr{V}} Y \subseteq A$. Because $A \subseteq M = Cl_\tau Y$ it follows $A = M$.

If $A \cap Y = \emptyset$, then if $B = C_M A$, $B \cap Y = Y$ and since B is also a $\mathscr{V}$-closed set, using the same demonstration it follows that $B = M$ and $A = \emptyset$. Hence M is connected.

Proposition 3. *Let (X, τ), $(Y, \mathscr{U})$ be two topological spaces and let $g : X \mapsto Y$ be a function. Then the following statements are equivalent.*

$C_1 \cdot$ *g is continuos.*

$C_2 \cdot gCl_\tau A \subseteq Cl_{\mathscr{U}} gA$ *for every* $A \in 2^X$.

$C_3 \cdot gA \cup gFr_\tau A \subseteq gA \cup Fr_{\mathscr{U}} gA$, *for every* $A \in 2^X$.

$C_4 \cdot Cl_\tau g^{-1}B \subseteq g^{-1}Cl_{\mathscr{U}} B$, *for every* $B \in 2Y$.

$C_5 \cdot g^{-1}B \cup Fr_\tau g^{-1}B \subseteq g^{-1}B \cup g^{-1}Fr_{\mathscr{U}} B$, *for every* $B \in 2^Y$.

Proof. For the equivalences $C_1 \Leftrightarrow C_2 \Leftrightarrow C_4$ see [2]. Since for every $C, D \in 2^X$, $g(C \cup D) = gC \cup gD$ and for every $E, G \in 2^Y$, $g^{-1}(E \cup G) = g^{-1}E \cup g^{-1}F$, it follows that $C_2 \Leftrightarrow C_3$ and $C_4 \Leftrightarrow C_5$.

5. THE FRONTIER AND COMPLEMENT PROBLEM

The Kuratowski closure and complement problem is as follows: if A is subset of a topological space, then at most 14 sets can be constructed from A by complementation and closure. We shall study what happens if we replace the closure operator by the frontier operator.

Proposition 4. *If (X, τ) is a topological space and $A \subseteq X$, then at most 6 sets can be constructed from A by applying the frontier and the complement operator.*

Proof. Let us denote $C_X A$ by CA and $Fr_\tau A$ by fA. First notice that $fffA = ffA$ (see [3]). The six sets which can be obtained are the following: $A_1 = A$, $A_2 = fA$, $A_3 = CA$, $A_4 = ffA$, $A_5 = CfA$, $A_6 = CffA$. It can be very easy verified that any other set constructed from A by applying the frontier and the complement operator is equal of one of the sets $A_1, A_2, \ldots, A_6$, because $ffB = fffB$, $fB = fCB$ for every $B \in 2^X$.

E x a m p l e 5. Let Q and R be the set of all rational, respectively, real numbers; $I = C_R Q$. Let τ be the usual topology on R and let a, b in R with $a < b$. Let $A = Q \cap \{z : a \leqslant z \leqslant b\}$;

Then the sets $A_1, A_2, \ldots, A_6$ are distinct. ($A_1 = A$; $A_2 = \{z : a \leqslant z \leqslant b\}$; $A_3 = I \cap \{z : a \leqslant z \leqslant b\}$; $A_4 = \{a, b\}$; $A_5 = \{z : z \in R$ and $z < a$ or $z > b\}$; $A_6 = C_R\{a, b\}$).

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