ON UNBOUNDED, NON-TRIVIAL HOCHSCHILD COHOMOLOGY IN FINITE VON NEUMANN ALGEBRAS AND HIGHER ORDER BEREZIN'S QUANTIZATION

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We introduce a class of densely defined, unbounded, 2-Hochschild cocycles [14] on finite von Neumann algebras M. Our cocycles admit a coboundary, determined by an unbounded operator on the standard Hilbert space associated to the von Neumann algebra M. For the cocycles associated to the Γ-equivariant deformation [17] of the upper half-plane ($\Gamma = \mathrm{PSL}_2(\mathbb{Z})$), the "imaginary" part of the coboundary operator is a cohomological obstruction – in the sense that it can not be removed by a "large class" of closable derivations, with non-trivial real part, that have a joint core domain, with the given coboundary.

As a byproduct, we prove a strengthening of the non-triviality of the Euler cocycle in the bounded cohomology $H^2_{\mathrm{bound}}(\Gamma, \mathbb{Z})$ [2].

INTRODUCTION

In this paper, we analyze from an abstract point of view a cohomological obstruction that appears in the study of the Γ-equivariant Berezin deformation quantization of the upperhalf plane. In this paper $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, or a congruence subgroup.

In the case of the Γ-equivariant Berezin type quantization of the upper half-plane $\mathbb{H} \subseteq \mathbb{C}$ considered in [16, 17] (see also [1, 12]), the deformation consists in a family of type II_1 von Neumann algebras $(\mathcal{A}_t)_{t>1}$ indexed by the parameter $t \in (1, \infty)$, and a symbol map which we describe below. Each algebra $(\mathcal{A}_t)_{t>1}$ is embedded in the bounded operators acting on the Hilbert space H_t, consisting of analytic functions on the upper half-plane, square integrable with respect to a measure depending on $t > 1$.

These Hilbert spaces are endowed with projective unitary representations $\pi_t : \mathrm{PSL}_2(\mathbb{R}) \to B(H_t)$, $t > 1$, from the extended discrete series [15], of projective unitary representations of $\mathrm{PSL}_2(\mathbb{R})$. In this representation, the algebra $\mathcal{A}_t$ is the commutant algebra $\{\pi_t(\Gamma)\}'$. The algebras $\mathcal{A}_t$ are finite von Neumann algebras (factors, *i.e.* they have trivial centers), as shown in [8, 16].

As observed in [8], the automorphic forms correspond to intertwiners for the representations π_t, when restricted to Γ. Using the canonical branch for the logarithm of the Dedekind Δ function, the multiplication operators by the powers Δ^ε, $\varepsilon > 0$, of Δ, give intertwining operators for the representations $\pi_t|\Gamma$, $\pi_{t+\varepsilon}|\Gamma$ acting on the Hilbert spaces H_t, and respectively $H_{t+\varepsilon}$. We denote these injective, linear, bounded, intertwining multiplication operators by $S_{t+\varepsilon,t}$. These operators are compared with the canonical inclusions $I_{t+\varepsilon,t}$, $H_t \to H_{t+\varepsilon}$.

Out of this data (see [17]), one obtains a family of completely positive maps $\Psi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$, $s \geq t > 1$, the Berezin symbol map.

Concomitantly, using the intertwiners described above, one obtains a family

$$\beta_{s,t} : \mathcal{A}_t \to \mathcal{A}_s, \beta_{s,t}(a) = S_{s,t}a(S_{s,t})^*, \quad s \geq t > 1, a \in \mathcal{A}_t$$

of completely positive maps. Consequently, the linear map on $\mathcal{A}_t$ with values in $\mathcal{A}_s$, defined by the formula:

$$\alpha_{s,t}(a) = \beta_{s,t}(1)^{-1/2}\beta_{s,t}(a)\beta_{s,t}^{-1/2}(1), \quad a \in \mathcal{A}_s$$

is an isomorphism from $\mathcal{A}_t$ onto $E_t^s \mathcal{A}_s E_t^s$, where E_t^s is a projection, of trace $\frac{\chi_t}{\chi_s}$ in $\mathcal{A}_s$. More precisely E_t^s is the orthogonal projection on the closure of the image of $S_{s,t}$. Note that $\beta_{s,t}(1)^{1/2}$ is the absolute value of the intertwining operator $S_{s,t}$. Clearly, $\alpha_{s,t}$ is an isomorphism onto its image. Here χ_t, $t > 1$, is a linear, increasing function, with positive values, depending on t, related to the Plancherel dimension (coefficient) of π_t (see *e.g.* [15]).

In this paper we initiate the axiomatization of the properties of the Berezin deformation quantization, properties which were used in [17] to construct a cohomological data for the deformation. This cohomological data consists of a family of unbounded Hochschild 2-cocycles $(c_t)_{t>1}$ (see [5, 10, 14, 17] for definitions) defined on dense unital $*-$ subalgebras of $\mathcal{A}_t$ (see [17]).

The cocycles $(c_t)_{t>1}$ are associated to the deformation; they represent the obstruction to construct isomorphisms between the algebras corresponding to different values of the deformation parameter t. Formally, this 2-cocycle is obtained by differentiating the multiplication operation for operators with fixed symbols.

We prove in Theorem 3 that the "imaginary part" c_t^0 of the unbounded Hochschild cocycle c_t (see Definition 2, and see also [6]), associated to this

deformation data, is implemented, for a fixed t, by the "infinitesimal generator", at t, of the family $(\alpha_{s,t})_s$.

We prove that this generator is of the form $Z_t = \lambda_t 1 + iY_t$, where Y_t is a symmetric, unbounded operator, acting on the standard Hilbert space $L^2(\mathcal{A}_t)$ associated to the algebra $\mathcal{A}_t$ and to its unique trace, with the additional property that $iY_t^* 1 = -\lambda_t 1$, where $\lambda_t = \frac{1}{2}\frac{\chi_t'}{\chi_t} > 0$. This clearly prevents the identity element 1 to belong to the domain of Y_t. Thus, $(c_t^0, \lambda_t 1 + iY_t)$ is an invariant "characterizing" the algebra $\mathcal{A}_t$ and which depends on the domain for the unbounded operators considered.

In terms of the space of Berezin symbols for operators in $B(H_t)$ this construction is described as follows: an operator $A \in B(H_t)$ is represented by a kernel $k_A = k_A(z, \overline{\eta})$ on $\mathbb{H} \times \mathbb{H}$, analytic in the first variable and antianalytic in the second, so that

$$Af(z) = \chi_t \int_{\mathbb{H}} \frac{k_A(z, \overline{\eta})}{(z - \overline{\eta})^t} f(\eta) \mathrm{d}\nu_t(\eta), \quad z, \eta \in \mathbb{H},$$

for $f \in H_t = H^2(\mathbb{H}, \mathrm{d}\nu_t)$, $t > 1$. Here, for $t \geq 0$ we let $\mathrm{d}\nu_t$ be the measure $(\mathrm{Im}\, z)^{t-2}\mathrm{d}z\mathrm{d}\overline{z}$ on $\mathbb{H}$. Moreover, we let $\chi_t = \frac{t-1}{\pi}$. To indicate the antianalytic dependence in the second variable, we put the conjugation symbol on the second variable. We identify A with its reproducing kernel k_A.

We obtain the following formula representing the kernel of the product in $B(H_t)$ of the operators defined by kernels k, l

$$(k *_t l)(z, \overline{\zeta}) = \chi_t \int_{\mathbb{H}} k(z, \overline{\eta}) l(\eta, \overline{\zeta}) [z, \overline{\eta}, \eta, \overline{\zeta}]^t \mathrm{d}\nu_0(\eta), \quad z, \zeta \in \mathbb{H},$$

where, for $z, \eta, \zeta \in \mathbb{H}$, the expression $[z, \overline{\eta}, \eta, \overline{\zeta}] = \frac{(z - \overline{\zeta})(\eta - \overline{\eta})}{(z - \overline{\eta})(\eta - \overline{\zeta})}$ is the four point function.

In [16] (see also the review in the introduction of [17]) we constructed a dense unital *-algebra domain $\mathcal{D}_{c_t} \subseteq \mathcal{A}_t$, so that right-differentiating in the t parameter, the formula for the product of two symbols $k, l \in \mathcal{D}_{c_t}$ yields an element $c_t(k, l)$ in $\mathcal{A}_t$, whose reproducing kernel on $\mathbb{H} \times \mathbb{H}$ is given by the formula:

$$c_t(k, l)(z, \zeta) = \frac{\mathrm{d}}{\mathrm{d}s}\left[(k \star_s l)(z, \zeta)\right]\Big|_{s \to t_+}$$
$$= \frac{\chi_t'}{\chi_t}(k \star_t l)(z, \zeta) + \chi_t \int_{\mathbb{H}} k(z, \overline{\eta}) l(\eta, \overline{\zeta}) [z, \overline{\eta}, \eta, \overline{\zeta}]^t \ln[z, \overline{\eta}, \eta, \overline{\zeta}] \mathrm{d}\nu_0(\eta).$$

The space $L^2(\mathcal{A}_t)$ is then a space of analytic-antianalytic functions $\mathbb{H}^2$, with norm

$$\|k\|_{L^2(\mathcal{A}_t)}^2 = \chi_t \iint_{F \times \mathbb{H}} |k(z, \overline{\eta})|^2 \mathrm{d}_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta),$$

where

$$d_{\mathbb{H}}(z, \eta) = \frac{|z - \overline{\eta}|}{(\operatorname{Im} z)^{1/2}(\operatorname{Im} \eta)^{1/2}}$$

is a function depending only on the hyperbolic distance between $z, \eta \in \mathbb{H}$.

The "imaginary" part c_t^0 of the deformation Hochschild cocycle (see Definition 2) is then implemented by $Z_t = \frac{1}{2}\lambda_t + i\mathcal{T}_{\operatorname{Im}(\ln \varphi)}^{t,\Gamma}$ where $Y_t = \mathcal{T}_{\operatorname{Im}(\ln \varphi)}^{t,\Gamma}$ is the unbounded Toeplitz operator on $L^2(\mathcal{A}_t)$ (see Chapter 2, Definition 10) with symbol $\operatorname{Im}(\ln \varphi)$, and $\varphi(z, \overline{\zeta}) = \Delta(z)\overline{\Delta(\zeta)}(z - \overline{\zeta})^{1/2}$, $z, \zeta \in \mathbb{H}$, and $\lambda_t = \frac{\chi_t'}{\chi_t} > 0$ (here we use a principal value for the logarithm).

As a corollary, we prove in Theorem 3 that the operator $\mathcal{T}_{i\operatorname{Im}(\ln \varphi)}^{t,\Gamma}$ is antisymmetric with deficiency indices $(0, 1)$. This is in fact another way to express the fact that the Euler cocycle is non-trivial [2] in the bounded cohomology group $H_{bd}^2(\Gamma, \mathbb{Z})$. This statement remains valid for modular subgroups of $\operatorname{PSL}_2(\mathbb{Z})$.

Finally we consider the space of operators on $L^2(\mathcal{A}_t)$ with Toeplitz symbols, identifying $L^2(\mathcal{A}_t)$ with a Hilbert space of analytic functions. Assume that δ is a derivation, densely defined on $\mathcal{A}_t$, that admits a measurable function as a Toeplitz symbol, as in Definition 10, (and one mild condition that it extends with zero on a dense domain $\mathcal{D}$ in $B(H_t)$, so that $\mathcal{D} \cap L^2(\mathcal{A}_t) \subseteq \mathcal{D}(Y_t)$ is a core for Y_t, the coboundary operator defined in the previous paragraph).

We prove in Theorem 19 that the obstruction for the cocycle c_t^0 of having a coboundary with nontrivial real part, cannot be removed by perturbing the coboundary operator Z_t with the derivation δ.

To prove all these results we consider an abstract framework, assuming that we are given a family $(\mathcal{A}_t)_{t>1}$ of finite von Neumann algebras. Then, the symbol map is described by a Chapman-Kolmogorov system of linear maps $(\Psi_{s,t})_{s \geq t > 1} : \mathcal{A}_t \to \mathcal{A}_s$ (see also [17]).

We assume that the Chapman-Kolmogorov system $(\Psi_{s,t})_{s \geq t > 1}$ consists of completely positive, unit preserving maps. Thus, we are given a family of finite von Neumann algebras $\mathcal{A}_t$, and we assume that $\Psi_{s,t}$ maps $\mathcal{A}_t$ injectively into $\mathcal{A}_s$, for $s > t$, with dense range.

The cocycle c_t is in this case defined on a dense domain (if it exists) as $c_t(k, l) = \frac{d}{ds}\Psi_{s,t}^{-1}(\Psi_{s,t}(k)\Psi_{s,t}(l))|_{s \to t_+}$ for k, l in a dense domain. Obviously, $c_t(1, 1) = 0$, as $(\Psi_{s,t})_{s > t}$ are unital.

It is clear that if there exists a single von Neumann algebra $\mathcal{A}$, and isomorphisms $\alpha_t : \mathcal{A}_t \to \mathcal{A}$, $t > 1$, so that $\widetilde{\Psi_{s,t}} = (\alpha_t^{-1}\Psi_{s,t}\alpha_s)_{s,t}$ is still compatible with the differential structure, then the Hochschild cocycles $\widetilde{c}_t$, associated to $(\widetilde{\Psi_{s,t}})_{s \geq t > 1}$, are (up to the domain) equal to $\alpha_t^{-1}c_t\alpha_t$. Obviously, in this case the generator $\widetilde{\alpha}_t = \frac{d}{ds}\widetilde{\Psi_{s,t}}|_{s \to t_+}$ would contain the identity element 1 in its domain.

Hence, the problem described above, about the structure of the (unbounded) Hochschild cocycles $(c_t)_{t>1}$, is related to the obstruction to deform the system $(\Psi_{s,t})_{s>t}$ into a Chapman-Kolmogorov system of completely positive maps, "living" on a single algebra.

Based on this generalization of the obstruction in realizing the family of the symbol maps on a single algebra, we introduce a new invariant for finite von Neumann algebras M, consisting of a pair (c, Z), where c is an unbounded 2-Hochschild cohomology cocycle with domain $\mathcal{D}$, a unital $*$-algebra, and Z is an unbounded coboundary operator for c, with domain $\mathcal{D}_0 \subseteq \mathcal{D}$ not containing the space of multiples of the identity operator 1.

We assume that $c(1, 1) = 0$ and that $Z = \alpha + X + iY$ has antisymmetric, unbounded part Y, so that Y has deficiency indices $(0, 1)$ and moreover $(iY^*)1 = (-\alpha)1$, $\alpha > 0$, and X is selfadjoint with $X1 = 0$. Therefore $Z^*1 = 0$, and hence, 1 is forbidden to belong to the domain of Y since $(iY^*)1 = (-\alpha)1$. We also assume reality for the 2-cocycle c_t, that is we assume that $c(k^*, l^*) = c(l, k)^*$, $X(k^*) = X(k)^*$, $Y(k^*) = Y(k)^*$, for $l, k \in \mathcal{D}$.

The new invariant for the algebra M and the data (c, Z) consists of the existence (or not) of an unbounded derivation δ on M, $\delta(k^*) = k^*$ for $k \in \text{Dom}\,\delta$, with non-trivial real part $\text{Re}\,\delta = \alpha 1$, so that $Z + \delta$ is defined at 1 with $(Z + \delta)1 = 0$, and $iY + (\delta - \text{Re}\,\delta)$ is selfadjoint (here we require that $\text{Dom}(Z) \cap \text{Dom}(\delta)$ is a core for $Z + \delta$). Here one therefore asks the question of whether one can (or can not) perturb Z by a derivation so that the imaginary part of the cocycle c admits a coboundary which is also an antisymmetric operator, that is also defined at 1.

In our example of the Γ-equivariant quantization, for every $t>1$ there exists a 1-parameter semigroup $(\Phi_\varepsilon^t)_{\epsilon \geq 0}$ of densely defined and completely positive maps on their domain, with the property that the coboundary operator Z_t is the infinitesimal generator $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Phi_\varepsilon^t|_{\varepsilon=0}$, and the domain of Φ_ε^t is controlled by another commuting family D_ε of bounded completely positive maps on $\mathcal{A}_t$. Moreover, $\overline{(\text{range}\Phi_\varepsilon^t(1))}_{\varepsilon>0}$ is an increasing family of projections, increasing to 1 as $\varepsilon \to 0$.

The semigroup Φ_ε^t has the remarkable property that

$$\Phi_\varepsilon^t(1)^{-1/2}\Phi_\varepsilon^t\Phi_\varepsilon^t(1)^{-1/2}$$

is an isomorphism onto its range, scaling the trace and increasing the support as ε tends to 0. This is a feature that is rather common for finite von Neumann algebras having non-trivial (full) fundamental group.

This is interesting because (see Proposition 20) the finite von Neumann algebra $L(F_\infty)$ associated to the free group F_∞ with infinitely many generators admits such a derivation, that is it is acting as the derivative of a dilation on a specific subalgebra.

1. THE QUANTUM DYNAMICAL SYSTEM ASSOCIATED TO THE SYMBOL MAP, AND ITS ASSOCIATED DIFFERENTIAL STRUCTURE

In this chapter, we formalize the properties of the Berezin symbol map, and derive some consequences.

Assume we are given a family $(\mathcal{A}_t)_{t>1}$ of finite von Neumann algebras with faithful traces $\tau_{\mathcal{A}_t}$ (τ simply when no confusion in possible). The index set $(1, \infty)$ is taken here as a reference, it could be replaced by any other left open interval.

To introduce a "differentiable" type structure on the family $(\mathcal{A}_t)_{t>1}$, we consider a system $(\Psi_{s,t})_{s \geq t > 1}$ of unital, completely positive, trace preserving maps $\Psi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$.

This family of completely positive maps is, in fact, an abstract framework for the Berezin symbol. We assume that the maps $(\Psi_{s,t})_{s \geq t > 1}$ are injective, with weakly dense ranges. By $\Psi_{s,t}^{-1}$ we denote the unbounded inverse of the continuous extension of $\Psi_{s,t}$ to $L^2(\mathcal{A}_t)$. We assume that all operations of differentiation in the t-parameter, involving the maps $\Psi_{s,t}^{-1}$, have dense domains. This will be the case in our main example coming from the Berezin Γ-equivariant quantization. Being completely positive maps, we have that $\Psi_{s,t}(k^\star) = \Psi_{s,t}(k)^\star$, for all k in $\mathcal{A}_s$, $s \geq t > 1$.

In addition, we assume that the system $(\Psi_{s,t})_{s \geq t > 1}$ verifies the Chapman-Kolmogorov relations $\Psi_{r,s}\Psi_{s,t} = \Psi_{r,t}$, if $r \geq s \geq t > 1$.

The algebras $(\mathcal{A}_t)_{t>1}$ are also assumed to be Morita equivalent, in a functorial way, with the growth of the comparison maps (for the Morita equivalence) controlled by the absolute value $|\Psi_{s,t}|$ of the completely positive maps $(\Psi_{s,t})_{s \geq t > 1}$, viewed as contractions acting on $L^2(\mathcal{A}_t)$.

We introduce a set of assumptions. The first assumption describes the Morita equivalence between the algebras $(\mathcal{A}_s)_{s>1}$. In the sequel we will seldom identify a completely positive map on $\mathcal{A}_t$ with its extension to $L^2(\mathcal{A}_t)$.

ASSUMPTION F.M. We are given an increasing, differentiable function χ_t, $t > 1$, with strictly positive values. In the particular situation of the Berezin quantization (see [16]) the function χ_t, depending on t, is a linear function depending on the Plancherel coefficient of the projective unitary representation π_t.

We assume that we have a family of injective completely positive maps

$$\beta_{s,t} : \mathcal{A}_t \to E_t^s \mathcal{A}_s E_t^s, \quad s \geq t > 1,$$

where, for fixed s, $(E_t^s)_{1 < t \leq s}$ is a family of increasing (selfadjoint) projections in $\mathcal{A}_s$ of size (trace) $\frac{\chi_t}{\chi_s}$.

Observe that, for $1 < t \leq s$, $\beta_{s,t}(1) \in \mathcal{A}_s$ is a positive element, which we denote by $X_t^s \in (\mathcal{A}_s)_+$, $0 \leq X_t^s \leq 1$, $X_t^s \in E_t^s \mathcal{A}_s E_t^s$. We assume that X_t^s has

zero kernel for all $s \geq t$. By $(X_t^s)^{-1}$ we denote the (eventually unbounded) inverse of the operator X_t^s on its support.

We define, for $1 < t \leq s$, $a \in \mathcal{A}_t$,

$$\alpha_{s,t}(a) = (X_t^s)^{-1/2} \beta_{s,t}(a) (X_t^s)^{-1/2}.$$

We are also assuming that $\alpha_{s,t}$ is a surjective isomorphism from $\mathcal{A}_t$ onto $E_t^s \mathcal{A}_s E_t^s$. In the specific case of the Berezin quantization, the operator X_t^s is the absolute value of an intertwiner for the corresponding Hilbert spaces, so that $\alpha_{s,t}$ is automatically an isomorphism onto its image (see Chapter 6 of [17]). Thus, $\beta_{s,t}(a) = (X_t^s)^{1/2} \alpha_{s,t}(a) (X_t^s)^{1/2}$, $a \in \mathcal{A}_t$. If $Y_t^s \in \mathcal{A}_t$ is the pre-image of X_t^s through the morphism $\alpha_{s,t}$, $s \geq t$, then

$$\beta_{s,t}(a) = \alpha_{s,t}((Y_t^s)^{1/2} a (Y_t^s)^{1/2}), \quad a \in \mathcal{A}_t.$$

We assume that the family of completely positive maps $(\beta_{s,t})_{1<t\leq s}$ verifies the Chapman-Kolmogorov condition. Thus, we are assuming that

$$\beta_{r,s} = \beta_{r,t} \circ \beta_{t,s}, \quad \text{for } r \geq t \geq s > 1.$$

In addition, we are assuming that the two families of completely positive maps are correlated by a semigroup of "quasi positive" maps, as explained in the next assumption. By "quasi positive" map we mean a densely defined map which is completely positive on its domain. More precisely we make:

ASSUMPTION SQP. We assume that the diagram

$$
\begin{array}{ccccc}
\mathcal{A}_{t-\varepsilon} & \xrightarrow{\Psi_{t,t-\varepsilon}} & \mathcal{A}_t & \xrightarrow{\Psi_{t+\varepsilon,t}} & \mathcal{A}_{t+\varepsilon} \\
\Phi_\varepsilon^{t-\varepsilon} \uparrow & \nearrow \beta_{t,t-\varepsilon} & \uparrow \Phi_\varepsilon^t & \nearrow \beta_{t+\varepsilon,t} & \uparrow \Phi_\varepsilon^{t+\varepsilon} \\
\mathcal{A}_{t-\varepsilon} & \xrightarrow{\Psi_{t,t-\varepsilon}} & \mathcal{A}_t & \xrightarrow{\Psi_{t+\varepsilon,t}} & \mathcal{A}_{t+\varepsilon}
\end{array}
$$

is commutative, where Φ_ε^t is densely defined (the domain will be made explicit in the Assumption SQP1), and it is uniquely defined by the formula

$$\Phi_\varepsilon^t = \Psi_{t+\varepsilon,t}^{-1} \circ \beta_{t+\varepsilon,t} = \beta_{t,t-\varepsilon} \circ \Psi_{t,t-\varepsilon}^{-1}, \quad \varepsilon > 0.$$

Thus, we are also requiring the above equality.

We implicitly assume that $\Phi_\varepsilon^{t+\varepsilon}$ defined as $\beta_{t+\varepsilon,t} \circ \Psi_{t+\varepsilon,t}^{-1}$ and similarly for $\Phi_\varepsilon^{t-\varepsilon}$, Φ_ε^t, are making the above diagram commutative.

The next two assumptions describe the growth of the absolute value of the completely positive maps in the Chapman-Kolmogorov system.

ASSUMPTION D. For $s \geq t > 1$, the linear positive maps $D_{s-t}^t = \Psi_{s,t}^* \Psi_{s,t}$ on $L^2(\mathcal{A}_t)$, where the completely positive maps $\Psi_{s,t}$ are uniquely extended by continuity to $L^2(\mathcal{A}_t)$, have the property that $(D_\varepsilon^t)_{\varepsilon \geq 0}$ is a commuting family of positive operators acting on $L^2(\mathcal{A}_t)$. In the case of the Berezin quantization this

is simply the Toeplitz operator $\mathcal{T}^{t,\Gamma}_{d^\varepsilon_{\mathbb{H}}}$ on $L^2(\mathcal{A}_t)$ with symbol $d^\varepsilon_{\mathbb{H}}$, with $d_{\mathbb{H}}$ as in the introduction (see Definition 10 for the definition of the Toeplitz operator).

ASSUMPTION D$'$. For $r \leq t$, there exists a family of unbounded, densely defined, closed, injective, positive maps D^t_{r-t} acting on a dense subspace of $L^2(\mathcal{A}_t)$, that have the property that the domain of D^t_{r-t} is the image of $\Psi_{t,r}$. In addition $((D^t_{-\varepsilon})^{-1})_{\varepsilon\geq 0}$ is a commuting family of bounded operators commuting with $(D^t_\varepsilon)_{\varepsilon\geq 0}$. Thus, the image of $D^t_{-\varepsilon}$ coincides with the image of $\Psi_{t-\varepsilon,t}$ inside $L^2(\mathcal{A}_t)$.

In the case of the Berezin quantization, the operator $D^t_{-\varepsilon}, \varepsilon > 0$ is simply the (unbounded) Toeplitz operator $\mathcal{T}^{t,\Gamma}_{d^{-\varepsilon}_{\mathbb{H}}}$ on $L^2(\mathcal{A}_t)$ with symbol $d^{-\varepsilon}_{\mathbb{H}}$.

The Toeplitz operators considered in Assumptions D, D$'$ in the case of the Berezin quantization, are simply functions of the invariant Laplacian, when using the Berezin transform (see [16])).

We complete the Assumption SQP by the Assumption SQP$_1$, which describes the domain of the semigroup $(\Phi^t_\varepsilon)_{\varepsilon\geq 0}$ for fixed $t > 1$.

ASSUMPTION SQP$_1$. The composition $\Phi^t_\varepsilon \circ D^t_{-\varepsilon}$ is continuous and maps positive into positive elements. This assumption will be proven later in this chapter to hold true in the case of the Berezin quantization.

In particular, the domain of Φ^t_ε is the image of $D^t_{-\varepsilon}$.

If all Assumptions FM, D, D$'$, SQP, SQP$_1$ are verified, we introduce the following definition.

Definition 1. The symbol system $(\mathcal{A}_t)_{t>1}$, $(\Psi_{s,t})_{s\geq t>1}$, $(\beta_{s,t})_{s\geq t>1}$ with all assumptions listed above (FM, D, D$'$, SQP, SQP$_1$) is regular if the following operations, obtained by differentiation, have a dense domain $\mathcal{D} = \mathcal{D}_{c_t}$. We define the Hochschild cocycle associated to the deformation by the formula

$$c_t(k,l) = \frac{\mathrm{d}}{\mathrm{d}s}\Psi^{-1}_{s,t}(\Psi_{s,t}(k)\Psi_{s,t}(l))\Big|_{s\to t_+}, \quad k,l \in \mathcal{D}.$$

(by $s \to t_+$ we denote the right derivative at t of the above expression).

The Dirichlet form associated to the deformation is

$$\mathcal{E}_t(k,l) = \frac{\mathrm{d}}{\mathrm{d}s}\tau_{\mathcal{A}_s}(\Psi_{s,t}(k)\Psi_{s,t}(l))\Big|_{s\to t_+}, \quad k,l \in \mathcal{D},$$

and the real part of the cocycle is

$$\langle R_t k, l\rangle_{L^2(\mathcal{A}_t)} = -\frac{1}{2}\mathcal{E}_t(k,l^*) = \frac{d}{ds}\langle \Psi_{s,t}(k), \Psi_{s,t}(l)\rangle_{L^2(\mathcal{A}_s,\tau_{\mathcal{A}_s})}, \quad k,l \in \mathcal{D}.$$

Note that, as proved in [17], the bilinear form $\mathcal{E}_t$ is indeed a Dirichlet form in the sense considered in the papers [4, 19].

Definition 2. The imaginary part of the cocycle associated to the deformation is defined by the formula

$$c_t^0(k,l) = c_t(k,l) - [R_t(kl) - kR_t(l) - R_t(k)l], \quad k,l \in \mathcal{D}.$$

All these equations are assumed satisfied on the dense domain $\mathcal{D}$.

Note that the definition for $c_t^0(k,l)$ is so that c_t^0 remains a Hochschild cocycle, and in addition $\tau(c_t^0(k,l)) = 0$ for all k,l in the domain. Moreover, as proved in [16, 17], the trilinear form defined by

$$\psi_t(k,l,m) = \tau_{\mathcal{A}_t}(c_t^0(k,l)m)$$

is densely defined and has the property that

$$\psi_t(k,l,m) = \psi_t(m,l,k) = \overline{\psi_t(l^*,k^*,m^*)},$$

and is a 2-cyclic cohomology cocycle [5].

Both c_t, c_t^0 have the property $c(k^*,l^*) = c(l,k)^*$ and similarly for c_t^0, as also $R_t(k^*) = R_t(k)^*$.

The analysis between the two types of completely positive maps, $\Psi_{s,t}$ and $\beta_{s,t}$, allows us to construct an unbounded coboundary for the cocycle c_t. The domain of this coboundary, which we denote by $\mathcal{D}^0$, is slightly smaller than the previous domain $\mathcal{D}$. In the case of the Berezin quantization [17] we explicitly constructed $\mathcal{D}_0$ as a $*$- algebra that does not contain the identity. The domain $\mathcal{D}_0$ will be the common domain for the derivation operations.

However, by definition, the cocycles c_t, c_t^0, ψ_t naturally contain the identity element in their natural domain, and $c_t(1,1)=0$, $c_t^0(1,1)=0$, $\psi_t(1,1,1)=0$.

We do the analysis of the relation between the two families of completely positive maps in the following theorem.

THEOREM 3. *Assume all the above assumptions on the system* $(\mathcal{A}_t, \Psi_{s,t}, \beta_{s,t})_{s \geq t}$. *In addition, assume that the generator* $\mathcal{L}_t$ *of the semigroup* Φ_ε^t *at* $\varepsilon = 0$ *has a dense domain, which is a dense $*$-algebra* $\mathcal{D}^0 = \mathcal{D}_{\mathcal{L}_t} \subseteq \mathcal{D}$. *Then,*

$$c_t(k,l) = \mathcal{L}_t(kl) - k\mathcal{L}_t(l) - \mathcal{L}_t(k)l - kT^t l, \quad k,l \in \mathcal{D}_{\mathcal{L}_t}.$$

Here T^t *is the derivative* $\frac{d}{d\varepsilon}Y_t^{t+\varepsilon}|_{\varepsilon=0}$, *an unbounded operator affiliated with* $\mathcal{A}_t$. *Assume that* T^t *has a dense domain in* H_t *with the property that* $kT^t l$ *is bounded for* k,l *in* $\mathcal{D}_{\mathcal{L}_t}$. *Recall that the operator* $Y_t^{t+\varepsilon}$ *was introduced in Assumption F. M.*

Note that as a consequence, the cocycle c_t *is implemented by an unbounded 1-cocycle, defined on* $\mathcal{D}_{\mathcal{L}_t}$, *of the form*

$$\Lambda_t = \mathcal{L}_t - \frac{1}{2}\{T^t, \cdot\}.$$

In the case of the Berezin quantization this is the Lindblad form (see [3, 11]) *of a generator of a quantum dynamical semigroup* (see also the references in [17]).

Then $\mathcal{L}_t^0 = \mathcal{L}_t - R_t$ *has the property that*

$$c_t^0(k,l) = \mathcal{L}_t^0(kl) - k\mathcal{L}_t^0(l) - \mathcal{L}_t^0(k)l, \quad k,l \in \mathcal{D}_{\mathcal{L}_t}$$

and

$$\mathcal{L}_t^0 = \lambda_t 1 + iY_t,$$

where Y_t *is symmetric,* $(\mathcal{L}_t^0)^*1 = 0$, *and hence,* $(iY_t)^*1 = \lambda_t 1$. *In addition,* Y_t *has deficiency indices* $(0,1)$. *Here* $\lambda_t = -\frac{1}{2}\frac{\chi_t'}{\chi_t}$.

Proof. The domain of the semigroup Φ_ε^t is defined abstractly, thus it contains the range of integrals of the form $\int_\alpha^\beta \Phi_\varepsilon^t d\varepsilon$.

Note that the family $X_r^t = \beta_{r,t}(1)$ is an increasing family of positive elements. Indeed, if $r_1 < r_2 < t$, then

$$X_{r_1}^t = \beta_{r_1,t}(1) = \beta_{r_2,t}(\beta_{r_1,r_2}(1)) = \beta_{r_2,t}(X_{r_2}^{r_1}) \leq \beta_{r_2,t}(1) = X_{r_2}^t.$$

The rest of the statement was essentially proved, for the particular case of the Berezin quantization, in [17].

We prove again the statement here, in the general framework. All derivatives are computed on the domain $\mathcal{D}_{\mathcal{L}_t}$.

Since $\beta_{s,t}(a) = \alpha_{s,t}\left((Y_s^t)^{1/2}a(Y_s^t)^{1/2}\right)$, $a \in \mathcal{A}_t$, where $Y_s^t \in \mathcal{A}_t$ is the positive element previously defined, it follows that for $s \geq t$ the following equality holds:

$$\beta_{s,t}(k)\beta_{s,t}(l) = \alpha_{s,t}\left((Y_s^t)^{1/2}kY_s^t l(Y_s^t)^{1/2}\right) = \beta_{s,t}(kY_s^t l)$$

for k,l in the maximal domain, on which the differentiation is possible (which a posteriori is $\mathcal{D}_{\mathcal{L}_t}$).

We apply $\Psi_{s,t}^{-1}$ and obtain that

$$\Psi_{s,t}^{-1}(\beta_{s,t}(kY_s^t l)) = \Psi_{s,t}^{-1}([\Psi_{s,t}(\Psi_{s,t}^{-1}(\beta_{s,t}(k)))][\Psi_{s,t}(\Psi_{s,t}^{-1}(\beta_{s,t}(l)))]),$$

and hence,

$$\Phi_{s-t}^t(kY_s^t l) = \Psi_{s,t}^{-1}([\Psi_{s,t} \circ \Phi_{s-t}^t(k)][\Psi_{s,t} \circ \Phi_{s-t}^t(l)]).$$

Differentiating in the parameter s when $s \to t_+$, we obtain

$$\mathcal{L}_t(kl) + kT^t l = c_t(k,l) + k\mathcal{L}_t(l) + \mathcal{L}_t(k)l.$$

Thus, for k,l in the domain of $\mathcal{L}_t$, we have that

$$c_t(k,l) = \mathcal{L}_t(kl) - k\mathcal{L}_t(l) - \mathcal{L}_t(k)l + kT^t l.$$

We analyze the real and imaginary part of $\mathcal{L}_t$. By definition $\mathcal{L}_t$ is the generator of the semigroup $\Phi_\varepsilon^t = \Psi_{t+\varepsilon,t}^{-1} \circ \beta_{t+\varepsilon,t}$. Thus,

$$\langle \operatorname{Re} \mathcal{L}_t k, l \rangle_{L^2(\mathcal{A}_t)} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \left[\langle \Psi_{s,t}^{-1}(\beta_{ts}(k)), \Psi_{s,t}^{-1}(\beta_{ts}(l)) \rangle_{L^2(\mathcal{A}_t)} \right] \Big|_{s \to t_+}$$

$$= \frac{1}{2} \left[-\frac{\mathrm{d}}{\mathrm{d}s} \langle \Psi_{s,t}(k), \Psi_{s,t}(l) \rangle_{L^2(\mathcal{A}_s)} + \frac{\mathrm{d}}{\mathrm{d}s} \langle \beta_{s,t}(k), \beta_{s,t}(l) \rangle_{L^2(\mathcal{A}_s)} \right] \Big|_{s \to t_+}.$$

Using further that $\beta_{s,t}(k) = \alpha_{s,t}\left((Y_s^t)^{1/2} k (Y_s^t)^{1/2} \right)$ we obtain that this is further equal to

$$\frac{1}{2} \left[-\mathcal{E}_t(k, l^*) + \langle T^t k + k T^t, l \rangle - \frac{\chi_t'}{\chi_t} \langle k, l \rangle_{L^2(\mathcal{A}_t)} \right].$$

The last term is due to the fact that $\langle \alpha_{s,t}(k), \alpha_{s,t}(l) \rangle$ is $\frac{\chi_t}{\chi_s} \langle k, l \rangle$, because of the trace scaling property ($\alpha_{s,t}$ is non-unital).

Moreover, the remaining part is the antisymmetric part, which is $\operatorname{Im} \mathcal{L}_t$.

We have thus obtained that

$$\operatorname{Re} \mathcal{L}_t = R_t + \frac{1}{2} \{T^t, \cdot\} - \frac{1}{2} \frac{\chi_t'}{\chi_t} \operatorname{Id},$$

and hence, $c_t(k, l)$ is implemented by $\mathcal{L}_t - \frac{1}{2}\{T^t, \cdot\} = R_t - \frac{1}{2}\frac{\chi_t'}{\chi_t} + i \operatorname{Im} \mathcal{L}_t$.

Consequently, the unbounded linear map $\Lambda_t^0 = -\frac{1}{2}\frac{\chi_t'}{\chi_t} + i \operatorname{Im} \mathcal{L}_t$ implements c_t^0. For simplicity, we denote $\mathcal{L}_t^0 = i \operatorname{Im} \mathcal{L}_t$, which is an antisymmetric form.

Moreover, we note that $\mathcal{L}_t^0$ is in fact obtained from the derivative of $\Psi_{s,t}^{-1}(\alpha_{s,t})$ after taking into account the rescaling due to the variation of $\langle \cdot, \cdot \rangle_{L^2(\mathcal{A}_s)}$. Since $\left(\frac{\chi_t}{\chi_s}\right)^{1/2} \alpha_{s,t}(k)$ is an isometry form $L^2(\mathcal{A}_t)$ onto $E_t^s L^2(\mathcal{A}_s) E_t^s$, it follows that $\mathcal{L}_t^0$ is antisymmetric, $(\mathcal{L}_t^0)^* 1 = -\frac{1}{2}\frac{\chi_t'}{\chi_t}$, and the deficiency indices are $(0, 1)$ (since $\alpha_{s,t}$ is surjective onto E_t^s). In particular,

$$(\Lambda_t^0)^* = -\frac{1}{2}\frac{\chi_t'}{\chi_t} - i \operatorname{Im} \mathcal{L}_t^* = -\frac{1}{2}\frac{\chi_t'}{\chi_t} + (\mathcal{L}_t^0)^*,$$

and hence, $(\Lambda_t^0)^* 1 = 0$. $\square$

We explain all the above results and assumptions in the case of the Berezin Γ-equivariant quantization of the upper half-plane.

In this case we have $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and we let $\pi_t : \mathrm{PSL}_2(\mathbb{R}) \to B(H_t)$ be the discrete series (extended, for $t > 1$, to the continuous family of projective unitary representations of $\mathrm{PSL}_2(\mathbb{R})$ as in [15]).

By taking $\mathcal{A}_t = \{\pi_t(\Gamma)\}' \subseteq B(H_t)$, it is proved in [8, 16] that $\mathcal{A}_t$ is isomorphic to the von Neumann algebra associated to the group Γ with cocycle ε_t, reduced by $\frac{t-1}{12}$ (see [8]), that is the algebra $L(\Gamma, \varepsilon_t)_{\frac{t-1}{12}}$ (here ε_t is the cocycle coming from the projective representation π_t).

Every kernel (function) $k = k(z, \overline{\zeta})$ on $\mathbb{H} \times \mathbb{H}$, analytic in the first variable antianalytic in the second variable, corresponds, if certain growth conditions are verified (see [16, Chapter 1]), to a bounded operator on $B(H_t)$, which has exactly the reproducing kernel k. If $k(\gamma z, \overline{\gamma \zeta}) = k(z, \overline{\zeta})$, $\gamma \in \Gamma$, $z, \zeta \in \mathbb{H}$, then the corresponding operator is in $\mathcal{A}_t$.

The following data is then computed directly in the space of kernels. We identify kernels k, l with the corresponding operators and denote by $k \star_t l$ the kernel (symbol) of the product.

The formulae for the various operations in $\mathcal{A}_t$ are computed as follows:

For k, l kernels (symbols) of operators in $\mathcal{A}_t$, we have

$$\tau_{\mathcal{A}_t}(k) = \frac{1}{\mathrm{ha}(\mathrm{F})} \int_F k(z, \overline{z}) \mathrm{d}\nu_0(z).$$

Here, F is a fundamental domain for the action of Γ in $\mathbb{H}$, and $\mathrm{ha}(\mathrm{F})$ is the hyperbolic area of F. Moreover, the product formula for the kernel of the product operator, of the operators represented by the kernels k, l is:

$$(k \star_t l)(z, \overline{\zeta}) = \chi_t (z - \overline{\zeta})^t \int_{\mathbb{H}} \frac{k(z, \overline{\eta}) l(\eta, \overline{\zeta})}{(z - \overline{\eta})^t (\eta \overline{\zeta})^t} \mathrm{d}\nu_t(\eta)$$

$$= \chi_t \int_{\mathbb{H}} k(z, \overline{\eta}) l(\eta, \overline{\zeta}) [z, \overline{\eta}, \eta, \overline{\zeta}]^t \mathrm{d}\nu_0(\eta),$$

where $[z, \overline{\eta}, \eta, \overline{\zeta}] = \frac{(z - \overline{\zeta})(\eta - \overline{\zeta})}{(z - \overline{\eta})(\eta - \overline{\zeta})}$, $z, \zeta, \eta \in \mathbb{H}$, is the four point function.

The Hochschild cocycle, obtained by derivation of the product formula, for k, l in the domain $\mathcal{D}_{c_t}$ is given by

$$c_t(k, l)(z, \zeta) = \frac{\chi'_t}{\chi_t} + c_t \int_{\mathbb{H}} k(z, \overline{\eta}) l(\eta, \overline{\zeta}) [z, \overline{\eta}, \eta, \overline{\zeta}]^t \ln([z, \overline{\eta}, \eta, \overline{\zeta}]) \mathrm{d}\nu_0(\eta).$$

Here, we use a standard branch of the logarithm for $\ln(z - \overline{\zeta})$, $z, \zeta \in \mathbb{H}$. Consequently,

$$\tau_{\mathcal{A}_t}(c_t(k, l)) = \frac{\mathrm{d}}{\mathrm{d}s} \left[\tau_{\mathcal{A}_s}(k \star_s l) \right] \Big|_{s \to t_+}$$

$$= \frac{\chi'_t}{\chi_t} \langle k, l^* \rangle_{L^2(\mathcal{A}_t)} + \int_F \int_{\mathbb{H}} k(z, \overline{\eta}) l(\eta, \overline{z}) d_{\mathbb{H}}^t(z, \eta) \ln d_{\mathbb{H}}(z, \eta) \, \mathrm{d}\nu_0^2(z, \eta).$$

Moreover:

$$\langle k, l \rangle_{L^2(\mathcal{A}_t)} = \chi_t \iint_{F \times \mathbb{H}} k(z, \overline{\eta}) \overline{l(\eta, \overline{z})} d_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta),$$

so

$$\|k\|_{L^2(\mathcal{A}_t)}^2 = \chi_t \iint_{F \times \mathbb{H}} |k(z, \overline{\eta})|^2 d_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta).$$

In particular $L^2(\mathcal{A}_t)$ is a space of functions with analytic structure, analytic in the first variable, antianalytic in the second, and diagonally Γ-invariant, square summable, with respect to the measure $d_{\mathbb{H}}^t(z,\eta)\mathrm{d}\nu_0^2(z,\eta)$.

To construct the rest of the data, we let $\Psi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ be simply the symbol map, associating to a bounded operator with kernel k in $\mathcal{A}_t$, the bounded operator with the same kerned k in $\mathcal{A}_s$ (see [16]).

The family of completely positive maps $\beta_{s,t}$ are constructed from intertwiners between various representations π_t, π_{t+n}, $t > 1$, $n \in \mathbb{N}$, obtained from automorphic cusp forms.

Namely, if f, g are automorphic cusp forms of order n for the group $\mathrm{PSL}_2(\mathbb{Z})$, then let $M_f^t : H_t \to H_{t+n}$ be the operator of multiplication by f. Because of the results in [8] this is a bounded intertwiner between the two representations π_t, π_{t+n}, and hence, we obtain a map

$$\beta_{f,g} : \mathcal{A}_t \to \mathcal{A}_{t+n},$$

defined by the formula

$$\beta_{\overline{f},g}(k) = M_g^t k (M_f^t)^*.$$

This is a bounded operator from $\mathcal{A}_t$ into $\mathcal{A}_{t+n}$. We recall the following fact.

PROPOSITION 4 ([16, 17]). *Given f, g as above, of order n, and $t > 1$, the symbol of $\beta_{\overline{f},g}(k)$ is*

$$\frac{\chi_t}{\chi_{t+n}} f(z)\overline{g(\zeta)}k(z,\overline{\zeta}), \quad z, \zeta \in \mathbb{H}.$$

In particular, if $f = g$, then $\beta_{\overline{f},f}$ is a completely positive map, and if $x_f v_f$ is the polar decomposition of M_f^t, then $\beta_f(k) = x_f(v_f k v_f^*)x_f$ and $\alpha_f(k) = v_f k v_f^*$ defines an isomorphism from $\mathcal{A}_t$ into $E_f \mathcal{A}_{t+n} E_f$, where $E_f \in \mathcal{A}_{t+n}$ is the range of v_f.

We do this construction for Δ^ε, $\varepsilon > 0$, where Δ is the Dedekind function (a cusp form of order 12). Since Δ has no zeros, $\ln \Delta, \Delta^\varepsilon$ are uniquely defined, and we define

$$\beta_\varepsilon = \beta_{\overline{\Delta^\varepsilon},\Delta^\varepsilon} : \mathcal{A}_t \to E_t^{t+\varepsilon} \mathcal{A}_{t+\varepsilon} E_t^{t+\varepsilon},$$

where $E_t^{t+\varepsilon}$ is the closure of the range of M_{Δ^ε}. Since $M_{\Delta^\varepsilon}^t$ is injective, as Δ^ε has no zeros, it follows that $\tau(E_t^{t+\varepsilon}) = \frac{\chi_t}{\chi_{t+\varepsilon}}$.

We note the following interesting corollary.

COROLLARY 5. *The algebras $L(\Gamma, \varepsilon_t)$, $t > 1$, are Morita equivalent.*

Proof. By the above we have that $(\mathcal{A}_{t+\varepsilon})_{\frac{\chi_t}{\chi_{t+\varepsilon}}} = \mathcal{A}_t$ since $c_s = \frac{s-1}{\pi}$ and $\mathcal{A}_s = L(\Gamma, \varepsilon_s)_{\frac{s-1}{12}}$ for $s > 1$, so the result follows. $\square$

We return to the description of the Γ-equivariant Berezin quantization and the description of the constitutive elements, as used in the assumptions made for Theorem 3. It follows that the formula for Φ_ε^t is

$$\Phi_\varepsilon^t(k) = \frac{\chi_t}{\chi_{t+\varepsilon}}(k\varphi^\varepsilon), \quad k \in \mathcal{D}_{\Phi_t},$$

where

$$\varphi(z,\overline{\zeta}) = \ln((z - \overline{\zeta})^{12}) + \ln\Delta(z) + \ln(\overline{\Delta(\zeta)}), \quad z,\zeta \in \mathbb{H},$$

and where we use the standard analytic branch of the logarithm of the non-zero, analytic function Δ. The domain of the maps Φ_ε^t, $\varepsilon \geq 0$, was constructed explicitly in Chapter 6 in [17], but it can also be described in a more axiomatic way, as in Assumption SQP1.

The condition that $\Phi_\varepsilon^t \circ D_{-\varepsilon}^t$ is bounded follows from

$$|\Delta(z)\Delta(\overline{\zeta})(\overline{z} - \zeta)^{12}|^2 = |\Delta(z)|^2(\operatorname{Im} z)^{12}|\Delta(\zeta)|^2(\operatorname{Im}\zeta)^{12}\left(\frac{|(\overline{z} - \zeta)|^2}{(\operatorname{Im} z)\operatorname{Im}\zeta}\right)^{12},$$

which is a quantity bounded by $d_{\mathbb{H}}^{12}$.

Finally, we note (see also Chapter 6 in [17]) that the domain of

$$\mathcal{L}_t = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\Psi_{t+\varepsilon,t}^{-1}(\beta_{t+\varepsilon,t})\right]\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\beta_{t,t-\varepsilon}(\Psi_{t,t-\varepsilon}^{-1}(k))\right]\Big|_{\varepsilon=0}$$

contains all integrals of the form

$$\int_{\varepsilon_1}^{\varepsilon_2} M_{\Delta^\varepsilon} k M_{\Delta^\varepsilon}\mathrm{d}\varepsilon$$

if $\varepsilon_0 < \varepsilon_1 < \varepsilon_2$ and k belongs to $\mathcal{A}_{t-\varepsilon_0}$ (rigorously, in the integral we are using $\Psi_{t-\varepsilon,t-\varepsilon_0}(k)$ instead of k).

The formula for $\mathcal{L}_t$ as described in ([17], Chapter 6) corresponds to a Toeplitz operator of multiplication with an analytic function, as in Definition 10.

More precisely we have that

$$\mathcal{L}_t = -\frac{\chi_t'}{\chi_t}1 + \mathcal{M}_\varphi,$$

where by $\mathcal{M}_\varphi$ we denote the "analytic" Toeplitz operator of multiplication by φ. In the terminology of Definition 10 this is $\mathcal{T}_\varphi^{t,\Gamma}$.

It follows that $\mathcal{L}_t^0$ (the imaginary part of $\mathcal{L}_t$) is a Toeplitz operator, in the sense of Definition 10 on the Hilbert space $L^2(\mathcal{A}_t)$ identified with a space of analytic, antianalytic functions -respectively in the first and in second variable.

Thus, $\langle\mathcal{L}_t^0 k, l\rangle = \langle\mathcal{M}_{\varphi_0}k, l\rangle_{L^2(\mathcal{A}_t)}$ where $\mathcal{M}_{\varphi_0}$ is the multiplication operator with the function φ_0, acting on $L^2(\mathcal{A}_t)$. Consequently $\mathcal{L}_t^0$ is the Toeplitz operator with symbol

$$\varphi_0(z,\overline{\zeta}) = i[\arg(z - \overline{\zeta})^{12} + \arg(\Delta)(z) - \arg(\Delta)(\zeta)].$$

COROLLARY 6. *Recall that, for $t > 1$, the Hilbert space $L^2(\mathcal{A}_t)$ is identified with the Hilbert space of analytic-antianalytic (in the first, respectively the second variable) functions $k = k(z, \overline{\eta})$ on $\mathbb{H} \times \mathbb{H}$, diagonally Γ-invariant, with norm*

$$\|k\|_{L^2(\mathcal{A}_t)}^2 = \chi_t \iint_{F \times \mathbb{H}} |k(z, \overline{\eta})|^2 d_{\mathbb{H}}^t(z, \eta) \, d\nu_0^2(z, \eta).$$

Then the Toeplitz operator $T_0 = \mathcal{T}_{\varphi_0}^{t,\Gamma} = iY^t$ (see Definition 10), having a dense domain as constructed in Corollary 6.6 of [17], with symbol

$$\varphi_0 = i(\arg(z - \overline{\zeta})^{12} + \arg \Delta(z) - \arg \Delta(\zeta)),$$

is antisymmetric, with deficiency indices $(0, 1)$, and

$$T_0^* 1 = -\frac{\chi_t'}{\chi_t} 1 = -\frac{1}{t-1} 1.$$

We note the following problem that arises naturally, that we formulate below in abstract setting: determine when can one perturb the operator $\Lambda_t^0 = -\frac{1}{2}\frac{\chi_t'}{\chi_t} 1 + iY_t$ by a densely defined derivation, with the same domain, so that the real part (which is a multiple of the identity) in the coboundary operator Λ_t^0, for c_t^0, is removed.

Problem 7. Let M be a type II_1 factor with trace τ, let Y be a symmetric operator with dense domain a $*-$algebra $\mathcal{D}^0 = \mathcal{D}_Y^0$ in $L^2(M, \tau)$ such that $(iY)^* 1 = -\alpha 1$, where $\alpha > 0$, and so that Y has deficiency indices $(0, 1)$. Here the adjoint is taken with respect to the action on the Hilbert space $L^2(M, \tau)$. We also assume that Y takes values in M rather than $L^2(M, \tau)$.

Let $Z = \alpha 1 + iY$, so that $Z^* 1 = 0$. Let

$$c^0(k, l) = Z(kl) - kZ(l) - Z(k)l, \quad k, l \in \mathcal{D}_Y^0.$$

Assume in addition that the identity element 1 may be adjoined to the domain of c^0 by taking graph closure and that $c^0(1, 1) = 0$. This is an additional hypothesis, as Y is not defined at 1.

Note that here we are automatically assuming that c^0 coincides with its imaginary part in the sense of Definition 2.

The problem is then stated as follows:

Determine if there exists a densely defined derivation δ with dense $*-$ algebra domain $\mathcal{D}(\delta)$, so that

  i) $Y_1 = Y + \delta$ is densely defined and $\mathcal{D}_Y^0 \cap \mathcal{D}(\delta)$ is a core for Y_1.
 ii) The real part of δ, in the sense of Hilbert space operators acting on $L^2(M, \tau)$, is $\operatorname{Re} \delta = -\alpha 1$ (note that in this case $\alpha 1 + iY_1 + \delta$ is antisymmetric).
iii) $(iY_1)^*(1) = 0$, and 1 is in the domain of the closure of Y_1.

Note this amounts to find a coboundary for c^0 that, as an operator acting on $L^2(M, \tau)$, is antisymmetric with (0,1) deficiency indices.

For the cocycle c_t^0 constructed explicitly in Chapter 1 for the Berezin quantization, this represents the obstruction to replace the operator $\mathcal{L}_t^0$ by an antisymmetric operator, that is to find $\mathcal{L}_t$ so that $\mathcal{L}_t 1 = \mathcal{L}_t^* 1 = 0$ and so that $\mathcal{L}_t$ implements c_t and 1 in the domain of $\mathcal{L}_t$.

This would happen, for example, if a unique finite von Neumann algebra $\mathcal{A}$ and isomorphisms $\alpha_t : \mathcal{A}_t \to \mathcal{A}$ would exist. In this case we may transfer the maps $\Psi_{s,t}$, $s \geq t > 1$, using the isomorphisms α_t into a Chapman-Kolmogorov system of unital completely positive maps $\widetilde{\Psi_{s,t}}$ acting on $\mathcal{A}$.

Then the corresponding Hochschild cocycle $\widetilde{c_t}$ is simply implemented by $\widetilde{\mathcal{L}_t} = \frac{\mathrm{d}}{\mathrm{d}s} \widetilde{\Psi_{s,t}}|_{s=t}$, which obviously is 0 at 1 and 1 is in the domain of $\widetilde{\mathcal{L}_t}$. To see this, one could apply again the argument in Theorem 3 , with $\widetilde{\beta_{s,t}} = \mathrm{Id}$, as all algebras are now equal.

The following remark shows why the condition $c(1,1) = 0$ is necessary.

Remark 8. Assume c on M is an unbounded cocycle with $\tau(c(k,l)) = 0$. If 1 is in the domain of c, then $c(1,1) = 0$.

Proof. From the cocycle condition we have that $m\, c(1,1) = c(m,1)$. Hence, $\tau(m(c(1,1)) = 0$ for all m in the domain, and thus, $c(1,1) = 0$, as we assumed that the domain is dense. $\square$

Finally we show that the structure and the properties of the deformation $(\mathcal{A}_t)_{t>1}$ and of the corresponding maps $\beta_{s,t}$, $\Psi_{s,t}$, $s \geq t > 1$, can be deduced from the "rigged" Hilbert space structure on $(H_t)_{t>1}$, determined by a specific chain of embeddings, as described below.

For simplicity, we use the model of the unit disk, so that the Berezin quantization is realized using the Hilbert spaces:

$$H_t = H^2(\mathbb{D}, (1 - |z|^2)^{t-2} d\bar{z} dz).$$

We describe the properties of this "rigged" Hilbert space structure. Note that this could be used to obtain an even more general abstract formulation of the Berezin quantization.

Definition 9. Assume we have a family of unitary, projective, representations π_t of Γ on a family of Hilbert spaces $(H_t)_{t>1}$, that are finite multiples [8, 16] of the left regular representation, taken with the cocycle corresponding to the projective unitary representation. Thus, we assume, using the notations from [8], that

$$\dim_{\{\pi_t(\Gamma)\}''} H_t = \chi_t, \quad t > 1.$$

We also assume the existence of two collections of bounded maps $j_{s,t}$, $\Delta_{s,t} : H_t \to H_s$, $s \geq t > 1$, verifying Chapman-Kolmogorov conditions for $s \geq t \geq r$.

We assume that maps $\Delta_{s,t}$ intertwine the representations π_s and π_t of the group Γ. In the particular case of Berezin's quantization, with the Hilbert spaces considered above, the maps $j_{s,t}$ are the obvious embeddings, while $\Delta_{s,t}$ are the multiplication operators by Δ^{s-t}.

In addition, we assume that the following diagram is commutative, for $s \geq t > 1, \varepsilon > 0$:

$$
\begin{array}{ccc}
H_t & \xrightarrow{\ \Delta_{s,t}\ } & H_s \\[4pt]
{\scriptstyle j_{t,t-\varepsilon}}\big\uparrow & & \big\uparrow{\scriptstyle j_{s,s-\varepsilon}} \\[4pt]
H_{t-\varepsilon} & \xrightarrow[\Delta_{s-\varepsilon,t-\varepsilon}]{} & H_{s-\varepsilon}
\end{array}
$$

Furthermore, we assume that $j_{s,t}$, $\Delta_{s,t}$ are injective and that $j_{s,t}$ has dense range.

It automatically follows that the orthogonal projection E_t^s onto the closure of the range of $\Delta_{s,t}$ belongs to $\mathcal{A}_s$, and its Murray von Neumann dimension (trace) is equal to $\frac{\chi_t}{\chi_s}$.

Also, it automatically follows that the linear, densely defined maps Φ_ε^t from H_t into $E_{t-\varepsilon}^t H_t$, defined on $\operatorname{Im} j_{t,t-\varepsilon}$ by the formula

$$
\Delta_{t-\varepsilon,t} \circ j_{t,t-\varepsilon}^{-1} = j_{t,t+\varepsilon}^{-1} \circ \Delta_{t,t+\varepsilon}
$$

form a semigroup. We observe that the ranges of $j_{t-\varepsilon,t}$, $\Delta_{t-\varepsilon,t}$ are increasing with ε, by the Chapman-Kolmogorov property. In particular χ_t is an increasing function of t.

We call a structure as above a "rigged" chain of projective, unitary representations of the group Γ.

Given such a structure, we may define the maps $j_{t,t-\varepsilon} \cdot j_{t,t-\varepsilon}^*$ from $\mathcal{A}_{t-\varepsilon}$ onto $\mathcal{A}_t$, by mapping $a \in \mathcal{A}_{t-\varepsilon}$ into $j_{t,t-\varepsilon} a j_{t,t-\varepsilon}^* \in \mathcal{A}_t$ for $1 < t - \varepsilon$.

Note that in the case of the Berezin quantization, the generator of Φ_ε^t is $T_{\ln \Delta}^t$, the unbounded Toeplitz operator on H_t with analytic symbol $\ln \Delta$.

In this case, for a kernel k representing an operator in $\mathcal{A}_t$, for $s \geq t$ the operator $j_{s,t} k j_{s,t}^* \in \mathcal{A}_s$ has symbol $\frac{\chi_s}{\chi_t}(1 - z\bar{\zeta})^{t-s}k$.

Consequently the infinitesimal generator

$$
\frac{\mathrm{d}}{\mathrm{d}s}\left[j_{s,t} \cdot j_{s,t}^* \right]\Big|_{s \to t_+}
$$

is exactly the Toeplitz operator $\mathcal{T}_{-\frac{1}{2}\frac{\chi_t'}{\chi_t} + \ln(1 - z\bar{\zeta})}$ constructed in the next section in Definition 11.

The operators $j_{s,t}^* j_{s,t}$ are exactly the Toeplitz operators with symbol $(1 - |z|^2)^{s-t}$, acting on on H_t. Likewise the operator $j_{s,t} \, j_{s,t}^*$, acting on H_s, is the inverse of the unbounded Toeplitz operator on H_t with symbol $(1 - |z|^2)^{t-s}$.

In particular, most of the properties of the domain of $\mathcal{L}_t$, can be read from properties of the semigroup Φ_ε^t at the level of Hilbert spaces, because of the following formula:

$$\Phi_\varepsilon^t(a) = (j_{t+\varepsilon,t}^{-1} \circ \Delta_{t,t+\varepsilon}) a (\Delta_{t,t+\varepsilon}^* \circ (j_{t,t+\varepsilon}^{-1})^*), \quad a \in \mathcal{A}_t.$$

For fixed $t > 1$, consider the completely positive semigroup of operators $(\Theta_\varepsilon^t)_{\varepsilon \geq 0}$ acting on $B(H_t)$, defined by multiplying the reproducing kernel of an operator with the positive definite kernel on $\mathbb{H}^2$, which in this model is identified with the unit disk $\mathbb{D}$, defined on $\mathbb{D} \times \mathbb{D}$ by:

$$(z, \eta) \to \frac{1}{(1 - z\overline{\eta})^\varepsilon}, \quad z, \eta \in \mathbb{D}.$$

Thus, for a kernel k representing an operator in $B(H_t)$, the formula for the kernel of the operator $\Theta_\varepsilon^t(k) \in B(H_t)$ is

$$\Theta_\varepsilon^t(k)(z, \overline{\eta}) = k(z, \overline{\eta}) \frac{1}{(1 - z\overline{\eta})^\varepsilon}, \quad z, \eta \in \mathbb{D}, \varepsilon \geq 0.$$

The positivity properties mentioned above were proved in the second chapter of [17]. Note that Θ_ε^t is the Toeplitz operator introduced in Definition 11, with symbol $\frac{1}{(1 - z\overline{\eta})^\varepsilon}$.

Using this semigroup, it is easily observed that we may also reconstruct the symbol map. Indeed, for $s \geq t > 1$, we have the following equality:

$$\Psi_{s,t}(a) = \frac{\chi_s}{\chi_t} j_{s,t} \Theta_{s-t}^t(a) j_{s,t}^*, \quad a \in \mathcal{A}_t.$$

2. HIGHER ORDER BEREZIN SYMBOLS

In this section, we use the machinery of Berezin quantization, by using the symbols not just to define linear operators, but rather to define (eventually unbounded) linear operators on spaces of operators. We will also explain how the model can be applied to multilinear operators.

Recall, from last section, that any bounded operator A is $B(H_t)$ is represented by a kernel $k_A = k_A(z, \overline{\eta})$ on $\mathbb{H} \times \mathbb{H}$, in such a way that

$$Af(z) = \chi_t \int_{\mathbb{H}} \frac{k_A(z, \overline{\eta})}{(z - \overline{\eta})^t} f(\eta) \mathrm{d}\nu_t(\eta), \quad z \in \mathbb{H}, f \in H_t.$$

Recall that k_A is analytic in z and antianalytic in η. Here $\chi_t = \frac{t-1}{\pi}$.

The Hilbert space $\mathcal{C}_2(H_t)$ of Hilbert-Schmidt operators induces the scalar product on the space of reproducing kernels k, l, representing operators $\mathcal{C}_2(H_t)$ given by the formula

$$\langle k, l \rangle = \langle k, l \rangle_{\mathcal{C}_2(H_t)} = \chi_t \iint_{\mathbb{H}^2} k(z, \overline{\eta}) \overline{l(z, \overline{\eta})} d_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta).$$

For the space $L^2(\mathcal{A}_t)$, if $k, l \in L^2(\mathcal{A}_t)$, the scalar product formula is

$$\langle k, l \rangle = \chi_t \iint_{F \times \mathbb{H}} k(z, \overline{\eta}) \overline{l(z, \overline{\eta})} d_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta),$$

where F is a fundamental domain for the action of the group Γ on $\mathbb{H}$.

Definition 10. Note that both $\mathcal{C}_2(H_t), L^2(\mathcal{A}_t)$ are spaces of functions analytic in the first variable, antianalytic in the second. Hence, for every measurable function d on $\mathbb{H} \times \mathbb{H}$, we define an unbounded Toeplitz operator (bounded if d is bounded) by the formula

$$\langle \mathcal{T}_d^t k, l \rangle = \chi_t \iint_{\mathbb{H}^2} d(z, \eta) k(z, \overline{\eta}) \overline{l(z, \overline{\eta})} d_{\mathbb{H}}^t(z, \eta) \mathrm{d}^2\nu_0(z, \eta).$$

In the case of $L^2(\mathcal{A}_t)$, if $d(z, \zeta) = d(\gamma z, \gamma \zeta)$, $\gamma \in \Gamma$, $z, \zeta \in \mathbb{H}$, by replacing, in the previous formula, the integration domain $\mathbb{H} \times \mathbb{H}$ by the integration domain $F \times \mathbb{H}$, we obtain a Toeplitz operator $\mathcal{T}_d^{t,\Gamma}$ densely defined on $L^2(\mathcal{A}_t)$. Recall that F is a fundamental domain for the action of Γ on $\mathbb{H}$.

Certainly, the two different types of Toeplitz operators are related, and we also define a third type of Toeplitz operator as follows:

Definition 11. Let $d : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ be a measurable function. We define a linear operator $Q_d = Q_d^t$ on a subspace $\mathcal{D}(Q_d)$ of $B(H_t)$, by the pairing formula

$$\langle Q_d(k), l \rangle = \mathrm{Tr}(Q_d(k) l^*) = \chi_t \iint_{\mathbb{H}^2} d(z, \eta) k(z, \overline{\eta}) \overline{l(z, \overline{\eta})} d_{\mathbb{H}}^t(z, \eta) \mathrm{d}\nu_0^2(z, \eta).$$

Here l runs over the nuclear operators $\mathcal{C}_1(H_t)$, and the domain $\mathcal{D}(Q_d)$ of Q_d consists in all $k \in B(H_t)$, such that $Q_d(k)$ defined by the above formula is a bounded operator on H_t. So Q_d is an unbounded operator on the Banach space $B(H_t)$.

It is not necessary that d bounded implies that Q_d is bounded. If d is diagonally Γ-invariant, then $Q_d|_{L^2(\mathcal{A}_t) \cap \mathcal{D}(Q_d)}$ is the Toeplitz operator $\mathcal{T}_d^{t,\Gamma}$ introduced in the previous definition. Similarly Q_d, by restriction to $\mathcal{C}_2(H_t)$, is a Toeplitz operator if the domain intersection is non-zero.

Clearly, $Q_{\overline{d}}$ restricts to the adjoint, but, apparently, there is no functorial method of constructing $Q_{\overline{d}}$ in an operator theoretic method, starting from Q_d (see more on this below).

Note that most of the above definitions could also be introduced for distributions $d(z, \zeta)$ acting on a dense subspace of the linear span of functions of the form $k(z, \overline{\eta})l(z, \overline{\eta})$. In this case we will use the notation

$$\langle d(z, \zeta), k(z, \overline{\eta})\overline{l(\overline{z}, \eta)} \rangle$$

for the evaluation form associated to the distribution.

We collect below a number of elementary properties of these operators. In the next proposition the notation from the previous two definitions is used.

PROPOSITION 12. 1) *If we restrict the operator Q_d to $\mathcal{C}_2(H_t)$ or $L^2(\mathcal{A}_t)$, then $Q_{\overline{d}}|_{\mathcal{C}_2(H_t)}$ or respectively $Q_{\overline{d}}|_{L^2(\mathcal{A}_t)}$ is contained in the adjoint of Q_d.*

2) *Recall that given a kernel k representing the symbol of a bounded operator on H_t, the symbol k^* of the adjoint operator is given by the formula $k^*(z, \overline{\eta}) = \overline{k(\eta, \overline{z})}$, $z, \eta \in \mathbb{H}$.*

Given d as in the previous definitions, we introduce $s(d)(z, \zeta) = \overline{d(\zeta, z)}$ for $z, \zeta \in \mathbb{H}$.

Let $C_d(k)$ be defined by the formula $C_d(k) = Q_d(k^*)^*$. Assuming that the domain $\mathcal{D}_{Q_d}$ of the operator Q_d is closed under the adjoint operation, it follows that $C_d = Q_{s(d)}$.

3) *If d is a distribution as above, then the condition that Q_d is a derivation is that the measures*

$$\int_{\mathbb{H}} d_{13}\mathrm{d}\nu_t^2 \quad and \quad \int_{\mathbb{H}} d_{12}\mathrm{d}\nu_t^3 + \int_{\mathbb{H}} d_{23}\mathrm{d}\nu_t^1$$

coincide when evaluated on functions on $\mathbb{H}^3$, in the variables $z, \eta, \zeta \in \mathbb{H}$, belonging to the space

$$\mathcal{M} = Sp\left\{ \frac{k(z, \overline{\eta})}{(z - \overline{\eta})^t} \cdot \frac{l(\eta, \overline{\zeta})}{(\eta - \overline{\zeta})^t} \cdot \frac{m(\zeta, \overline{z})}{(\zeta - \overline{z})^t} \;\middle|\; k, l, m \in \mathcal{D}(Q_d) \right\}.$$

By d_{13} we mean the distribution d acting on first and third variable, and the integral is evaluated in the second variable, and similarly for the other two integrals. In the above formula, by integration with respect to the measure $\mathrm{d}\nu_t^j$, $j = 1, 2, 3$, we designate integration by the jth variable from the sequence $z, \eta, \zeta \in \mathbb{H}$.

In general, if Q_d is a derivation it does not follow that $Q_{\overline{d}}$ is a derivation. The above formula proves that this would happen if the spaces $\mathcal{M}$ and $\overline{\mathcal{M}}$ coincide ($\overline{\mathcal{M}}$ is the complex conjugate subspace of $\mathcal{M}$).

4) *If $Q_d|_{L^2(\mathcal{A}_t) \cap \mathcal{D}(Q_d)}$ is a derivation, and if d is diagonally Γ-invariant (so that the range is in $\mathcal{A}_t$), then the condition on d for Q_d to be a derivation is similar, the only difference being that we replace the space $\mathcal{M}$ by the space*

$\mathcal{M}_\Gamma$ *of diagonally* Γ*-invariant functions on* $\mathbb{H}$*, obtained by requiring that* k, l, m *correspond to diagonally* Γ*-invariant operators.*

The Γ *invariance property for functions* $\Theta \in \mathcal{M}_\Gamma$*, is*

$$\Theta(\gamma z, \gamma \zeta, \zeta \eta) = \Theta(z, \zeta, \eta) |j(\gamma, z)|^2 |j(\gamma, \zeta)^2| \, |j(\gamma, \eta)^2|,$$

for all $\gamma \in \Gamma$*,* z, ζ, η *in* $\mathbb{H}$*.*

5) *The conditions in 3), 4) above are verified if*

$$d(z, \zeta) = d_1(z) - d_1(\zeta), \quad z, \eta \in \mathbb{H},$$

for some measurable function d_1 *on* $\mathbb{H}$*. If* Q_d *is real, that is* $Q_d(k^*) = Q_d(k)^*$*, for all* k *in the domain, then* d *takes only imaginary values. In particular,* Q_d *is antisymmetric in this case.*

6) *For* k *in the domain of* Q_d *the formula for* $\langle Q_d(k)v_1, v_2\rangle$*, is*

$$\langle Q_d(k)v_1, v_2 \rangle_{H_t} = \langle d(z, \zeta), k(z, \overline{\zeta})v_1(z)\overline{v_2(\zeta)}\rangle, \quad v_1, v_2 \in H_t.$$

7) *If* d *depends only on the first (or second) variable and is a measurable function, then* $Q_d(k) = T_d^t k$ *(respectively* $Q_d(k) = kT_d^t$*), where* T_d^t *is the (unbounded) Toeplitz operator acting on* H_t *with symbol* d*.*

8) *Let* $a(z, \zeta) = \arg(z - \overline{\zeta})$*,* $z, \zeta \in \mathbb{H}$*. Then* $(iQ_a)1 = -\frac{1}{2}\frac{\chi'_t}{\chi_t}1$*. However on* $\mathcal{C}_2(H_t)$*,* iQ_a *is antisymmetric.*

The properties 1), 2) are obvious. The property 3) is deduced as follows: we start with the equality

$$Q_d(kl) = kQ_d(l) + Q_d(l)k, \quad k, l \in \mathcal{D}_{Q_d},$$

and pair this with $m \in \mathcal{C}_1(H_t)$. We obtain an identity of the form

$$\langle Q_d(kl), m\rangle = \langle Q_d(l), k^*m\rangle + \langle Q_d(l), km^*\rangle,$$

which by writing explicitly the multiplication gives the identity in the statement.

Property **4** follows similarly, using the trace on $\mathcal{A}_t$ instead of the pairing.

Property **5** is a consequence of the definition, taking $m \in \mathcal{C}_1(H)$ to be the kernel associated to $\langle v_1, \cdot \rangle v_2$.

Property **6** is a consequence of Property 5).

Property **7** is deduced by differentiating in t the identity

$$\chi_t \int_{\mathbb{H}} \frac{1}{(z - \overline{\zeta})^t} \mathrm{d}\nu_t(\zeta) = (\mathrm{Im}\, z)^t.$$

We observe that some of the previous definitions may be extended to multilinear operators. In particular, we have:

Proposition 13. 1) *If* $\Theta : \mathbb{H}^{n+1} \to \mathbb{C}$ *is an Alexander-Spanier n-th cocycle on $\mathbb{H}^{n+1}$ [7] and Θ is diagonally Γ-invariant, then the following formula $(z, \zeta \in \mathbb{H}, \ k_1, ..., k_n \in \mathcal{D})$:*

$$c_\Theta(k_1, k_2, \ldots, k_n)(z, \overline{\zeta}) = \chi_t^{n+1}(z - \overline{\zeta})^t$$

$$\cdot \iint \cdots \int_{\mathbb{H}^{n+1}} \frac{\Theta(\eta_1 \ldots \eta_{n+1}) k_1(\eta_1, \overline{\eta_2}) \ldots k_n(\eta_n, \overline{\eta_{n+1}})}{(z - \overline{\eta_1})^t (\eta_1 - \overline{\eta_2})^t \ldots (\eta_{n+1} - \overline{\zeta})^t} d\nu_t^{n+1}(\eta_1, \ldots, \eta_{n+1})$$

defines a densely defined n-Hochschild cocycle on a dense subspace $\mathcal{D}$ of $\mathcal{A}_t$.

2) *When $n + 1 = 3$ we obtain a densely defined Hochschild 2-cocycle*

$$c_\Theta(k, l)(z, \overline{\zeta}) = \chi_t^3 (z - \overline{\zeta})^t \iiint_{\mathbb{H}^3} \frac{\Theta(\eta_1, \eta_2, \eta_3) k(\eta_1, \overline{\eta_2}) l(\eta_2, \overline{\eta_3})}{(z - \overline{\eta_1})^t (\eta_1 - \overline{\eta_2})^t (\eta_2 - \overline{\eta_3})^t (\eta_3 - \overline{\zeta})^t} d\nu_t^3(\eta_1, \eta_2, \eta_3).$$

3) *If $\Theta(\eta_1, \eta_2, \eta_3) = \phi(\eta_1, \eta_3) - \phi(\eta_1, \eta_2) - \phi(\eta_2, \eta_3)$, for a measurable, Γ-diagonally invariant function ϕ on $\mathbb{H}$, then let*

$$X_\phi(k)(z, \overline{\zeta}) = c_t^2 \iint_{\mathbb{H}^2} \frac{\phi(\eta_1, \eta_2) k(\eta_1, \overline{\eta_2})}{(z - \overline{\eta_1})^t (\eta_1 - \overline{\eta_2})^t (\eta_2 - \overline{\zeta})^t} d\nu_t^2(\eta_1, \eta_2).$$

Here the domain consists in all k such that the above integral defines an element in $\mathcal{A}_t$. Then on the joint domain for c_Θ, X_ϕ we have

$$c_\Theta(k_1, k_2) = X_\phi(k_1 k_2) - k_1 X_\phi(k_2) - X_\phi(k_1) k_2.$$

Note that in fact for k, l in the domain, we have

$$\langle X_\phi k, l \rangle_{L^2(\mathcal{A}_t)} = \chi_t \iint_{F \times \mathbb{H}} \phi(\eta_1, \eta_2) k(\eta_1, \overline{\eta_2}) \overline{l(\overline{\eta_1}, \eta_2)} d\nu_t(\eta_1, \eta_2).$$

Thus, X_ϕ is in fact the Toeplitz operator $\mathcal{T}_\phi^{t, \Gamma}$ introduced in Definition 10.

4) *Clearly we have the following equality:*

$$c_\Theta(k_n^*, k_{n-1}^*, \ldots, k_1^*) = c_{\overline{\Theta}}(k_1, k_2, \ldots, k_n)^*,$$

if the elements $k_1, \ldots, k_n$ belong to the domain of the corresponding cocycles and $\overline{\Theta}$ is the conjugate of the function Θ.

5) *In particular, if*

$$\Theta(\eta_1, \eta_2, \ldots, \eta_{n+1}) = \Theta(\eta_2, \eta_3, \ldots, \eta_{n+1}, \eta_1) = \overline{\Theta(\overline{\eta_{n+1}}, \overline{\eta_n}, \ldots, \overline{\eta_1})},$$

then $\Psi_\Theta(k_1, \ldots, k_{n+1}) = \tau_{\mathcal{A}_t}(c_\Theta(k_1, \ldots, k_n) k_{n+1})$ is a cyclic cohomology cocycle defined on the domain of the corresponding cocycles.

We particularize to the case of the cocycles that arise in the Γ-equivariant Berezin quantization.

Remark 14. Using a principal branch for the logarithmic function, let $a(z, \eta)$ be the argument of $(z - \bar{\eta})$, $z, \eta \in \mathbb{H}$. The cocycle c_t^0 introduced in the last chapter corresponds to the Alexander-Spanier cocycle

$$\Theta_0(\eta_1, \eta_2, \eta_3) = -\frac{1}{2}\frac{\chi_t'}{\chi_t} + i[a(\eta_1, \eta_3) - a(\eta_1, \eta_2) - a(\eta_2, \eta_3)].$$

In particular, $c_t^0(k, l) = c_{\Theta_0}(k, l)$. Moreover, because of 3) in the previous statement, we have that

$$c_t^0(k, l) = c_{\Theta_0}(k, l) = Z^t(kl) - kZ^t(l) - Z^t(k)l,$$

for kernels k, l representing operators in the domain of c_t^0. Here Z^t is the operator

$$-\frac{1}{2}\frac{\chi_t'}{\chi_t}\,\mathrm{Id} + \mathcal{T}_{i\varphi_0}^{t,\Gamma}.$$

Recall that

$$\varphi_0(z, \zeta) = a(z, \zeta) + \arg \Delta(z) - \arg \Delta(\zeta), \quad z, \zeta \in \mathbb{H}.$$

Here we are using a principal, analytic branch of the logarithm of the non-zero function Δ, and by $\arg \Delta$ we denote the imaginary part of the logarithm.

Remark 15. The cocycle c_t^0 makes also sense as a densely defined cocycle on $B(H_t)$. It is the cocycle $\widetilde{c_t^0}$ associated to the deformation consisting in the family of algebras $B(H_t)_{t>1}$. In this case

$$\widetilde{c_t^0}(k, l) = \widetilde{\Lambda_t^0}(kl) - \widetilde{\Lambda_t^0}(k)l - k\widetilde{\Lambda_t^0}(l),$$

where $\widetilde{\Lambda_t^0} = -\frac{1}{2}\frac{\chi_t'}{\chi_t} + iQ_a$, and the domain of $\widetilde{\Lambda_t^0}$ and $\widetilde{c_t^0}$ is the domain of Q_a. Here Q_a is the unbounded operator introduced in Definition 11, and the function a was defined in the previous definition. Note that in this case $\widetilde{\Lambda_t^0}(1) = 0$, by Property 7), from the previous statement.

Recall that the modular factor for an element $\gamma \in \Gamma$, with

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z},$$

is

$$j(\gamma, z) = (cz + d), \quad z \in \mathbb{H}.$$

Since $j(\gamma, z) \neq 0$, for all z, we choose a an analytic branch of the logarithm and taking the imaginary part we define $\arg(j(\gamma, z))$, $z \in \mathbb{H}$.

COROLLARY 16. *Any densely defined derivation D on $B(H_t)$, or on $\mathcal{A}_t$ that has the property that the unbounded operator $\widetilde{Z_t} = D + \widetilde{\Lambda_t^0}$ maps a dense $*$-subalgebra $\widetilde{\mathcal{D}}$ of $\mathcal{A}_t$ into $\mathcal{A}_t$ (so that, on its domain intersected with $L^2(\mathcal{A}_t)$,*

the operator $\widetilde{Z_t}$ implements the cocycle c_t^0), is a solution to the 1-cohomology [9, 13, 19] problem

$$\gamma D - D = -i[T_{\arg(j(\gamma,z))}^t, \cdot] = K(\gamma), \quad \gamma \in \Gamma.$$

In this formula, by γD we denote the derivation defined by $(\gamma D)(k) = \pi_t(\gamma) D(k) \pi_t(\gamma)$, k in the domain of D. Here $T_{\arg(j(\gamma,z))}^t$ is the Toeplitz operator acting on H_t with symbol $\arg(j(\gamma, z))$.

Note that K is 1-cocycle for the group Γ [9, 13], with values in the bounded derivations on $B(H_t)$. The condition that K is 1-cocycle is that

$$K(\gamma_1 \gamma_2) = \gamma_1 K(\gamma_2) + K(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.$$

We are therefore looking for a derivation that implements this 1-cocycle.

Proof. This is simply a consequence of the fact that $\gamma \widetilde{\Lambda_0^t} - \widetilde{\Lambda_0^t}$ has the property that $\gamma \widetilde{\Lambda_0^t} - \widetilde{\Lambda_0^t}$ is a Toeplitz operator of the form introduced in Definition 11 corresponding to the unbounded symbol $i(\arg(j(\gamma, z)) - \arg(j(\gamma, \zeta))$, $z, \zeta \in \mathbb{H}$. Consequently, the coboundary operators for the Hochschild 2-cocycle c_t^0 are in one-to-one correspondence with the derivations implementing the 1-cocycle K on Γ, with values in bounded derivations. $\square$

Remark 17. The canonical solution D_0 of the 1-cocycle problem $\gamma D - D = K(\gamma)$ is in fact the derivation $D_0 = [T_{i \arg \Delta}^t, \cdot]$, where $T_{i \arg \Delta}^t$ is the Toeplitz operator with symbol $i \arg \Delta$ acting on H_t. This corresponds to the canonical solution we constructed in Chapter 1. Moreover, if D is any other derivation, with the same domain as D_0, then if $D = [T_{\sqrt{-1}d}, \cdot]$, thus if D is given as the form $Q_{i(d(z)-d(\zeta))}$ for some measurable real valued function d on $\mathbb{H}$, then the function on $\mathbb{H}$ defined by $d(z) - \arg \Delta(z)$, $z \in \mathbb{H}$, is a Γ-invariant function, and hence, $T_d^t - T_{\arg \Delta(z)}^t$ is affiliated to $\mathcal{A}_t$.

Proof. The fact that D_0 is a solution to the 1-cocycle problem is simply a consequence of the fact that $\arg \Delta(\gamma z) - \arg \Delta(z) = \arg(j(\gamma, z))$.

On the other hand, if D is any other derivation on the same domain as $[M_{\arg \Delta(z)}, \cdot]$ that verifies the 1-cohomology problem, and D is of the form $T_{i(d(z)-d(\zeta))}$, then for any $z, \zeta \in \mathbb{H}, \gamma \in \Gamma$, we have

$$d(\gamma z) - d(\gamma \zeta) - [d(z) - d(\zeta)] = \arg(j(\gamma, z)) - \arg(j(\gamma, \zeta)).$$

Because Γ has no characters, this implies that $d(z) - \arg \Delta(z)$ is Γ-invariant. $\square$

Finally, we recall that the domain of the form $\mathcal{L}_t^0$, introduced in Chapter 1, is in fact the domain of the generator semigroup of unbounded maps Φ_ε^t on $\mathcal{A}_t$. Recall that $\Phi_\varepsilon^t(k)$ is the operator with symbol

$$k(z, \zeta) \Delta(z) \overline{\Delta(\zeta)} ((\overline{z} - \zeta)^{12})^\varepsilon \frac{\chi_{t-\varepsilon}}{\chi_t}, \quad \varepsilon > 0.$$

As proved in [17], this generator, has a dense $*$−algebra domain $\mathcal{D}_t^0 \subseteq L^2(\mathcal{A}_t)$.

In fact our computation proves that this domain contains the intersection between the domain $\mathcal{D}([T_{\ln\Delta}^t, \cdot]) \subseteq B(H_t)$ of the unbounded derivation $[T_{\ln\Delta}^t, \cdot]$ and the domain $\mathcal{D}(\mathcal{T}_{\arg(1-z\overline{\zeta})}^t)$ of the Toeplitz operator with symbol $\arg(1 - z\overline{\zeta})$ considered in Definition 11.

We consider the problem of finding derivations Q as in Definition 11, densely defined on a dense $*$-algebra domain $\mathcal{D}_t^0 \subseteq \mathcal{A}_t$, so that Q admits an extension to a derivation $\widetilde{Q}$ with domain $\mathcal{D}(\widetilde{Q})$ containing $\mathcal{D}([T_{\ln\Delta}^t, \cdot])$, and taking value 0 on $\mathcal{A}_t'$, that is $\widetilde{Q}(k\pi_t(\gamma)) = \widetilde{Q}(k)\pi_t(\gamma)$ for $k \in \mathcal{D}(\widetilde{Q})$, $\gamma \in \Gamma$.

We first prove the following

PROPOSITION 18. *Assume $Q = Q_d$ is a Toeplitz operator as in Definition 11, having as symbol, a measurable function $d = d(z, \zeta)$ on $\mathbb{H} \times \mathbb{H}$, i.e.*

$$\langle Q_d k, m \rangle = \chi_t \iint_{\mathbb{H}^2} d(z, \zeta) k(z, \overline{\zeta}) \overline{m(z, \overline{\zeta})} d_{\mathbb{H}}^t(z, \zeta) \mathrm{d}\nu_0^2(z, \zeta).$$

We assume that Q admits a domain $\mathcal{D}$, that is a dense $*$-subalgebra of $B(H_t)$ so that $\mathcal{D} \cap \mathcal{A}_t$ is dense in $L^2(\mathcal{A}_t)$, and such that $Sp\{\pi_t(\gamma) \mid \gamma \in \Gamma\}$ is contained in $\mathcal{D}$.

Assume that $Q(k\pi_t(\gamma)) = Q(k)\pi_t(\gamma)$ and that $\mathcal{D}$ is sufficiently large so that the span of $k(z, \overline{\zeta}) m(z, \overline{\zeta})$, $k \in \mathcal{D}$, $m \in \mathcal{C}_1(H_t)$, is dense in the continuous functions $C(\mathbb{H} \times \mathbb{H})$.

Then d is $(\Gamma \times \Gamma)$-invariant. In particular, $Q_d(\mathcal{D} \cap \mathcal{A}_t) \subseteq \mathcal{A}_t$.

Proof. We use this for symbols $m(z, \zeta)$ of the form $\overline{v_1(\zeta)} v_2(z)$, where v_1, v_2 are vectors in H_t. We have

$$\langle Q_d(k)v_1, v_2 \rangle = \chi_t \iint_{\mathbb{H}^2} d(z, \zeta) k(z, \overline{\zeta}) v_1(\zeta) \overline{v_2(z)} d_{\mathbb{H}}^t(z, \zeta) \mathrm{d}\nu_0^2(z, \zeta).$$

Then the identity $Q_d(k\pi_t(\gamma)) = Q_d(k)\pi_t(\gamma)$ implies

$$\langle Q_d(k\pi_t(\gamma^{-1}))\pi_t(\gamma)v_1, v_2 \rangle = \langle Q_d(k)v_1, v_2 \rangle,$$

and hence,

$$\iint_{\mathbb{H}^2} d(z, \zeta) k(z, \overline{\gamma\zeta}) v_1(\gamma^{-1}\zeta) \overline{v_2(z)} d_{\mathbb{H}}^t(z, \zeta) \mathrm{d}\nu_0^2(z, \zeta) =$$
$$= \iint_{\mathbb{H}^2} d(z, \zeta) k(z, \overline{\zeta}) v_1(\zeta) \overline{v_2(z)} d_{\mathbb{H}}^t(z, \zeta) \mathrm{d}\nu_0^2(z, \zeta).$$

We obtain:

$$d(z, \gamma\zeta) = d(z, \zeta)$$

for all z, ζ in $\mathbb{H}$, $\gamma \in \Gamma$. $\square$

We prove that for a large class of derivations δ, densely defined on $\mathcal{A}_t$, we can not perturb by δ the operator $\lambda_t 1 + \mathcal{L}_t^0$, so as to get rid of the real part.

THEOREM 19. *Let c_t^0 the unbounded Hochschild cocycle constructed in Chapter 1 of [17], associated to the Γ-invariant Berezin quantization deformation. This cocycle is densely defined on the domain $\mathcal{D}_t^0 \times \mathcal{D}_t^0$, where $\mathcal{D}_t^0$ is the dense $*$-subalgebra of $\mathcal{A}_t$ constructed in Corollary 6.6 of [17].*

Recall that c_t^0 admits a dense coboundary operator $Z_t = \lambda_t 1 + \mathcal{L}_t^0$ with domain $\mathcal{D}_t^0$, where $\lambda_t > 0$ and $\mathcal{L}_t^0$ an antisymmetric operator with $(\mathcal{L}_t^0)^* 1 = -\lambda_t 1$ (so $\mathcal{L}_t^0$ is not defined at 1).

Let δ be a densely defined derivation on $\mathcal{D}_t^0 \subseteq L^2(\mathcal{A}_t)$, and so that δ admits an extension $\widetilde{Q}$ to a domain $\widetilde{\mathcal{D}} \subseteq B(H_t)$, verifying the conditions of the previous lemma.

Then the operator $Z_t + \delta$ is not defined at 1.

Proof. Because of the previous lemma, δ is of the form $\mathcal{T}_d^{t,\Gamma}$ with d a function on $\mathbb{H} \times \mathbb{H}$ that is $(\Gamma \times \Gamma)$-invariant.

We let F be the canonical fundamental domain for the action of Γ on $\mathbb{H}$. We select a function, denoted $\arg_0 \Delta(z)$, which we assume to be a measurable choice of the argument of $\Delta(z)$, taking values interval $(-\pi, \pi]$ and so that the function $\arg \Delta(z) - \arg_0 \Delta(z), z \in \mathbb{H}$, is almost everywhere constant, on the translates of F by elements in the group Γ.

Then, using the identification of the Hilbert space $L^2(\mathcal{A}_t)$ with the Hilbert space $H^2(F \times \mathbb{H}, d_{\mathbb{H}}^t d\nu_0^2)$, we observe that the bilinear form associated to the operator $\mathcal{T}_\varphi^{t,\Gamma}$, where

$$\varphi(z, \zeta) = \arg(z - \overline{\zeta}) + \arg \Delta(z) - \arg \Delta(\zeta),$$

breaks into the bounded linear form

$$\iint_{F \times \mathbb{H}} \left[\arg_0 \Delta(z) - \arg_0 \Delta(\zeta) + \arg(z - \overline{\zeta}) \right] k(z, \overline{\zeta}) \overline{l(\overline{z}, \zeta)} d_{\mathbb{H}}^t(z, \zeta) d\nu_0^2(z, \zeta)$$

and the unbounded bilinear form

$$\iint_{F \times \mathbb{H}} \left(\sum \theta_\gamma \chi_{\gamma F}(z) - \sum \theta_\gamma \chi_{\gamma F}(\zeta) \right) k(z, \overline{\zeta}) \overline{l(\zeta, \overline{z})} d_{\mathbb{H}}^t(z, \zeta) d\nu_0^2(z, \zeta).$$

In this expression we have $\theta_\gamma = \arg \Delta(z) - \arg_0 \Delta(z)$ for $z \in \gamma F$, $\gamma \in \Gamma$. Hence,

$$\sum \theta_\gamma \chi_{\gamma F}(z) = \arg \Delta(z) - \arg_0 \Delta(z), \quad z \in \mathbb{H}.$$

Hence, if we consider the unbounded bilinear form associated to

$$\langle (Q_d + \mathcal{L}_t^0) k, l \rangle_{L^2(\mathcal{A}_t)},$$

then the unbounded part is

$$\iint_{F \times \mathbb{H}} \Big(\sum \theta_\gamma \chi_{\gamma F}(\zeta) - d(z, \zeta) \Big) k(z, \overline{\zeta}) \overline{l(\zeta, \overline{z})} d_{\mathbb{H}}^t(z, \zeta) \mathrm{d}\nu_0^2(z, \zeta).$$

But θ_γ is the unique (unbounded) 1-cocycle coboundary of the Euler co-cycle, which is rapidly increasing to ∞, and d is $(\Gamma \times \Gamma)$-invariant, and hence, this expression can not contain the element $k = 1$ in the domain of its form closure. $\square$

We finally note that on the von Neumann algebra $L(F_\infty)$ associated to the free group F_∞ with infinitely many generators, there exists a closed derivation δ, so that $\mathrm{Re}\, \delta = \alpha > 0$. In addition there is an abelian δ-invariant algebra $L^\infty([0, 1])$, and so that $\mathrm{Im}\, \delta$ has deficiency indices $(0, 1)$ when restricted to the Hilbert space of the smaller algebra, and has the same indices on the Hilbert space associated to the full algebra.

PROPOSITION 20. *We use the description of $L(F_\infty)$ as the infinite corner $\chi_{[0,1]}(L^\infty([0, \infty]) \star \mathbb{C}(X))\chi_{[0,1]}$ introduced in [18].*

Here X is an "infinite semicircular" element (see [18]), and α_t acts on $L^\infty([0, \infty)) \star \mathbb{C}(X)$ by dilation on $L^\infty([0, \infty])$ and multiplying X by $t^{-1/2}$.

Let $\mathcal{D} \subseteq \chi_{[0,1]}(L^\infty([0, \infty)) \star \mathbb{C}(X))\chi_{[0,1]}$ be the domain generated by the span of $f_0 X f_1 X \ldots X f_{n-1} X f_n$, where $f_0, f_n \in C_0^1([0, 1))$ and $f_1, \ldots, f_n$ are differentiable with compact support.

Then $\delta = \frac{\mathrm{d}}{\mathrm{d}_t} \chi_{[0,1]} \alpha_t(x) \chi_{[0,1]}$ is a derivation on $\mathcal{D}$, closable, with $\delta^* 1 = \alpha > 0$. Moreover δ leaves invariant a dense domain in $L^\infty([0, 1])$.

Proof. This is simply a consequence of the fact that α_t is scaling the trace and acts as a dilation on $L^\infty([0, 1])$. $\square$

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