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DUALITY FEATURES OF LEFT HOPF ALGEBROIDS

SOPHIE CHEMLA, FABIO GAVARINI, AND NIELS KOWALZIG

ABSTRACT. We explore special features of the pair (U^*, U_*) formed by the right and left dual over a (left) bialgebroid U in case the bialgebroid is, in particular, a left Hopf algebroid. It turns out that there exists a bialgebroid morphism S^* from one dual to another that extends the construction of the antipode on the dual of a Hopf algebra, and which is an isomorphism if U is both a left and right Hopf algebroid. This structure is derived from Phùng's categorical equivalence between left and right comodules over U without the need of a (Hopf algebroid) antipode, a result which we review and extend. In the applications, we illustrate the difference between this construction and those involving antipodes and also deal with dualising modules and their quantisations.

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1. INTRODUCTION

A characteristic feature in standard Hopf algebra theory is its self-duality, that is, the dual of a (finite-dimensional) Hopf algebra (over a field) is a Hopf algebra again. In particular, the antipode of this dual is nothing but the transpose of the original antipode; see,

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for example, [Sw]. In the broader setup of (*left* or *full*) Hopf algebroids over possibly noncommutative rings, this peculiar property appears to be more intricate; see [B] or §2 for the precise definitions of these objects, we only mention here that, in contrast to full Hopf algebroids, there is no notion of antipode for left Hopf algebroids: one rather considers the inverse of a certain Hopf-Galois map and its associated *translation map*. Nevertheless, left Hopf algebroids appear as the correct generalisation of Hopf algebras over noncommutative rings, whereas full Hopf algebroids generalise Hopf algebras twisted by a character, see, for example, [Ko, §4.1.2].

Recently (after the first posting of this article), Schauenburg [Sch2] showed that the (skew) dual of a left Hopf algebroid (under a suitable finiteness assumption) carries some Hopf structure as well without giving an explicit expression for the inverse of the respective Hopf-Galois map or the associated translation map.

However, instead of one dual, a left bialgebroid U rather possesses *two*, the *right dual* U^* and the *left dual* U_*, which, on top, live in a different category compared to U as they are both (under certain finiteness assumptions) right bialgebroids [KadSz]. There is no reason why one should prefer one of the duals to the other. Hence, any question concerning "the dual of U" should be converted into a question about the pair (U^*, U_*).

Dealing with *full* Hopf algebroids (see §5.2.1) does notably worsen the situation as there are actually *four* duals to be taken into account, two of which are left and two of which are right bialgebroids. In this case, an answer to the question of the nature of the Hopf structure on the dual(s) has only been given in certain cases, more precisely, in the presence of integrals [BSz, §5].

1.1. **Aims and objectives.** As mentioned a moment ago, the object one should investigate to discover the limits of self-duality in (left) Hopf algebroid theory is a *pair* of duals. In short, our question reads as follows: if a left bialgebroid U is, in particular, a left (or right) Hopf algebroid, what extra structure can be found on the pair (U^*, U_*) of duals?

1.2. **Main results.** After highlighting in §3 a multitude of module structures that exist on Hom-spaces and tensor products in presence of a left or right Hopf algebroid structure and that will be used in the sequel, in §4 we review (and extend) Phùng's equivalence (cf. [Phù]) of comodule categories (see the main text for all definitions and conventions used hereafter):

Theorem A. *Let (U, A) be a left bialgebroid.*

(*i*) *Let (U, A) be additionally a left Hopf algebroid such that $U_\triangleleft$ is projective. Then there exists a (strict) monoidal functor $\mathbf{Comod}\text{-}U \to U\text{-}\mathbf{Comod}$: if M is a right U-comodule with coaction $m \mapsto m_{(0)} \otimes_A m_{(1)}$, then*

$$M \to U_\triangleleft \otimes_A M, \quad m \mapsto m_{(1)-} \otimes_A m_{(0)}\epsilon(m_{(1)+}),$$

defines a left comodule structure on M over U.

(*ii*) *Let (U, A) be a right Hopf algebroid such that ${}_\triangleright U$ is projective. Then there exists a (strict) monoidal functor $U\text{-}\mathbf{Comod} \to \mathbf{Comod}\text{-}U$: if N is a left U-comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then*

$$N \to N \otimes_A {}_\triangleright U, \quad n \mapsto \epsilon(n_{(-1)[+]})n_{(0)} \otimes_A n_{(-1)[-]},$$

defines a right comodule structure on N over U.

(*iii*) *If U is both a left and right Hopf algebroid and if both $U_\triangleleft$ and ${}_\triangleright U$ are A-projective, then the functors mentioned in (i) and (ii) are quasi-inverse to each other and we have an equivalence*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U$$

of monoidal categories.

Note that this equivalence works without the help of an antipode as there are objects that are both left and right Hopf algebroids but not full Hopf algebroids (cocommutative left Hopf algebroids, for example).

Starting from this result, under suitable finiteness hypotheses on U, one can construct functors $\mathbf{Mod}\text{-}U_* \to \mathbf{Mod}\text{-}U^*$ resp. $\mathbf{Mod}\text{-}U^* \to \mathbf{Mod}\text{-}U_*$, and from this we isolate maps $U^* \to U_*$ resp. $U_* \to U^*$, which even make sense without any finiteness assumptions as proven in §5, and which are our main object of interest.

In §5.1 we can then give the following answer to the problem mentioned in §1.1, that is, elucidate the relation between the left and the right dual:

Theorem B. *Let (U, A) be a left bialgebroid.*

(i) *If (U, A) is moreover a left Hopf algebroid, there is a morphism $S^* : U^* \to U_*$ of A^e-rings with augmentation; if, in addition, both ${}_\triangleright U$ and $U_\triangleleft$ are finitely generated A-projective, then (S^*, id_A) is a morphism of right bialgebroids.*

(ii) *If (U, A) is a right Hopf algebroid instead, there is a morphism $S_* : U_* \to U^*$ of A^e-rings with augmentation; if, in addition, both ${}_\triangleright U$ and $U_\triangleleft$ are finitely generated A-projective, then (S_*, id_A) is a morphism of right bialgebroids.*

(iii) *If (U, A) is simultaneously both a left and a right Hopf algebroid, then the two morphisms are inverse to each other; hence, if both ${}_\triangleright U$ and $U_\triangleleft$ are finitely generated A-projective, then $U^* \simeq U_*$ as right bialgebroids.*

Now, as said before, for a left Hopf algebroid (which is finitely generated projective with respect to both source and target map) there is no canonical choice for which dual to consider but in view of Theorem B, in case the left Hopf algebroid is simultaneously a right Hopf algebroid, both duals are isomorphic and hence can be seen as *its* dual, which carries a Hopf structure by Schauenburg's recent result [Sch2]. This seems to be as close as one can get to self-duality.

Theorem B is a straight analogue of the construction on the dual for a (finite-dimensional) Hopf algebra H (over a field) with antipode S in the following sense: here, one has $H^* = (H_*)^{\mathrm{op}}_{\mathrm{coop}}$ and S^* is exactly the transpose of S and therefore the antipode for the dual Hopf algebra.

Observe that this last case in Theorem B, *i.e.*, the presence of both a left and right Hopf structure is given, for example, when U is a full Hopf algebroid with bijective antipode but also in weaker cases such as for the universal enveloping algebra of a Lie-Rinehart algebra. In the situation of a full Hopf algebroid, U^* and U_* are additionally linked (in both directions) by the transposition ${}^tS$ of the antipode $S : U \to U^{\mathrm{op}}_{\mathrm{coop}}$. However, in Theorem 5.2.4 we show that the map ${}^tS$ in general does not coincide with S^* or S_*, in contrast to the Hopf algebra case mentioned above. Moreover, if a left Hopf algebroid U is cocommutative with both ${}_\triangleright U$ and $U_\triangleleft$ finitely generated A-projective, then $U^* = (U_*)_{\mathrm{coop}}$ is a full Hopf algebroid (with antipode precisely given by S^*), though U might be not.

We shall also see in §6 that Theorem B actually extends to a larger setup, in particular, it applies to some interesting cases (coming from geometry), where neither ${}_\triangleright U$ nor $U_\triangleleft$ are finitely generated projective but U^* and U_* are still right bialgebroids in a suitable (topological) sense, such as when U is the universal enveloping of a Lie-Rinehart algebra, or a quantisation of it.

In §6, we illustrate these results by considering some examples related to Lie-Rinehart algebras (or Lie algebroids) and their jet spaces, as well as their quantised versions. Moreover, in §6.4 we consider further duality phenomena related to dualising modules, which appear in Poincaré duality, along with their quantisations.

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2. Preliminaries

We list here those preliminaries with respect to bialgebroids and their duals that are needed in this article; see, *e.g.*, [B] and references therein for an overview on this subject.

Fix an (associative, unital, commutative) ground ring k. Unadorned tensor products will always be meant over k. All other algebras, modules etc. will have an underlying structure of a k-module. Secondly, fix an associative and unital k-algebra A, *i.e.*, a ring with a ring homomorphism $\eta_A : k \to Z(A)$ to its centre. Denote by A^{op} the opposite and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra of A, and by A-$\mathbf{Mod}$ the category of left A-modules. Recall that an A-*ring* is a monoid in the monoidal category $(A^{\mathrm{e}}\text{-}\mathbf{Mod}, \otimes_A, A)$ of (A, A)-bimodules fulfilling the usual associativity and unitality axioms, whereas dually an A-*coring* is a comonoid in this category that is coassociative and counital.

2.1. **Bialgebroids.** For an A^e-ring U given by the k-algebra map $\eta : A^{\mathrm{e}} \to U$, consider the restrictions $s := \eta(- \otimes 1_U)$ and $t := \eta(1_U \otimes -)$, called *source* and *target* map, respectively. Thus an A^{e}-ring U carries two A-module structures from the left and two from the right, namely

$$a \triangleright u \triangleleft b := s(a)t(b)u, \qquad a \blacktriangleright u \blacktriangleleft b := ut(a)s(b), \qquad \forall\, a, b \in A, u \in U.$$

If we let $U_{\triangleleft} \otimes_{A\, \triangleright} U$ be the corresponding tensor product of U (as an A^{e}-module) with itself, we define the *(left) Takeuchi-Sweedler product* as

$$U_{\triangleleft} \times_{A\, \triangleright} U \ := \ \big\{ \textstyle\sum_i u_i \otimes u_i' \in U_{\triangleleft} \otimes_{A\, \triangleright} U \mid \sum_i (a \blacktriangleright u_i) \otimes u_i' = \sum_i u_i \otimes (u_i' \blacktriangleleft a), \ \forall a \in A \big\}.$$

By construction, $U_{\triangleleft} \times_{A\, \triangleright} U$ is an A^{e}-submodule of $U_{\triangleleft} \otimes_{A\, \triangleright} U$; it is also an A^{e}-ring via factorwise multiplication, with unit $1_U \otimes 1_U$ and $\eta_{U_{\triangleleft} \times_{A\, \triangleright} U}(a \otimes \tilde{a}) := s(a) \otimes t(\tilde{a})$.

Symmetrically, one can consider the tensor product $U_{\blacktriangleleft} \otimes_{A\, \blacktriangleright} U$ and define the *(right) Takeuchi-Sweedler product* as $U_{\blacktriangleleft} \times_{A\, \blacktriangleright} U$, which is an A^{e}-ring inside $U_{\blacktriangleleft} \otimes_{A\, \blacktriangleright} U$.

Definition 2.1.1. A *left bialgebroid* (U, A) is a k-module U with the structure of an A^{e}-ring (U, s^ℓ, t^ℓ) and an A-coring $(U, \Delta_\ell, \epsilon)$ subject to the following compatibility relations:

(*i*) the A^{e}-module structure on the A-coring U is that of ${}_{\triangleright}U_{\triangleleft}$;

(*ii*) the coproduct Δ_ℓ is a unital k-algebra morphism taking values in $U_{\triangleleft} \times_{A\, \triangleright} U$;

(*iii*) for all $a, b \in A$, $u, u' \in U$, one has:

$$\epsilon(a \triangleright u \triangleleft b) = a\epsilon(u)b, \quad \epsilon(uu') = \epsilon\big(u \blacktriangleleft \epsilon(u')\big) = \epsilon\big(\epsilon(u') \blacktriangleright u\big). \tag{2.1}$$

A *morphism* between left bialgebroids (U, A) and (U', A') is a pair (F, f) of maps $F : U \to U'$, $f : A \to A'$ that commute with all structure maps in an obvious way.

As for any ring, we can define the categories U-$\mathbf{Mod}$ and $\mathbf{Mod}$-U of left and right modules over U. Note that U-$\mathbf{Mod}$ forms a monoidal category but $\mathbf{Mod}$-U usually does not. However, in both cases there is a forgetful functor U-$\mathbf{Mod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$, resp. $\mathbf{Mod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$: whereas we denote left and right action of a bialgebroid U on $M \in U$-$\mathbf{Mod}$ or $N \in \mathbf{Mod}$-U usually by juxtaposition, for the resulting A^{e}-module structures the notation

$$a \triangleright m \triangleleft b := s^\ell(a)t^\ell(b)m, \qquad a \blacktriangleright m \blacktriangleleft b := ns^\ell(b)t^\ell(a)$$

for $m \in M,\ n \in N,\ a, b \in A$ is used instead. For example, the base algebra A itself is a left U-module via the left action $u(a) := \epsilon(u \blacktriangleleft a) = \epsilon(a \blacktriangleright u)$ for $u \in U$ and $a \in A$, but in most cases there is no right U-action on A.

Dually, one can introduce the categories U-$\mathbf{Comod}$ and $\mathbf{Comod}$-U of left resp. right U-comodules, both of which are monoidal; here again, one has forgetful functors U-$\mathbf{Comod} \to A^{\mathrm{e}}$-$\mathbf{Mod}$ and $\mathbf{Comod}$-$U \to A^{\mathrm{e}}$-$\mathbf{Mod}$. More precisely (see, *e.g.*, [B]), a (say) right comodule is a right comodule of the coring underlying U, *i.e.*, a right A-module M and a right A-module map ${}_M\Delta : M \to M \otimes_A {}_{\triangleright}U$, $m \mapsto m_{(0)} \otimes_A m_{(1)}$, satisfying the usual coassociativity and counitality axioms. On any $M \in \mathbf{Comod}$-U there is an induced *left* A-action given by

$$am := m_{(0)}\varepsilon(a \blacktriangleright m_{(1)}), \tag{2.2}$$

and ${}_M\Delta$ is then an A^{e}-module morphism $M \to M \times_A {}_{\triangleright}U$, where $M \times_A {}_{\triangleright}U$ is the A^{e}-submodule of $M \otimes_A {}_{\triangleright}U$ whose elements $\sum_i m_i \otimes_A u_i$ fulfil

$$\textstyle\sum_i am_i \otimes_A u_i = \sum_i m_i \otimes_A u_i \blacktriangleleft a, \ \forall a \in A. \tag{2.3}$$

The notion of a *right bialgebroid* is obtained if one starts with the A^{e}-module structure given by $\blacktriangleright$ and $\blacktriangleleft$ instead of $\triangleright$ and $\triangleleft$. We will refrain from giving the details here and refer to [KadSz] instead.

Remark 2.1.2. The *opposite* of a left bialgebroid $(U, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ yields a *right* bialgebroid $(U^{\mathrm{op}}, A, t^\ell, s^\ell, \Delta_\ell, \epsilon)$. The *coopposite* of a left bialgebroid is the *left* bialgebroid given by $(U, A^{\mathrm{op}}, t^\ell, s^\ell, \Delta_\ell^{\mathrm{coop}}, \epsilon)$.

2.2. **Pairings of U-modules and dual bialgebroids.** Let (U, A) be a left bialgebroid, $M, M' \in U$-$\mathbf{Mod}$ be left U-modules, and $N, N' \in \mathbf{Mod}$-U be right U-modules. Define

$$\begin{array}{rclrcl} \mathrm{Hom}_{A^{\mathrm{op}}}(M, M') &:=& \mathrm{Hom}_{A^{\mathrm{op}}}(M_\triangleleft, M'_\triangleleft), & \mathrm{Hom}_A(M, M') &:=& \mathrm{Hom}_A({}_\triangleright M, {}_\triangleright M'), \\ \mathrm{Hom}_{A^{\mathrm{op}}}(N, N') &:=& \mathrm{Hom}_{A^{\mathrm{op}}}(N_\blacktriangleleft, N'_\blacktriangleleft), & \mathrm{Hom}_A(N, N') &:=& \mathrm{Hom}_A({}_\blacktriangleright N, {}_\blacktriangleright N'). \end{array}$$

In particular, for $M' := A$ we set $M_* := \mathrm{Hom}_A(M, A)$ and $M^* := \mathrm{Hom}_{A^{\mathrm{op}}}(M, A)$, called, respectively, the *left* and *right* dual of M.

The notion of *pairing* between A^{e}-bimodules is also useful (see, for instance, [ChGa]):

Definition 2.2.1. Let U and W be two A^{e}-bimodules.

(*i*) A *left A^{e}-pairing* is a k-bilinear map $\langle\ ,\ \rangle : U \times W \to A$ such that for any $u \in U$, $w \in W$, and $a \in A$, one has

$$\begin{array}{llll} \langle u, a \triangleright w\rangle = \langle u \triangleleft a, w\rangle, & \langle u, w \triangleleft a\rangle = \langle a \blacktriangleright u, w\rangle, & \langle u, a \blacktriangleright w\rangle = \langle u \blacktriangleleft a, w\rangle, \\ \langle u, w \blacktriangleleft a\rangle = \langle u, w\rangle a, & \langle a \triangleright u, w\rangle = a\langle u, w\rangle. & \end{array}$$

(*ii*) A *right A^{e}-pairing* is a k-bilinear map $\langle\ ,\ \rangle : U \times W \to A$ such that for any $u \in U$, $w \in W$, and $a \in A$, one has

$$\begin{array}{llll} \langle u, w \triangleleft a\rangle = \langle a \triangleright u, w\rangle, & \langle u, a \triangleright w\rangle = \langle u \blacktriangleleft a, w\rangle, & \langle u, w \blacktriangleleft a\rangle = \langle a \blacktriangleright u, w\rangle, \\ \langle u, a \blacktriangleright w\rangle = a\langle u, w\rangle, & \langle u \triangleleft a, w\rangle = \langle u, w\rangle a. & \end{array}$$

2.2.2. Duals of bialgebroids. Let U_* resp. U^* be the left resp. right dual of a left bialgebroid. If ${}_\triangleright U$ is finitely generated projective, then U_* is canonically endowed with a *right* bialgebroid structure [KadSz] such that the evaluation pairing between U and U_* is a (nondegenerate) *left* pairing; similarly, if $U_\triangleleft$ is finitely generated projective, then U^* has a canonical *right* bialgebroid structure for which the natural pairing between U and U^* is a *right* pairing. If instead in either case the above finitely generated projective assumption is not satisfied, then both U^* and U_* are nevertheless A^{e}-rings endowed with a "counit" map, or augmentation.

2.3. **Left and right Hopf algebroids.** For any left bialgebroid U, define the *Hopf-Galois maps*

$$\begin{array}{rclrcl} \alpha_\ell : {}_\blacktriangleright U \otimes_{A^{\mathrm{op}}} U_\triangleleft &\to& U_\triangleleft \otimes_A {}_\triangleright U, & u \otimes_{A^{\mathrm{op}}} v &\mapsto& u_{(1)} \otimes_A u_{(2)} v, \\ \alpha_r : U_\blacktriangleleft \otimes^A {}_\triangleright U &\to& U_\triangleleft \otimes_A {}_\triangleright U, & u \otimes^A v &\mapsto& u_{(1)} v \otimes_A u_{(2)}. \end{array}$$

With the help of these maps, we make the following definition due to Schauenburg [Sch1]:

Definition 2.3.1. A left bialgebroid U is called a *left Hopf algebroid* if α_ℓ is a bijection. Likewise, it is called a *right Hopf algebroid* if α_r is so. In either case, we adopt for all $u \in U$ the following (Sweedler-like) notation

$$u_+ \otimes_{A^{\mathrm{op}}} u_- := \alpha_\ell^{-1}(u \otimes_A 1), \qquad u_{[+]} \otimes^A u_{[-]} := \alpha_r^{-1}(1 \otimes_A u), \tag{2.4}$$

and call both maps $u \mapsto u_+ \otimes_{A^{\mathrm{op}}} u_-$ and $u \mapsto u_{[+]} \otimes^A u_{[-]}$ *translation maps*.

Analogous notions exist with respect to an underlying *right* bialgebroid structure, but we will not give the details here.

Remark 2.3.2.

(i) In case $A = k$ is central in U, one can show that α_ℓ is invertible if and only if U is a Hopf algebra, and the translation map reads $u_+ \otimes u_- := u_{(1)} \otimes S(u_{(2)})$, where S is the antipode of U. On the other hand, U is a Hopf algebra with invertible antipode if and only if both α_ℓ and α_r are invertible, and then $u_{[+]} \otimes u_{[-]} := u_{(2)} \otimes S^{-1}(u_{(1)})$.

(ii) The underlying left bialgebroid in a *full* Hopf algebroid with bijective antipode is both a left and right Hopf algebroid (but not necessarily vice versa); see [BSz, Prop. 4.2] for the details of this construction.

The following proposition collects some properties of the translation maps [Sch1]:

Proposition 2.3.3. *Let U be a left bialgebroid.*

(i) *If U is a left Hopf algebroid, the following relations hold:*

$$\begin{aligned}
u_+ \otimes_{A^{\mathrm{op}}} u_- &\in U \times_{A^{\mathrm{op}}} U, && (2.5)\\
u_{+(1)} \otimes_A u_{+(2)} u_- &= u \otimes_A 1 \quad \in U_{\triangleleft} \otimes_A {}_{\triangleright}U, && (2.6)\\
u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} &= u \otimes_{A^{\mathrm{op}}} 1 \quad \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, && (2.7)\\
u_{+(1)} \otimes_A u_{+(2)} \otimes_{A^{\mathrm{op}}} u_- &= u_{(1)} \otimes_A u_{(2)+} \otimes_{A^{\mathrm{op}}} u_{(2)-}, && (2.8)\\
u_+ \otimes_{A^{\mathrm{op}}} u_{-(1)} \otimes_A u_{-(2)} &= u_{++} \otimes_{A^{\mathrm{op}}} u_- \otimes_A u_{+-}, && (2.9)\\
(uv)_+ \otimes_{A^{\mathrm{op}}} (uv)_- &= u_+ v_+ \otimes_{A^{\mathrm{op}}} v_- u_-, && (2.10)\\
u_+ u_- &= s^\ell(\varepsilon(u)), && (2.11)\\
\varepsilon(u_-) \blacktriangleright u_+ &= u, && (2.12)\\
(s^\ell(a) t^\ell(b))_+ \otimes_{A^{\mathrm{op}}} (s^\ell(a) t^\ell(b))_- &= s^\ell(a) \otimes_{A^{\mathrm{op}}} s^\ell(b), && (2.13)
\end{aligned}$$

where in (2.5) we mean the Takeuchi-Sweedler product

$$U \times_{A^{\mathrm{op}}} U := \big\{ \textstyle\sum_i u_i \otimes v_i \in {}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft} \mid \sum_i u_i \triangleleft a \otimes v_i = \sum_i u_i \otimes a \blacktriangleright v_i, \ \forall a \in A \big\}.$$

(ii) *Analogously, if U is a right Hopf algebroid, one has:*

$$\begin{aligned}
u_{[+]} \otimes^A u_{[-]} &\in U \times^A U, && (2.14)\\
u_{[+](1)} u_{[-]} \otimes_A u_{[+](2)} &= 1 \otimes_A u \quad \in U_{\triangleleft} \otimes_A {}_{\triangleright}U, && (2.15)\\
u_{(2)[-]} u_{(1)} \otimes^A u_{(2)[+]} &= 1 \otimes^A u \quad \in U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U, && (2.16)\\
u_{[+](1)} \otimes^A u_{[-]} \otimes_A u_{[+](2)} &= u_{(1)[+]} \otimes^A u_{(1)[-]} \otimes_A u_{(2)}, && (2.17)\\
u_{[+][+]} \otimes^A u_{[+][-]} \otimes_A u_{[-]} &= u_{[+]} \otimes^A u_{[-](1)} \otimes_A u_{[-](2)}, && (2.18)\\
(uv)_{[+]} \otimes^A (uv)_{[-]} &= u_{[+]} v_{[+]} \otimes^A v_{[-]} u_{[-]}, && (2.19)\\
u_{[+]} u_{[-]} &= t^\ell(\varepsilon(u)), && (2.20)\\
u_{[+]} \blacktriangleleft \varepsilon(u_{[-]}) &= u, && (2.21)\\
(s^\ell(a) t^\ell(b))_{[+]} \otimes^A (s^\ell(a) t^\ell(b))_{[-]} &= t^\ell(b) \otimes^A t^\ell(a), && (2.22)
\end{aligned}$$

where in (2.14) we mean the Sweedler-Takeuchi product

$$U \times^A U := \big\{ \textstyle\sum_i u_i \otimes v_i \in U_{\blacktriangleleft} \otimes^A {}_{\triangleright}U \mid \sum_i a \triangleright u_i \otimes v_i = \sum_i u_i \otimes v_i \blacktriangleleft a, \ \forall a \in A \big\}.$$

These two structures are not entirely independent:

Lemma 2.3.4. *The following mixed relations hold among left and right translation maps:*

$$u_{+[+]} \otimes_{A^{\mathrm{op}}} u_- \otimes^A u_{+[-]} = u_{[+]+} \otimes_{A^{\mathrm{op}}} u_{[+]-} \otimes^A u_{[-]}, \tag{2.23}$$
$$u_+ \otimes_{A^{\mathrm{op}}} u_{-[+]} \otimes^A u_{-[-]} = u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} \otimes^A u_{(2)}, \tag{2.24}$$
$$u_{[+]} \otimes^A u_{[-]+} \otimes_{A^{\mathrm{op}}} u_{[-]-} = u_{(2)[+]} \otimes^A u_{(2)[-]} \otimes_{A^{\mathrm{op}}} u_{(1)}, \tag{2.25}$$

where, for example, in the first equation (2.23) the second tensor product relates the first component with the third, and mutatis mutandis *for the other identities.*

Proof. In order to prove (2.23), we apply $\alpha_\ell \otimes \mathrm{id}$ to both sides (note that this operation is well-defined on the considered tensor products); for the right hand side we obtain, by definition,

$$(\alpha_\ell \otimes \mathrm{id})(u_{[+]+} \otimes_{A^{\mathrm{op}}} u_{[+]-} \otimes^A u_{[-]}) = (u_{[+]} \otimes_A 1) \otimes^A u_{[-]},$$

and for the left hand side we have

$$\begin{aligned}(\alpha_\ell \otimes \mathrm{id})(u_{+[+]} \otimes_{A^{\mathrm{op}}} u_- \otimes^A u_{+[-]}) &= (u_{+[+](1)} \otimes_A u_{+[+](2)} u_-) \otimes^A u_{+[-]} \\ = (u_{+(1)[+]} \otimes_A u_{+(2)} u_-) \otimes^A u_{+(1)[-]} &= (u_{[+]} \otimes_A 1) \otimes^A u_{[-]},\end{aligned}$$

using (2.17) and (2.6). Since α_ℓ is assumed to be an isomorphism, this proves (2.23).

Let us also prove (2.24); the remaining identity will be left to the reader. To this end, apply $\mathrm{id} \otimes \alpha_r$ to both sides in (2.24): for the left hand side, we obtain

$$\begin{aligned}(\mathrm{id} \otimes \alpha_r)(u_+ \otimes_{A^{\mathrm{op}}} u_{-[+]} \otimes^A u_{-[-]}) &= u_+ \otimes_{A^{\mathrm{op}}} (u_{-[+](1)} u_{-[-]} \otimes_A u_{-[+](2)}) \\ &= u_+ \otimes_{A^{\mathrm{op}}} (1 \otimes_A u_-)\end{aligned}$$

by (2.15), and where in the second equation the first tensor product relates the first component with the third. As for the right hand side, we compute:

$$\begin{aligned}(\mathrm{id} \otimes \alpha_r)(u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} \otimes^A u_{(2)}) &= u_{(1)+} \otimes_{A^{\mathrm{op}}} (u_{(1)-(1)} u_{(2)} \otimes_A u_{(1)-(2)}) \\ = u_{(1)++} \otimes_{A^{\mathrm{op}}} (u_{(1)-} u_{(2)} \otimes_A u_{(1)+-}) &= u_+ \otimes_{A^{\mathrm{op}}} (1 \otimes_A u_-),\end{aligned}$$

using (2.9) and (2.7) in the last step as follows: Eq. (2.7) yields $u_{(1)+} \otimes_{A^{\mathrm{op}}} u_{(1)-} u_{(2)} \otimes_A 1 = u \otimes_{A^{\mathrm{op}}} 1 \otimes_A 1$ and applying α_ℓ^{-1} to the first and the third component gives the required equality. □

3. Modules over left or right Hopf algebroids

In this section we collect some general results about modules over left and right Hopf algebroids. Some of them are known, while others seem to have passed unnoticed so far (see Note 3.1.2 below).

3.1. **Module structures on** Hom**-spaces and tensor products.** Similarly as for bialgebras, the tensor product $M_\triangleleft \otimes_A {}_\triangleright M'$ of two left U-modules with left U-module structure given by

$$u(m \otimes_A m') := u_{(1)} m \otimes_A u_{(2)} m' \tag{3.1}$$

equips the category U-$\mathbf{Mod}$ for a left bialgebroid U with a monoidal structure. On the other hand, for $M \in U$-$\mathbf{Mod}$ and $N \in \mathbf{Mod}$-U, the A^{e}-module $\mathrm{Hom}_{A^{\mathrm{op}}}(M_\triangleleft, N_\blacktriangleleft)$ is a right U-module via

$$(fu)(m) := f(u_{(1)} m) u_{(2)}.$$

The existence of a translation map if U is, on top, a left or right Hopf algebroid makes it possible to endow Hom-spaces and tensor products of U-modules with further natural U-module structures. The proof of the following proposition is straightforward.

Proposition 3.1.1. *Let (U, A) be a left bialgebroid, $M, M' \in U$-$\mathbf{Mod}$ and $N, N' \in \mathbf{Mod}$-U be left resp. right U-modules, denoting the respective actions by juxtaposition.*

(*i*) *Let (U, A) be additionally a left Hopf algebroid.*

(a) *The* A^{e}*-module* $\mathrm{Hom}_{A^{\mathrm{op}}}(M, M')$ *carries a left* U*-module structure given by*

$$(uf)(m) := u_+\big(f(u_- m)\big). \tag{3.2}$$

In particular, M^* *is endowed with a left* U*-module structure.*

(b) *The* A^{e}*-module* $\mathrm{Hom}_A(N, N')$ *carries a left* U*-module structure via*

$$(u \rhd f)(n) := \big(f(nu_+)\big)u_-. \tag{3.3}$$

(c) *The* A^{e}*-module* ${}_\blacktriangleright N \otimes_{A^{\mathrm{op}}} M_\triangleleft$ *carries a right* U*-module structure via*

$$(n \otimes_{A^{\mathrm{op}}} m) \lhd u := nu_+ \otimes_{A^{\mathrm{op}}} u_- m. \tag{3.4}$$

(*ii*) *Let* (U, A) *be a right Hopf algebroid instead.*

(a) *The* A^{e}*-module* $\mathrm{Hom}_A(M, M')$ *carries a left* U*-module structure given by*

$$(uf)(m) := u_{[+]}\big(f(u_{[-]} m)\big). \tag{3.5}$$

In particular, M_* *is naturally endowed with a left* U*-module structure.*

(b) *The* A^{e}*-module* $\mathrm{Hom}_{A^{\mathrm{op}}}(N, N')$ *carries a left* U*-module structure given by*

$$(u \rhd f)(n) := \big(f(nu_{[+]})\big)u_{[-]}. \tag{3.6}$$

(c) *The* A^{e}*-module* $N_\blacktriangleleft \otimes^A {}_\triangleright M$ *carries a right* U*-module structure given by*

$$(n \otimes^A m) \lhd u := nu_{[+]} \otimes^A u_{[-]} m. \tag{3.7}$$

Note 3.1.2. These structures are well-known for D-modules (that is, when $U = D_X$, see [Bo, Ka]) and were later extended to $V^\ell(L)$-modules in [Ch1], [Ch3]. The results about tensor products can be found in [KoKr], whereas (3.2) serves in [Sch1, Thm. 3.5] to characterise a possible (left) Hopf structure on a bialgebroid.

3.2. **Switching left and right modules: dualising modules.** We investigate now conditions which imply an equivalence between the categories of left and of right U-modules for a left bialgebroid U which is simultaneously a left and right Hopf algebroid. As in other frameworks, this is guaranteed by the existence of a suitable *dualising module*. This is the content of the next result, which generalises the well-known equivalence of categories between left and right $\mathcal{D}$-modules (due to Borel [Bo] and Kashiwara [Ka]). It also generalises the equivalence between left and right modules over a Lie-Rinehart algebra, *cf.* [Ch1].

Proposition 3.2.1. *Let* (U, A) *be simultaneously a left and right Hopf algebroid. Assume that there exists a right* U*-module* P*, where* $P_\blacktriangleleft$ *is finitely generated projective over* A^{op}*, such that*

(*i*) *the left* U*-module morphism*

$$A \to \mathrm{Hom}_{A^{\mathrm{op}}}(P, P), \quad a \mapsto \{p \mapsto a \blacktriangleright p\}$$

is an isomorphism of k*-modules;*

(*ii*) *the evaluation map*

$${}_\blacktriangleright P \otimes_{A^{\mathrm{op}}} \mathrm{Hom}_{A^{\mathrm{op}}}(P, N)_\triangleleft \to N, \quad p \otimes_{A^{\mathrm{op}}} \phi \mapsto \phi(p) \tag{3.8}$$

is an isomorphism for any $N \in \mathbf{Mod}\text{-}U$*.*

Then

$$U\text{-}\mathbf{Mod} \to \mathbf{Mod}\text{-}U, \quad M \mapsto {}_\blacktriangleright P \otimes_{A^{\mathrm{op}}} M_\triangleleft$$

is an equivalence of categories with quasi inverse given by $N' \mapsto \mathrm{Hom}_{A^{\mathrm{op}}}(P, N')$.

Proof. For $M \in U\text{-}\mathbf{Mod}$ and $N, N' \in \mathbf{Mod}\text{-}U$, one checks with (2.25) that the map

$$M_\triangleleft \otimes_{A} {}_\triangleright \mathrm{Hom}_{A^{\mathrm{op}}}(N, N') \to \mathrm{Hom}_{A^{\mathrm{op}}}(N, {}_\blacktriangleright N' \otimes_{A^{\mathrm{op}}} M_\triangleleft), \ m \otimes_A \chi \mapsto \{n \mapsto \chi(n) \otimes_{A^{\mathrm{op}}} m\}$$

is a morphism of left U-modules, where the left U-module structure on the left hand side is given by (3.1) combined with (3.6), and on the right hand side by (3.6) combined with (3.4). It is even an isomorphism if $N_\blacktriangleleft$ is finitely generated projective over A. On the other hand, using (2.24) and (2.11), one easily sees that the evaluation (3.8) is a morphism of right U-modules; it is then an isomorphism by hypothesis, which finishes the proof. $\square$

Remark 3.2.2. A right U-module P with the properties as in the above proposition appeared in various contexts in the literature: we shall call it a *dualising module*. We refer to [Ch1, KoKr, Hue] for applications and details, and in particular to the situation in §6.4.

4. Comodule equivalences and induced maps between duals

The aim of this section is to construct a map between the left and right dual of a left Hopf algebroid, which in some sense replaces the missing antipode on either of the duals. This can be essentially done in two ways, either by a quite straightforward generalisation of the antipode construction on the dual of a cocommutative left Hopf algebroid as in [KoP], or by considering Phùng's comodule equivalence in [Phù] as a starting point, as suggested by the referee of the present paper. To pursue the latter approach, we will review and slightly extend the results in *op. cit.*

4.1. **A categorical equivalence for comodules.** The following theorem, originally due to [Phù], shows that under the given conditions every right U-comodule can be transformed into a left one (resp. vice versa in the second case). We repeat it here for future use and also slightly extend it by saying that the two given functors are quasi-inverse to each other and that they are (strict) monoidal:

Theorem 4.1.1. *Let (U, A) be a left bialgebroid.*

(i) *Let (U, A) be additionally a left Hopf algebroid such that $U_{\triangleleft}$ is projective. Then there exists a (strict) monoidal functor $F : \mathbf{Comod}\text{-}U \to U\text{-}\mathbf{Comod}$; namely, if M is a right U-comodule with coaction $m \mapsto m_{(0)} \otimes_A m_{(1)}$, then*

$$\lambda_M : M \to U_{\triangleleft} \otimes_A M, \quad m \mapsto m_{(1)-} \otimes_A m_{(0)}\epsilon(m_{(1)+}), \tag{4.1}$$

defines a left comodule structure on M over U.

(ii) *Let (U, A) be a right Hopf algebroid such that ${}_{\triangleright}U$ is projective. Then there exists a (strict) monoidal functor $G : U\text{-}\mathbf{Comod} \to \mathbf{Comod}\text{-}U$; namely, if N is a left U-comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then*

$$\rho_N : N \to N \otimes_A {}_{\triangleright}U, \quad n \mapsto \epsilon(n_{(-1)[+]})n_{(0)} \otimes_A n_{(-1)[-]}, \tag{4.2}$$

defines a right comodule structure on N over U.

(iii) *If U is both a left and right Hopf algebroid and if both $U_{\triangleleft}$ and ${}_{\triangleright}U$ are A-projective, then the functors mentioned in (i) and (ii) are quasi-inverse to each other and we have an equivalence*

$$U\text{-}\mathbf{Comod} \simeq \mathbf{Comod}\text{-}U$$

of monoidal categories.

Proof. Let us first prove that (4.1) is well defined. For any right U-comodule M with coaction $\rho : M \to M \otimes_A U$, there is a well-defined map $\mathrm{id}_M \otimes_A \epsilon : M \otimes_A U \to M$. Its restriction to the Takeuchi product $M \times_A U$ is a left A-module map as shows the following equation: for any $\sum_i m_i \otimes u_i \in M \times_A U$ and any $a \in A$, one has

$$\sum_i m_i\epsilon(a \blacktriangleright u_i) = \sum_i m_i\epsilon(u_i \blacktriangleleft a) = \sum_i a m_i\epsilon(u_i).$$

Thus, there is a well-defined map

$$\mathrm{id}_M \times_A \epsilon : M \times_A U \to M, \quad \sum_i m_i \otimes u_i \mapsto \sum_i m_i\epsilon(u_i),$$

and hence, in particular, the map

$$\phi := (\mathrm{id}_M \times_A \epsilon) \otimes_{A^{\mathrm{op}}} \mathrm{id}_U : (M \times_A U) \times_{A^{\mathrm{op}}} U \to M \times_{A^{\mathrm{op}}} U \tag{4.3}$$

is well-defined, too.

On the other hand, any right coaction corestricts to a map $M \to M \times_A U$; similarly, the translation map $\beta^{-1}(u \otimes_A 1) = u_+ \otimes_{A^{\rm op}} u_-$ of U corestricts to a map $U \to U \times_{A^{\rm op}} U$. Combining these two maps gives a map

$$\psi : M \to M \times_A (U \times_{A^{\rm op}} U), \tag{4.4}$$

and it is clear that if we could combine ϕ in (4.3) with ψ in (4.4) followed by a tensor flip, this would yield the map (4.1).

Now the problem is that usually $(M \times_A U) \times_{A^{\rm op}} U$ and $M \times_A (U \times_{A^{\rm op}} U)$ are different, hence the two maps might not be composable. Let us introduce as in [T, Def. 1.4] the triple Takeuchi product

$$M \times_A U \times_{A^{\rm op}} U := \{\textstyle\sum_i m_i \otimes u_i \otimes v_i \in M \otimes_A U \otimes_{A^{\rm op}} U \mid \textstyle\sum_i am_i \otimes u_i \triangleleft b \otimes v_i = \textstyle\sum_i m_i \otimes u_i \blacktriangleleft a \otimes b \blacktriangleright v_i, \ \forall\, a, b \in A\}.$$

It can be seen that ψ actually maps to $M \times_A U \times_{A^{\rm op}} U$ but it is a priori not clear whether ϕ can be directly defined on $M \times_A U \times_{A^{\rm op}} U$ so as to make the two maps composable.

However, in any case there are always maps

$$\gamma : M \times_A (U \times_{A^{\rm op}} U) \to M \times_A U \times_{A^{\rm op}} U, \quad m \otimes_A u \otimes_{A^{\rm op}} v \mapsto m \otimes_A u \otimes_{A^{\rm op}} v$$

and

$$\alpha : (M \times_A U) \times_{A^{\rm op}} U \to M \times_A U \times_{A^{\rm op}} U, \quad m \otimes_A u \otimes_{A^{\rm op}} v \mapsto m \otimes_A u \otimes_{A^{\rm op}} v.$$

If now $U_\triangleleft$ is projective, α is an isomorphism [T, Prop. 1.7]; then the composition $\tau \circ \phi \circ \alpha^{-1} \circ \gamma \circ \psi$ of well-defined maps (where τ is the tensor flip) yields a well-defined map again, and on an element $m \in M$ it is an easy check that this gives the map λ_M in (4.1).

That the so-defined map λ_M is $A^{\rm e}$-linear follows from the $A^{\rm e}$-linearity of the right coaction along with (2.13). That λ_M indeed defines a left U-coaction is an easy check using (2.9) and (2.8), the counitality of the bialgebroid U, and the coassociativity with the $A^{\rm e}$-linearity of the right U-coaction on M again: we have for $m \in M$

$$\begin{aligned}
(\Delta_\ell \otimes {\rm id})\lambda_M(m) &= m_{(1)-(1)} \otimes_A m_{(1)-(2)} \otimes_A m_{(0)}\epsilon(m_{(1)+}) \\
&= m_{(1)-} \otimes_A m_{(1)+-} \otimes_A m_{(0)}\epsilon(m_{(1)++}) \\
&= m_{(1)-} \otimes_A \big(t^\ell\epsilon(m_{(1)+(2)})m_{(1)+(1)}\big)_- \otimes_A m_{(0)}\epsilon\big(\big(t^\ell\epsilon(m_{(1)+(2)})m_{(1)+(1)}\big)_+\big) \\
&= m_{(2)-} \otimes_A \big(t^\ell\epsilon(m_{(2)+})m_{(1)}\big)_- \otimes_A m_{(0)}\epsilon\big(\big(t^\ell\epsilon(m_{(2)+})m_{(1)}\big)_+\big) \\
&= ({\rm id} \otimes \lambda_M)\lambda_M(m).
\end{aligned}$$

The counitality of λ_M follows from (2.3) along with the second equation in (2.1).

As for the claim that the so-given functor $F : \textbf{Comod}\text{-}U \to U\text{-}\textbf{Comod}$ is (strict) monoidal, observe first that for any two M, M' in the monoidal category $\textbf{Comod}\text{-}U$, their tensor product $M \otimes_A M'$ is a right U-comodule by means of the codiagonal coaction $m \otimes_A m' \mapsto (m_{(0)} \otimes_A m'_{(0)}) \otimes_A m'_{(1)}m_{(1)}$, that is, with a flip in the factors in U. On the other hand, the tensor product of two N, N' in the monoidal category $U\text{-}\textbf{Comod}$ becomes a left U-comodule again via $n \otimes_A n' \mapsto n_{(-1)}n'_{(-1)} \otimes_A (n_{(0)} \otimes_A n'_{(0)})$. By the bialgebroid properties, (2.10), and (2.3) it is then simple to see that

$$\begin{aligned}
&(m'_{(1)}m_{(1)})_- \otimes_A (m_{(0)} \otimes_A m'_{(0)})\epsilon\big((m'_{(1)}m_{(1)})_+\big) \\
&\qquad = m_{(1)-}m'_{(1)-} \otimes_A \big(m_{(0)} \otimes_A m'_{(0)}\epsilon\big(m'_{(1)+}s^\ell(\epsilon(m_{(1)+}))\big)\big) \\
&\qquad = m_{(1)-}m'_{(1)-} \otimes_A \big(m_{(0)}\epsilon(m_{(1)+}) \otimes_A m'_{(0)}\epsilon(m'_{(1)+})\big),
\end{aligned}$$

that is, $F(M \otimes_A M') = F(M) \otimes_A F(M')$. Also, the unit object in both $\textbf{Comod}\text{-}U$ and $U\text{-}\textbf{Comod}$ is given by A with coaction $a \mapsto t^\ell(a)$ resp. $a \mapsto s^\ell(a)$, and $F(A) = A$ now follows from (2.13). Moreover, note that F does not affect the underlying $A^{\rm e}$-module structures of the comodules in question, and hence its (strict) monoidality follows.

The proof of *(ii)* is similar, and the last claim follows by the preceding two combined with a direct computation: applying GF to a right comodule $M \in \mathbf{Comod}\text{-}U$, the resulting right coaction on M reads

$$M \to M \otimes_A {}_{\triangleright}U\,, \quad m \mapsto \epsilon(m_{(1)-[+]})m_{(0)}\epsilon(m_{(1)+}) \otimes_A m_{(1)-[-]}.$$

By using (2.24), the coassociativity and counitality of the original right coaction on M, (2.3), (2.1), and (2.12) one obtains

$$\begin{aligned}\epsilon(m_{(1)-[+]})m_{(0)}\epsilon(m_{(1)+}) \otimes_A m_{(1)-[-]} &= \epsilon(m_{(1)-})m_{(0)}\epsilon(m_{(1)+}) \otimes_A m_{(2)} \\ &= m_{(0)}\epsilon(\epsilon(m_{(1)-}) \blacktriangleright m_{(1)+}) \otimes_A m_{(2)} \\ &= m_{(0)}\epsilon(m_{(1)}) \otimes_A m_{(2)} = m_{(0)} \otimes_A m_{(1)},\end{aligned}$$

that is, the right coaction on M we started with. An analogous consideration holds for FG using (2.25), (2.21), and the Takeuchi property that holds for left U-comodules analogous to (2.3). □

Remark 4.1.2. Note that the equivalence in Theorem 4.1.1 does *not* boil down to the usual equivalence of left and right comodules via the antipode (as there is no antipode for left or right Hopf algebroids, not even if the bialgebroid is simultaneously both). Even if we dealt with a full Hopf algebroid, this is still a different kind of equivalence (compared to the construction in [B, Remark 4.6]), as follows from the considerations in §5.2 and §6.2 below. For example, if the left Hopf algebroid U is considered a right comodule over itself via the coproduct, the left U-coaction on U from (4.1) is given by

$$U \to U_{\triangleleft} \otimes_A {}_{\blacktriangleright}U\,, \quad u \mapsto u_- \otimes_A u_+,$$

that is, the "flipped" translation map. On the other hand, for Hopf *algebras* the construction in Theorem 4.1.1 is exactly the equivalence induced by the antipode.

4.2. **Constructing maps between the duals.** We now want to construct a map between the right and the left dual of a left Hopf algebroid. To this end, we first need to recall from [Ko, Theorem 3.1.11] the following bialgebroid generalisation of the classical bialgebra module-comodule correspondence, which, however, in its first part comes somewhat unexpected at first sight:

Proposition 4.2.1. *Let (U, A) be a left bialgebroid.*

(i) *There exists a functor $\mathbf{Comod}\text{-}U \to \mathbf{Mod}\text{-}U_*$; namely, if M is a right U-comodule with coaction $m \mapsto m_{(0)} \otimes_A m_{(1)}$, then*

$$M \otimes_k U_* \to M, \quad m \otimes_k \psi \mapsto m_{(0)}\psi(m_{(1)}), \tag{4.5}$$

defines a right module structure over the A^e-ring U_*. If ${}_{\triangleright}U$ is finitely generated A-projective (so that U_* is a right bialgebroid), this functor is monoidal and has a quasi-inverse $\mathbf{Mod}\text{-}U_* \to \mathbf{Comod}\text{-}U$ such that there is an equivalence $\mathbf{Comod}\text{-}U \simeq \mathbf{Mod}\text{-}U_*$ of categories.

(ii) *Likewise, there exists a functor $U\text{-}\mathbf{Comod} \to \mathbf{Mod}\text{-}U^*$; namely, if N is a left U-comodule with coaction $n \mapsto n_{(-1)} \otimes_A n_{(0)}$, then*

$$N \otimes_k U^* \to N, \quad n \otimes_k \phi \mapsto \phi(n_{(-1)})n_{(0)}, \tag{4.6}$$

defines a right module structure over the A^e-ring U^*. If $U_{\triangleleft}$ is finitely generated A-projective (so that U^* is a right bialgebroid), this functor is monoidal and has a quasi-inverse $\mathbf{Mod}\text{-}U^* \to U\text{-}\mathbf{Comod}$ such that there is an equivalence $U\text{-}\mathbf{Comod} \simeq \mathbf{Mod}\text{-}U^*$ of categories.

The case (ii) of the above Proposition 4.2.1 can also be found in [Sch1, §5]. An explicit proof and a description of all involved functors is given in [Ko, §3.1], along with the respective structure maps of the right bialgebroids $(U_*, A, s^r_*, t^r_*, \Delta^r_*, \partial_*)$ and $(U^*, A, s^*_r, t^*_r, \Delta^*_r, \partial^*)$, in case the respective mentioned finiteness assumptions are met.

Observe that when (U, A) is both a left and a right Hopf algebroid and both $U_{\triangleleft}$ as well as ${}_{\triangleright}U$ are finitely generated projective over A, then (4.8) here below is a commutative diagram of monoidal equivalences.

We shall also need an explicit expression of the induced coaction on $M \in \mathbf{Mod}\text{-}U_*$ in case ${}_{\triangleright}U$ is finitely generated projective as in *(i)*: let $m \otimes_k \psi \mapsto m\psi$ denote the right U_*-action on M and $\{e_i\}_{1 \leqslant i \leqslant n} \in U$, $\{e^i\}_{1 \leqslant i \leqslant n} \in U_*$ a dual basis (see, for example, [AnFu, p. 202] for the notion of dual basis of a finitely generated projective module). Then the resulting right U-coaction on M can be expressed as

$$m \mapsto \sum_i m e^i \otimes_A e_i, \tag{4.7}$$

see [Ko, Eq. (3.1.23)]. Consider now the diagram

$$\begin{array}{ccc} \mathbf{Comod}\text{-}U & \longrightarrow & \mathbf{Mod}\text{-}U_* \\ \big\downarrow & & \vdots \\ U\text{-}\mathbf{Comod} & \longrightarrow & \mathbf{Mod}\text{-}U^* \end{array} \tag{4.8}$$

of categories, where the left vertical arrow is that from Theorem 4.1.1 *(i)*. Under the finiteness assumption for ${}_{\triangleright}U$, the upper horizontal arrow is invertible. One therefore obtains a functor that corresponds to the dotted arrow if $U_{\triangleleft}$ is A-projective and ${}_{\triangleright}U$ is finitely generated A-projective. Explicitly, by using (4.7), (4.1), and (4.6) one obtains on a right U_*-module M with U_*-action $m \otimes_k \psi \mapsto m\psi$ the following right U^*-action:

$$M \otimes_k U^* \to M, \quad m \mapsto m \prec \phi := \phi(e_{i-}) m e^i \epsilon(e_{i+}) = m e^i \epsilon\big(e_{i+} s^\ell(\phi(e_{i-}))\big), \tag{4.9}$$

where the second expression follows by taking the Takeuchi property (2.3) of the right coaction (4.7) into account, along with (2.13).

Consider now the case $M = U_*$ as right module over itself by right multiplication; then as in (4.9) it also carries a right U^*-action, which is equivariant with respect to the regular left U_*-action, that is

$$(\psi'\psi'') \prec \phi = \psi'(\psi'' \prec \phi). \tag{4.10}$$

In particular, this implies $\psi \prec \phi = \psi(1_{U_*} \prec \phi)$, which leads us to consider

$$S^*\phi := 1_{U_*} \prec \phi = \epsilon \prec \phi. \tag{4.11}$$

With (4.9), we see that $S^*\phi = \epsilon \prec \phi = e^i s^r_*\big(\epsilon\big(e_{i+} s^\ell(\phi(e_{i-}))\big)\big)$. Hence, for any $u \in U$,

$$\begin{aligned} S^*\phi(u) &= \langle \epsilon \prec \phi, u \rangle = \big\langle e^i s^r_*\big(\epsilon\big(e_{i+} s^\ell(\langle \phi, e_{i-}\rangle)\big)\big), u \big\rangle \\ &= \langle e^i, u \rangle \langle \epsilon, e_{i+} s^\ell(\langle \phi, e_{i-}\rangle)\rangle = \langle \epsilon, s^\ell(\langle e^i, u\rangle) e_{i+} t^\ell(\langle \phi, e_{i-}\rangle)\rangle, \end{aligned} \tag{4.12}$$

where we used [Ko, Eq. (3.1.3)] in the third step and (2.1) in the fourth. Inserting now into (4.12) the identity

$$u_+ \otimes_{A^{\mathrm{op}}} u_- = s^\ell(\langle e^i, u \rangle) e_{i+} \otimes_{A^{\mathrm{op}}} e_{i-},$$

which is seen by applying the bijective Hopf-Galois map α_ℓ from (2.4) to both sides (as we assumed U to be a left Hopf algebroid), one further obtains

$$S^*\phi(u) = \langle \epsilon, s^\ell(\langle e^i, u \rangle) e_{i+} t^\ell(\langle \phi, e_{i-} \rangle) = \epsilon\big(u_+ t^\ell(\phi(u_-))\big). \tag{4.13}$$

As will be discussed at length in the next section, this yields a map $S^* : U^* \to U_*$ (as is seen using (2.13) and (2.1)) of A^{e}-rings that even makes sense without any projectiveness or finiteness assumptions.

By means of (4.5) and (4.13), the action (4.9) can then be written as

$$m \prec \phi := m S^*(\phi), \tag{4.14}$$

which, without assuming any finiteness conditions on U, still leads to a functor $\mathbf{Mod}\text{-}U_* \to \mathbf{Mod}\text{-}U^*$ between the categories of modules over A^{e}-rings.

If instead U is a *right* Hopf algebroid, where $U_{\triangleleft}$ is finitely generated A-projective and ${}_{\triangleright}U$ is A-projective, one obtains by analogous steps a map $S_* : U_* \to U^*$ given by

$$S_*\psi(u) = \epsilon\big(u_{[+]}s^{\ell}(\psi(u_{[-]}))\big)$$

for any $u \in U$, to which analogous comments apply as above.

We will discuss the properties of these maps in detail in the subsequent §5

5. Linking structure for the duals of left Hopf algebroids

In this section — the core of the present work —, we find that the map S^* constructed in the previous section is linking the right dual to the left dual of a left Hopf algebroid, which is apparently as close as one can get to an explicit formula of an antipode kind-of structure on the dual. Note, however, that even in the case of a full Hopf algebroid this map is not simply the transpose of the antipode, as discussed in §5.2. In some sense, this special map amounts to sort of a generalisation of (the antipode in) a full Hopf algebroid as explained in Remark 5.2.5.

As mentioned before, the definition of the map S^* (and S_*) actually makes sense even without any finiteness or projectiveness assumptions. Indeed, one can trace their first appearance already in [KoP] in the rôle of the antipode in the example of the bialgebroid of jet spaces.

In what follows, we will prove the fact that S^* and S_* are morphisms of A^{e}-rings in a direct way, whereas the fact that under suitable finiteness assumptions they are bialgebroid morphisms is shown by using the comodule equivalence discussed in the previous section (note, however, that even the latter can be achieved by direct computation).

In particular, since the finiteness assumptions are not needed for all properties stated below, we will be able to apply S^* and S_* in greater generality to the examples in §6.

5.1. **Morphisms between left and right duals.** Let (U, A) be a left bialgebroid. If it is additionally a left Hopf algebroid, its right dual U^* (see §2.2) carries a left U-module structure as in (3.2); (re-)define

$$S^*(\phi)(u) := (u\phi)(1_U) = \epsilon_U\big(u_+t^{\ell}(\phi(u_-))\big), \qquad \forall \phi \in U^*,\ u \in U. \tag{5.1}$$

Likewise, if the left bialgebroid (U, A) is a right Hopf algebroid instead, its left dual U_* (see §2.2 again) carries a left U-module structure as in (3.5), with the help of which one (re-)defines

$$S_*(\psi)(u) := (u\psi)(1_U) = \epsilon_U\big(u_{[+]}s^{\ell}(\psi(u_{[-]}))\big), \qquad \forall \psi \in U_*,\ u \in U. \tag{5.2}$$

The following result presents the key properties of the maps S^* and S_*:

Theorem 5.1.1. *Let (U, A) be a left bialgebroid.*

(*i*) *If (U, A) is moreover a left Hopf algebroid, (5.1) defines a morphism $S^* : U^* \to U_*$ of A^{e}-rings with augmentation (the "counit"); if in addition both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated projective as A-modules, then (S^*, id_A) is a morphism of right bialgebroids. In any case, S^* is also a morphism of left U-modules for the action (3.3) on U^* and the left action on U_* given by right multiplication in U.*

(*ii*) *If (U, A) is a right Hopf algebroid instead, (5.2) defines a morphism $S_* : U_* \to U^*$ of A^{e}-rings with augmentation (the "counit"); if in addition both ${}_{\triangleright}U$ and $U_{\triangleleft}$ are finitely generated projective as A-modules, then (S_*, id_A) is a morphism of right bialgebroids. In any case, S_* is also a morphism of left U-modules for the action (3.6) on U_* and the left action on U^* given by right multiplication in U.*

Proof. We only prove part *(i)* as *(ii)* follows *mutatis mutandis*. For the explicit computations, we will again use the notation and description of the structure maps of the two right bialgebroids $(U_*, A, s^r_*, t^r_*, \Delta^r_*, \partial_*)$ and $(U^*, A, s^*_r, t^*_r, \Delta^*_r, \partial^*)$ — where the coproduct Δ^r_* or Δ^*_r only make sense if $U_{\triangleleft}$ resp. ${}_{\triangleright}U$ is finitely generated A-projective — as given in detail in [Ko, §3.1], together with the respective properties of left and right pairings $\langle .,. \rangle$ as in Definition 2.2.1. Direct verification shows that S^* takes values in U_*. Besides, for S^* to be a bialgebroid morphism, we need to show the following properties:

$$\textit{(a)} \qquad S^* s_r^* = s_*^r, \quad S^* t_r^* = t_*^r, \quad \partial_* S^* = \partial^*,$$

$$\textit{(b)} \qquad S^*\big(\phi\phi'\big) = S^*(\phi) S^*(\phi')$$

$$\textit{(c)} \qquad \Delta_*^r S^* = (S^* \otimes S^*) \Delta_r^*,$$

(where, as said before, *(c)* only makes sense if $U_\triangleleft$ and ${}_\triangleright U$ are finitely generated A-projective).

As for *(a)*, we find for $u \in U$, $a \in A$ by direct computation using (2.12) and (2.13):

$$S^*\big(s_r^*(a)\big)(u) = \epsilon\big(u_+ t^\ell\big(s_r^*(a)(u_-)\big)\big) = \epsilon\big(u_+ t^\ell\big(\epsilon(u_- s^\ell(a))\big)\big) = \epsilon(u)a = s_*^r(a)(u).$$

Likewise, the second identity follows from

$$S^*\big(t_r^*(a)\big)(u) = \epsilon\big(u_+ t^\ell\big(t_r^*(a)(u_-)\big)\big) = \epsilon\big(u_+ t^\ell(a\epsilon(u_-))\big) = \epsilon(ut^\ell(a)) = t_*^r(a)(u).$$

The last identity in *(a)* regarding the respective counits is for $\phi \in U^*$ proven by the line

$$\partial_* S^*(\phi) = S^*(\phi)(1_U) = \phi(1_U) = \partial^* \phi.$$

As for *(b)*, let us first more generally compute an element $S^*(\phi)\psi$ for $\phi \in U^*$ and $\psi \in U_*$: by [Ko, Eq. (3.1.1)], Eq. (2.8), and the properties of a bialgebroid counit, we have

$$\begin{aligned}
\langle S^*(\phi)\psi, u\rangle &= \langle \psi, t^\ell(\langle u_{(2)}, S^*(\phi)\rangle) u_{(1)}\rangle = \langle \psi, t^\ell(\langle \epsilon, u_{(2)+} t^\ell(\langle \phi, u_{(2)-}\rangle)\rangle) u_{(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)} t^\ell(\langle \phi, u_-\rangle)\rangle) u_{+(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)} s^\ell(\langle \phi, u_-\rangle)\rangle) u_{+(1)}\rangle \\
&= \langle \psi, t^\ell(\langle \epsilon, u_{+(2)}\rangle) u_{+(1)} t^\ell(\langle \phi, u_-\rangle)\rangle = \langle \psi, u_+ t^\ell(\langle \phi, u_-\rangle)\rangle.
\end{aligned}$$

With the help of this property, by [Ko, Eq. (3.1.2)] along with (2.9), (2.13), and the fact that the counit in U gives the unit in U_*, one sees that for all $\phi, \phi' \in U^*$

$$\begin{aligned}
\langle S^*(\phi\phi'), u\rangle &= \langle \epsilon, u_+ t^\ell(\langle \phi\phi', u_-\rangle)\rangle = \langle \epsilon, u_+ t^\ell(\langle \phi', s^\ell \phi(u_{-(1)}) u_{-(2)}\rangle)\rangle \\
&= \langle \epsilon, u_{++} t^\ell(\langle \phi', s^\ell \phi(u_-) u_{+-}\rangle)\rangle \\
&= \langle \epsilon, (u_+ t^\ell \phi(u_-))_+ t^\ell(\langle \phi', (u_+ t^\ell \phi(u_-))_-\rangle)\rangle \\
&= \langle S^*(\phi')\epsilon, u_+ t^\ell \phi(u_-)\rangle = \langle S^*(\phi) S^*(\phi'), u\rangle.
\end{aligned}$$

Observe that if ${}_\triangleright U$ is finitely generated A-projective, then *(b)* follows by the fact that (4.14) defines an action, but in general we do not want to assume this at this point.

For proving *(c)* — when $U_\triangleleft$ and ${}_\triangleright U$ are finitely generated A-projective —, one could equally do this by a straightforward somewhat technical computation. A quicker way is to use the results in §4: denoting the right coproduct on U_* resp. U^* by Sweedler superscripts, one has

$$\begin{aligned}
S^*(\phi)^{(1)} \otimes_A S^*(\phi)^{(2)} &= (\epsilon \otimes_A \epsilon) S^*(\phi) = (\epsilon \otimes_A \epsilon) \prec \phi \\
&= (\epsilon \prec \phi^{(1)}) \otimes_A (\epsilon \prec \phi^{(2)}) = S^*(\phi^{(1)}) \otimes_A S^*(\phi^{(2)}),
\end{aligned}$$

where in the first equation we used the monoidal structure on $\mathbf{Mod}$-U_*, and in the third the fact that all functors in (4.8) are strict monoidal.

The second part in *(i)* — about the U-linearity of S^* —, which is straightforward, is left to the reader. □

Remark 5.1.2. When U is just a Hopf algebra over $A = k$ with antipode S, we have $U^* = (U_*)^{\mathrm{op}}_{\mathrm{coop}}$, and S^* is nothing but the transpose of S. If U^* itself is in turn a Hopf algebra — namely, if the transpose of the multiplication m_U in U takes values in the tensor square of U^* —, then S^* is just the antipode of this dual Hopf algebra U^*. In this context, Theorem

5.1.1 simply expresses the fact that the antipode in a Hopf algebra is an antimorphism of algebras and of coalgebras.

In particular, in case U is both a left and right Hopf algebroid we have:

Theorem 5.1.3. *Let (U, A) be simultaneously a left and a right Hopf algebroid. Then the maps S^* and S_* are inverse to each other. Hence, if both A-modules ${}_{\triangleright}U$ and $U_{\triangleleft}$ are, in addition, finitely generated projective, (S^*, id_A) and (S_*, id_A) are isomorphisms of right bialgebroids which are inverse to each other.*

Proof. As for the first statement, we directly compute by means of the bialgebroid axioms along with (2.25) and (2.20), for any $\phi \in U^*$:

$$\begin{aligned}(S_* S^* \phi)(u) &= \epsilon\big(u_{[+]} s^\ell (S^*\phi(u_{[-]}))\big) = \epsilon\big(u_{[+]} s^\ell\big(\epsilon_U(u_{[-]+} t^\ell \phi(u_{[-]-}))\big)\big)\\ &= \epsilon\big(u_{[+]} u_{[-]+} t^\ell \phi(u_{[-]-})\big) = \epsilon\big(u_{(2)[+]} u_{(2)[-]} t^\ell \phi(u_{(1)})\big)\\ &= \phi(u_{(1)})\epsilon(u_{(2)}) = \phi(u),\end{aligned}$$

which proves that $S_* \circ S^* = \mathrm{id}_{U^*}$. Likewise, one shows that $S^* \circ S_* = \mathrm{id}_{U_*}$. □

5.2. **The case of a full Hopf algebroid.** If H is a full Hopf algebroid with bijective antipode S in the sense of [BSz], then it is, in particular, both a left and right bialgebroid (see the short summary below): therefore — still assuming that ${}_{\triangleright}H$ and $H_{\triangleleft}$ are both finitely generated projective as A-modules —, there is a right bialgebroid analogue to the previous constructions concerning the maps S^* and S_*. On the other hand, the antipode S induces by transposition new maps S^t, ${}^tS$, etc., for the dual spaces. Hereafter we discuss links between these various maps, in particular showing that, while for the Hopf algebra case one has identities like $S^* = {}^tS$ (*cf.* Remark 5.1.2), this is no longer the case for the general setup of full Hopf algebroids as illustrated in §6.2 below.

5.2.1. Reminder on full Hopf algebroids. Recall that a full Hopf algebroid structure (see, for example, [B]) on a k-module H consists of the following data:

(*i*) a left bialgebroid structure $H^\ell := (H, A, s^\ell, t^\ell, \Delta_\ell, \epsilon)$ over a k-algebra A;
(*ii*) a right bialgebroid structure $H^r := (H, B, s^r, t^r, \Delta_r, \partial)$ over a k-algebra B;
(*iii*) the assumption that the k-algebra structures for H in *(i)* and in *(ii)* be the same;
(*iv*) a k-module map $S : H \to H$;
(*v*) some compatibility relations between the previously listed data for which we refer to *op. cit.*

We shall denote by lower Sweedler indices the left coproduct Δ_ℓ and by upper indices the right coproduct Δ_r, that is, $\Delta_\ell(h) =: h_{(1)} \otimes_A h_{(2)}$ and $\Delta_r(h) =: h^{(1)} \otimes_B h^{(2)}$ for any $h \in H$. As said before, a full Hopf algebroid (with bijective antipode) is both a left and right Hopf algebroid but not necessarily vice versa (as illustrated in §6.2). In this case, the translation maps in (2.4) are given by

$$h_+ \otimes_{A^{\mathrm{op}}} h_- = h^{(1)} \otimes_{A^{\mathrm{op}}} S(h^{(2)}) \quad \text{and} \quad h_{[+]} \otimes_{B^{\mathrm{op}}} h_{[-]} = h^{(2)} \otimes_{B^{\mathrm{op}}} S^{-1}(h^{(1)}), \tag{5.3}$$

formally similar as for Hopf algebras.

The following lemma [B, BSz] will be needed to prove the main result in this subsection.

Lemma 5.2.2. *Let H be any Hopf algebroid. Then*

(*i*) *the maps $\nu := \partial s^\ell : A \to B^{\mathrm{op}}$ and $\mu := \epsilon s^r : B \to A^{\mathrm{op}}$ are isomorphisms of k-algebras;*
(*ii*) *the pair of maps $(S, \nu) : H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$ gives an isomorphism of left bialgebroids;*
(*iii*) *the pair of maps $(S, \mu) : H^r \to (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}$ gives an isomorphism of right bialgebroids.*

The next observation might let us consider S^* and S_* as sort of an analogue of the antipode on the dual:

Proposition 5.2.3. *Let (U, A) be a cocommutative left bialgebroid (in particular, A is commutative and $s^\ell = t^\ell$). Then (U, A) is a left Hopf algebroid if and only if it is a right Hopf algebroid; in this case, assuming in addition that ${}_\triangleright U$ and $U_\triangleleft$ are finitely generated A-projective, $(U^*, A) = ((U_*)_{\mathrm{coop}}, A)$ is a full Hopf algebroid with involutive antipode $\mathscr{S} := S^* = S_*$.*

Proof. The first claim directly holds true by the very definitions. The rest of the proof follows *verbatim* in the footsteps of the one of Theorem 3.17 in [KoP], which considers the special case for $U = V^\ell(L)$. □

As mentioned before, one can also link the duals of a Hopf algebroid (H, S) by transposed maps ${}^tS$, which usually do not coincide with S^* or S_* (see also §6.2). The next result explains a relation between them.

Theorem 5.2.4. *Let H be a Hopf algebroid such that ${}_\triangleright H$ and $H_\triangleleft$ are finitely generated A-projective. Then the diagram*

$$\begin{array}{ccc} \big((H^r)^{\mathrm{op}}_{\mathrm{coop}}\big)^* & \xrightarrow{\ {}^tS\ } & (H^\ell)^* \\ {\scriptstyle S_r^*}\big\downarrow & & \big\downarrow{\scriptstyle S_\ell^*} \\ \big((H^r)^{\mathrm{op}}_{\mathrm{coop}}\big)_* & \xrightarrow[\ {}^tS\ ]{} & (H^\ell)_* \end{array}$$

of right bialgebroid morphisms is commutative.

Proof. Let us identify B^{op} and A by means of the k-algebra isomorphism $\nu : A \to B^{\mathrm{op}}$ mentioned above; then the left algebroid $(H^r)^{\mathrm{op}}_{\mathrm{coop}}$ is described by the sextuple

$$\big((H^r)^{\mathrm{op}}, \widehat{s^\ell} := s^r\nu, \widehat{t^\ell} := t^r\nu, \Delta_r^{\mathrm{coop}}, \widehat{\epsilon} := \nu^{-1}\partial\big).$$

Moreover, the Hopf algebroid $\big((H^r)^{\mathrm{op}}_{\mathrm{coop}}, (H^\ell)^{\mathrm{op}}_{\mathrm{coop}}, (S, \mu) : (H^r)^{\mathrm{op}}_{\mathrm{coop}} \to H^\ell\big)$ is the one we have to consider to compute S_r^*. For $\phi \in \big((H_r)^{\mathrm{op}}_{\mathrm{coop}}\big)_*$ and $h \in H$ we have

$$\begin{aligned} \langle({}^tS \circ S_r^*)(\phi), h\rangle &= \widehat{\epsilon}\big(S(h)_{(2)}\widehat{t^\ell}\big(\langle\phi, S(S(h)_{(1)})\rangle\big) = \big(\nu^{-1}\partial S\big)\big(h^{(1)}t^\ell\big(\langle\phi, S^2(h^{(2)})\rangle\big) \\ &= \epsilon\big(h^{(1)}t^\ell\big(\langle\phi, S^2(h^{(2)})\rangle\big)\big) \\ &= \epsilon\big(h^{(1)}t^\ell\big(\langle{}^tS(\phi), S(h^{(2)})\rangle\big)\big) = \langle(S_\ell^* \circ {}^tS)(\phi), h\rangle, \end{aligned}$$

where we used the explicit form (5.3) of the translation map and the fact that S is an anti-coring morphism between left and right coproduct, which proves ${}^tS \circ S_r^* = S_\ell^* \circ {}^tS$ as claimed. □

Remark 5.2.5. In general, both maps S^* or S_* can be thought of as an extension of the notion of antipode for a full Hopf algebroid, in the following sense. As mentioned in Lemma 5.2.2, the antipode in a full Hopf algebroid H yields a bialgebroid morphism $S : H^\ell \to (H^r)^{\mathrm{op}}_{\mathrm{coop}}$. On the other hand, if U is a left Hopf algebroid, for which ${}_\triangleright U$ and $U_\triangleleft$ are finitely generated projective as A-modules, then we have a similar situation replacing (H^ℓ, H^r, S) with the triple $((U^*)^{\mathrm{op}}, (U_*)_{\mathrm{coop}}, S^*)$, and one might be tempted to define a Hopf algebroid as a triple (U, V, S) of a left resp. right bialgebroid U resp. V, where the underlying ring structure is *not* the same: this way, the apparent asymmetry of a Hopf algebroid consisting of two coring structures but only one ring structure (that makes it difficult to obtain self-duality) would be somewhat attenuated. On the other hand, in case a left Hopf algebroid is simultaneously a right Hopf algebroid, by Theorem 5.1.3 both duals are isomorphic and hence can be seen (under the stated finiteness conditions) as *its* dual (right) bialgebroid, which carries a Hopf structure by the results in [Sch2].

6. Examples and applications

In this section we present some further developments and some applications to specific examples.

6.1. **Mixed distributive law between duals.** A direct application of the existence of the bialgebroid morphism S^* (or S_*) is to the setup of *distributive laws*. Indeed, a particular kind of mixed distributive law (or entwining) in the sense of Beck [Be] can be constructed via the following recipe. Combining a morphism $(\phi_1, \phi_0) : (V, B) \to (V', B')$ of right (say) bialgebroids with a Hopf-Galois map yields

$$\chi : V'_{\blacktriangleleft} \otimes^B {}_{\triangleright}V \to V_{\blacktriangleleft} \otimes_B {}_{\blacktriangleright}V', \quad v' \otimes^B v \mapsto v^{(1)} \otimes_B v'\phi(v^{(2)}),$$

which can be easily seen to define a mixed distributive law between V' (thought of as a coalgebra) and V (thought of as an algebra, although its coproduct appears in χ). Applying this to the two duals of a left bialgebroid U along with S^*, one obtains

$$\chi : U_{*\blacktriangleleft} \otimes^A {}_{\triangleright}U^* \to U^*{}_{\blacktriangleleft} \otimes_A {}_{\blacktriangleright}U_*, \qquad \psi \otimes^A \phi \mapsto \phi^{(1)} \otimes_A \psi S^*(\phi^{(2)})$$

as a mixed distributive law between U^* and U_*, to which any standard construction based on mixed distributive laws can be applied.

6.2. **Lie-Rinehart algebras and their jet spaces.** Let (A, L) be a Lie-Rinehart algebra (*cf.* [Ri], geometrically a Lie algebroid). Then its (left) universal enveloping algebra $V^\ell(L)$ carries not only the structure of a left bialgebroid over the commutative algebra A (see [Xu]) but also of a left Hopf algebroid [KoKr]: on generators $a \in A$ and $X \in L$, its translation map is given by

$$a_+ \otimes_{A^{\mathrm{op}}} a_- = a \otimes_{A^{\mathrm{op}}} 1, \quad X_+ \otimes_{A^{\mathrm{op}}} X_- = X \otimes_{A^{\mathrm{op}}} 1 - 1 \otimes_{A^{\mathrm{op}}} X. \tag{6.1}$$

Moreover, as $V^\ell(L)$ is cocommutative, it is also a right Hopf algebroid.

Full Hopf algebroid structures on $V^\ell(L)$ are in bijection with right $V^\ell(L)$-module structures on A which play the rôle of possible right counits, expressed by suitable maps $\partial : V^\ell(L) \to A$ (*cf.* [Ko, §4.2] or [KoP] for more information). The corresponding antipode $S : V^\ell(L) \to V^\ell(L)^{\mathrm{op}}_{\mathrm{coop}}$ is then uniquely determined by the prescriptions

$$S(a) = a, \quad S(X) = -X + \partial(X), \qquad \forall a \in A, \ \forall X \in L, \tag{6.2}$$

on generators. For a general Lie-Rinehart algebra (which does not arise from a Lie algebroid), such a map ∂ and hence the antipode might or might not exist.

Let us consider the (right) *jet spaces* $J^r(L) := V^\ell(L)^*$ and ${}^rJ(L) := V^\ell(L)_*$. If L is finitely generated projective as an A-module, then $J^r(L)$ and ${}^rJ(L)$ are right bialgebroids *in a suitable topological sense*, as their coproduct takes values in a *topological* tensor product; concerning this, we quickly recall some non-trivial key facts, referring to [KoP, CaVdB] for further details.

First, $V^\ell(L)$ is the direct limit of an increasing bialgebroid filtration (*i.e.*, the strict analogue of a bialgebra filtration) of finitely generated projective modules $V^\ell(L)_n$; it follows that $J^r(L)$ in turn is the inverse limit of all the $J^r(L)_n := (V^\ell(L)_n)^*$, which are finitely generated projective as well. Similar remarks apply to ${}^rJ(L)$. As $V^\ell(L)_p \cdot V^\ell(L)_q \subseteq V^\ell(L)_{p+q}$ (for all $p, q \in \mathbb{N}$), the recipe used to define the coproduct in U^* when U is a left bialgebroid such that $U_{\triangleleft}$ is finitely generated A-projective (see §2.2.2) can be applied again and yields maps

$$J^r(L)_n = (V^\ell(L)_n)^* \xrightarrow{\Delta_n^{J^r}} \sum_{p+q=n} (V^\ell(L)_p)^*{}_{\blacktriangleleft} \otimes_A {}_{\blacktriangleright}(V^\ell(L)_q)^* = \sum_{p+q=n} J^r(L)_{p\blacktriangleleft} \otimes_A {}_{\blacktriangleright}J^r(L)_q$$

whose inverse limit $\Delta^{J^r} := \varprojlim \Delta_n^{J^r}$ is the coproduct of $J^r(L)$. Similarly, one constructs "coproduct-like maps" $\Delta_n^{{}^rJ}$ for the ${}^rJ(L)_n := (V^\ell(L)_n)_*$ and then takes their inverse limit $\Delta^{{}^rJ} := \varprojlim \Delta_n^{{}^rJ}$ as a coproduct for ${}^rJ(L)$.

Now, because of the very definition of the $V^\ell(L)_n$ and of the explicit form (6.1) of the translation map of $V^\ell(L)$, one easily finds that the translation map itself (much like the coproduct) maps every $V^\ell(L)_n$ into $\sum_{p+q=n} V^\ell(L)_p \otimes_{A^{\mathrm{op}}} V^\ell(L)_q$. Then formula (5.1) makes sense again, and thus can be used to produce a well-defined map

$$S^*_n : J^r(L)_n = (V^\ell(L)_n)^* \longrightarrow \big(V^\ell(L)_n\big)_* = {}^rJ(L)_n.$$

Moreover, the arguments used in the proof of Theorem 5.1.1 to show that S^* preserves the coproduct apply again in the present situation, and yield a commutative diagram

$$\begin{array}{ccc} J^r(L)_n & \xrightarrow{\ \Delta_n^{J^r}\ } & \sum\limits_{p+q=n} J^r(L)_p \,{}_\blacktriangleleft\!\otimes_A {}_\blacktriangleright J^r(L)_q \\ {\scriptstyle S^*_n}\big\downarrow & & \big\downarrow {\scriptstyle \sum_{p+q=n} S^*_p \otimes S^*_q} \\ {}^rJ(L)_n & \xrightarrow[\ \Delta_n^{{}^rJ}\ ]{} & \sum\limits_{p+q=n} {}^rJ(L)_p \,{}_\blacktriangleleft\!\otimes_A {}_\blacktriangleright {}^rJ(L)_q \end{array} \tag{6.3}$$

Taking the inverse limit of all these S^*_n we get a well-defined (continuous) map

$$S^* : J^r(L) = V^\ell(L)^* \longrightarrow V^\ell(L)_* = {}^rJ(L).$$

It follows by construction that this map necessarily coincides with the same name map in §5.1, hence it respects all A^{e}-ring structure maps of $J^r(L)$ and ${}^rJ(L)$ as well as their counits; from (6.3) follows that this map also respects the coproduct on both sides. All in all, this means that S^* is a morphism of (topological) bialgebroids. As $V^\ell(L)$ is also a right Hopf algebroid, §5.1 also provides a map $S_* : {}^rJ(L) \to J^r(L)$, which again turns out to be a morphism of (topological) bialgebroids, inverse to S^*. The outcome is that

Theorem 5.1.1 holds true (in full strength) for $U = V^\ell(L)$

(replacing the formulation "morphism of right bialgebroids" by "morphism of topological right bialgebroids"), although the left bialgebroid $V^\ell(L)$ does not comply with the finiteness assumptions required (in general) for that result.

Finally, note that both $J^r(L)$ and ${}^rJ(L)$ are commutative (because $V^\ell(L)$ is cocommutative), so they are also left bialgebroids. Identifying $J^r(L)$ as the coopposite of ${}^rJ(L)$ and with the cocommutativity of $V^\ell(L)$, one finds that S^* and S_* are equal and yield an *antipode* for $J^r(L)$, which in this way becomes a full Hopf algebroid. In other words, Proposition 5.2.3 holds true for $U = V^\ell(L)$ and $U^* = J^r(L) = {}^rJ(L)_{\mathrm{coop}} = (U_*)_{\mathrm{coop}}$, although $V^\ell(L)$ is *not* finitely generated projective.

6.2.1. *Difference between S^* and ${}^tS$.* In this specific example, one can explicitly observe the difference between S^* and the transpose of the antipode S on $V^\ell(L)$ in (6.2). Apart from the fact mentioned above that S^* always exists while ${}^tS$ does not, this is already clear on an abstract level since these are maps of different nature as pointed out in Theorem 5.2.4. Nevertheless, one directly sees here that with respect to the A-module structures coming from left and right multiplication in $V^\ell(L)$, the map $S^*(\phi)$ is left A-linear whereas ${}^tS(\phi)$ is A-linear from the right, for $\phi \in V^\ell(L)^*$. Evaluating both maps on an element in $L \subset V^\ell(L)$, one obtains

$${}^tS(\phi)(X) = -\phi(X) + \partial(X)\phi(1) \qquad \forall \phi \in V^\ell(L)^*, X \in L,$$

on one hand, and on the other hand

$$S^*(\phi)(X) = -\phi(X) + X(\phi(1)) \qquad \forall \phi \in V^\ell(L)^*, X \in L,$$

where $L \to \mathrm{Der}(A, A)$, $X \mapsto \{a \mapsto X(a)\}$ denotes the anchor of the Lie-Rinehart algebra (A, L). Using the property $Xa - aX = X(a)$ with respect to the product in $V^\ell(L)$ as well as the right A-linearity of ∂, one obtains $\partial(aX) = \partial(X)a - X(a)$ and therefore ${}^tS(\phi)(X) - S^*(\phi)(X) = \partial(\phi(1)X)$, which in general does not vanish.

6.3. **Examples from quantisation.** In this section, we adapt our main constructions and results to a different setup, that of quantisations of universal enveloping algebras (of Lie-Rinehart algebras) and other associated objects. In particular, this means that we deal with yet another kind of topological bialgebroids, so that we have to clarify the nature of these objects and how the analysis and results of the preceding sections fits into this modified context. Hereafter, k is assumed to be a field.

Definition 6.3.1. Let $\big(U, A, s^\ell, t^\ell, m, \Delta, \epsilon\big)$ be a left (resp. right) bialgebroid. A *quantisation of* U (or *quantum bialgebroid*) is a *topological* left (resp. right) bialgebroid $\big(U_h, A_h, s^\ell_h, t^\ell_h, m_h, \Delta_h, \epsilon_h\big)$ over a topological $k[[h]]$-algebra A_h such that:

(*i*) A_h is isomorphic to $A[[h]]$ as a topological $k[[h]]$-module, and this isomorphism induces an algebra isomorphism $A_h\big/hA_h \cong A[[h]]\big/hA[[h]] \cong A$;

(*ii*) U_h is isomorphic to $U[[h]]$ as a topological $k[[h]]$-module;

(*iii*) $U_h\big/hU_h \cong U[[h]]\big/hU[[h]]$ is isomorphic to U as a left A-bialgebroid via the isomorphism $A_h\big/hA_h \cong A[[h]]\big/hA[[h]] \cong A$ mentioned in (i);

(*iv*) the coproduct Δ_h of U_h takes values in $U_h \,\widehat{\times}_{A_h} U_h$, where

$$U_h \,\widehat{\times}_{A_h} U_h := \big\{ \textstyle\sum_i u_i \otimes u'_i \in U_{h\triangleleft} \,\widehat{\otimes}_{A_h}\, {}_{\triangleright}U_h \;\mid\; \sum_i (a \blacktriangleright u_i) \otimes u'_i = \sum_i u_i \otimes (u'_i \blacktriangleleft a) \big\}$$

is the *Takeuchi-Sweedler product*, and where $U_{h\triangleleft} \,\widehat{\otimes}_{A_h}\, {}_{\triangleright}U_h$ denotes the completion of $U_{h\triangleleft} \otimes_{A_h} {}_{\triangleright}U_h$ with respect to the h-adic topology.

In this setting, we shall say that U_h is a *quantisation*, or *quantum deformation*, of U.

Remark 6.3.2.

(a) The notions of quantum left or right Hopf algebroid are defined replacing the ordinary tensor product by a suitable completion, just as for $J^r(L)$ above.

(b) When dealing with $k[[h]]$-modules, any morphism (*i.e.*, $k[[h]]$-linear map) is automatically continuous for the h-adic topology on the source and target $k[[h]]$-module; we shall tacitly use this fact with no further mention. In particular, for a quantum bialgebroid U_h both its (full linear) duals $(U_h)^*$ and $(U_h)_*$ are also *topological duals*.

(c) For a left bialgebroid U with a quantisation U_h, assume that U is also a left Hopf algebroid. Then U_h is automatically a left Hopf algebroid (in a topological sense) as well by a standard argument in deformation theory. By assumption, we have $U_h \cong U[[h]]$ as modules over $A_h \cong A[[h]]$; from this isomorphism one deduces similar isomorphisms for modules of homomorphisms or tensor products of modules. Moreover — because $U_h\Big/hU_h \cong U$ as bialgebroids —, all bialgebroid structure maps of U_h taken modulo h reduce to the same name structure maps of U. Now, for the (topological) left bialgebroid U_h we have a well-defined Hopf-Galois map

$$(\alpha_\ell)_h : {}_{\blacktriangleright}U_h \,\widehat{\otimes}_{A_h^{\mathrm{op}}}\, U_{h\triangleleft} \to U_{h\triangleleft} \,\widehat{\otimes}_{A_h}\, {}_{\triangleright}U_h\,, \quad u \,\widehat{\otimes}_{A_h^{\mathrm{op}}} v \mapsto u_{(1)} \,\widehat{\otimes}_{A_h}\, u_{(2)} v,$$

which belongs to $\mathrm{Hom}_{k[[h]]}\big({}_{\blacktriangleright}U_h \,\widehat{\otimes}_{A_h^{\mathrm{op}}} U_{h\triangleleft}, U_{h\triangleleft} \widehat{\otimes}_{A_h} {}_{\triangleright}U_h\big)$: as mentioned above, this module is isomorphic to $\mathrm{Hom}_k\big({}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, U_{\triangleleft} \otimes_A {}_{\triangleright}U\big)[[h]]$, so that $(\alpha_\ell)_h$ expands as $(\alpha_\ell)_h = \sum_{n\in\mathbb{N}} a_n h^n$ for some $a_n \in \mathrm{Hom}_k\big({}_{\blacktriangleright}U \otimes_{A^{\mathrm{op}}} U_{\triangleleft}, U_{\triangleleft} \otimes_A {}_{\triangleright}U\big)$. In addition, as all structure maps of U_h modulo h are just those of U, one has $\alpha_\ell = (\alpha_\ell)_h \mod h = a_0$. But U was a left Hopf algebroid, hence $\alpha_\ell = a_0$ is invertible, and therefore $(\alpha_\ell)_h = \sum_{n\in\mathbb{N}} a_n h^n$ is invertible too, so that U_h is a left Hopf algebroid as well.

6.3.3. Universal enveloping algebras and deformations. As in [ChGa], one can consider a quantum deformation $V^\ell(L)_h$ of $V^\ell(L)$: as the latter is both a left and right Hopf algebroid, the same holds true for $V^\ell(L)_h$ as well, by Remark 6.3.2 *(c)* above.

On the other hand, the dual (right) bialgebroids $J^r(L)_h := (V^\ell(L)_h)^*$ and ${}^rJ(L)_h = (V^\ell(L)_h)_*$ are deformations of $J^r(L) = V^\ell(L)^* = (V^\ell(L)_*)_{\mathrm{coop}}$. This common "limit" is a full Hopf algebroid (with bijective antipode) by the above, hence in particular it is a left and right Hopf algebroid with respect to the underlying right bialgebroid structure. It then follows that the same is true for the right bialgebroids $J^r(L)_h$ and ${}^rJ(L)_h$, but usually they are not full Hopf algebroids. Nonetheless, we can apply our constructions of §5.1 to $U_h := V^\ell(L)_h$ and find the maps S^* and S_*, as we now shortly explain.

By construction, the maps S^* and S_* as in (5.1) and (5.2) are given in terms of structure maps and translation maps of the (non-topological) bialgebroid U: when U is replaced by U_h, all those maps are continuous, hence both definitions still make sense and provide maps $S^* : (U_h)^* \to (U_h)_*$ and $S_* : (U_h)_* \to (U_h)^*$ as announced. Once these maps are properly defined (for $U_h = V^\ell(L)_h$), the proof of all their properties still works untouched (all arguments and calculations make sense and go through in the proper setup of topological bialgebroids). In particular, Theorem 5.1.3 then assures that the two deformations $J^r(L)_h := (U_h)^*$ and ${}^rJ(L)_h := (U_h)_*$ of $V^\ell(L)^* = (V^\ell(L)_*)_{\mathrm{coop}}$ are isomorphic (as right bialgebroids) via S^* and S_*.

6.4. **Cases where a dualising module exists.** In this section, we will come back to the situation of dualising modules as in §3.2 by investigating their (deformation) quantisation. To this end, we first need to introduce some extra notation, terminology, and definitions with respect to decreasing filtrations; see, for example, [Ch2, Schn] for further basic results and details.

Let A be an algebra endowed with a decreasing filtration $(F_nA)_{n\in\mathbb{N}}$ and consider a filtered FA-module denoted by FM, whereas its underlying A-module will be denoted by M. If FM and FN are two filtered FA-modules, then a filtered morphism $Fu : FM \to FN$ is a morphism $u : M \to N$ of the underlying A-modules such that $u(F_sM) \subset F_sN$. A filtered morphism $Fu : FM \to FN$ is *strict* if it satisfies $u(F_sM) = u(M) \cap F_sN$. An exact sequence of FA-modules is a sequence

$$FM \xrightarrow{Fu} FN \xrightarrow{Fv} FP \tag{6.4}$$

such that $\mathrm{Ker}\,(F_sv) = \mathrm{Im}\,(F_su)$, where $F_sv := v\big|_{F_sN}$ and $F_su := u\big|_{F_sM}$; hence Fu is strict. If moreover Fv is also strict, then (6.4) is a called a *strict exact sequence*.

The filtration of a filtered module gives rise to a topology and even a metric if the filtered module is separated, that is, if $\bigcap_{n\in\mathbb{N}} F_nM = \{0\}$. For any $r \in \mathbb{Z}$ and for any FA-module FM, we define the *shifted module* $FM(r)$ as the module M endowed with the filtration $(F_{s+r}M)_{s\in\mathbb{Z}}$. An FA-module is called *finite free* if isomorphic to an FA-module of the type $\bigoplus_{i=1}^p FA(-d_i)$, where $d_1, \ldots, d_p \in \mathbb{Z}$. An FA-module FM is called *of finite type* if one can find $m_1 \in F_{d_1}M, \ldots, m_p \in F_{d_p}M$ such that any $m \in F_dM$ may be written as

$$m = \sum_{i=1}^p a_{d-d_i} m_i,$$

where $a_{d-d_i} \in F_{d-d_i}A$. We will be dealing with the case where M is a $k[[h]]$-module and $F_nM = h^nM$, the so-called *h-adic filtration*.

Remark 6.4.1. The existence of a translation map if U_h is a left or right Hopf algebroid makes it possible to endow

- Hom-spaces with values in a h-adic complete space, and
- complete tensor products of U_h-modules

with further natural U_h-module structures. Let us make this explicit for the cases we will use, *i.e.*, adapt Proposition 3.1.1.

If $\mathcal{P}_h$ is a right U_h-module and N_h is a left U_h-module, then ${}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\mathrm{op}}} N_{h\triangleleft}$ is endowed with a right U_h-module structure as follows: if $u \in U_h$, then $u_+ \otimes_{A_h^{\mathrm{op}}} u_- \in {}_\blacktriangleright U_h \widehat{\otimes}_{A_h^{\mathrm{op}}} U_{h\triangleleft}$ can be written as $u_+ \otimes_{A_h^{\mathrm{op}}} u_- = \lim_{n\to\infty} u_{+,n} \otimes_{A_h^{\mathrm{op}}} u_{-,n}$. For $x \otimes_{A_h^{\mathrm{op}}} y \in {}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\mathrm{op}}} N_{h\triangleleft}$, one defines

$$(x \otimes_{k[[h]]} y)u := \lim_{n\to\infty} xu_{+,n} \otimes_{A_h^{\mathrm{op}}} u_{-,n}y \in {}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\mathrm{op}}} N_{h\triangleleft}.$$

As $\lim_{n\to\infty} t^\ell(a)u_{+,n} \otimes_{A_h^{\rm op}} u_{-,n} = \lim_{n\to\infty} u_{+,n} \otimes_{A_h^{\rm op}} u_{-,n}t^\ell(a)$, we have thus defined a right action of U_h on ${}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\rm op}} N_{h\triangleleft}$.

If $\mathcal{P}_h$ and N_h are right U_h-modules, then $\mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)$ is endowed with a left U_h-module structure as follows: if $u_{[+]} \otimes^{A_h} u_{[-]} = \lim_{n\to\infty} u_{[+],n} \otimes^{A_h} u_{[-],n} \in U_{h\blacktriangleleft} \widehat{\otimes}^{A_h} {}_\triangleright U_h$, one sets for $\phi \in \mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)$ and $u \in U_h,\ p \in P_h$,

$$u_n\phi(p) := \phi\left(pu_{[+],n}\right) u_{[-],n},$$

and argues similarly as above that this defines, indeed, a left U_h-action on $\mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)$.

Lemma 6.4.2. *Let (U_h, A_h) be a quantum left Hopf algebroid, and let $\mathcal{P}_h$ be a right U_h-module such that $\mathcal{P}_{h\blacktriangleleft}$ (respectively ${}_\blacktriangleright\mathcal{P}_h$) is a finitely generated projective $A_h^{\rm op}$-module (resp. A_h-module). Then*

- (i) *$\mathcal{P}_h$ is complete for the h-adic topology.*
- (ii) *For a right U_h-module N_h, any element of $\mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)$ is continuous if we endow both modules with the h-adic topology.*
- (iii) *If N_h is a left U_h-module that is complete in the h-adic topology, then so is the right U_h-module ${}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\rm op}} N_{h\triangleleft}$.*
- (iv) *If N_h is a right U_h-module that is complete in the h-adic topology, then so is the left U_h-module $\mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)$.*

Proof. If N_h is a right U_h-module endowed with the h-adic topology, then the h-adic topology on $(N_h)^p$ coincides with the product topology. Thus, if N_h is complete for the h-adic topology, then so is $(N_h)^p$.

- (i) As $\mathcal{P}_h$ is a finitely generated projective $A_h^{\rm op}$-module, it is a summand of a finite free module, which is complete for the h-adic topology as A_h is so. Hence $\mathcal{P}_h$ is complete for the h-adic topology.
- (ii) This is obvious as such a morphism is $k[[h]]$-linear.
- (iii) $\mathcal{P}_h$ is a direct summand of a rank r free $A_h^{\rm op}$-module F_h. Thus ${}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\rm op}} N_\triangleleft$ is a summand of $(N_h)^r$, which is complete, hence it is itself complete.
- (iv) The proof of this part is analogous to the proof of (iii).

□

In the following, denote by $\mathbf{cMod}$-U_h resp. U_h-$\mathbf{cMod}$ the category of right resp. left U_h-modules which are complete for the h-adic topology. We then have the following result, analogous to Proposition 3.2.1:

Proposition 6.4.3. *Let (U_h, A_h) be simultaneously a quantum left and right Hopf algebroid. Assume that there exists a right U_h-module $\mathcal{P}_h$, where $\mathcal{P}_{h\blacktriangleleft}$ (resp. ${}_\blacktriangleright\mathcal{P}_h$) is finitely generated projective over $A_h^{\rm op}$ (resp. A_h), such that*

- (i) *the left U_h-module morphism*

$$A_h \to \mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, \mathcal{P}_h), \quad a \mapsto \{p \mapsto a \blacktriangleright p\}$$

  is an isomorphism of $k[[h]]$-modules;
- (ii) *the evaluation map*

$${}_\blacktriangleright\mathcal{P}_h \otimes_{A_h^{\rm op}} \mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N_h)_\triangleleft \to N_h, \quad p \otimes_{A_h^{\rm op}} \phi \mapsto \phi(p)$$

  is an isomorphism for any $N_h \in \mathbf{cMod}$-U_h.

Then

$$U_h\text{-}\mathbf{cMod} \to \mathbf{cMod}\text{-}U_h, \quad M_h \mapsto {}_\blacktriangleright\mathcal{P}_h \otimes_{A^{\rm op}} M_{h\triangleleft}$$

is an equivalence of categories with quasi inverse given by $N'_h \mapsto \mathrm{Hom}_{A_h^{\rm op}}(\mathcal{P}_h, N'_h)$.

We will now give an example of such a situation. Consider a left bialgebroid (U, A) and a quantisation (U_h, A_h) of it. Observe that the natural left U_h-module structure on A_h quantises that of U on A.

Theorem 6.4.4. *Let (U, A) be a left bialgebroid, where U is assumed to be a k-Noetherian algebra. Assume that there exists an integer d satisfying*

$$\mathrm{Ext}^i_U(A, U) = \begin{cases} 0 & \text{if } i \neq d, \\ \Lambda & \text{if } i = d. \end{cases}$$

Then there exists an A_h-module Λ_h that is a quantisation of Λ such that

$$\mathrm{Ext}^i_{U_h}(A_h, U_h) = \begin{cases} 0 & \text{if } i \neq d, \\ \Lambda_h & \text{if } i = d, \end{cases}$$

where the right action of U_h on $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is a quantisation of the right action of U on $\mathrm{Ext}^d_U(A, U)$ given by right multiplication.

We remind the reader here that Λ_h is $\Lambda[[h]]$ as a $k[[h]]$-module. This theorem is proven in [Ch2] in the case where $A_h = k[[h]]$. For the proof of the general case, we will need the following auxiliary statement:

Lemma 6.4.5. *There exists a resolution of the U_h-module A_h by finite rank free (filtered) FU_h-modules*

$$\ldots \xrightarrow{\partial_{i+1}} FL^i \xrightarrow{\partial_i} \ldots \xrightarrow{\partial_2} FL^1 \xrightarrow{\partial_1} FL^0 \longrightarrow A_h \longrightarrow \{0\},$$

where FL^i is $(U_h)^{d_i}$ endowed with the h-adic filtration such that the associated graded complex

$$\ldots GL^i \xrightarrow{G\partial_i} \ldots \to GL^1 \xrightarrow{G\partial_1} GL^0 \longrightarrow A[h] \longrightarrow \{0\}$$

is a resolution of the $U[h]$-module $A[h]$.

Proof. We will construct the p-th module FL^p by induction on p: for $p = 0$, one may take $FL^0 := U_h$ and $\partial_0 := \epsilon$, endowed with the h-adic topology. Assume then that $FL^0, FL^1, \ldots, FL^p$ are already constructed along with $\partial_0, \partial_1, \ldots, \partial_p$. As FL^p is topologically free, the induced filtration and the h-adic filtration coincide on $\mathrm{Ker}\,\partial_p$. As $\mathrm{Ker}\,\partial_p$ is closed in FL^p, it is also complete. This $k[[h]]$-module is topologically free as it is complete for the h-adic topology and also torsion free; set $\mathrm{Ker}\,\partial_p := V_p[[h]]$. Since $GU_h = U[h]$ is Noetherian, the (filtered) algebra U_h is (filtered) Noetherian [Ch2, Prop. 3.0.7] and the U_h-module $\mathrm{Ker}\,\partial_p$ is finitely generated so that the U-module V_p is finitely generated as well. Let $(\overline{v_1}, \ldots, \overline{v_{d_{p+1}}})$ be a generating system of the U-module V_p and let $(v_1, \ldots, v_{d_{p+1}}) \in (\mathrm{Ker}\,\partial_p)^{d_{p+1}}$ be a lift of $(\overline{v_1}, \ldots, \overline{v_{d_{p+1}}})$. Moreover, introduce the U_h-module morphism

$$\partial_{p+1} : (U_h)^{d_{p+1}} \to \mathrm{Ker}\,\partial_p, \quad (u_1, \ldots, u_{p+1}) \mapsto \sum u_i v_i,$$

which is a strict morphism of filtered modules. The filtered exact sequence

$$(U_h)^{p+1} \xrightarrow{\partial_{p+1}} (U_h)^p \xrightarrow{\partial_p} (U_h)^{p-1}$$

is strict exact so that the sequence

$$(GU_h)^{p+1} \xrightarrow{G\partial_{p+1}} (GU_h)^p \xrightarrow{G\partial_p} (GU_h)^{p-1}$$

is exact (*cf.* [Ch2, Prop. 3.0.2]). □

Proof of Theorem 6.4.4. The $\mathrm{Ext}^\bullet_{U_h}(A_h, U_h)$-groups can be computed via the complex $M^\bullet := \big(\mathrm{Hom}_{U_h}(L^\bullet, U_h), \partial_\bullet\big)$. Its components are endowed with the natural filtration

$$F_s \mathrm{Hom}_{U_h}(L^i, U_h) := \{\lambda \in \mathrm{Hom}_{U_h}(L^i, U_h) \mid \lambda(F_p L^i) \subset F_{s+p} U_h\},$$

and the right FA-modules $F\,\mathrm{Hom}_{U_h}(L^i, U_h)$ are isomorphic to $(U_h)^{d_i}$ endowed with the h-adic filtration. On the other hand, the filtration of the $M^i := \mathrm{Hom}_{U_h}(L^i, U_h)$ induces a filtration on $\mathrm{Ext}^i_{U_h}(A_h, U_h)$ as follows:

$$F_s\mathrm{Ext}^i_{U_h}(A_h, U_h) := \frac{\mathrm{Ker}\,{}^t\partial_i \cap F_sM^i + \mathrm{Im}{}^t\partial_{i-1}}{\mathrm{Im}{}^t\partial_{i-1}} \simeq \frac{\mathrm{Ker}\,{}^t\partial_i \cap F_sM^i}{\mathrm{Im}{}^t\partial_{i-1} \cap F_sM^{i-1}}.$$

The filtration on the $\mathrm{Ext}^i_{U_h}(A_h, U_h)$-groups is nothing but the h-adic filtration. Reproducing the proof of [Ch2], one can see that:

– if $i \neq d$, then $\mathrm{Ext}^i_{U_h}(A_h, U_h) = \{0\}$;
– the maps ${}^t\partial_i$ are strict filtered morphisms;
– $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is complete for the h-adic filtration (as it is a finitely generated U_h^{op}-module, see [Ch2]). Moreover, $\mathrm{Ext}^d_{U_h}(A_h, U_h)/h\mathrm{Ext}^d_{U_h}(A_h, U_h) \simeq \mathrm{Ext}^d_U(A, U)$ as U^{op}-modules.

Let us show that $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is h-torsion free. Let $[\sigma_d] \in \mathrm{Ext}^d_{U_h}(A_h, U_h)$, where $\sigma_d \in \mathrm{Ker}\,{}^t\partial_d$, be an h-torsion element in $\mathrm{Ext}^d_{U_h}(A_h, U_h)$. There exists a minimal $n \in \mathbb{N}^*$ such that $h^n[\sigma_d] = 0$. Let $\sigma_{d-1} \in \mathrm{Hom}_{U_h}(L^{d-1}, U_h)$ be such that $h^n\sigma_d = {}^t\partial_{d-1}(\sigma_{d-1})$. Then, by reduction modulo h, one obtains $\overline{{}^t\partial_{d-1}}(\overline{\sigma_{d-1}}) = 0$ and there exists $\overline{\sigma_{d-2}}$ such that $\overline{\sigma_{d-1}} = \overline{\partial_{d-2}}\,(\overline{\sigma_{d-2}})$. Let σ_{d-2} be a lift of $\overline{\sigma_{d-2}}$. Then there exists τ_{d-1} such that

$$\sigma_{d-1} = {}^t\partial_{d-2}(\sigma_{d-2}) + h\tau_{d-1}.$$

Hence $h^n\sigma_d = h{}^t\partial_{d-1}(\tau_{d-1})$, which gives (using the fact that $\mathrm{Hom}_{U_h}(L^d, U_h)$ is topologically free) $h^{n-1}\sigma_d = {}^t\partial_{d-1}(\tau_{d-1})$. This contradicts the minimality of n so that $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is h-torsion free. As $\mathrm{Ext}^d_{U_h}(A_h, U_h)$ is complete for the h-adic topology and h-torsion free, it is topologically free. ☐

Combining this result with the more general structure theory as in Proposition 3.2.1 resp. Proposition 6.4.3, one obtains:

Proposition 6.4.6. *Let U satisfy the conditions of Theorem 6.4.4. Assume moreover that*

(*i*) *A is noetherian;*
(*ii*) *$\mathrm{Ext}_U(A, U)$ is a dualising module for (U, A),* i.e.*, satisfies the hypothesis of Proposition 3.2.1;*
(*iii*) *${}_\blacktriangleright\mathrm{Ext}_U(A, U)$ is a finitely generated projective A-module.*

Then $\mathcal{P}_h = \mathrm{Ext}^d_{U_h}(A_h, U_h)$ is a dualising module for (U_h, A_h) and produces an equivalence between the categories of left resp. right complete U_h-modules.

Remark 6.4.7. Let $M_h := M[[h]]$ and $N_h := N[[h]]$ be two A_h^{op}-modules which are topologically free with respect to the h-adic topology. Assume moreover that M_h is finitely generated projective over A_h^{op}; then $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is topologically free and, as said before, is isomorphic to $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)[[h]]$ as a $k[[h]]$-module: observe that $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is complete for the induced topology as it is a closed subset of the topologically free $k[[h]]$-module $\mathrm{Hom}_{k[[h]]}(M_h, N_h)$. On the other hand, on $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$, the induced topology coincides with the h-adic topology. Hence $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is complete for the h-adic topology and since it is also torsion free, it is topologically free. Let us now show that $\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)/h\,\mathrm{Hom}_{A_h^{\mathrm{op}}}(M_h, N_h)$ is isomorphic to $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)$: in fact, there exists an A_h^{op}-module M_h' and a finitely generated free A_h^{op}-module F_h such that $M_h \oplus M_h' = F_h$. Any element ϕ of $\mathrm{Hom}_{A^{\mathrm{op}}}(M, N)$ can be extended to an element of $\mathrm{Hom}_{A^{\mathrm{op}}}(F_h/hF_h, N)$, which, in turn, can be lifted to an element of $\mathrm{Hom}_{A_h^{\mathrm{op}}}(F_h, N_h)$ and produces (by restriction) a lift of ϕ.

Proof of Proposition 6.4.6. The module $\mathcal{P}_{h\blacktriangleleft}$ is a finitely generated A_h^{op}-module as $\mathcal{P}_\blacktriangleleft := \mathrm{Ext}_U(A, U)_\blacktriangleleft$ is a finitely generated A^{op}-module (see Proposition 3.0.5 of the preprint version of [Ch2]).

Let N_h be a finitely generated A_h^{op}-module. It can be considered as a filtered FA_h^{op}-module as follows: one has an epimorphism $\big(A_h^{\mathrm{op}}\big)^n \xrightarrow{p} N_h \longrightarrow 0$, and we endow N_h with the filtration $p\big(F\big(A_h^{\mathrm{op}}\big)^n\big)$. As $\mathcal{P}_\blacktriangleleft$ is a projective A^{op}-module, $\mathcal{P}[h]_\blacktriangleleft$ is a projective $A[h]^{\mathrm{op}}$-module, and Proposition 3.0.11 of the preprint version of [Ch2] shows that $\mathrm{Ext}^i_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N_h) = \{0\}$ if $i > 0$.

Let now N_h be any A_h^{op}-module. We have $N_h = \varinjlim N'_h$, where N'_h runs over all finitely generated A_h^{op}-submodules of N_h. Let $F^\bullet$ be a resolution of $\mathcal{P}$ by finitely generated free A_h^{op}-modules. We have

$$\begin{aligned}\mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N_h) &= \mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, \varinjlim N'_h) = H^j\big(\mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, \varinjlim N'_h)\big)\\ &= H^j\big(\varinjlim \mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, N'_h)\big) = \varinjlim H^j\big(\mathrm{Hom}_{A_h^{\mathrm{op}}}(F^\bullet, N'_h)\big)\\ &= \varinjlim \mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N'_h) = \{0\},\end{aligned}$$

where we used the fact that the functor $\varinjlim$ is exact because the set of finitely generated submodules of M is a directed set, *cf.* [Ro, Prop. 5.33]. Thus we have proven that if N_h is any A_h^{op}-module, then

$$\mathrm{Ext}^j_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N_h) = \{0\} \quad \text{if } j > 0.$$

Consequently, $\mathcal{P}_{h\blacktriangleleft}$ is a projective A_h^{op}-module; similarly, ${}_\blacktriangleright\mathrm{Ext}_{U_h}(A_h, U_h)$ is a projective A_h^{op}-module.

The assertion with respect to the evaluation map yet is true if N_h is a topologically free U_h-module as it is true modulo h, see Remark 6.4.7. Furthermore, the functor $N_h \mapsto \mathcal{P}_h \otimes_{A_h} \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N_h)$ is exact as $\mathcal{P}_{h\blacktriangleleft}$ resp. ${}_\blacktriangleright\mathcal{P}_h$ is a projective A_h^{op}-module resp. A_h-module.

Let now N_h be a finitely generated U_h-module. Using a finite free resolution of N_h, one can show (by a diagram chase argument) that the evaluation map is an isomorphism (as it is an isomorphism for any component of the resolution). If N_h is any U_h-module instead, one can write $N_h = \varinjlim N'_h$, where N'_h runs over all finitely generated submodules of N_h. Since $\mathcal{P}_h$ is a finitely generated A_h^{op}-module, any element $\phi \in \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N_h)$ can be considered as an element of $\mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, N'_h)$ for a well-chosen finitely generated A_h^{op}-module N'_h. Using the finitely generated case, one can see that the evaluation map is an isomorphism for any U_h-module N_h.

As $\mathcal{P}_h$ is a finitely generated projective A_h^{op}-module, the natural left U_h-module map

$$A_h \to \mathrm{Hom}_{A_h^{\mathrm{op}}}(\mathcal{P}_h, \mathcal{P}_h), \quad a \mapsto (p \mapsto a \blacktriangleright p)$$

of Proposition 6.4.3 is an isomorphism as it is an isomorphism modulo h. This concludes the proof. □

Example 6.4.8. For example, if A is the algebra of regular functions on a smooth affine variety X and L is the Lie-Rinehart algebra of vector fields over X, then $U = V^\ell(L)$ satisfies the conditions of Theorem 6.4.4. More generally, for any Lie-Rinehart algebra (A, L), where L is finitely generated projective of constant rank d over a Noetherian algebra A, the pair $\big(A, V^\ell(L)\big)$ fulfils the conditions of Theorem 6.4.4 and one obtains $\mathrm{Ext}^d_{V^\ell(L)}(A, V^\ell(L)) = \bigwedge_A^d \mathrm{Hom}_A(L, A)$ for the dualising module (see [Ch1, Hue] for more details in this direction). Then, for any quantisation $V^\ell(L)_h$ of $V^\ell(L)$, Proposition 6.4.6 leads to an equivalence of categories between left and right complete $V^\ell(L)_h$-modules. Examples of quantisations of $V^\ell(L)$ are given in [ChGa].

REFERENCES

[AnFu] F. Anderson and K. Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992.

[Be] J. Beck, *Distributive laws*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 119–140.

[B] G. Böhm, *Hopf algebroids*, Handbook of algebra, Vol. 6, North-Holland, Amsterdam, 2009, pp. 173–236.

[BSz] G. Böhm and K. Szlachányi, *Hopf algebroids with bijective antipodes: axioms, integrals, and duals*, J. Algebra **274** (2004), no. 2, 708–750.

[Bo] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, vol. 2, Academic Press, Inc., Boston, MA, 1987.

[CaVdB] D. Calaque and M. Van den Bergh, *Hochschild cohomology and Atiyah classes*, Adv. Math. **224** (2010), 1839–1889.

[Ca] P. Cartier, *Cohomologie des coalgèbres, exposés 4, 5*, Séminaire Sophus Lie, tome 2 (1955–1956), Faculté des Sciences de Paris, 1957.

[Ch1] S. Chemla, *Poincaré duality for k-A Lie superalgebras*, Bull. Soc. Math. France **122** (1994), no. 3, 371–397.

[Ch2] ———, *Duality properties for quantum groups*, Pacific J. Math. **252** (2011), no. 2, 313–341 (a more detailed version is available at `arXiv:0911.2860`).

[Ch3] ———, *Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators*, J. Algebra **276** (2004), 80–102.

[ChGa] S. Chemla and F. Gavarini, *Duality functors for quantum groupoids*, J. Noncomm. Geom. **9** (2015), no. 2, 287–358.

[Hue] J. Huebschmann, *Duality for Lie-Rinehart algebras and the modular class*, J. Reine Angew. Math. **510** (1999), 103–159.

[KadSz] L. Kadison and K. Szlachányi, *Bialgebroid actions on depth two extensions and duality*, Adv. Math. **179** (2003), no. 1, 75–121.

[Ka] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[Ko] N. Kowalzig, *Hopf algebroids and their cyclic theory*, Ph. D. thesis, Universiteit Utrecht and Universiteit van Amsterdam, 2009.

[KoKr] N. Kowalzig and U. Krähmer, *Duality and products in algebraic (co)homology theories*, J. Algebra **323** (2010), no. 7, 2063–2081.

[KoP] N. Kowalzig and H. Posthuma, *The cyclic theory of Hopf algebroids*, J. Noncomm. Geom. **5** (2011), no. 3, 423–476.

[Phù] H. H. Phùng, *Tannaka-Krein duality for Hopf algebroids*, Israel J. Math. **167** (2008), 193–225.

[Ri] G. Rinehart, *Differential forms on general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.

[Ro] J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009.

[Sch1] P. Schauenburg, *Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)*, New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., vol. 267, Amer. Math. Soc., Providence, RI, 2000, pp. 273–299.

[Sch2] P. Schauenburg, *The dual and the double of a Hopf algebroid are Hopf algebroids*, preprint `arXiv:1504.05057` (2015).

[Schn] J.-P. Schneiders, *An introduction to $\mathscr{D}$-modules*, Bull. Soc. Roy. Sci. Liège **63** (1994), no. 3-4, 223–295, Algebraic Analysis Meeting (Liège, 1993).

[Sw] M. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.

[T] M. Takeuchi, *Groups of algebras over $A \otimes \overline{A}$*, J. Math. Soc. Japan **29** (1977), no. 3, 459–492.

[Xu] P. Xu, *Quantum groupoids*, Comm. Math. Phys. **216** (2001), no. 3, 539–581.

S.C.: INSTITUT DE MATHÉMATIQUES DE JUSSIEU, UMR 7586, UNIVERSITÉ PIERRE ET MARIE CURIE, 75005 PARIS, FRANCE

E-mail address: `sophie.chemla@imj-prg.fr`

F.G.: DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA TOR VERGATA, VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALIA

E-mail address: `gavarini@mat.uniroma2.it`

N.K.: ISTITUTO NAZIONALE DI ALTA MATEMATICA, P.LE ALDO MORO 5, 00185 ROMA, ITALIA

E-mail address: `kowalzig@mat.uniroma1.it`