ANNALES

DE

L'INSTITUT FOURIER

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Tome 52, nº 3 (2002), p. 809-834.

<http://aif.cedram.org/item?id=AIF_2002__52_3_809_0>

Ann. Inst. Fourier, Grenoble
52, 3 (2002), 809–834

THE QUANTUM DUALITY PRINCIPLE

by Fabio GAVARINI

*"Dualitas dualitatum
et omnia dualitas"*

N. Barbecue, "Scholia"

Introduction.

The *quantum duality principle* is known in literature under at least two formulations. One claims that quantum function algebras associated to dual Poisson groups can be considered to be dual — in the Hopf sense — to each other; and similarly for quantum enveloping algebras (cf. [FRT] and [Se]). The second one, due to Drinfeld (cf. [Dr]), states that any quantisation of the universal enveloping algebra of a Poisson group can also be understood — in some sense — as a quantisation of the dual formal Poisson group, and, conversely, any quantisation of a formal Poisson group also "serves" as a quantisation of the universal enveloping algebra of the dual Poisson group: this is the point of view we are interested in. I am now going to describe this result more in detail.

Let $\mathbf{k}$ be a field of zero characteristic. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbf{k}$, $U(\mathfrak{g})$ its universal enveloping algebra: then $U(\mathfrak{g})$ has a natural structure of Hopf algebra. Let $F[[\mathfrak{g}]]$ be the (algebra of regular functions on the) formal group associated to $\mathfrak{g}$: it is a complete topological Hopf algebra (the coproduct taking values in a suitable topological tensor

Keywords: Quantum groups – Topological Hopf algebras.
Math. classification: 17B37 – 20G42 – 81R50 – 16W30.

product of the algebra with itself), which has two realisations. The first one is as follows: if G is an affine algebraic group with tangent Lie algebra $\mathfrak{g}$, and $F[G]$ is the algebra of regular functions on G, then $F[[\mathfrak{g}]]$ is the $\mathfrak{m}_e$-completion of $F[G]$ at the maximal ideal $\mathfrak{m}_e$ of the identity element $e \in G$, endowed with the $\mathfrak{m}_e$-adic topology. The second one is $F[[\mathfrak{g}]] := U(\mathfrak{g})^*$, the linear dual of $U(\mathfrak{g})$), endowed with the weak topology. In any case, $U(\mathfrak{g})$ identifies with the *topological dual* of $F[[\mathfrak{g}]]$, i.e. the set of all $\mathbf{k}$-linear continuous maps from $F[[\mathfrak{g}]]$ to $\mathbf{k}$, where $\mathbf{k}$ is given the discrete topology; similarly $F[[\mathfrak{g}]] = U(\mathfrak{g})^*$ is also the topological dual of $U(\mathfrak{g})$ if we take on the latter space the discrete topology: in particular, a (continuous) biduality theorem relates $U(\mathfrak{g})$ and $F[[\mathfrak{g}]]$, and evaluation yields a natural Hopf pairing among them. Now assume $\mathfrak{g}$ is a *Lie bialgebra*: then $U(\mathfrak{g})$ is a *co-Poisson Hopf* algebra, $F[[\mathfrak{g}]]$ is a topological *Poisson* Hopf algebra, and the above pairing is compatible with these additional co-Poisson and Poisson structures. Further, the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is a Lie bialgebra as well, so we can consider also $U(\mathfrak{g}^*)$ and $F[[\mathfrak{g}^*]]$.

Let $\mathfrak{g}$ be a Lie bialgebra. A quantisation of $U(\mathfrak{g})$ is, roughly speaking, a topological Hopf $\mathbf{k}[[h]]$-algebra which for $h = 0$ is isomorphic, as a co-Poisson Hopf algebra, to $U(\mathfrak{g})$: these objects form a category, called $\mathcal{QUEA}$. Similarly, a quantisation of $F[[\mathfrak{g}]]$ is, in short, a topological Hopf $\mathbf{k}[[h]]$-algebra which for $h = 0$ is isomorphic, as a topological Poisson Hopf algebra, to $F[[\mathfrak{g}]]$: we call $\mathcal{QFSHA}$ the category formed by these objects.

The quantum duality principle (after Drinfeld) states that there exist two functors, namely $(\ )' \colon \mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$ and $(\ )^\vee \colon \mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$, which are inverse of each other, and if $U_h(\mathfrak{g})$ is a quantisation of $U(\mathfrak{g})$ and $F_h[[\mathfrak{g}]]$ is a quantisation of $F[[\mathfrak{g}]]$, then $U_h(\mathfrak{g})'$ is a quantisation of $F[[\mathfrak{g}^*]]$, and $F_h[[\mathfrak{g}]]^\vee$ is a quantisation of $U(\mathfrak{g}^*)$.

This paper provides an explicit thorough proof (seemingly, the first one in the literature) of this result. I also point out some further details and what is true when $\mathbf{k}$ has positive characteristic, and sketch a plan for generalizing all this to the infinite dimensional case.

Note that several properties of the objects I consider have been discovered and exploited in the works by Etingof and Kazhdan (see [EK1], [EK2]), by Enriquez (cf. [E]) and by Kassel and Turaev (cf. [KT]), who deal with some *special cases* of quantum groups, arising from a specific construction, and also applied Drinfeld's results. The analysis in the present paper shows that that those properties are often direct consequences of more general facts.

I point out that Drinfeld's result is essentially *local* in nature, as it deals with quantisations over the ring of formal series and ends up only with infinitesimal data, i.e. objects attached to Lie bialgebras; a *global* version of the principle, dealing with quantum groups over a ring of Laurent polynomials, which give information on the global data of the underlying Poisson groups will be provided in a forthcoming paper (cf. [Ga2]): this is useful in applications, e.g. it yields a quantum duality principle for Poisson homogeneous spaces, cf. [CG].

Acknowledgements. — The author thanks Pierre Baumann and Alessandro D'Andrea for their valuable remarks.

1. Notation and terminology.

1.1. **Topological $\mathbf{k}[[h]]$-modules and topological Hopf $\mathbf{k}[[h]]$-algebras.** — Let $\mathbf{k}$ be a fixed field, h an indeterminate. The ring $\mathbf{k}[[h]]$ will always be considered as a topological ring w.r.t. the h-adic topology. Let X be any $\mathbf{k}[[h]]$-module. We set $X_0 := X/hX = \mathbf{k} \otimes_{\mathbf{k}[[h]]} X$, a $\mathbf{k}$-module (via scalar restriction $\mathbf{k}[[h]] \to \mathbf{k}[[h]]/h\mathbf{k}[[h]] \cong \mathbf{k}$) which we call the *specialisation* of X at $h = 0$, or *semiclassical limit* of X; we shall also use notation $X \xrightarrow{h \to 0} \overline{Y}$ to mean $X_0 \cong \overline{Y}$. Note that if X is a topological $\mathbf{k}[[h]]$-module which is torsionless, complete and separated w.r.t. the h-adic topology then there is a natural isomorphism of $\mathbf{k}[[h]]$-modules $X \cong X_0[[h]]$: indeed, choose any $\mathbf{k}$-basis $\{b_i\}_{i \in I}$ of X_0, and pick any subset $\{\beta_i\}_{i \in I} \subseteq X$ such that $\beta_i \bmod h = b_i (\forall i)$: then an isomorphism as required is given by $\beta_i \mapsto b_i$ (however, topologies on either side may be different).

For later use, we also set ${}^F X := \mathbf{k}((h)) \otimes_{\mathbf{k}[[h]]} X$, a vector space over $\mathbf{k}((h))$, which is not equipped with any topology.

If X is a topological $\mathbf{k}[[h]]$-module, we let its *full dual* to be $X^* := \mathrm{Hom}_{\mathbf{k}[[h]]}\big(X, \mathbf{k}[[h]]\big)$, and its *topological dual* to be $X^\star := \big\{ f \in X^* \,\big|\, f \text{ is continuous} \big\}$. Note that $X^* = X^\star$ when the topology on X is the h-adic one.

We introduce now two tensor categories of topological $\mathbf{k}[[h]]$-modules, $\mathcal{T}_{\widehat{\otimes}}$ and $\mathcal{P}_{\widetilde{\otimes}}$: the first one is modeled on the tensor category of *discrete* topological $\mathbf{k}$-vector spaces, the second one is modeled on the category of *linearly compact* topological $\mathbf{k}$-vector spaces.

Let $\mathcal{T}_{\widehat{\otimes}}$ be the category whose objects are all topological $\mathbf{k}[[h]]$-modules which are topologically free (i.e. isomorphic to $V[[h]]$ for some

$\mathbf{k}$-vector space V, with the h-adic topology) and whose morphisms are the $\mathbf{k}[[h]]$-linear maps (which are automatically continuous). This is a tensor category w.r.t. the tensor product $T_1 \widehat{\otimes} T_2$ defined to be the separated h-adic completion of the algebraic tensor product $T_1 \otimes_{\mathbf{k}[[h]]} T_2$ (for all T_1, $T_2 \in \mathcal{T}_{\widehat{\otimes}}$).

Let $\mathcal{P}_{\widetilde{\otimes}}$ be the category whose objects are all topological $\mathbf{k}[[h]]$-modules isomorphic to modules of the type $\mathbf{k}[[h]]^E$ (the Cartesian product indexed by E, with the Tikhonov product topology) for some set E: these are complete w.r.t. to the weak topology, in fact they are isomorphic to the projective limit of their finite free submodules (each one taken with the h-adic topology); the morphisms in $\mathcal{P}_{\widetilde{\otimes}}$ are the $\mathbf{k}[[h]]$-linear continuous maps. This is a tensor category w.r.t. the tensor product $P_1 \widetilde{\otimes} P_2$ defined to be the completion of the algebraic tensor product $P_1 \otimes_{\mathbf{k}[[h]]} P_2$ w.r.t. the weak topology: therefore $P_i \cong \mathbf{k}[[h]]^{E_i}$ $(i = 1, 2)$ yields $P_1 \widetilde{\otimes} P_2 \cong \mathbf{k}[[h]]^{E_1 \times E_2}$ (for all P_1, $P_2 \in \mathcal{P}_{\widetilde{\otimes}}$).

Note that the objects of $\mathcal{T}_{\widehat{\otimes}}$ and of $\mathcal{P}_{\widetilde{\otimes}}$ are complete and separated w.r.t. the h-adic topology, so by the previous remark one has $X \cong X_0[[h]]$ for each of them.

We denote by $\mathcal{HA}_{\widehat{\otimes}}$ the subcategory of $\mathcal{T}_{\widehat{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{T}_{\widehat{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{T}_{\widehat{\otimes}}$. Similarly, we call $\mathcal{HA}_{\widetilde{\otimes}}$ the subcategory of $\mathcal{P}_{\widetilde{\otimes}}$ whose objects are all the Hopf algebras in $\mathcal{P}_{\widetilde{\otimes}}$ and whose morphisms are all the Hopf algebra morphisms in $\mathcal{P}_{\widetilde{\otimes}}$. Moreover, we define $\mathcal{HA}_{\widetilde{\otimes}}^{w-I}$ to be the full subcategory of $\mathcal{HA}_{\widetilde{\otimes}}$ whose objects are all the $H \in \mathcal{HA}_{\widetilde{\otimes}}$ whose (weak) topology coincides with the I_H-adic topology, where $I_H := hH + \mathrm{Ker}(\epsilon) = \epsilon^{-1}\big(h\mathbf{k}[[h]]\big)$.

As a matter of notation, when dealing with a (possibly topological) Hopf algebra H, I shall denote by m its product, by 1 its unit element, by Δ its coproduct, by ϵ its counit and by S its antipode; subscripts H will be added whenever needed for clarity. Note that the objects of $\mathcal{HA}_{\widehat{\otimes}}$ and of $\mathcal{HA}_{\widetilde{\otimes}}$ are *topological* Hopf algebras, not standard ones: in particular, in σ-notation $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ the sum is understood in topological sense.

Definition 1.2 (cf. [Dr], § 7).

(a) *We call* quantized universal enveloping algebra *(in short, QUEA) any $H \in \mathcal{HA}_{\widehat{\otimes}}$ such that $H_0 := H/hH$ is a co-Poisson Hopf algebra* *isomorphic to* $U(\mathfrak{g})$ *for some finite dimensional Lie bialgebra* $\mathfrak{g}$ *(over* $\mathbf{k}$*); in this case we write* $H = U_h(\mathfrak{g})$*, and say* H *is a* quantisation *of* $U(\mathfrak{g})$*. We call* $\mathcal{QUEA}$ *the full subcategory of* $\mathcal{HA}_{\widehat{\otimes}}$ *whose objects are QUEA, relative to all possible* $\mathfrak{g}$ *(see also Remark 1.3* (a) *below).*

(b) *We call* quantized formal series Hopf algebra *(in short, QFSHA) any* $K \in \mathcal{HA}_{\widetilde{\otimes}}$ *such that* $K_0 := K\big/hK$ *is a topological Poisson Hopf algebra isomorphic to* $F[[\mathfrak{g}]]$ *for some finite dimensional Lie bialgebra* $\mathfrak{g}$ *(over* $\mathbf{k}$*); then we write* $H = F_h[[\mathfrak{g}]]$*, and say* K *is a* quantisation *of* $F[[\mathfrak{g}]]$*. We call* $\mathcal{QFSHA}$ *the full subcategory of* $\mathcal{HA}_{\widetilde{\otimes}}$ *whose objects are QFSHA, relative to all possible* $\mathfrak{g}$ *(see also Remark 1.3* (a) *below).*

(c) *If* H_1*,* H_2*, are two quantisations of* $U(\mathfrak{g})$*, resp. of* $F[[\mathfrak{g}]]$ *(for some Lie bialgebra* $\mathfrak{g}$*), we say that* H_1 is equivalent to H_2*, and we write* $H_1 \equiv H_2$*, if there is an isomorphism* $\varphi\colon H_1 \cong H_2$ *(in* $\mathcal{QUEA}$*, resp. in* $\mathcal{QFSHA}$*) such that* $\varphi = \mathrm{id} \bmod h$.

Remarks 1.3. — (a) If $H \in \mathcal{HA}_{\widehat{\otimes}}$ is such that $H_0 := H\big/hH$ as a Hopf algebra is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, then $H_0 = U(\mathfrak{g})$ is also a *co-Poisson* Hopf algebra w.r.t. the Poisson cobracket δ defined as follows: if $x \in H_0$ and $x' \in H$ gives $x = x' + hH$, then $\delta(x) := \big(h^{-1}\big(\Delta(x') - \Delta^{\mathrm{op}}(x')\big)\big) + hH\widehat{\otimes}H$; then (by [Dr], §3, Theorem 2) the restriction of δ makes $\mathfrak{g}$ into a Lie bialgebra. Similarly, if $K \in \mathcal{HA}_{\widetilde{\otimes}}$ is such that $K_0 := K\big/hK$ is a topological Poisson Hopf algebra isomorphic to $F[[\mathfrak{g}]]$ for some Lie algebra $\mathfrak{g}$ then $K_0 = F[[\mathfrak{g}]]$ is also a topological *Poisson* Hopf algebra w.r.t. the Poisson bracket $\{\ ,\ \}$ defined as follows: if x, $y \in K_0$ and x', $y' \in K$ give $x = x' + hK$, $y = y' + hK$, then $\{x, y\} := \big(h^{-1}(x'y' - y'x')\big) + hK$; then $\mathfrak{g}$ is a bialgebra again, and $F[[\mathfrak{g}]]$ is (the algebra of regular functions on) a *Poisson* formal group. These natural co-Poisson and Poisson structures are the ones considered in Definition 1.2 above.

In fact, specialisation gives a tensor functor from $\mathcal{QUEA}$ to the tensor category of universal enveloping algebras of Lie bialgebras and a tensor functor from $\mathcal{QFSHA}$ to the tensor category of (algebras of regular functions on) formal Poisson groups.

(b) Clearly $\mathcal{QUEA}$, resp. $\mathcal{QFSHA}$, is a *tensor* subcategory of $\mathcal{HA}_{\widehat{\otimes}}$, resp. of $\mathcal{HA}_{\widetilde{\otimes}}$.

(c) Let H be a QFSHA. Then H is complete w.r.t. the weak topology, and $H_0 \cong F[[\mathfrak{g}]]$ for some finite dimensional Lie bialgebra $\mathfrak{g}$, and the weak topology on $H_0 \cong F[[\mathfrak{g}]]$ coincides with the $\mathrm{Ker}(\epsilon_{H_0})$-adic topology. It follows that the weak topology in H coincides with the I_H-adic topology, so $\mathcal{QFSHA}$ is a subcategory of $\mathcal{HA}_{\widetilde{\otimes}}^{w-I}$. In particular, if $H \in \mathcal{QFSHA}$ then $H\widetilde{\otimes}H$ equals the completion of $H \otimes_{\mathbf{k}[[h]]} H$ w.r.t. the $I_{H\times H}$-adic topology.

DEFINITION 1.4. — *Let H, K be Hopf algebras (in any category) over a ring R. A pairing $\pi = \langle\ ,\ \rangle : H \times K \longrightarrow R$ is called* perfect *if it is non-degenerate; it is called a* Hopf pairing *if for all x, x_1, $x_2 \in H$, y, y_1, $y_2 \in K$, the elements $\big\langle \Delta(x), y_1 \otimes y_2 \big\rangle := \sum_{(x)} \langle x_{(1)}, y_1 \rangle \cdot \langle x_{(2)}, y_2 \rangle$ and $\big\langle x_1 \otimes x_2, \Delta(y) \big\rangle := \sum_{(y)} \langle x_1, y_{(1)} \rangle \cdot \langle x_2, y_{(2)} \rangle$ are well defined and we have*

$$\big\langle x, y_1 \cdot y_2 \big\rangle = \big\langle \Delta(x), y_1 \otimes y_2 \big\rangle, \qquad \big\langle x_1 \cdot x_2, y \big\rangle = \big\langle x_1 \otimes x_2, \Delta(y) \big\rangle$$
$$\langle x, 1 \rangle = \epsilon(x), \qquad \langle 1, y \rangle = \epsilon(y), \qquad \big\langle S(x), y \big\rangle = \big\langle x, S(y) \big\rangle.$$

1.5. **Drinfeld's functors**. — Let H be a Hopf algebra (of any type) over $\mathbf{k}[[h]]$. For each $n \in \mathbb{N}$, define $\Delta^n : H \longrightarrow H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_H$, and $\Delta^n := \big(\Delta \otimes \mathrm{id}_H^{\otimes (n-2)}\big) \circ \Delta^{n-1}$ if $n \geqslant 2$. For any ordered subset $E = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < \ldots < i_k$, define the morphism $j_E : H^{\otimes k} \longrightarrow H^{\otimes n}$ by $j_E(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_{i_m} := a_m$ for $1 \leqslant m \leqslant k$; then set

$$\Delta_E := j_E \circ \Delta^k, \quad \Delta_\emptyset := \Delta^0, \quad \text{and} \quad \delta_E := \sum_{E' \subset E} (-1)^{n-|E'|} \Delta_{E'}, \quad \delta_\emptyset := \epsilon.$$

By the inclusion-exclusion principle, the inverse formula $\Delta_E = \sum_{\Psi \subseteq E} \delta_\Psi$ holds. We shall also use the notation $\delta_0 := \delta_\emptyset$, $\delta_n := \delta_{\{1,2,\ldots,n\}}$. Then we define

$$H' := \big\{ a \in H \big| \delta_n(a) \in h^n H^{\otimes n} \forall n \in \mathbb{N} \big\} \qquad \big(\subseteq H \big).$$

Note that the useful formula $\delta_n = (\mathrm{id}_H - \epsilon)^{\otimes n} \circ \Delta^n$ holds, for all $n \in \mathbb{N}_+$. Then H splits as $H = \mathbf{k}[[h]] \cdot 1_H \oplus J_H$, and $(\mathrm{id} - \epsilon)$ projects H onto $J_H := \mathrm{Ker}(\epsilon)$: so $(\mathrm{id} - \epsilon)^{\otimes n}$ projects $H^{\otimes n}$ onto ${J_H}^{\otimes n}$; therefore $\delta_n(a) = (\mathrm{id} - \epsilon)^{\otimes n}\big(\Delta^n(a)\big) \in {J_H}^{\otimes n}$ for any $a \in H$.

Now let $I_H := \epsilon^{-1}\big(h\mathbf{k}[[h]]\big)$; set

$$H^\times := \sum_{n \geqslant 0} h^{-n} {I_H}^n = \sum_{n \geqslant 0} \big(h^{-1} I_H\big)^n = \bigcup_{n \geqslant 0} \big(h^{-1} I_H\big)^n$$

(the $\mathbf{k}[[h]]$-subalgebra of ${}^F\! H$ generated by $h^{-1} I_H$; the second identity follows immediately from $\big(h^{-1} I_H\big)^n \subseteq \big(h^{-1} I_H\big)^m$ for all $n < m$), and define

$$H^\vee := h\text{-adic completion of the } \mathbf{k}[[h]]\text{-module } H^\times$$

(*Warning:* $H^\times$ naturally embeds into ${}^F\! H$, whereas $H^\vee$ a priori *does not*, for the completion procedure may "lead outside" ${}^F\! H$). Note also that

$I_H = J_H + h \cdot H$ (with J_H as above), so $H^\times = \sum_{n \geqslant 0} h^{-n} {J_H}^n$ and $H^\vee = h$-adic completion of $\sum_{n \geqslant 0} h^{-n} {J_H}^n$.

We are now ready to state the main result we are interested in:

THEOREM 1.6 ("The quantum duality principle"; cf. [Dr], §7). *Assume* char($\mathbf{k}$) $= 0$.

The assignments $H \mapsto H^\vee$ and $H \mapsto H'$ respectively define functors of tensor categories $\mathcal{QFSHA} \longrightarrow \mathcal{QUEA}$ and $\mathcal{QUEA} \longrightarrow \mathcal{QFSHA}$. These functors are inverse to each other. Indeed, for all $U_h(\mathfrak{g}) \in \mathcal{QUEA}$ and all $F_h[[\mathfrak{g}]] \in \mathcal{QFSHA}$ one has (cf. §1.2)

$$\circledast \qquad U_h(\mathfrak{g})' \Big/ h U_h(\mathfrak{g})' = F[[\mathfrak{g}^*]], \qquad F_h[[\mathfrak{g}]]^\vee \Big/ h F_h[[\mathfrak{g}]]^\vee = U(\mathfrak{g}^*)$$

that is, if $U_h(\mathfrak{g})$ is a quantisation of $U(\mathfrak{g})$ then $U_h(\mathfrak{g})'$ is a quantisation of $F[[\mathfrak{g}^*]]$, and if $F_h[[\mathfrak{g}]]$ is a quantisation of $F[[\mathfrak{g}]]$ then $F[[\mathfrak{g}^*]]^\vee$ is a quantisation of $U(\mathfrak{g}^*)$.

Moreover, the functors preserve equivalence, that is $H_1 \equiv H_2$ implies ${H_1}^\vee \equiv {H_2}^\vee$ or ${H_1}' \equiv {H_2}'$.

Our analysis also moves us to set the following (half-proved)

CONJECTURE 1.7. — *The quantum duality principle holds as well for* char($\mathbf{k}$) > 0.

2. General properties of Drinfeld's functors.

The rest of this paper will be devoted to prove Theorem 1.6. In this section we establish some general properties of Drinfeld's functors. The first step is entirely standard.

LEMMA 2.1. — *The assignments $H \mapsto H^* = H^\star$ and $H \mapsto H^\star$ define contravariant functors of tensor categories $(\ )^\star \colon \mathcal{T}_{\widehat{\otimes}} \longrightarrow \mathcal{P}_{\widetilde{\otimes}}$ and $(\ )^\star \colon \mathcal{P}_{\widetilde{\otimes}} \longrightarrow \mathcal{T}_{\widehat{\otimes}}$ which are inverse to each other. Their restriction gives antiequivalences of tensor categories $\mathcal{HA}_{\widehat{\otimes}} \xrightarrow{(\ )^*} \mathcal{HA}_{\widetilde{\otimes}}$, $\mathcal{HA}_{\widetilde{\otimes}} \xrightarrow{(\ )^*} \mathcal{HA}_{\widehat{\otimes}}$, and, if* char($\mathbf{k}$) $= 0$, *$\mathcal{QUEA} \xrightarrow{(\ )^*} \mathcal{QFSHA}$, $\mathcal{QFSHA} \xrightarrow{(\ )^*} \mathcal{QUEA}$.* □

The following key fact shows that, in a sense, Drinfeld's functors are *dual* to each other:

Proposition 2.2. — *Let $H \in \mathcal{HA}_{\widehat{\otimes}}$, $K \in \mathcal{HA}_{\widetilde{\otimes}}$, and let $\pi = \langle\ ,\ \rangle\colon H \times K \longrightarrow \mathbf{k}[[h]]$ be a Hopf pairing. Then π induces a bilinear pairing $\langle\ ,\ \rangle\colon H' \times K^\vee \longrightarrow \mathbf{k}[[h]]$.*

If in addition π is perfect, and the induced $\mathbf{k}$-valued pairing $\pi_0\colon H_0 \times K_0 \longrightarrow \mathbf{k}$ is still perfect, then $H' = (K^\times)^\circ := \{\eta \in {}^F\!H \mid \langle \eta, K^\times \rangle \subseteq \mathbf{k}[[h]]\}$ (w.r.t. the natural $\mathbf{k}((h))$-valued pairing induced by scalar extension). In particular, if $H = K^\star$ and $K = H^*$ the evaluation pairing yields $\mathbf{k}[[h]]$-module isomorphisms $H' \cong \left(K^\vee\right)^*$ and $K^\vee \cong \left(H'\right)^\star$.

Proof. — First note that, for all x, x_1, $x_2 \in H$, y, y_1, $y_2 \in K$, the elements $\left\langle \Delta(x), y_1 \otimes y_2 \right\rangle := \sum_{(x)} \langle x_{(1)}, y_1 \rangle \cdot \langle x_{(2)}, y_2 \rangle$ and $\left\langle x_1 \otimes x_2, \Delta(y) \right\rangle := \sum_{(y)} \langle x_1, y_{(1)} \rangle \cdot \langle x_2, y_{(2)} \rangle$ (cf. Definition 1.4) are well defined: in fact K acts – via π – as a Hopf subalgebra of H^*, hence $K \widetilde{\otimes} K$ acts – via $\pi \otimes \pi$ – as a Hopf subalgebra of $H^* \widetilde{\otimes} H^* = \left(K \widehat{\otimes} K\right)^*$, due to Lemma 2.1. Therefore it is perfectly meaningful to require π to be a *Hopf* pairing.

Now, scalar extension gives a Hopf pairing $\langle\ ,\ \rangle\colon {}^F\!H \times {}^F\!K \longrightarrow \mathbf{k}((h))$ which restricts to a similar pairing $\langle\ ,\ \rangle\colon H' \times K^\times \longrightarrow \mathbf{k}((h))$: we have to prove that the latter takes values in $\mathbf{k}[[h]]$, that is $\left\langle H', K^\times \right\rangle \subseteq \mathbf{k}[[h]]$, for then it will extend by continuity to a pairing $\langle\ ,\ \rangle\colon H' \times K^\vee \longrightarrow \mathbf{k}[[h]]$; in addition, this will also imply $H' \subseteq \left(K^\times\right)^\circ$.

Take $c_1, \dots, c_n \in I_K$; then $\langle 1, c_i \rangle = \epsilon(c_i) \in h\mathbf{k}[[h]]$. Now, given $y \in H'$, consider

$$\Big\langle y, \prod_{i=1}^n c_i \Big\rangle = \langle \Delta^n(y), \otimes_{i=1}^n c_i \rangle = \Big\langle \sum_{\Psi \subseteq \{1,\dots,n\}} \delta_\Psi(y), \otimes_{i=1}^n c_i \Big\rangle$$
$$= \sum_{\Psi \subseteq \{1,\dots,n\}} \left\langle \delta_\Psi(y), \otimes_{i=1}^n c_i \right\rangle$$

(using formulas in §1.5) and look at the generic summand in the last expression above. Let $|\Psi| = t$ ($t \leqslant n$): then $\left\langle \delta_\Psi(y), \otimes_{i=1}^n c_i \right\rangle = \left\langle \delta_t(y), \otimes_{i \in \Psi} c_i \right\rangle \cdot \prod_{j \not\in \Psi} \langle 1, c_j \rangle$, by definition of δ_Ψ. Thanks to the previous analysis, we have $\prod_{j \not\in \Psi} \langle 1, c_j \rangle \in h^{n-t}\mathbf{k}[[h]]$, and $\left\langle \delta_t(y), \otimes_{i \in \Psi} c_i \right\rangle \in h^t\mathbf{k}[[h]]$ because $y \in H'$; thus we get $\left\langle \delta_t(y), \otimes_{i \in \Psi} c_i \right\rangle \cdot \prod_{j \not\in \Psi} \langle 1, c_j \rangle \in h^n\mathbf{k}[[h]]$, whence $\left\langle y, \prod_{i=1}^n c_i \right\rangle \in h^n\mathbf{k}[[h]]$. The outcome is that $\langle y, \psi \rangle \in h^n\mathbf{k}[[h]]$ for all $y \in H'$, $\psi \in {I_K}^n$, and therefore $\left\langle H', h^{-n}{I_K}^n \right\rangle \subseteq \mathbf{k}[[h]]$ for all $n \in \mathbb{N}$, whence $\left\langle H', K^\times \right\rangle \subseteq \mathbf{k}[[h]]$, q.e.d.

We are now left with proving $\left(K^\times\right)^\circ \subseteq H'$: we do it by reverting the previous argument.

Let $\eta \in \left(K^\times\right)^\circ$: then $\left\langle \eta, h^{-s}{I_K}^s \right\rangle \in \mathbf{k}[[h]]$ hence $\left\langle \eta, {I_K}^s \right\rangle \in h^s\mathbf{k}[[h]]$, for all $s \in \mathbb{N}$. In particular, for $s = 0$ this gives $\langle \eta, K \rangle \in \mathbf{k}[[h]]$, whence – thanks to non-degeneracy of π_0 – we get $\eta \in H$. Let now $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I_K$; then

$$\Big\langle \delta_n(\eta), \otimes_{k=1}^n i_k \Big\rangle = \Big\langle \sum_{\Psi \subseteq \{1,\ldots,n\}} (-1)^{n-|\Psi|} \Delta_\Psi(\eta), \otimes_{k=1}^n i_k \Big\rangle$$
$$= \sum_{\Psi \subseteq \{1,\ldots,n\}} (-1)^{n-|\Psi|} \cdot \Big\langle \eta, \prod_{k \in \Psi} i_k \Big\rangle \cdot \prod_{k \notin \Psi} \langle 1, i_k \rangle$$
$$\in \sum_{\Psi \subseteq \{1,\ldots,n\}} \big\langle \eta, I_K^{|\Psi|} \big\rangle \cdot h^{n-|\Psi|} \mathbf{k}[[h]]$$
$$\subseteq \sum_{s=0}^{n} h^s \cdot h^{n-s} \mathbf{k}[[h]] = h^n \mathbf{k}[[h]],$$

therefore $\Big\langle \delta_n(\eta), I_K^{\otimes n} \Big\rangle \subseteq h^n \mathbf{k}[[h]]$. In addition, H splits as $K = \mathbf{k}[[h]] \cdot 1_K \oplus J_K$, so $K^{\otimes n}$ splits into the direct sum of $J_K^{\otimes n}$ and of other direct summands which are again tensor products but in which at least one tensor factor is $\mathbf{k}[[h]] \cdot 1_K$. Since $J_H := \mathrm{Ker}(\epsilon_H) = \big\langle \mathbf{k}[[h]] \cdot 1_K \big\rangle^{\perp} = \big\{ y \in H \,\big|\, \langle y, 1_K \rangle = 0 \big\}$ (the subspace of H orthogonal to $\big\langle \mathbf{k}[[h]] \cdot 1_K \big\rangle$), we have $\Big\langle J_H^{\otimes n}, K^{\otimes n} \Big\rangle = \Big\langle J_H^{\otimes n}, J_K^{\otimes n} \Big\rangle$. Since $\delta_n(\eta) \in J_H^{\otimes n}$ (cf. §1.5), this analysis yields $\Big\langle \delta_n(\eta), K^{\otimes n} \Big\rangle \subseteq h^n \mathbf{k}[[h]]$, whence – due to the non-degeneracy of the specialised pairing – we get $\delta_n(\eta) \in h^n H^{\otimes n}$. Therefore $\eta \in H'$; hence we get $\big(K^\times\big)^\circ \subseteq H'$, q.e.d.

For the last part of the statement, since $K^\vee$ is the h-adic completion of $K^\times$ one has $\big(K^\vee\big)^* = \big(K^\times\big)^*$, so now we show that the latter is equal to $\big(K^\times\big)^\circ = H'$. On the one hand, it is clear that $H' = \big(K^\times\big)^\circ \subseteq \big(K^\times\big)^*$. On the other hand, pick $f \in \big(K^\times\big)^*$: then f is uniquely determined by $f\big|_K$, and by construction $f\big|_K \in K^*$ and $f\big|_K \big(I_K^n\big) \subseteq h^n \mathbf{k}[[h]]$ because $f\big(h^{-n} I_K^n\big) \subseteq f\big(K^\times\big) \subseteq \mathbf{k}[[h]]$. Therefore $f\big|_K \in K^\star = \big(H^*\big)^\star = H$ (by Lemma 2.1), thus $f\big|_K \in H$ and $f\big|_K \big(K^\times\big) \subseteq \mathbf{k}[[h]]$ yields $f\big|_K \in \big(K^\times\big)^\circ = H'$, whence $f \in H'$. □

LEMMA 2.3. — *Let $H_1, H_2 \in \mathcal{HA}_{\widetilde{\otimes}}^{w-I}$. Then $\big(H_1 \widetilde{\otimes} H_2\big)^\vee = H_1^\vee \widehat{\otimes} H_2^\vee$. In particular this holds true for any $H_1, H_2 \in \mathcal{QFSHA}$.*

Proof. — Clearly $I_{H_1 \widetilde{\otimes} H_2} = I_{H_1} \widetilde{\otimes} H_2 + H_1 \widetilde{\otimes} I_{H_2}$, and the assumption on topologies implies that $H_1 \widetilde{\otimes} H_2$ is the $I_{H_1 \widetilde{\otimes} H_2}$-adic completion of

$H_1 \otimes_{\mathbf{k}[[h]]} H_2$. Then, for each $\eta_{\widetilde{\otimes}} \in H \widetilde{\otimes} H$ we can find an expression $\eta_{\widetilde{\otimes}} = \sum_{m \in \mathbb{N}} \eta_{\widetilde{\otimes}}^{(m)}$ such that $\eta_{\widetilde{\otimes}}^{(m)} \in (I_{H \otimes H})^m$ for all m; as $(I_{H \widetilde{\otimes} H})^m$ is the completion $\sum_{r+s=m} I^r \widetilde{\otimes} I^s$ of $\sum_{r+s=m} I^r \otimes I^s$, we can in fact write $\eta_{\widetilde{\otimes}} = \sum_{m \in \mathbb{N}} \sum_{r+s=m} \eta_m^{(r)} \otimes \eta_m^{(s)}$ for some $\eta_m^{(r)} \in I^r$, $\eta_m^{(s)} \in I^s$ (for all m, r, s), with $\sum_{r+s=m} \eta_m^{(r)} \otimes \eta_m^{(s)} = 0$ for all $m < n$ if $\eta_{\widetilde{\otimes}} \in {I_{H \widetilde{\otimes} H}}^n$. Thus for any $n \in \mathbb{N}$ and $\eta_{\widetilde{\otimes}} \in {I_{H \widetilde{\otimes} H}}^n$

$$\begin{aligned} h^{-n} \eta_{\widetilde{\otimes}} &= h^{-n} \sum_{m \geqslant n} \sum_{r+s=m} \eta_m^{(r)} \otimes \eta_s^{(s)} \in h^{-n} \sum_{m \geqslant n} \sum_{r+s=m} I^r \otimes I^s \\ &= \sum_{m \geqslant n} \sum_{r+s=m} h^{m-n} h^{-r} I^r \otimes h^{-s} I^s \subseteq \sum_{\ell \in \mathbb{N}} h^\ell H^\times \otimes H^\times, \end{aligned}$$

from which one argues that the natural morphism $H^\times \longrightarrow H^\vee$ induces a similar map $\big(H \widetilde{\otimes} H\big)^\vee \longrightarrow H^\vee \widehat{\otimes} H^\vee$. Conversely, $\sum_{r+s=m} {I_{H_1}}^r \otimes {I_{H_2}}^s \subseteq {I_{H_1 \widetilde{\otimes} H_2}}^m$ (for all m) implies ${H_1}^\times \otimes {H_2}^\times \subseteq \big(H_1 \widetilde{\otimes} H_2\big)^\times$, whence one gets by completion a continuous morphism ${H_1}^\vee \widehat{\otimes} {H_2}^\vee \longrightarrow \big(H_1 \widetilde{\otimes} H_2\big)^\vee$, inverse of the previous one. This gives the equality in the claim.

Finally, by Remark 1.3 (c) any $H_1, H_2 \in \mathcal{QFSHA}$ fulfills the hypotheses. $\square$

Proposition 2.4.

(a) *Let $H \in \mathcal{HA}_{\widetilde{\otimes}}$. Then $H^\vee$ is a unital (topological)* $\mathbf{k}[[h]]$*-algebra in $\mathcal{T}_{\widehat{\otimes}}$.*

(b) *Let $H \in \mathcal{HA}_{\widetilde{\otimes}}^{w-I}$. Then $H^\vee \in \mathcal{HA}_{\widehat{\otimes}}$, and the* $\mathbf{k}$*-Hopf algebra $H^\vee_0$ is cocommutative and connected; if* $\mathrm{char}(\mathbf{k}) = 0$, *it is a universal enveloping algebra, and $H^\vee \in \mathcal{QUEA}$.*

Proof.

(a) We must prove that $H^\vee$ is topologically free: by the criterion in [KT], §4.1, this is equivalent to $H^\vee$ being a torsionless, separated and complete topological $\mathbf{k}[[h]]$-module. Now, H is torsionless, so the same is true for ${}^F\!H$ hence for $H^\times$ too; as $H^\vee$ is the h-adic completion of $H^\times$, it is torsionless as well, and by definition it is complete and separated. Furthermore, by construction $H^\vee$ is a (topological) $\mathbf{k}[[h]]$-algebra, unital since $1_H \in H^\times$.

(b) Let $I := I_H$ (cf. §1.5). The definition yields $S_H(I) = I$, whence $S_H\big(h^{-n} I^n\big) = h^{-n} I^n$ for all $n \in \mathbb{N}$, so $S_H\big(H^\times\big) = H^\times$, so one can define

$S_{H^\vee}$ by continuous extension. As for Δ, the assumption on topologies implies that $\big(H\widetilde{\otimes}H\big)^\vee = H^\vee\widehat{\otimes}H^\vee$, by Lemma 2.3. Moreover, definitions yield $\Delta_H\big(I^n\big) \subseteq \sum_{r+s=n} I^r\widetilde{\otimes}I^s = {I_{H\widetilde{\otimes}H}}^n$ (for all n), hence

$$\Delta_H\big(h^{-n}I^n\big) \subseteq h^{-n}\sum_{r+s=n} I^r\widetilde{\otimes}I^s = h^{-n}{I_{H\widetilde{\otimes}H}}^n \subseteq \big(H\widetilde{\otimes}H\big)^\times$$

so that $\Delta_H\big(H^\times\big) \subseteq \big(H\widetilde{\otimes}H\big)^\times$, thus one gets $\Delta_{H^\vee}$ by continuity. Finally, by construction ϵ_H extends to a counit for $H^\vee$. It is clear that all axioms of a Hopf algebra in $\mathcal{T}_{\widehat{\otimes}}$ are then fulfilled, therefore $H^\vee \in \mathcal{HA}_{\widehat{\otimes}}$. Now, since $H^\times = \sum_{n\geqslant 0}\big(h^{-1}J_H\big)^n$, the unital topological algebra $H^\vee$ is generated by ${J_H}^\times := h^{-1}J_H$. Consider $j^\vee \in {J_H}^\times$, and $j := hj^\vee \in J_H$; then $\Delta = \delta_2 + \mathrm{id}\otimes 1 + 1\otimes\mathrm{id} - \epsilon\cdot 1\otimes 1$ and $\mathrm{Im}(\delta_2) \subseteq J_H\otimes J_H$ (cf. §1.5) give

$$\Delta(j) = \delta_2(j) + j\otimes 1 + 1\otimes j - \epsilon(j)\cdot 1\otimes 1 \in j\otimes 1 + 1\otimes j + J_H\widetilde{\otimes}J_H.$$

Therefore

$$\Delta\big(j^\vee\big) = \delta_2\big(j^\vee\big) + j^\vee\otimes 1 + 1\otimes j^\vee - \epsilon\big(j^\vee\big)1\otimes 1 = j^\vee\otimes 1 + 1\otimes j^\vee + \delta_2\big(j^\vee\big)$$

which maps (through completion) into

$$j^\vee\otimes 1 + 1\otimes j^\vee + h^{-1}J_H\widetilde{\otimes}J_H = j^\vee\otimes 1 + 1\otimes j^\vee + h^{+1}{J_H}^\vee\widehat{\otimes}{J_H}^\vee,$$

whence we conclude that

$$\Delta_{H^\vee}\big(j^\vee\big) \equiv j^\vee\otimes 1 + 1\otimes j^\vee \mod hH^\vee\widehat{\otimes}H^\vee, \quad \forall j^\vee \in {J_H}^\vee.$$

Thus ${J_H}^\vee \bmod hH^\vee$ is contained in $P\big(H^\vee\big)$, the set of *primitive elements* of ${H^\vee}_0$; since ${J_H}^\vee \bmod hH^\vee$ generates ${H^\vee}_0$ – as ${J_H}^\vee$ generates $H^\vee$ – this proves *a fortiori* that $P\big(H^\vee\big)$ generates ${H^\vee}_0$, and also shows that ${H^\vee}_0$ is cocommutative. In addition, we can also apply Lemma 5.5.1 in [M] to the Hopf algebra ${H^\vee}_0$, with $A_0 = \mathbf{k}\cdot 1$ and $A_1 = {J_H}^\vee \bmod hH^\vee$, to argue that ${H^\vee}_0$ is connected, q.e.d.

If $\mathrm{char}(\mathbf{k}) = 0$ by Kostant's Theorem (cf. for instance [A], Theorem 2.4.3) we have ${H^\vee}_0 = U(\mathfrak{g})$ for the Lie (bi)algebra $\mathfrak{g} = P\big({H^\vee}_0\big)$. We conclude that $H \in \mathcal{QUEA}$. □

LEMMA 2.5 ([KT], Lemma 3.2). — *Let H be a Hopf $\mathbf{k}[[h]]$-algebra, let $a,\ b \in H$, and let Φ be a finite subset of $\mathbb{N}$. Then $\delta_\Phi(ab) = \sum_{\Lambda\cup Y=\Phi}\delta_\Lambda(a)\delta_Y(b)$. In addition, if $\Phi \neq \emptyset$ then*

$$\delta_\Phi(ab - ba) = \sum_{\substack{\Lambda\cup Y=\Phi \\ \Lambda\cap Y\neq\emptyset}} \big(\delta_\Lambda(a)\delta_Y(b) - \delta_Y(b)\delta_\Lambda(a)\big). \qquad \square$$

PROPOSITION 2.6. — *Let $H \in \mathcal{HA}_{\widehat{\otimes}}$. Then $H' \in \mathcal{HA}_{\widetilde{\otimes}}$, and the $\mathbf{k}$-Hopf algebra ${H'}_0$ is commutative.*

Proof. — First, H' is a $\mathbf{k}[[h]]$-submodule of H, for the maps δ_n ($n \in \mathbb{N}$) are $\mathbf{k}[[h]]$-linear; to see it lies in $\mathcal{P}_{\widetilde{\otimes}}$, we resort to a duality argument. Let $K := H^* \in \mathcal{HA}_{\widetilde{\otimes}}$, so $H = K^\star$ (cf. Lemma 2.1), and let $\pi \colon H \times K \longrightarrow \mathbf{k}[[h]]$ be the natural Hopf pairing given by evaluation. Then Proposition 2.2 gives $H' \cong \big(K^\vee\big)^* \in \mathcal{P}_{\widetilde{\otimes}}$, thus since $K^\vee$ is a unital algebra we have that H' is a counital coalgebra in $\mathcal{P}_{\widetilde{\otimes}}$, with $\Delta_{H'} = \Delta_H\big|_{H'}$ and $\epsilon_{H'} = \epsilon_H\big|_{H'}$. In addition, by Lemma 2.5 one easily sees that H' is a $\mathbf{k}[[h]]$-subalgebra of H, and by construction it is unital for $1_H \in H'$. The outcome is $H' \in \mathcal{HA}_{\widetilde{\otimes}}$, q.e.d.

Finally, the very definitions give $x = \delta_1(x) + \epsilon(x)$ for all $x \in H$. If $x \in H'$ we have $\delta_1(x) \in hH$, hence there exists $x_1 \in H$ such that $\delta_1(x) = hx_1$. Now for $a, b \in H'$, write $a = ha_1 + \epsilon(a)$, $b = hb_1 + \epsilon(b)$, hence $ab - ba = hc$ with $c = h(a_1b_1 - b_1a_1)$; we show that $c \in H'$. For this we must check that $\delta_\Phi(c)$ is divisible by $h^{|\Phi|}$ for any finite subset Φ of $\mathbb{N}_+$: as multiplication by h is injective (for H is topologically free), it is enough to show that $\delta_\Phi(ab - ba)$ is divisible by $h^{|\Phi|+1}$. Let Λ and Y be subsets of Φ such that $\Lambda \cup Y = \Phi$ and $\Lambda \cap Y \neq \emptyset$: then $|\Lambda| + |Y| \geqslant |\Phi| + 1$. Now, $\delta_\Lambda(a)$ is divisible by $h^{|\Lambda|}$ and $\delta_Y(b)$ is divisible by $h^{|Y|}$: from this and the second part of Lemma 2.5 it follows that $\delta_\Phi(ab - ba)$ is divisible by $h^{|\Phi|+1}$. The outcome is $ab \equiv ba \mod hH'$, so ${H'}_0$ is commutative. $\square$

LEMMA 2.7. — *Let H_1, $H_2 \in \mathcal{QUEA}$. Then $\big(H_1 \widehat{\otimes} H_2\big)' = {H_1}' \widetilde{\otimes} {H_2}'$.*

Proof. — Proceeding as in the proof of Proposition 2.6, let $K_i := {H_i}^* \in \mathcal{QFSHA}$ ($i = 1, 2$); then $K_1 \widetilde{\otimes} K_2 = {H_1}^* \widetilde{\otimes} {H_2}^* = \big(H_1 \widehat{\otimes} H_2\big)^*$ (by Lemma 2.1), and ${H_i}' = \big({K_i}^\vee\big)^*$ ($i = 1, 2$), and similarly $\big(H_1 \widehat{\otimes} H_2\big)' = \Big(\big(K_1 \widetilde{\otimes} K_2\big)^\vee\Big)^*$. Then applying Lemma 2.3 we get

$$\begin{aligned}\big(H_1 \widehat{\otimes} H_2\big)' &= \Big(\big(K_1 \widetilde{\otimes} K_2\big)^\vee\Big)^* = \big({K_1}^\vee \widehat{\otimes} {K_2}^\vee\big)^* \\ &= \big({K_1}^\vee\big)^* \widetilde{\otimes} \big({K_2}^\vee\big)^* = {H_1}' \widetilde{\otimes} {H_2}'.\end{aligned}$$ $\square$

LEMMA 2.8. — *The assignment $H \mapsto H^\vee$, resp. $H \mapsto H'$, gives a well-defined functor $\mathcal{HA}_{\widetilde{\otimes}}^{w-I} \longrightarrow \mathcal{HA}_{\widehat{\otimes}}$, resp. $\mathcal{HA}_{\widehat{\otimes}} \longrightarrow \mathcal{HA}_{\widetilde{\otimes}}$.*

Proof. — In order to define the functors, we only have to set them on morphisms. Let $H, K \in \mathcal{HA}_{\widetilde{\otimes}}^{w-I}$ and $\phi \in \mathrm{Mor}_{\mathcal{HA}_{\widetilde{\otimes}}^{w-I}}(H, K)$; by scalar extension it gives a morphism ${}^F\!H \to {}^F\!K$ of $\mathbf{k}((h))$-Hopf algebras, which maps $h^{-1}I_H$ into $h^{-1}I_K$, hence $H^\times$ into $K^\times$: extending it by continuity we get the morphism $\phi^\vee \in \mathrm{Mor}_{\mathcal{HA}^{\frown}_{\otimes}}\big(H^\vee, K^\vee\big)$ we were looking for. Similarly, let $H, K \in \mathcal{HA}_{\widehat{\otimes}}$ and $\varphi \in \mathrm{Mor}_{\mathcal{HA}^{\frown}_{\otimes}}(H,K)$: then $\delta_n \circ \varphi = \varphi^{\otimes n} \circ \delta_n$ (for all $n \in \mathbb{N}$), so $\varphi\big(H'\big) \subseteq K'$: thus as $\varphi' \in \mathrm{Mor}_{\mathcal{HA}^{\sim}_{\otimes}}\big(H', K'\big)$ we simply take $\varphi\big|_{H'}$. $\square$

3. Drinfeld's functors on quantum groups.

We focus now on the effect of Drinfeld's functors on quantum groups. The first result is an explicit description of $F_h[[\mathfrak{g}]]^\vee$ when $F_h[[\mathfrak{g}]]$ is a QFSHA.

3.1. **An explicit description of $F_h[[\mathfrak{g}]]^\vee$.** — Let $F_h[[\mathfrak{g}]] \in \mathcal{QFSHA}$, and set for simplicity $F_h := F_h[[\mathfrak{g}]]$, $F_0 := F_h \big/ hF_h = F[[\mathfrak{g}]]$, $F_h^\vee := F_h[[\mathfrak{g}]]^\vee$, and $F_0^\vee := F_h^\vee \big/ hF_h^\vee$. Then $F_0 \cong F[[\mathfrak{g}]] = \mathbf{k}[[\bar{x}_1, \dots, \bar{x}_n]]$ (for some $n \in \mathbb{N}$) as topological $\mathbf{k}$-algebras. Letting $\pi \colon F_h \twoheadrightarrow F_0$ be the natural projection, if we pick an $x_j \in \pi^{-1}(\bar{x}_j)$ for any j, then F_h is generated by $\{x_1, \dots, x_n\}$ as a topological $\mathbf{k}[[h]]$-algebra, that is to say $F_h = F_h[[\mathfrak{g}]] = \mathbf{k}[[x_1, \dots, x_n, h]]$. In this description we have $I := I_{F_h} = \big(x_1, \dots, x_n, h\big)$, and I^ℓ identifies with the space of all formal series whose degree (that is, the degree of the lowest degree monomials occurring in the series with non-zero coefficient) is at least ℓ, that is

$$I^\ell = \Big\{ f = \sum_{\mathbf{d} \in \mathbb{N}^{n+1}} c_{\mathbf{d}} \cdot h^{d_0} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} \,\Big|\, c_{\mathbf{d}} \in \mathbf{k}, \mathbf{d} \in \mathbb{N}^{n+1}, |\mathbf{d}| \geqslant \ell \Big\}$$

for all $\ell \in \mathbb{N}$ (hereafter, we set $|\mathbf{d}| := \sum_{s=0}^{n} d_s$ for any $\mathbf{d} = (d_0, d_1, \dots, d_n) \in \mathbb{N}^{n+1}$). Then ${}^{F}\!F_h \cong \mathbf{k}[[x_1, \dots, x_n]]((h))$, and

$$h^{-\ell} I^\ell = \Big\{ f = \sum_{\mathbf{d} \in \mathbb{N}^{n+1}} P_n(\underline{\check{x}}) \cdot h^n \,\Big|\, P_n(\underline{X}) \in \mathbf{k}[X_1, \dots, X_n], \\ n - \partial_{\mathbf{X}}(P_n) \geqslant \ell \,\forall n \in \mathbb{N} \Big\}$$

where $\check{x}_j := h^{-1} x_j$ and $\partial_{\mathbf{X}}(f)$ denotes the degree of a polynomial or a series f in the X_j $(j = 1, \dots, n)$, whence we get

$$F_h^\times = \bigcup_{\ell \in \mathbb{N}} h^{-\ell} I^\ell = \Big\{ f = \sum_{\mathbf{d} \in \mathbb{N}^{n+1}} P_n(\check{x}_1, \dots, \check{x}_n) \cdot h^n \,\Big| \\ \Big|\, P_n(X_1, \dots, X_n) \in \mathbf{k}[X_1, \dots, X_n], \exists \ell_f \in \mathbb{N} : n - \partial_{\mathbf{X}}(P_n) \geqslant \ell_f \forall n \in \mathbb{N} \Big\}.$$

Moreover, we easily see that $\bigcap_{\ell \in \mathbb{N}} h^\ell F_h^\times = \{0\}$, hence the natural completion map $F_h^\times \longrightarrow F_h^\vee$ is an embedding. Finally, when taking the h-adic completion we get

$$F_h^\vee = \Big\{ f = \sum_{\mathbf{d}\in\mathbb{N}^{n+1}} P_n(\check{x}_1,\dots,\check{x}_n)\cdot h^n \Big| \\ \Big| P_n(X_1,\dots,X_n) \in \mathbf{k}[X_1,\dots,X_n] \forall n \in \mathbb{N} \Big\}$$

that is $F_h^\vee = \mathbf{k}[\check{x}_1,\dots,\check{x}_n][[h]]$ as topological $\mathbf{k}[[h]]$-modules.

PROPOSITION 3.2. — *If $F_h[[\mathfrak{g}]] \in \mathcal{QFSHA}$, then $F_h[[\mathfrak{g}]]^\vee \in \mathcal{QUEA}$. Namely, we have $F_h[[\mathfrak{g}]]^\vee = U_h(\mathfrak{g}^*)$ (where $\mathfrak{g}^*$ is the dual Lie bialgebra to $\mathfrak{g}$), that is*

$$F_h[[\mathfrak{g}]]^\vee \Big/ hF_h[[\mathfrak{g}]]^\vee = U(\mathfrak{g}^*).$$

Proof. — Let $F_h[[\mathfrak{g}]] \in \mathcal{QFSHA}$; set for simplicity $F_h := F_h[[\mathfrak{g}]]$, $F_0 := F_h/hF_h = F[[\mathfrak{g}]]$, $F_h^\vee := F_h[[\mathfrak{g}]]^\vee$, $F_0^\vee := F_h^\vee/hF_h^\vee$, and let $\pi\colon F_h \twoheadrightarrow F_0$ be the natural projection.

From the discussion in §3.1, we recover the identification $F_h = \mathbf{k}[[x_1,\dots,x_n,h]]$ (for some $n \in \mathbb{N}$) as topological $\mathbf{k}[[h]]$-modules, where $x_j \in F_h$ for all j and the $\bar{x}_j = \pi(x_j)$ gives $F_0 = \mathbf{k}[[\bar{x}_1,\dots,\bar{x}_n]]$ and generate $\mathfrak{m} := \mathrm{Ker}(\epsilon_{\mathfrak{F}[[\mathfrak{g}]]})$. Taking if necessary $x_j - \epsilon(x_j)$ instead of x_j (for any j), we can assume in addition that the x_j belong to $J := \mathrm{Ker}(\epsilon_{F_h})$, so this kernel is the set of all formal series f whose degree in the x_j, call it $\partial_{\mathbf{x}}(f)$, is positive. From §3.1 we also have $F_h^\vee = \mathbf{k}[\check{x}_1,\dots,\check{x}_n][[h]]$ as topological $\mathbf{k}[[h]]$-modules.

Since F_0 is commutative, we have $x_i x_j - x_j x_i = h\chi$ for some $\chi \in F_h$, and in addition we must have $\chi \in J$ too, thus $\chi = \sum_{j=0}^n c_j(h)\cdot x_j + f(x_1,\dots,x_n,h)$ where $c_j(h) \in \mathbf{k}[[h]]$ for all j and $f(x_1,\dots,x_n,h) \in \mathbf{k}[[x_1,\dots,x_n,h]]$ with $\partial_{\mathbf{x}}(f) > 1$. Then

$$\check{x}_i\check{x}_j - \check{x}_j\check{x}_i = h^{-2}\cdot h\chi = \sum_{j=0}^n c_j(h)\cdot\check{x}_j + h^{-1}\check{f}(\check{x}_1,\dots,\check{x}_n,h),$$

where $\check{f}(\check{x}_1,\dots,\check{x}_n,h) \in k[\check{x}_1,\dots,\check{x}_n][[h]]$ is formally obtained from $f(x_1,\dots,x_n,h)$ simply by rewriting $x_j = h\check{x}_j$ for all j. Then since $\partial_{\mathbf{x}}(f) > 1$ we have $h^{-1}\check{f}(\check{x}_1,\dots,\check{x}_n,h) \in h\mathbf{k}[\check{x}_1,\dots,\check{x}_n][[h]]$, whence

$$\check{x}_i\check{x}_j - \check{x}_j\check{x}_i \equiv \sum_{j=0}^n c_j(h)\cdot\check{x}_j \mod hF_h^\vee.$$

This shows that *the $\mathbf{k}$-span of the set of cosets $\{\check{x}_j \bmod hF_h^\vee\}_{j=1,\dots,n}$ is a Lie algebra*, which we call $\mathfrak{h}$. Then the identification $F_h^\vee = \mathbf{k}[\check{x}_1,\dots,\check{x}_n][[h]]$ shows that $F_0^\vee = U(\mathfrak{h})$, so that $F_h^\vee \in \mathcal{QUEA}$, q.e.d.

Our purpose now is to prove that $\mathfrak{h} \cong \mathfrak{g}^*$ as Lie bialgebras. For this we have to improve a bit the previous analysis. Recall that(1) $\mathfrak{g} := \left(\mathfrak{m}/\mathfrak{m}^2\right)^*$, that $\mathfrak{m}$ (the unique maximal ideal of $F[[\mathfrak{g}]]$) is closed under the Poisson bracket of $F[[\mathfrak{g}]]$, and that the dual Lie bialgebra $\mathfrak{g}^*$ can be realized as $\mathfrak{g}^* = \mathfrak{m}/\mathfrak{m}^2$, its Lie bracket being induced by the Poisson bracket.

Consider $J^\vee := h^{-1}J \subset F_h^\vee$. Multiplication by h^{-1} yields a $\mathbf{k}[[h]]$-module isomorphism $\mu: J \overset{\cong}{\hookrightarrow\!\!\!\to} J^\vee$. Furthermore, the specialisation map $\pi^\vee: F_h^\vee \twoheadrightarrow F_0^\vee = U(\mathfrak{h})$ restricts to a similar map $\eta: J^\vee \twoheadrightarrow \mathcal{J}_0^\vee := J^\vee \big/ J^\vee \cap \left(hF_h^\vee\right)$. The latter has kernel $J^\vee \cap \left(hF_h^\vee\right)$: we contend that this is equal to $\left(J + J^\vee J\right)$. In fact, let $y \in J^\vee \cap hF_h^\vee$: then the series $\gamma := hy \in J$ has $\partial_{\mathbf{x}}(\gamma) > 0$. As above we write $y = h^{-1}\gamma$ as $y = h^{-1}\gamma \in F_h^\vee = \mathbf{k}[\check{x}_1, \dots, \check{x}_n][[h]]$: then $y = h^{-1}\gamma \in h\mathbf{k}[\check{x}_1, \dots, \check{x}_n][[h]]$ means $\partial_{\mathbf{x}}(\gamma) > 1$, or $\partial_{\mathbf{x}}(\gamma) = 1$ *and* $\partial_h(\gamma) > 0$ (i.e. $\gamma \in h\mathbf{k}[[x_1, \dots, x_n, h]]$), i.e. exactly $\gamma \in hJ + J^2$, so $y \in J + J^\vee J$, which proves our claim true. Note also that $\eta\left(J^\vee\right) = \eta\left(\oplus_{j=1}^n \mathbf{k}[[h]] \cdot x_j\right) = \oplus_{j=1}^n \mathbf{k} \cdot \check{x}_j = \mathfrak{h}$.

Now, recall that $\mathfrak{g}^* = \mathfrak{m}/\mathfrak{m}^2$: we fix a $\mathbf{k}$-linear section $\nu: \mathfrak{g}^* \hookrightarrow \mathfrak{m}$ of the projection $\rho: \mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}^*$ such that $\gamma(\mathfrak{m}^2) \subseteq \mathfrak{h}\mathfrak{J} + \mathfrak{J}^2$. Moreover, the specialisation map $\pi: F_h \twoheadrightarrow F_0$ restricts to $\pi': J \twoheadrightarrow J\big/(J \cap hF_h) = J\big/hJ = \mathfrak{m}$; we fix a $\mathbf{k}$-linear section $\gamma: \mathfrak{m} \hookrightarrow \mathfrak{J}$ of π'. Now consider the composition map $\sigma := \eta \circ \mu \circ \gamma \circ \nu: \mathfrak{g}^* \longrightarrow \mathfrak{h}$. This is well-defined, i.e. it is independent of the choice of ν and γ. Indeed, if $\nu, \nu': \mathfrak{g}^* \hookrightarrow J_F$ are two sections of ρ, and σ, σ' are defined correspondingly (with the same fixed γ for both), then $\mathrm{Im}(\nu - \nu') \subseteq \mathrm{Ker}(\rho) = {J_F}^2 \subseteq \mathrm{Ker}(\eta \circ \mu \circ \gamma)$, so that $\sigma = \eta \circ \mu \circ \gamma \circ \nu = \eta \circ \mu \circ \gamma \circ \nu' = \sigma'$. Similarly, if $\gamma, \gamma': J_F \hookrightarrow J$ are two sections of π', and σ, σ' are defined correspondingly (with the same ν for both), we have $\mathrm{Im}(\gamma - \gamma') \subseteq \mathrm{Ker}(\pi') = hJ \subseteq hJ + J^2 = \mathrm{Ker}(\eta \circ \mu)$, thus $\sigma = \eta \circ \mu \circ \gamma \circ \nu = \eta \circ \mu \circ \gamma' \circ \nu = \sigma'$, q.e.d. In a nutshell, σ is the composition map

$$\mathfrak{g}^* \xhookrightarrow{\ \bar{\nu}\ } J_F\big/{J_F}^2 \xhookrightarrow{\ \bar{\gamma}\ } J\big/\left(J^2 + hJ\right) \xhookrightarrow{\ \bar{\mu}\ } J^\vee\big/\left(J + J^\vee J\right) \xhookrightarrow{\ \bar{\eta}\ } \mathfrak{h}$$

where the maps $\bar{\nu}$, $\bar{\gamma}$, $\bar{\mu}$, $\bar{\eta}$, are the ones canonically induced by ν, γ, μ, η, and $\bar{\nu}$, *resp.* $\bar{\gamma}$, *does not depend on the choice of* ν, *resp.* γ, as it is the inverse of the isomorphism $\bar{\rho}: J_F/{J_F}^2 \overset{\cong}{\hookrightarrow\!\!\!\to} \mathfrak{g}^*$, resp. $\overline{\pi'}: J\big/\left(J^2 + hJ\right) \overset{\cong}{\hookrightarrow\!\!\!\to} J_F/{J_F}^2$, induced by ρ, resp. by π'. We use this remark to show that σ is also an isomorphism of the Lie bialgebra structure.

(1) Hereafter, the product of ideals in a topological algebra will be understood as the *closure* of their algebraic product.

Using the vector space isomorphism $\sigma\colon \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{h}$ we pull-back the Lie bialgebra structure of $\mathfrak{h}$ onto $\mathfrak{g}^*$, and denote it by $\big(\mathfrak{g}^*, [\ ,\ ]_\bullet, \delta_\bullet\big)$; on the other hand, $\mathfrak{g}^*$ also carries its natural structure of Lie bialgebra, dual to that of $\mathfrak{g}$ (e.g., the Lie bracket is induced by restriction of $\{\ ,\ \}$), denoted by $\big(\mathfrak{g}^*, [\ ,\ ]_\times, \delta_\times\big)$: we must prove that these two structures coincide.

First, *for all x_1, $x_2 \in \mathfrak{g}^*$ we have $\big[x_1, x_2\big]_\bullet = \big[x_1, x_2\big]_\times$.*

Indeed, let $f_i := \nu(x_i)$, $\varphi_i := \gamma(f_i)$, $\varphi_i^\vee := \mu(\varphi_i)$, $y_i := \eta\big(\varphi_i^\vee\big)$ $(i = 1, 2)$. Then

$$\begin{aligned}
\big[x_1, x_2\big]_\bullet &:= \sigma^{-1}\Big(\big[\sigma(x_1), \sigma(x_2)\big]_{\mathfrak{h}}\Big) = \sigma^{-1}\big([y_1, y_2]\big) \\
&= \big(\rho \circ \pi' \circ \mu^{-1}\big)\Big(\big[\varphi_1^\vee, \varphi_2^\vee\big]\Big) \\
&= \big(\rho \circ \pi'\big)\big(h^{-1}[\varphi_1, \varphi_2]\big) = \rho\big(\{f_1, f_2\}\big) \\
&=: \big[x_1, x_2\big]_\times . \qquad\qquad \text{q.e.d.}
\end{aligned}$$

The case of cobrackets can be treated similarly; but since they take values in tensor squares, we make use of suitable maps $\nu_\otimes := \nu^{\otimes 2}$, $\gamma_\otimes := \gamma^{\otimes 2}$, etc; we set also $\chi_\otimes := \eta_\otimes \circ \mu_\otimes = (\eta \circ \mu)^{\otimes 2}$ and $\nabla := \Delta - \Delta^{\mathrm{op}}$. Then *for all $x \in \mathfrak{g}^*$ we have $\delta_\bullet(x) = \delta_\times(x)$.*

Indeed, let $f := \nu(x)$, $\varphi := \gamma(f)$, $\varphi^\vee := \mu(\varphi)$, $y := \eta\big(\varphi^\vee\big)$. Then we have

$$\begin{aligned}
\delta_\bullet(x) &:= {\sigma_\otimes}^{-1}\big(\delta_{\mathfrak{h}}(\sigma(x))\big) = {\sigma_\otimes}^{-1}\big(\delta_{\mathfrak{h}}\big(\eta\big(\varphi^\vee\big)\big)\big) \\
&= \sigma_\otimes^{-1}\Big(\eta_\otimes\big(h^{-1}\nabla\big(\varphi^\vee\big)\big)\Big) = \sigma_\otimes^{-1}\Big(\big(\eta \circ \mu\big)_\otimes\big(\nabla(\varphi)\big)\Big) \\
&= \sigma_\otimes^{-1}\Big(\big(\eta \circ \mu \circ \gamma\big)_\otimes\big(\nabla(f)\big)\Big) = \rho\big(\nabla(f)\big) = \rho\big(\nabla(\nu(x))\big) = \delta_\times(x)
\end{aligned}$$

where the last equality holds because $\delta_\times(x)$ is uniquely defined as the unique element in $\mathfrak{g}^* \otimes \mathfrak{g}^*$ such that $\big\langle u_1 \otimes u_2, \delta_\times(x)\big\rangle = \big\langle [u_1, u_2], x\big\rangle$ for all $u_1, u_2 \in \mathfrak{g}$, and we have

$$\big\langle [u_1, u_2], x\big\rangle = \big\langle [u_1, u_2], \rho(f)\big\rangle = \big\langle u_1 \otimes u_2, \nabla(f)\big\rangle = \big\langle u_1 \otimes u_2, \rho\big(\nabla(\nu(x))\big)\big\rangle .$$

□

Now we need one more technical lemma. From now on, if $\mathfrak{g}$ is any Lie algebra and $\bar{x} \in U(\mathfrak{g})$, we denote by $\partial(\bar{x})$ the degree of $\bar{x}$ w.r.t. the standard filtration of $U(\mathfrak{g})$.

LEMMA 3.3. — *Let U_h be a QUEA, let $x' \in {U_h}'$, and let $x \in U_h \setminus hU_h$, $n \in \mathbb{N}$, be such that $x' = h^n x$. Set $\bar{x} := x \bmod hU_h \in \big(U_h\big)_0$. Then $\partial(\bar{x}) \leqslant n$.*

Proof (cf. [EK], Lemma 4.12). — By hypothesis $\delta_{n+1}(x') \in h^{n+1} {U_h}^{\otimes(n+1)}$, hence $\delta_{n+1}(x) \in h{U_h}^{\otimes(n+1)}$, therefore $\delta_{n+1}(\bar{x}) = 0$, i.e. $\bar{x} \in \mathrm{Ker}\big(\delta_{n+1}\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes(n+1)}\big)$, where $\mathfrak{g}$ is the Lie bialgebra such that $\big(U_h\big)_0 := U_h \big/ hU_h = U(\mathfrak{g})$. But the latter kernel equals the subspace $U(\mathfrak{g})_n := \big\{\bar{y} \in U(\mathfrak{g}) \,\big|\, \partial(\bar{y}) \leqslant n\big\}$ (cf. [KT], §3.8), whence the claim follows. □

PROPOSITION 3.4. — *Let F_h be a QFSHA. Then $\big({F_h}^\vee\big)' = F_h$.*

Proof. — As a matter of notation, we set ${J_{F_h}}^\vee := h^{-1} J_{F_h}$, and we denote by $\bar{x} \in {F_0}^\vee$ the image of any $x \in {F_h}^\vee$ inside ${F_h}^\vee \big/ h{F_h}^\vee = {F_0}^\vee$.

Now, for any $n \in \mathbb{N}$, we have $\delta_n(F_h) \subseteq {J_{F_h}}^{\otimes n}$ (see §1.5); this can be read as $\delta_n(F_h) \subseteq {J_{F_h}}^{\otimes n} = h^n \big(h^{-1} J_{F_h}\big)^{\otimes n} \subseteq h^n \big({F_h}^\times\big)^{\otimes n} \subseteq h^n \big({F_h}^\vee\big)^{\otimes n}$, which gives $F_h \subseteq \big({F_h}^\vee\big)'$.

Conversely, let $x' \in \big({F_h}^\vee\big)' \setminus \{0\}$ be given; as ${F_h}^\vee \in \mathcal{T}_{\widehat{\otimes}}$, there are (unique) $n \in \mathbb{N}$, $x \in {F_h}^\vee \setminus h{F_h}^\vee$, such that $x' = h^n x$. By Proposition 3.2, ${F_h}^\vee$ is a QUEA, with semiclassical limit $U(\mathfrak{h})$ where $\mathfrak{h} = \mathfrak{g}^*$ if $F_0 = F[[\mathfrak{g}]]$. Fix an ordered basis $\{b_\lambda\}_{\lambda \in \Lambda}$ of $\mathfrak{h}$ and a subset $\{x_\lambda\}_{\lambda \in \Lambda}$ of ${F_h}^\vee$ such that $\overline{x}_\lambda = b_\lambda$ for all λ; in particular, since $\mathfrak{h} \subset \mathrm{Ker}(\epsilon_{U(\mathfrak{g})})$ we can choose the x_λ inside ${J_{F_h}}^\vee := h^{-1} J_{F_h}$: so $x_\lambda = h^{-1} x'_\lambda$ for some $x'_\lambda \in J_{F_h}$, for all $\lambda \in \Lambda$.

Since Lemma 3.3 gives $\partial(\bar{x}) \leqslant n$, that is $\bar{x} \in U(\mathfrak{g})_n := \big\{\bar{y} \in U(\mathfrak{g}) \,\big|\, \partial(\bar{y}) \leqslant n\big\}$, by the PBW theorem we can write $\bar{x}$ as a polynomial $P\big(\{b_\lambda\}_{\lambda \in \Lambda}\big)$ in the b_λ of degree $d \leqslant n$ (with coefficients in $\mathbf{k}$); then $x_0 := P\big(\{x_\lambda\}_{\lambda \in \Lambda}\big) \equiv x \mod h{F_h}^\vee$, that is $x = P\big(\{x_\lambda\}_{\lambda \in \Lambda}\big) + h x_{\langle 1 \rangle}$ for some $x_{\langle 1 \rangle} \in H^\vee$. Now we can write $x_0 := P\big(\{x_\lambda\}_{\lambda \in \Lambda}\big) = \sum_{s=0}^{d} h^{-s} j_s (\in {F_h}^\times)$, where every $j_s \in {J_{F_h}}^s$ is a homogeneous polynomial in the x'_λ of degree s, and $j_d \neq 0$; but then $h^n x_0 = \sum_{s=0}^{d} h^{n-s} j_s \in F_h$ because $d \leqslant n$. Since $F_h \subseteq \big({F_h}^\vee\big)'$ – thanks to the first part of the proof – we get also $h^{n+1} x_{\langle 1 \rangle} = h^n (x - x_0) = x' - h^n x_0 \in \big({F_h}^\vee\big)'$: thus

$$x' = h^n x_0 + h^{n+1} x_{\langle 1 \rangle}, \quad \text{with} \quad h^n x_0 \in F_h \quad \text{and} \quad h^{n+1} x_{\langle 1 \rangle} \in \big({F_h}^\vee\big)'.$$

If $x_{(1)} := h^{n+1} x_{\langle 1 \rangle}$ is zero we are done; if not, we can repeat the argument for $x_{(1)}$ in the role of $x_{(0)} := x'$: this will provide us with an $x_1 \in {F_h}^\times$ and an $x_{\langle 2 \rangle} \in {F_h}^\vee$ such that $x_{(1)} = h^{n+1} x_1 + h^{n+2} x_{\langle 2 \rangle}$, with $h^{n+1} x_1 \in F_h$ and $h^{n+2} x_{\langle 2 \rangle} \in \big({F_h}^\vee\big)'$. Iterating, we eventually find a sequence $\{x_\ell\}_{\ell \in \mathbb{N}} \subset {F_h}^\times$ such that $h^{n+\ell} x_\ell \in F_h$ for all $\ell \in \mathbb{N}$, and $x' = \sum_{\ell=0}^{+\infty} h^{n+\ell} x_\ell$, in the sense that the right-hand-side series does converge to x' inside ${F_h}^\vee$. Furthermore, this convergence takes place inside F_h as well: indeed, the very construction gives $h^{n+\ell} x_\ell = h^{n+\ell} P_{d_\ell}\big(\{x_\lambda\}_{\lambda \in \Lambda}\big) =$

$h^{n+\ell} P_{d_\ell}\big(\{h^{-1}x'_\lambda\}_{\lambda\in\Lambda}\big)$ (where P_{d_ℓ} is a suitable polynomial of degree $d_\ell \leqslant n+\ell$) and this last element belongs to ${I_{F_h}}^{n+\ell}$: but F_h is a QFSHA, hence it is complete w.r.t. the I_{F_h}-adic topology, so the series $x' = \sum_{\ell=0}^{+\infty} h^{n+\ell} x_\ell$ does converge (to x') inside F_h. □

3.5. **An explicit description of $U_h(\mathfrak{g})'$ (for $\mathrm{char}(\mathbf{k})=0$).** — When $\mathrm{char}(\mathbf{k}) = 0$, for any $U_h(\mathfrak{g}) \in \mathcal{QUEA}$ we can give an explicit description of $U_h(\mathfrak{g})'$, as follows.

Like in the proof of Proposition 2.6 consider $F_h[[\mathfrak{g}]] := U_h(\mathfrak{g})^* \in \mathcal{HA}_{\widetilde{\otimes}}$ and its natural Hopf pairing with $U_h(\mathfrak{g})$: then we showed that $U_h(\mathfrak{g})' = \Big(\big(F_h[[\mathfrak{g}]]\big)^\vee\Big)^*$. Note that this time we have in addition $F_h[[\mathfrak{g}]] \in \mathcal{QFSHA}$, with $F_0[[\mathfrak{g}]] := F_h[[\mathfrak{g}]]\big/hF_h[[\mathfrak{g}]] = U(\mathfrak{g})^* = F[[\mathfrak{g}]]$.

Pick any basis $\{\overline{x}_i\}_{i\in\mathcal{I}}$ of $\mathfrak{g}$, endowed with some total order; then (PBW theorem) the set of ordered monomials $\{\overline{x}^{\underline{e}}\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is a basis of $U(\mathfrak{g})$: hereafter, $\big(\mathbb{N}^{\mathcal{I}}\big)_0$ denotes the set of functions from $\mathcal{I}$ to $\mathbb{N}$ with finite support, and $\overline{x}^{\underline{e}} := \prod_{\imath\in\mathcal{I}} \overline{x}_i^{\underline{e}(i)}$ for all $\underline{e} \in \big(\mathbb{N}^{\mathcal{I}}\big)_0$ and all indeterminates $x_1, \dots, x_n$. Let $\{\overline{y}_i\}_{i\in\mathcal{I}}$ be the pseudobasis([2]) of $\mathfrak{g}^*$ dual to $\{\overline{x}_i\}_{\imath\in\mathcal{I}}$, endowed with the same total order; then the set of "rescaled" ordered monomials $\{\overline{y}^{\underline{e}}/\underline{e}!\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ (with $\underline{e}! := \prod_{i\in\mathcal{I}} \underline{e}(i)!$; that's where we need $\mathrm{char}(\mathbf{k}) = 0$) is the pseudobasis of $U(\mathfrak{g})^* = F_h[[\mathfrak{g}]]$ dual to the PBW basis $\{\overline{x}^{\underline{e}}\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ of $U(\mathfrak{g})$, namely $\big\langle \overline{x}^{\underline{e}}, \overline{y}^{\underline{e}'}/\underline{e}!\big\rangle = \delta_{\underline{e},\underline{e}'}$ for all $\underline{e}$, $\underline{e}' \in \big(\mathbb{N}^{\mathcal{I}}\big)_0$.

Lift $\{\overline{x}_i\}_{i\in\mathcal{I}}$ to a subset $\{x_i\}_{\imath\in\mathcal{I}} \subseteq U_h(\mathfrak{g})$ such that $\overline{x}_i = x_i \bmod hU_h(\mathfrak{g})$, and $\{\overline{y}_i\}_{\imath\in\mathcal{I}}$ to a subset $\{y_\imath\}_{i\in\mathcal{I}} \subseteq F_h[[\mathfrak{g}]]$ such that $\overline{y}_i = y_i \bmod hF_h[[\mathfrak{g}]]$: then $\{x^{\underline{e}}\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is a topological basis of $U_h(\mathfrak{g})$ (as a topological $\mathbf{k}[[h]]$-module) and similarly $\{y^{\underline{e}}/\underline{e}!\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is a topological pseudobasis of $F_h[[\mathfrak{g}]]$ (as a topological $\mathbf{k}[[h]]$-module), and they are dual to each other modulo h, i.e. $\big\langle x^{\underline{e}}, y^{\underline{e}'}/\underline{e}!\big\rangle \in \delta_{\underline{e},\underline{e}'} + h\mathbf{k}[[h]]$ for all $\underline{e}$, $\underline{e}' \in \big(\mathbb{N}^{\mathcal{I}}\big)_0$. In addition, $y^{\underline{e}'}/\underline{e}! \in \big(I_{F_h[[\mathfrak{g}]]}\big)^{|\underline{e}'|}$ for all $\underline{e}' \in \big(\mathbb{N}^{\mathcal{I}}\big)_0$, where $|\underline{e}'| := \sum_{i\in\mathcal{I}} \underline{e}'(i)$. Now, $U_h(\mathfrak{g})$ also contains a topological basis dual to $\{y^{\underline{e}}/\underline{e}!\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$, call it $\{\eta_{\underline{e}}\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$: indeed, from the previous analysis we see – by the "duality mod h" mentioned above – that such a basis is given by $\eta_{\underline{e}} = x^{\underline{e}} + \sum_{n=1}^{+\infty} h^n \cdot \sum_{\underline{e}'\in(\mathbb{N}^{\mathcal{I}})_0} c_{\underline{e},\underline{e}'}^{(n)} x^{\underline{e}'}$ for some $c_{\underline{e},\underline{e}'}^{(n)} \in \mathbf{k}$ ($\underline{e}$, $\underline{e}' \in \big(\mathbb{N}^{\mathcal{I}}\big)_0$), so $\{\eta_{\underline{e}}\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is

([2]) From now on, this means that each element of $\mathfrak{g}^*$ can be written uniquely as a (possibly infinite) linear combination of elements of the pseudobasis: such a (possibly infinite) sum will be convergent in the weak topology of $\mathfrak{g}^*$, so a pseudobasis is a *topological basis* w.r.t. the weak topology.

a lift of the PBW basis $\left\{\overline{x}^{\underline{e}}\right\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ of $U(\mathfrak{g})$. Since $\left\{y^{\underline{e}}/\underline{e}!\right\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is a topological pseudobasis of $F_h[[\mathfrak{g}]]$ and $y^{\underline{e}'}/\underline{e}! \in \left(J_{F_h[[\mathfrak{g}]]}\right)^{|\underline{e}'|}$ for all $\underline{e}'$, the set $\left\{h^{-|\underline{e}|}y^{\underline{e}}/\underline{e}!\right\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$ is a topological basis of the topologically free $\mathbf{k}[[h]]$-module $\left(F_h[[\mathfrak{g}]]\right)^{\vee}$: then the dual pseudobasis of $\left(\left(F_h[[\mathfrak{g}]]\right)^{\vee}\right)^* = U_h(\mathfrak{g})'$ to this basis is $\left\{h^{|\underline{e}|}\eta_{\underline{e}}\right\}_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0}$, so $U_h(\mathfrak{g})'$ is the set $\left\{\sum_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0} a_{\underline{e}} h^{|\underline{e}|}\eta_{\underline{e}} \;\middle|\; a_{\underline{e}} \in \mathbf{k}[[h]] \forall \underline{e} \in (\mathbb{N}^{\mathcal{I}})_0\right\}$.

Now observe that $(hx)^{\underline{e}} = h^{|\underline{e}|}x^{\underline{e}} \equiv h^{|\underline{e}|}\eta_{\underline{e}} \mod hU_h(\mathfrak{g})'$ by construction; therefore

$$\left\{\sum_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0} a_{\underline{e}}(hx)^{\underline{e}} \mid a_{\underline{e}} \in \mathbf{k}[[h]]\forall\underline{e}\right\} = \left\{\sum_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0} a_{\underline{e}}h^{|\underline{e}|}\eta_{\underline{e}} \mid a_{\underline{e}} \in \mathbf{k}[[h]]\forall\underline{e}\right\} = U_h(\mathfrak{g})'.$$

Finally, up to taking $x_i - \epsilon(x_i)$, one can also choose the x_i so that $\epsilon(x_i) = 0$.

To summarize, the outcome is the following:

Given any basis of $\mathfrak{g}$, there exists a lift $\{x_i\}_{i\in\mathcal{I}}$ of it in $U_h(\mathfrak{g})$ such that $\epsilon(x_i) = 0$ and $U_h(\mathfrak{g})'$ is nothing but the topological $\mathbf{k}[[h]]$-algebra in $\mathcal{P}_{\widetilde{\otimes}}$ generated (in topological sense) by $\{hx_i\}_{i\in\mathcal{I}}$, thus $U_h(\mathfrak{g})' = \left\{\sum_{\underline{e}\in(\mathbb{N}^{\mathcal{I}})_0} a_{\underline{e}}h^{|\underline{e}|}x^{\underline{e}} \;\middle|\; a_{\underline{e}} \in \mathbf{k}[[h]]\forall\underline{e}\right\}$ as a subset of $U_h(\mathfrak{g})$.

Remark. — This description of $U_h(\mathfrak{g})'$ implies that *the weak topology on $U_h(\mathfrak{g})'$*, which coincides with its $I_{U_h(\mathfrak{g})'}$-adic topology, *does coincide with the induced topology* (of $U_h(\mathfrak{g})'$ as a subspace of $U_h(\mathfrak{g})$, the latter being endowed with the h-adic topology). This defines the topology on $U_h(\mathfrak{g})'$ in an intrinsic way, i.e. without referring to any identification of $U_h(\mathfrak{g})'$ with the dual space to some $X \in \mathcal{T}_{\widehat{\otimes}}$ (as we did instead to prove Proposition 2.6).

PROPOSITION 3.6. — *Assume* char$(\mathbf{k}) = 0$. *If $U_h(\mathfrak{g}) \in \mathcal{QUEA}$, then $U_h(\mathfrak{g})' \in \mathcal{QFSHA}$. Namely, we have $U_h(\mathfrak{g})' = F_h[[\mathfrak{g}^*]]$ (where $\mathfrak{g}^*$ is the dual Lie bialgebra to $\mathfrak{g}$), that is*

$$U_h(\mathfrak{g})' \Big/ hU_h(\mathfrak{g})' = F[[\mathfrak{g}^*]].$$

Proof. — Consider $F_h[[\mathfrak{g}]] := U_h(\mathfrak{g})^* \in \mathcal{QFSHA}$ (cf. Lemma 2.1); then $U_h(\mathfrak{g}) \xrightarrow{h\to 0} U(\mathfrak{g})$ implies $F_h[[\mathfrak{g}]] := U_h(\mathfrak{g})^* \xrightarrow{h\to 0} U(\mathfrak{g})^* = F[[\mathfrak{g}]]$. By Proposition 3.2, $F_h[[\mathfrak{g}]]^{\vee}$ is a QUEA, with semiclassical limit $U(\mathfrak{g}^*)$; by Proposition 2.2 we have $U_h(\mathfrak{g})' = \left(F_h[[\mathfrak{g}]]^{\vee}\right)^*$, thus $U_h(\mathfrak{g})'$ is a QFSHA with semiclassical limit $F[[\mathfrak{g}^*]]$: indeed, $F_h[[\mathfrak{g}]]^\vee \xrightarrow{h\to 0} U(\mathfrak{g}^*)$ implies $U_h(\mathfrak{g})' = \big(F_h[[\mathfrak{g}]]^\vee\big)^* \xrightarrow{h\to 0} U(\mathfrak{g}^*)^* = F[[\mathfrak{g}^*]]$, i.e. $U_h(\mathfrak{g})' \big/ h U_h(\mathfrak{g})' = F[[\mathfrak{g}^*]]$ as claimed. □

Proposition 3.7. — *Assume* $\mathrm{char}(\mathbf{k}) = 0$. *Let* U_h *be a QUEA. Then* $\big({U_h}'\big)^\vee = U_h$.

Proof. — Consider $F_h := {U_h}^* \in \mathcal{QFSHA}$ (cf. Lemma 2.1): then Proposition 3.4 yields $\big(F_h^\vee\big)' = F_h$; furthermore, $U_h = \big({U_h}^*\big)^\star = {F_h}^\star$. Applying Proposition 2.2 to the pair $(H,K) = (U_h, F_h)$ we get ${U_h}' = \big(F_h^\vee\big)^*$ and $F_h^\vee = \big({U_h}'\big)^\star$. By Proposition 2.4 (as $F_h \in \mathcal{QFSHA} \subseteq \mathcal{HA}_{\widetilde{\otimes}}^{w-I}$, by Remark 1.3(c)) and by Proposition 2.6 we can apply Proposition 2.2 to the pair $(H,K) = \big(F_h^\vee, {U_h}'\big)$, thus getting $\big({U_h}'\big)^\vee = \Big(\big(F_h^\vee\big)'\Big)^\star = {F_h}^\star = U_h$. □

Lemma 3.8. — *Drinfeld's functors on quantum groups preserve equivalences: if* $H_1 \equiv H_2$ *in* $\mathcal{QFSHA}$, *resp. in* $\mathcal{QUEA}$, *then* ${H_1}^\vee \equiv {H_2}^\vee$ *in* $\mathcal{QUEA}$, *resp.* ${H_1}' \equiv {H_2}'$ *in* $\mathcal{QFSHA}$.

Proof. — Let H_1, $H_2 \in \mathcal{QFSHA}$ be two equivalent quantisations of some $F[[\mathfrak{g}]]$, and identify them – as $\mathbf{k}[[h]]$-modules – with $H := F[[\mathfrak{g}]][[h]]$, so that the equivalence $\phi\colon H = H_1 \equiv H_2 = H$ reads $\phi = \mathrm{id}_H + h\phi_+$ for some $\phi_+ \in \mathrm{End}_{\mathbf{k}[[h]]}(H)$. By definition, $\phi_+ = (\phi - \mathrm{id}_H)/h$; therefore, for all $n \in \mathbb{N}$, we have

$$\big(\phi^{\otimes n} - \mathrm{id}_H^{\otimes n}\big)/h = \Big(\sum_{k=0}^{n-1} \phi^{\otimes k} \otimes (\phi - \mathrm{id}_H) \otimes \mathrm{id}_H^{\otimes(n-k-1)}\Big)\Big/ h = \sum_{k=0}^{n-1} \phi^{\otimes k} \otimes \phi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}.$$

Now, let $J := \mathrm{Ker}(\epsilon_H)$: since ϕ is a Hopf isomorphism, it maps J into itself, hence also $\phi_+(J) = \big(\frac{\phi - \mathrm{id}_H}{h}\big)(J) \subseteq J$. Letting $m_n\colon H^{\otimes n} \to H$ be the n-fold multiplication, we have

$$\begin{aligned} \phi_+(J^n) &= \big((\phi - \mathrm{id}_H)/h\big)(J^n) = m_n\Big(\big((\phi^{\otimes n} - \mathrm{id}_H^{\otimes n})/h\big)(J^{\otimes n})\Big) \\ &= m_n\Big(\sum_{k=0}^{n-1} \phi^{\otimes k} \otimes \phi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}(J^{\otimes n})\Big) \subseteq m_n\big(J^{\otimes n}\big) = J^n, \end{aligned}$$

i.e. $\phi_+\big(J^n\big) \subseteq J^n$ for all n, so $\phi_+^\vee\big(H^\vee\big) \subseteq H^\vee$, where $\phi_+^\vee$ is the extension of ϕ_+ to $H^\vee$. Thus $\phi^\vee = \mathrm{id}_{H^\vee} + h\phi_+^\vee$ with $\phi_+^\vee \in \mathrm{End}_{\mathbf{k}[[h]]}\big(H^\vee\big)$, so $\phi^\vee$ is an equivalence in $\mathcal{QUEA}$.

Similarly, let $H_1, H_2 \in \mathcal{QUEA}$ be two equivalent quantisations of some $U(\mathfrak{g})$, and identify them – as $\mathbf{k}[[h]]$-modules – with $H := U(\mathfrak{g})[[h]]$. Then the equivalence $\varphi\colon H_1 \equiv H_2$ reads $\varphi = \mathrm{id}_H + h\varphi_+$ for some $\varphi_+ \in \mathrm{End}_{\mathbf{k}[[h]]}(H)$. As above, $\big(\varphi^{\otimes n} - \mathrm{id}_H^{\otimes n}\big)/h = \sum_{k=0}^{n-1} \varphi^{\otimes k} \otimes \varphi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}$, so $\delta_n \circ \varphi = \varphi^{\otimes n} \circ \delta_n$ (for φ is a *Hopf* isomorphism!), hence

$$\delta_n \circ \varphi_+ = \Big(\big(\varphi^{\otimes n} - \mathrm{id}_H^{\otimes n}\big)/h\Big) \circ \delta_n = \sum_{k=0}^{n-1} \big(\varphi^{\otimes k} \otimes \varphi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}\big) \circ \delta_n$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned}\delta_n\big(\varphi_+(H')\big) &= \sum_{k=0}^{n-1} \big(\varphi^{\otimes k} \otimes \varphi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}\big)\big(\delta_n(H')\big) \\ &\subseteq \sum_{k=0}^{n-1} \big(\varphi^{\otimes k} \otimes \varphi_+ \otimes \mathrm{id}_H^{\otimes(n-k-1)}\big)\big(h^n H^{\otimes n}\big) \subseteq h^n H^{\otimes n}\end{aligned}$$

that is $\delta_n\big(\varphi_+(H')\big) \subseteq h^n H^{\otimes n}$ for all $n \in \mathbb{N}$, so $\varphi_+(H') \subseteq H'$; hence $\varphi' := \varphi\big|_{H'} = \mathrm{id}_{H'} + h\varphi_+\big|_{H'}$ with $\varphi_+\big|_{H'} \in \mathrm{End}_{\mathbf{k}[[h]]}\big(H'\big)$, so that φ' is an equivalence in $\mathcal{QFSHA}$. □

Finally, our efforts are rewarded:

Proof of Theorem 1.6. — It is enough to collect together the previous results. Proposition 3.2 and 3.6 together with Lemma 2.8 ensure that the functors in the claim are well-defined, and that relations ⊛ do hold. Proposition 3.4 and 3.7 show these functors are inverse to each other. Finally, Lemma 3.8 prove that they preserve equivalence. □

3.9. **Generalizations.** — In this paper we dealt with finite dimensional Lie bialgebras. What about the infinite dimensional case? Hereafter we sketch a draft of an answer.

Let $\mathfrak{g}$ be an infinite dimensional Lie bialgebra; then its linear dual $\mathfrak{g}^*$ is a Lie bialgebra only in a *topological* sense: in fact, the natural Lie cobracket takes values in the "formal tensor product" $\mathfrak{g}^* \dot{\otimes} \mathfrak{g}^* := (\mathfrak{g} \otimes \mathfrak{g})^*$, which is the completion of $\mathfrak{g}^* \otimes \mathfrak{g}^*$ w.r.t. the weak topology. Note that a vector subspace $\mathfrak{g}^\times$ of $\mathfrak{g}^*$ is dense in $\mathfrak{g}^*$ w.r.t. the weak topology if and only if the restriction $\mathfrak{g} \times \mathfrak{g}^\times \to \mathbf{k}$ of the natural evaluation pairing is perfect.

If $\mathfrak{g}$ is a Lie bialgebra in the strict algebraic sense (i.e. $\delta_{\mathfrak{g}} \subseteq \mathfrak{g} \otimes \mathfrak{g}$) then $U(\mathfrak{g})$ is a co-Poisson Hopf algebra as usual; if instead $\mathfrak{g}$ is a Lie bialgebra in the topological sense (i.e. $\delta_{\mathfrak{g}} \subseteq \mathfrak{g} \dot{\otimes} \mathfrak{g}$) then $U(\mathfrak{g})$ is a topological co-Poisson Hopf algebra, whose co-Poisson bracket takes values in a suitable completion $U(\mathfrak{g})\dot{\otimes}U(\mathfrak{g})$ of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. On the other hand, for any Lie bialgebra $\mathfrak{g}$ (both algebraic or topological) we can consider *two* objects to play the role of $F[[\mathfrak{g}]]$, namely $F^{\circledast}[[\mathfrak{g}]] := U(\mathfrak{g})^*$ (the linear dual of $U(\mathfrak{g})$), endowed with the weak topology, and $F^{\infty}[[\mathfrak{g}]]$, the $\mathfrak{m}_e$-adic completion of $F[G]$ – provided the latter exists! – at the maximal ideal $\mathfrak{m}_e$ of $e \in G$, with the $\mathfrak{m}_e$-adic topology. Both $F^{\circledast}[[\mathfrak{g}]]$ and $F^{\infty}[[\mathfrak{g}]]$ are *topological* Poisson Hopf algebras (the coproduct taking values in a suitable topological tensor product), complete w.r.t. to their topology. Moreover, there are natural pairings of (topological) Hopf algebras between $U(\mathfrak{g})$ and $F^{\circledast}[[\mathfrak{g}]]$ and between $U(\mathfrak{g})$ and $F^{\infty}[[\mathfrak{g}]]$, compatible with the Poisson and co-Poisson structures. We still have $F^{\circledast}[[\mathfrak{g}]] \supseteq F^{\infty}[[\mathfrak{g}]]$, but contrary to the finite dimensional case we may have $F^{\circledast}[[\mathfrak{g}]] \neq F^{\infty}[[\mathfrak{g}]]$.

Let $\mathcal{HA}_{\widehat{\otimes}}$, resp. $\mathcal{HA}_{\widetilde{\otimes}}$, be defined as in §1.1. In addition, define $\mathcal{HA}_{\bar{\otimes}}$ to be the tensor category of all (topological) Hopf $\mathbf{k}[[h]]$-algebras H such that: (a) H is complete w.r.t. the I_H-adic topology; (b) the tensor product $H_1 \bar{\otimes} H_2$ is the completion of the algebraic tensor product $H_1 \otimes_{\mathbf{k}[[h]]} H_2$ w.r.t. the $I_{H_1 \otimes_{\mathbf{k}[[h]]} H_2}$-adic topology; in particular, the coproduct of H takes values in $H \bar{\otimes} H$. Then we call $\mathcal{QUEA}^{\wedge}$, resp. $\mathcal{QFSHA}^{\circledast}$, resp. $\mathcal{QFSHA}^{\infty}$, the subcategory of $\mathcal{HA}_{\widehat{\otimes}}$, resp. of $\mathcal{HA}_{\widetilde{\otimes}}$, resp. of $\mathcal{HA}_{\bar{\otimes}}$, composed of all objects whose specialisation at $h = 0$ is isomorphic to some $U(\mathfrak{g})$, resp. some $F^{\circledast}[[\mathfrak{g}]]$, resp. some $F^{\infty}[[\mathfrak{g}]]$; here $\mathfrak{g}$ is any Lie bialgebra. However, note that if $H \in \mathcal{QUEA}^{\wedge}$ then the Poisson cobracket δ of its semiclassical limit $H_0 = U(\mathfrak{g})$ (defined as in Remark 1.3(a)) takes values in $H_0 \otimes H_0$, so that H_0 is an *algebraic* (not topological) co-Poisson Hopf algebra hence $\mathfrak{g}$ is an *algebraic* Lie bialgebra; this means that if we start instead from a *topological* Lie bialgebra $\mathfrak{g}$ we cannot quantize $U(\mathfrak{g})$ in the category $\mathcal{QUEA}$: what's wrong is the tensor product $\widehat{\otimes}$ because, roughly, $H^{\vee} \widehat{\otimes} H^{\vee}$ is "too small"! Thus one must define a new category $\mathcal{T}_{\check{\otimes}}$ with the same objects than $\mathcal{T}_{\widehat{\otimes}}$ but with a "larger" tensor product $\check{\otimes}$ (a suitable completion of $\otimes_{\mathbf{k}[[h]]}$) and then consider the tensor category $\mathcal{HA}_{\check{\otimes}}$ of all (topological) Hopf algebras in $\mathcal{T}_{\check{\otimes}}$, and the subcategory $\mathcal{QUEA}^{\vee}$ whose objects have some $U(\mathfrak{g})$ as specialisation at $h = 0$: then in this case the Lie bialgebra $\mathfrak{g}$ might be of *topological* type as well.

Now let's have a look back. We review our previous work and, somewhat roughly, point out how far (and in which way) its results extend to the more generals setting.

Lemma 2.1: This turns into: *Dualisation $H \mapsto H^*$, resp. $H \mapsto H^{\star}$, defines a contravariant functor of tensor categories $\mathcal{HA}_{\widehat{\otimes}} \longrightarrow \mathcal{HA}_{\widetilde{\otimes}}$,* *resp.* $\mathcal{HA}_{\widetilde{\otimes}} \longrightarrow \mathcal{HA}_{\widehat{\otimes}}$, *which, if* $\mathrm{char}(\mathbf{k}) = 0$, *restrict to* $\mathcal{QUEA}^{\wedge} \longrightarrow \mathcal{QFSHA}^{\circledast}$, *resp.* $\mathcal{QFSHA}^{\circledast} \longrightarrow \mathcal{QUEA}^{\wedge}$.

Indeed, this suggest to define $\check{\otimes}$ in such a way that dualisation $H \mapsto H^*$, resp. $H \mapsto H^\star$, defines a functor of tensor categories $\mathcal{HA}_{\check{\otimes}} \longrightarrow \mathcal{HA}_{\bar{\otimes}}$, resp. $\mathcal{HA}_{\bar{\otimes}} \longrightarrow \mathcal{HA}_{\check{\otimes}}$; then, if $\mathrm{char}(\mathbf{k}) = 0$, this will restrict to $\mathcal{QUEA}^{\vee} \longrightarrow \mathcal{QFSHA}^{\infty}$, resp. $\mathcal{QFSHA}^{\infty} \longrightarrow \mathcal{QUEA}^{\vee}$.

Proposition 2.2: This still holds true for any pair $(H, K) \in \mathcal{HA}_{\widehat{\otimes}} \times \mathcal{HA}_{\widetilde{\otimes}}$. *Moreover, it should also holds true for any pair* $(H, K) \in \mathcal{HA}_{\check{\otimes}} \times \mathcal{HA}_{\bar{\otimes}}$ *of Hopf algebras in duality.*

Lemma 2.3 still holds true up to replacing $\mathcal{HA}_{\widetilde{\otimes}}^{w-I}$ *with* $\mathcal{HA}_{\bar{\otimes}}$.

Proposition 2.4 still holds true but for replacing $\mathcal{HA}_{\widetilde{\otimes}}^{w-I}$ *in part (b) with* $\mathcal{HA}_{\bar{\otimes}}$.

On the other hand, if we consider $H^\vee$ for any $H \in \mathcal{HA}_{\widetilde{\otimes}}$ then the sole thing which goes wrong is that $\Delta\big(H^\vee\big) \not\subseteq H^\vee \widehat{\otimes} H^\vee$, in general: indeed, $\Delta\big(H^\vee\big)$ will lie in something larger. Well, the definition of the category $\mathcal{HA}_{\check{\otimes}}$ above should fit in this frame to give exactly $H^\vee \in \mathcal{HA}_{\check{\otimes}}$. Once one has fixed this point, *our arguments still prove that* $H^\vee \in \mathcal{QUEA}^\vee$; thus one has a further version of Proposition 2.4, and similarly a proper version of Lemma 2.3 should hold with $\mathcal{HA}_{\widetilde{\otimes}}$, resp. $\mathcal{HA}_{\check{\otimes}}$, instead of $\mathcal{HA}_{\bar{\otimes}}$, resp. $\mathcal{HA}_{\widehat{\otimes}}$.

In any case, we can also drop at all the question of what kind of Hopf algebra $H^\vee$ is, for in any case *the proof of Proposition 2.4 will always prove* the following:

If $H \in \mathcal{HA}_{\widetilde{\otimes}}$, *then* ${H^\vee}_0 = U(\mathfrak{g})$ *for some Lie bialgebra (perhaps of topological type).*

Proposition 2.6, Lemma 2.7: The proofs we give actually show the following:
If $H \in \mathcal{HA}_{\widehat{\otimes}}$, *then* $H' \in \mathcal{QFSHA}^{\circledast}$. *If* H_1, $H_2 \in \mathcal{HA}_{\widehat{\otimes}}$, *then* $\big(H_1 \widehat{\otimes} H_2\big)' = {H_1}' \widetilde{\otimes} {H_2}'$.

To prove these results we used a duality argument, relying on Lemma 2.1. Alternatively, given $H \in \mathcal{HA}_{\widehat{\otimes}}$ or $H \in \mathcal{HA}_{\check{\otimes}}$ we can prove as before that H' is a unital $\mathbf{k}[[h]]$-subalgebra of H, and *also* that H' is complete w.r.t. the $I_{H'}$-adic topology and is closed for the antipode; then what one misses to have $H' \in \mathcal{HA}_{\bar{\otimes}}$ is a control on $\Delta\big(H'\big)$. Moreover, one proves as before that $\big(H'\big)_0$ is commutative.

Lemma 2.8: The way the action of Drinfeld's functors on morphisms is defined here still works for any one of the categories we are considering now.

Sections 3.1 through 3.4: These results also hold in a greater generality.

Indeed, changing a few details we can adapt the discussion in §3.1 and the (claim and) proof of Proposition 3.2 and of Proposition 3.4 (noting that Lemma 3.3, which still holds true untouched) to the case of $F_h^\infty[[\mathfrak{g}]] \in \mathcal{QFSHA}^\infty$. The outcome is

If $F_h^\infty[[\mathfrak{g}]] \in \mathcal{QFSHA}^\infty$, then $F_h^\infty[[\mathfrak{g}]]^\vee \in \mathcal{QUEA}^\wedge$, namely $F_h^\infty[[\mathfrak{g}]]^\vee = U_h(\mathfrak{g}^\times)$ where $\mathfrak{g}^\times$ is an algebraic Lie bialgebra which embeds in $\mathfrak{g}^*$ as a dense Lie sub-bialgebra. Moreover, $\left(F_h^\infty[[\mathfrak{g}]]^\vee\right)' = F_h^\infty[[\mathfrak{g}]]$.

Similarly, one can apply the same arguments to $F_h^\circledast[[\mathfrak{g}]] \in \mathcal{QFSHA}^\circledast$ and get essentially the same result but with $\mathcal{QUEA}^\vee$ instead of $\mathcal{QUEA}^\wedge$. Then again the sole real problem is to provide a proper definition for the category $\mathcal{HA}_{\hat{\otimes}}$ (or at least $\mathcal{QUEA}^\vee$) which fit well with these results. Once this (non-trivial...) point is set, the result would read

If $F_h^\circledast[[\mathfrak{g}]] \in \mathcal{QFSHA}^\circledast$, then $F_h^\circledast[[\mathfrak{g}]]^\vee \in \mathcal{QUEA}^\wedge$, namely $F_h^\circledast[[\mathfrak{g}]]^\vee = U_h(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual (topological) Lie bialgebra to $\mathfrak{g}$. Moreover, $\left(F_h^\circledast[[\mathfrak{g}]]^\vee\right)' = F_h^\circledast[[\mathfrak{g}]]$.

Sections 3.5 through 3.7: These again hold in a greater generality.

In this case, the main tool is the use duality functor to switch from QUEA to QFSHA and the property of Drinfeld's functors of being dual to each other ensured by Proposition 2.2. Therefore, our arguments apply *verbatim* to the case of $U_h(\mathfrak{g}) \in \mathcal{QUEA}^\wedge$. As for $U_h(\mathfrak{g}) \in \mathcal{QUEA}^\vee$, everything goes true as well the same *provided* Lemma 2.1 and Proposition 2.2 have been properly extended to deal with $\mathcal{QUEA}^\vee$ and $\mathcal{QFSHA}^\infty$, as mentioned above.

Lemma 3.8: Here again (as for Lemma 2.8) our analysis still works for any one of the categories we are considering now.

In a nutshell, we can say that, up to some details to be fixed,

The quantum duality principle holds, in a suitable formulation, also for infinite dimensional Lie bialgebras, both algebraic and topological.

3.10. **Examples.** — Several examples about finite dimensional Lie bialgebras can be found in [Ga2]: there we consider quantum groups "à la Jimbo-Lusztig", but one can easily translate all definitions and results into the language "à la Drinfeld" we use in the present paper.

We consider now some infinite dimensional samples. Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra, and $\hat{\mathfrak{g}}$ the associated untwisted affine Kac-Moody algebra, with the well-known Sklyanin-Drinfeld structure of Lie bialgebra; let also $\hat{\mathfrak{h}}$ be defined as in [Ga1], §1.2. Then both $\hat{\mathfrak{g}}^*$ and $\hat{\mathfrak{h}}$ are *topological* Lie bialgebras, with $\hat{\mathfrak{h}}$ dense inside $\hat{\mathfrak{g}}^*$.

Consider the quantum groups $\mathcal{U}^M(\hat{\mathfrak{g}})$ and $\mathfrak{U}^M(\hat{\mathfrak{g}})$ defined in [Ga1], §3.3. We can reformulate the definition of the first in Drinfeld's terms via the usual "dictionary": pick generators $H_i = \log(K_i)/\log(q_i)$ instead of the K_i's, take $h = \log(q)$ and fix $\mathbf{k}[[h]]$ as ground ring, and finally complete the resulting algebra w.r.t. the h-adic topology; then we have exactly $\mathfrak{U}^M(\hat{\mathfrak{g}}) \in \mathcal{HA}_{\widehat{\otimes}}$ and $\mathfrak{U}^M(\hat{\mathfrak{g}}) \xrightarrow{h\to 0} U(\hat{\mathfrak{g}})$, so $U_h(\hat{\mathfrak{g}}) := \mathfrak{U}^M(\hat{\mathfrak{g}}) \in \mathcal{QUEA}^\wedge$ (discarding the choice of the weight lattice M). On the other hand, doing the same "translations" for $\mathcal{U}^M(\hat{\mathfrak{g}})$ and completing w.r.t. the weak topology *or* w.r.t. the I-adic topology we obtain *two* different objects, in $\mathcal{HA}_{\widetilde{\otimes}}$ and in $\mathcal{HA}_{\bar{\otimes}}$ respectively, with semiclassical limit $F^{\circledast}[[\hat{\mathfrak{g}}^*]]$ and $F^{\infty}\big[\big[\hat{\mathfrak{h}}\big]\big]$ respectively; then we call them $F_h^{\circledast}[[\hat{\mathfrak{g}}^*]]$ and $F_h^{\infty}\big[\big[\hat{\mathfrak{h}}\big]\big]$ respectively, with $F_h^{\circledast}[[\hat{\mathfrak{g}}^*]] \in \mathcal{QFSHA}^{\circledast}$ and $F_h^{\infty}\big[\big[\hat{\mathfrak{h}}\big]\big] \in \mathcal{QFSHA}^{\infty}$. Now, acting as outlined in [Ga2], §3, one finds $F_h^{\infty}\big[\big[\hat{\mathfrak{h}}\big]\big]^\vee = U_h(\hat{\mathfrak{g}})$ and $U_h(\hat{\mathfrak{g}})' = F_h^{\infty}\big[\big[\hat{\mathfrak{h}}\big]\big]$, whilst $F_h^{\circledast}[[\hat{\mathfrak{g}}^*]]^\vee$ instead is a suitable completion of $U_h(\hat{\mathfrak{g}})$, which should be an object of $\mathcal{QUEA}^\vee$: indeed, we have $F_h^{\circledast}[[\hat{\mathfrak{g}}^*]]^\vee \xrightarrow{h\to 0} U\big(\hat{\mathfrak{h}}^*\big)$, and $\hat{\mathfrak{h}}^*$ is a topological Lie bialgebra in perfect duality with $\hat{\mathfrak{g}}^*$.

Dually, consider the quantum groups $\mathcal{U}^M\big(\hat{\mathfrak{h}}\big)$ and $\mathfrak{U}^M\big(\hat{\mathfrak{h}}\big)$ defined in [Ga1], §5; as above we can rephrase their definition, and then we find the following. First, the formulæ for the coproduct imply that $\mathcal{U}^M\big(\hat{\mathfrak{h}}\big) \notin \mathcal{HA}_{\bar{\otimes}}$ *but* $\mathcal{U}^M\big(\hat{\mathfrak{h}}\big) \in \mathcal{HA}_{\widetilde{\otimes}}$, and $\mathcal{U}^M\big(\hat{\mathfrak{h}}\big) \xrightarrow{h\to 0} F^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big]$, thus $F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big] := \mathcal{U}^M\big(\hat{\mathfrak{h}}\big) \in \mathcal{QFSHA}^{\circledast}$. Second, $\mathfrak{U}^M\big(\hat{\mathfrak{h}}\big) \notin \mathcal{HA}_{\widehat{\otimes}}$ *but* $\mathfrak{U}^M\big(\hat{\mathfrak{h}}\big) \in \mathcal{HA}_{\check{\otimes}}$, and $\mathfrak{U}^M\big(\hat{\mathfrak{h}}\big) \xrightarrow{h\to 0} U\big(\hat{\mathfrak{h}}\big)$, so in fact $U_h\big(\hat{\mathfrak{h}}\big) := \mathfrak{U}^M\big(\hat{\mathfrak{h}}\big) \in \mathcal{QUEA}^\vee$. By an analysis like that in [Ga2] one shows also that $F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big]^\vee$ is an object of $\mathcal{QUEA}^\vee$, it is a suitable completion of $U_h\big(\hat{\mathfrak{h}}\big)$, and $F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big]^\vee \xrightarrow{h\to 0} U\big(\hat{\mathfrak{g}}^*\big)$; moreover, $\Big(F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big]^\vee\Big)' = F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big]$. On the other hand, one has $U_h\big(\hat{\mathfrak{h}}\big)' = F_h^{\circledast}\big[\big[\hat{\mathfrak{h}}^*\big]\big] \in \mathcal{QFSHA}^{\circledast}$, and so $\Big(U_h\big(\hat{\mathfrak{h}}\big)'\Big)^\vee = U_h\big(\hat{\mathfrak{h}}\big)$.

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Manuscrit reçu le 5 mai 2001,
accepté le 24 juillet 2002.

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