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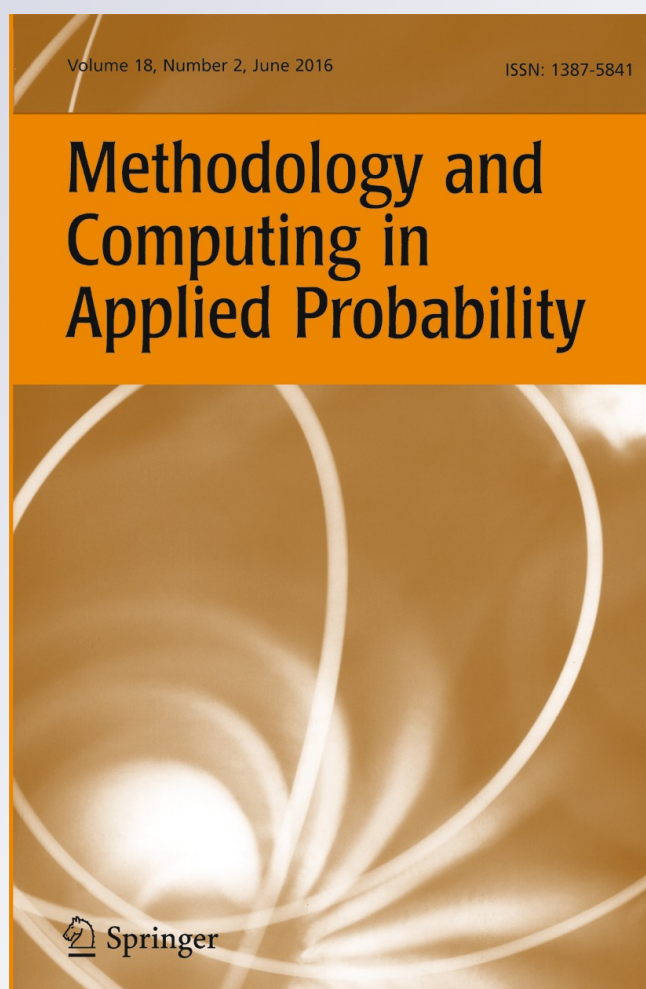
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On a Transform Method for the Efficient Computation of Conditional V@R (and V@R) with Application to Loss Models with Jumps and Stochastic Volatility

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Abstract In this paper we consider Fourier transform techniques to efficiently compute the Value-at-Risk and the Conditional Value-at-Risk of an arbitrary loss random variable, characterized by having a computable generalized characteristic function. We exploit the property of these risk measures of being the solution of an elementary optimization problem of convex type in one dimension. An application to univariate loss models driven by Lévy or stochastic volatility risk factors dynamic is finally reported.

Keywords V@R · CV@R · Fourier transform methods · Stochastic volatility · Jump-diffusion models

Mathematics Subject Classifications (2010) 91G60 · 91B30 · 60E10

1 Introduction

Measuring risks is certainly one of the core competence of any financial institution, even from a regulatory perspective. An efficient and reliable computation of risk measures is consequently a primary concern of any modern risk management activity, dramatically highlighted during the recent financial crisis. Value-at-Risk and Conditional (or Average) Value-at-Risk (henceforth V@R and CV@R) are without doubt among the best known monetary risk measures. Since its introduction, V@R rapidly has become a benchmark in the financial industry both for regulatory purposes and in the practice of risk management, mainly due to its simplicity. On the other side, it is sharply criticized for the lack of sub-additivity and the inability to quantify the severity of an exposure to rare events. For these

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reasons, alternative risk measures are considered as the CV@R, which turns out to be one of the well known example of coherent risk measure (see Acerbi and Tasche 2002; Artzner et al. 1999).

In this paper we present a method for their computation based on a Fourier transform technique. The Fourier representation of distribution functions and expected values of random variables is well-known in the financial and actuarial literature: Lèvy and Gil-Pelaez inversion formulas and the method introduced by Carr and Madan in their seminal paper (Carr and Madan 1999) for contingent claim valuation through Fast Fourier transform, nowadays are standard computational techniques to efficiently solve statistical and pricing problems. Here we consider a parametric framework, thus assuming a probabilistic description for the quantity we are interested in, namely profit or loss random variables. Aside the standard definitions for V@R and CV@R, in this paper we exploit an alternative characterizations of these risk measures in which transform representation can play an important tool for their efficient calculation. In particular, their property of being the solution of an elementary optimization problem of convex type in one dimension permits to evaluate both measures by solving numerically a unique simple univariate minimization problem, in which the objective function can be efficiently computed by means of Fourier representation. This is the main contribution of the present work. A “quick and dirty” solution is then available through (fractional) Fast Fourier Transform algorithms, that may also be useful to properly select a priori a feasible region in which the minimum could be located.

Analytical calculation of V@R by using inversion formulas has been considered in Duffie and Pan (2001) while the use of generalized Fourier transform and the FFT algorithm is more recent, see e.g. Le Courtois and Walter (2009) who applied such a technique in a Variance Gamma model, Kim et al. (2010) and Scherer et al. (2009) where the class of tempered stable and infinitely divisible distribution were considered or Bormetti et al. (2010) for an application to a stochastic volatility model. Even more recently, a Fourier-based representation approach has been exploited to build a numerical scheme (Fourier transform methods and deterministic root-finding) for the computation of a class of law-invariant risk measure, see Drapeau et al. (2014). Moreover, efficient numerical schemes based on wavelet approximation and Fourier-cosine series expansion have been used to compute V@R and CV@R for a non-linear portfolio in Ortiz-Gracia and Oosterlee (2014) by using a delta-gamma approximation in a gaussian framework.

The techniques considered here can be applied to all models having an analytic (and computable) characteristic function. This is the case of many financial dynamic models of returns emerged in the literature of the last decades for the analysis and the management of portfolio risks, such as Lévy finite/infinite activity and stochastic volatility models; an application to an univariate loss model in such a framework will be presented in the numerical section. But they can also be applied to that stochastic models which find their application in actuarial science, such as the compound Poisson loss distribution, used to model the aggregate claims of an insurance-risk business (see Dufresne et al. 2009).

The paper is organized as follows: V@R and CV@R are briefly introduced in Section 2, where their main characterizations are outlined. The transform technique is reviewed in Section 3 together with the use of fast and fractional Fourier transform algorithms and finally a set of numerical experiment is reported in Section 4 to show the effectiveness of the proposed procedures. In particular, the Fourier based techniques are applied to a univariate loss model of portfolio returns characterized by dynamical risk factors with jumps and stochastic volatility.

2 V@R and Conditional V@R

Let (Ω, F, \mathbb{P}) be a probability space, L a real-valued random variable and $F_L(x) = \mathbb{P}(L \leq x)$ its distribution function. In the following we suppose that $\mathbb{E}[|L|] < \infty$. To measure the risk of a financial position characterized by an uncertain future value over a given time horizon, the quantiles of its distribution function are commonly used. Given a confidence level $\alpha \in (0, 1)$, the set of α -quantiles of the random variable L is the interval $[q_\alpha^-(L), q_\alpha^+(L)]$ where

$$q_\alpha^-(L) = \inf\{q \in \mathbb{R} | \mathbb{P}(L \leq q) \geq \alpha\}, \quad q_\alpha^+(L) = \inf\{q \in \mathbb{R} | \mathbb{P}(L \leq q) > \alpha\}.$$

In this paper the random variable L describes the loss of a financial position. Given L , $V@R$ is defined as the lower α -quantile, $q_\alpha^-(L)$:

$$V@R_\alpha(L) := \inf\{q \in \mathbb{R} | \mathbb{P}(L \leq q) \geq \alpha\}.$$

In financial terms, $V@R$ is “the maximum possible loss which is not exceeded with probability α ”, or “the smallest amount of capital which, if added to the current position, keeps the probability of a non-negative outcome below the level $1 - \alpha$ ”. For a random variable having continuous and strictly increasing distributions function, $q_\alpha^-(L) = q_\alpha^+(L) \equiv q_\alpha(L) = F_L^{-1}(\alpha)$, the ordinary inverse of F , i.e. $V@R$ solves the equation

$$\mathbb{P}(L \leq V@R_\alpha(L)) = \alpha \quad (\text{or equivalently } \mathbb{P}(L \geq V@R_\alpha(L)) = 1 - \alpha).$$

Although widely used, it is well known that $V@R$ is not a coherent risk measure (see Acerbi and Tasche 2002; Artzner et al. 1999), in particular for being not sub-additive. To overcome the weakness of $V@R$, several alternative risk measures have been proposed in literature, among which the Conditional, or Average, Value-at-Risk, which does satisfy the axioms of coherence (see e.g. Acerbi and Tasche 2002; Artzner et al. 1999). Several equivalent definitions have been proposed in the literature: given the confidence level $\alpha \in (0, 1)$, the basic idea is to average all the possible losses exceeding $V@R_\alpha(L)$, that is

$$CV@R_\alpha(L) := \frac{1}{1 - \alpha} \int_\alpha^1 V@R_u(L) du.$$

Alternatively, it may be convenient to define the $CV@R$ as the expectation of L under the (scaled) distribution function (see Rockafellar and Uryasev 2002)

$$F_{L,\alpha}(x) = \begin{cases} 0 & \text{for } x < q_\alpha^-(L) \\ (F_L(x) - \alpha)/(1 - \alpha) & \text{for } x \geq q_\alpha^-(L). \end{cases}$$

When F is continuous and strictly increasing then $V@R_\alpha(L) = F_L^{-1}(\alpha)$ and

$$\int_\alpha^1 V@R_u(L) du = \int_{F_L^{-1}(\alpha)}^{+\infty} x dF_L(x) = \mathbb{E}[X \mathbb{I}_{X \geq F_L^{-1}(\alpha)}];$$

therefore $CV@R_\alpha(L) = \mathbb{E}[L | L \geq V@R_\alpha(L)]^1$. Furthermore, since $x I_{x \geq q} = (x - q)^+ + q I_{x \geq q}$ we get

$$CV@R_\alpha(L) = V@R_\alpha(L) + \frac{1}{1 - \alpha} \mathbb{E}[(X - V@R_\alpha(L))^+]. \tag{1}$$

¹For general distribution functions this is not true: see Rockafellar and Uryasev (2002) and Hürlimann (2002) for a detailed discussion about alternative definitions.

More generally, it is possible to prove that Eq. 1 is still valid for any α -quantile of L : that is

$$CV@R_\alpha(L) = q + \frac{1}{1 - \alpha} \mathbb{E}[(L - q)^+]$$

for any $q \in [q_\alpha^-(L), q_\alpha^+(L)]$, which evaluated for $q = V@R_\alpha(L)$, clearly gives (1). The quantity $SL_L(q) \equiv \mathbb{E}[(L - q)^+]$ is known as the *stop-loss* transform of L . The following approach can therefore be considered for computing $V@R$ and $CV@R$:

ALGORITHM 1 TWO STEPS

1. compute $V@R_\alpha(L)$, i.e. find q^* such that $\mathbb{P}(L \leq q^*) = \alpha$ (possibly not a unique solution);
2. compute $\mathbb{E}[(L - q^*)^+]$ to get $CV@R$ through formula (1).

The approximation error of $V@R$ and $CV@R$ through ALGORITHM 1 and the related speed of computation are due to the properties of the zero-finding algorithm and the numerical quadrature implemented.

An alternative characterization of $V@R$ and $CV@R$ for an arbitrary loss L , due to Rockafellar and Uryasev (2000) and Rockafellar and Uryasev (2002), is obtained as follows. For a given value $\alpha \in (0, 1)$ let us introduce the real function

$$G_{L,\alpha}(x) = x + \frac{1}{1 - \alpha} \mathbb{E}[(L - x)^+], \quad x \in \mathbb{R} \tag{2}$$

and let $\Gamma = \operatorname{argmin}_x G_{L,\alpha}(x)$ be the set of x for which the minimum value of $G_{L,\alpha}$ is attained, with Γ^- and Γ^+ respectively equal to the lower and the upper endpoint of Γ . The proof of the following Theorem can be found in Rockafellar and Uryasev (2002) (but see also Föllmer and Schied (2011), where an alternative proof based on the Fenchel-Legendre transform is proposed (Lemma 4.6)):

Theorem 1 *For any random variable L with $\mathbb{E}[|L|] < +\infty$, the function $G_{L,\alpha}(\cdot)$ is finite and convex with*

$$CV@R_\alpha(L) = \min_{x \in \mathbb{R}} G_{L,\alpha}(x).$$

Moreover Γ is a nonempty, closed, bounded interval with $q_\alpha^-(L) = \Gamma^-$ and $q_\alpha^+(L) = \Gamma^+$. In particular, one always has

$$q_\alpha(L) \in \Gamma, \quad CV@R_\alpha(L) = G_{L,\alpha}(q_\alpha(L)).$$

This theorem suggests to compute $V@R$ and $CV@R$ by solving a unique optimization problem:

ALGORITHM 2 NL-MIN

1. Solve the non-linear minimization problem $\min_x G_{L,\alpha}(x)$.

It is worth noting that for ALGORITHM 2 speed and approximation error in the computation of $V@R$ and $CV@R$ are due to the speed and precision of the univariate minimization algorithm used other than the numerical quadrature for the evaluation of the expectations.

In both the algorithms, the expectations required to compute $V@R$ and $CV@R$ must be evaluated for different values of a parameter q , namely $\mathbb{P}(L \leq q) = \mathbb{E}[\mathbb{I}_{\{L \leq q\}}]$ and $\mathbb{E}[(L - q)^+]$. This can be efficiently done by means of the Fourier transform techniques, which prove to be particularly useful also when the loss variable has

an exponential, form $L \propto \exp(X)$: in such a case, if $\mathbb{E}[e^X] < +\infty$, we define the function G as

$$G_{L,\alpha}^{(e)}(x) = x + \frac{1}{1-\alpha} \mathbb{E}[(e^X - e^x)^+] \text{ or } G_{L,\alpha}^{(e)}(x) = x + \frac{1}{1-\alpha} \mathbb{E}[(e^x - e^X)^+], \quad x \in \mathbb{R}. \quad (3)$$

Before recalling the basic properties of this class of computational methods, we now briefly introduce a simple loss model which will be used to test the computational procedures in the final Section.

An Univariate Loss Model Let us consider on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a stochastic process of the form $V_t = V_0 e^{X_t}$, $V_0 > 0$, modeling the value of a risky position for $t \in [0, T]$. The loss random variable we consider is

$$L = V_0 e^{rT} - V_T = V_0 e^{rT} - V_0 e^{X_T}$$

where r is the risk-free interest rate, that we can assume as a deterministic constant in the reference period for easiness of notation. In such a case the function $G_{L,\alpha}$ becomes

$$\begin{aligned} G_{L,\alpha} &\equiv G_{L,\alpha}^{(e)}(x) = x + \frac{1}{1-\alpha} \mathbb{E}[(V_0 e^{rT} - V_0 e^{X_T} - x)^+] \\ &= V_0 \left(\frac{x}{V_0} + \frac{1}{1-\alpha} \mathbb{E} \left[\left(\left(e^{rT} - \frac{x}{V_0} \right) - e^{X_T} \right)^+ \right] \right) \\ &= \begin{cases} x & x \geq V_0 e^{rT} \\ V_0 \left(\frac{x}{V_0} + \frac{1}{1-\alpha} \mathbb{E} \left[(e^{k(x/V_0)} - e^{X_T})^+ \right] \right) & x < V_0 e^{rT} \end{cases} \end{aligned} \quad (4)$$

where $k(v) = \log(e^{rT} - v)$. Theorem (1) still applies.

Example 2.1 Let us consider the classical log-normal model, where $X_T = (\mu - \sigma^2/2)T + \sigma W_T$, W_t being the Brownian process, $\mu \in \mathbb{R}$ and $\sigma > 0$ two given parameters. In such a case, standard calculations yield

$$V @ R_\alpha(X) = V_0 e^{rT} - V_0 e^{(\mu - \sigma^2/2)T + \sigma \sqrt{T} z_{1-\alpha}}$$

and

$$G_\alpha(x) = x + \frac{1}{1-\alpha} [(V_0 e^{rT} - x)N[-d_2(x)] - V_0 e^{\mu T} N[-d_1(x)]]$$

$$d_1(x) = \frac{1}{\sigma \sqrt{T}} \left(\log \left(\frac{V_0}{V_0 e^{rT} - x} \right) + (\mu + \sigma^2/2)T \right),$$

$$d_2(x) = \frac{1}{\sigma \sqrt{T}} \left(\log \left(\frac{V_0}{V_0 e^{rT} - x} \right) + (\mu - \sigma^2/2)T \right)$$

where $N[d]$ is the standard normal cumulative distribution function and $z_{1-\alpha}$ is the corresponding $(1 - \alpha)$ -quantile (or critical value). Finally,

$$CV @ R_\alpha(X) = V_0 (e^{rT} - e^{(\mu - \sigma^2/2)T + \sigma \sqrt{T} z_{1-\alpha}}) + \frac{V_0 e^{\mu T}}{1-\alpha} \left(e^{-\sigma^2 T/2 + \sigma \sqrt{T} z_{1-\alpha}} N[z_{1-\alpha}] - N[z_{1-\alpha} - \sigma \sqrt{T}] \right).$$

3 The Fourier Transform Method

Fourier transform methods are efficient techniques emerged in the last decades in the financial practice as one of the main methodology for the evaluation of derivatives. In fact, the no-arbitrage price of an European style contingent claim can be represented as the (conditional) expectation of the derivative payoff under a proper risk-neutral measure (see e.g. (Bjork 2004)). These methods essentially consist on the representation of such an expectation as the *convolution* of two Fourier transforms. Since the value of most derivatives depends on a trigger parameter, two main variants have been developed depending on which variable of the payoff is transformed into the Fourier space. In our setting, due to the functional (exponential) form of the transformed functions we consider, the formulas we get by applying the two approaches are essentially the same: we can pass from one to the other by simply changing the integration contour (see (Ramponi 2012)). In the following we choose to work with the generalized Fourier transform (GFT) w.r.t. the trigger parameter v . In essence, given a function $H(y, v)$, the quantity that we want to compute is $h(v) = \mathbb{E}[H(Y, v)]$, where the expectation is taken over a given probability measure \mathbb{P} . Let us consider the GFT with respect to v : formally we have for $z = u + iv \in \mathcal{C} \subset \mathbb{C}$

$$\hat{h}(z) = \int_{\mathbb{R}} e^{izv} h(v) dv = \int_{\mathbb{R}} e^{izv} \left(\int_{\mathbb{R}} H(y, v) \mathbb{P}(dy) \right) dv = \int_{\mathbb{R}} \hat{H}^{(v)}(z, y) \mathbb{P}(dy) = \mathbb{E}[\hat{H}^{(v)}(z, Y)]$$

where we have defined

$$\hat{H}^{(v)}(z, y) = \int_{\mathbb{R}} e^{izv} H(y, v) dv.$$

Notice that \hat{h} corresponds to the classical Fourier transform of the v -damped expectation, as introduced in Carr and Madan (1999): Fourier inversion gives

$$h(v) = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{-izv} \hat{h}(z) dz = \frac{1}{2\pi} \int_{iv-\infty}^{iv+\infty} e^{-izv} \mathbb{E}[\hat{H}^{(v)}(z, Y)] dz$$

for v in some strip of \mathbb{C} . The previous equalities must be justified under the appropriate conditions on the function H , its transform and the characteristic function of the underlying random variables (see e.g. (Lee 2004) for a thorough discussion on the subject). For our application we consider the following functions:

$$H_1(y, v) = (y - v)^+, \quad H_2(y, v) = \mathbb{I}_{\{y \leq v\}},$$

$$H_3(x, k) = (e^x - e^k)^+, \quad \bar{H}_3(x, k) = (e^k - e^x)^+.$$

The reason for considering an exponential transformation of the basic risk factor is the fact that many financial models are usually introduced in the form $\exp(X)$, as in Example (2.1). Their GFT are readily obtained by means of standard (complex) integration: we summarize such results in the following

Proposition 1 *Let $z = u + iv \in \mathbb{C}$, then*

$$\hat{H}_1^{(v)}(z, y) = \frac{e^{(iu-v)y}}{(iu - v)^2} = -\frac{e^{izy}}{z^2}, \quad v < 0 \tag{5}$$

$$\hat{H}_2^{(v)}(z, y) = \frac{e^{(iu-v)y}}{iu - v} = -\frac{i}{z} e^{izy}, \quad v > 0 \tag{6}$$

$$\widehat{H}_3^{(k)}(z, x) = \frac{e^{(iu-\nu+1)x}}{(iu-\nu)(iu-\nu+1)} = \frac{e^{i(z-i)x}}{iz-z^2}, \quad \nu < 0, \tag{7}$$

$$\widehat{H}_3^{(k)}(z, x) = \frac{e^{(iu-\nu+1)x}}{(iu-\nu)(iu-\nu+1)} = \frac{e^{i(z-i)x}}{iz-z^2}, \quad \nu > 1. \tag{8}$$

Let $\phi_Y(z) = \mathbb{E}[e^{izY}]$, $z \in \mathbb{C}$ be the (generalized) characteristic function of the r.v. Y . The following result can be proved as in Lee (2004).

Theorem 2 (a) *If $\mathbb{E}[e^{-\nu Y}] < \infty$, $\nu < 0$, then*

$$H_1 = \mathbb{E}[(Y - \nu)^+] = -\frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-iz\nu} \frac{\phi_Y(z)}{z^2} dz = -\frac{1}{\pi} \int_{i\nu-0}^{i\nu+\infty} \Im \left\{ e^{-iz\nu} \frac{\phi_Y(z)}{z^2} \right\} dz; \tag{9}$$

(b) *Let $\bar{H}_2 = \frac{1}{2}(\mathbb{P}(Y \leq \nu) + \mathbb{P}(Y < \nu))$. If $\mathbb{E}[e^{-\nu Y}] < \infty$, $\nu > 0$, then*

$$\bar{H}_2 = \frac{1}{2\pi} \lim_{M \rightarrow +\infty} \int_{i\nu-M}^{i\nu+M} e^{-iz\nu} \frac{i}{z} \phi_Y(z) dz = \frac{1}{\pi} \int_{i\nu-0}^{i\nu+\infty} \Im \left\{ e^{-iz\nu} \frac{i}{z} \phi_Y(z) \right\} dz; \tag{10}$$

(c) *If $\mathbb{E}[e^{(-\nu+1)X}] < \infty$, $\nu < 0$, then*

$$H_3 = \mathbb{E}[(e^X - e^k)^+] = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \frac{\phi_X(z-i)}{iz-z^2} dz = \frac{1}{\pi} \int_{i\nu-0}^{i\nu+\infty} \Im \left\{ e^{-izk} \frac{\phi_X(z-i)}{iz-z^2} \right\} dz; \tag{11}$$

(d) *If $\mathbb{E}[e^{(-\nu+1)X}] < \infty$, $\nu > 1$, then*

$$\bar{H}_3 = \mathbb{E}[(e^k - e^X)^+] = \frac{1}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izk} \frac{\phi_X(z-i)}{iz-z^2} dz = \frac{1}{\pi} \int_{i\nu-0}^{i\nu+\infty} \Im \left\{ e^{-izk} \frac{\phi_X(z-i)}{iz-z^2} \right\} dz. \tag{12}$$

Remark 3.1 It is worth noting that formula (10) is a slight generalization of the well-known Lévy’s Inversion (and Gil-Pelaez) formulas, that can be obtained by using the Residue Theorem. Furthermore, in the framework of the univariate loss model discussed in Example (21), we clearly have $\mathbb{P}(L \leq q) = 1 - \mathbb{P}(e^{X_T} < e^k)$, where $k = \log((V_0 e^{rT} - q)/V_0)$ for $q < V_0 e^{rT}$, and we notice that the probability of the events $\{e^X \leq e^k\}$ has the same integral representation as (10).

We may finally put together the previous results to obtain an integral representation for the functions $G_{L,\alpha}$ and $G_{L,\alpha}^{(e)}$:

Proposition 2 *Let L be a loss random variable with $\mathbb{E}[|L|] < +\infty$, $G_{L,\alpha}$ and $G_{L,\alpha}^{(e)}$ the functions defined in (2) and (3). Then, under the hypothesis of Proposition (1) and Theorem (2), we have*

$$\begin{aligned} G_{L,\alpha}(x) &= x - \frac{1}{(1-\alpha)2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{-izx} \frac{\phi_Y(z)}{z^2} dz \\ &= x - \frac{e^{\nu x}}{(1-\alpha)\pi} \int_0^{+\infty} \Im \left(e^{-iux} \frac{\phi_Y(u+i\nu)}{(u+i\nu)^2} \right) du, \quad \nu < 0 \end{aligned} \tag{13}$$

or, in the case of exponential models,

$$G_{L,\alpha}^{(e)}(x) = x + \frac{1}{(1-\alpha)2\pi} \int_{iv-\infty}^{iv+\infty} e^{-izx} \frac{\phi_X(z-i)}{iz-z^2} dz$$

$$= x + \frac{e^{\nu x}}{(1-\alpha)\pi} \int_0^{+\infty} \Re \left(e^{-iux} \frac{\phi_X(u+i(\nu-1))}{\nu^2-\nu-u^2+iu(1-2\nu)} \right) du, \quad \nu > 1, \text{ or } \nu < 0. \quad (14)$$

The computation of the functions $G_{L,\alpha}$ and $G_{L,\alpha}^{(e)}$ requires the use of quadrature algorithms, see Fusai and Roncoroni (2008) for a review of standard techniques in a financial framework. The development and the analysis of numerical methods to efficiently evaluate Fourier integrals have been an active field of research during the last years: the choices of the proper integral representation formula of the stop-loss transform of L through the Residue Theorem and the related integration contour (i.e. the damping parameter ν) and the quadrature algorithm permit to compute accurately the value of our target functions. Transformation of the integration domain, the use of adaptive quadrature algorithms, the optimal choice of the damping parameter have been studied in several papers (see e.g. Schmelzle 2010 and the reference therein). Even more recently, methods based on wavelets approximation and Fourier-cosine expansion have been exploited to efficiently implement Fourier inversion, see Ortiz-Gracia and Oosterlee (2014). These kind of methods permits to compute the value of the integral (or equivalently the stop-loss transform), for each value of the parameter x . On the other hand, the standard technique for fast pricing of options relies on the FFT as firstly introduced in the seminal paper by Carr and Madan (1999): essentially it permits to approximately compute a whole range of values in a unique run: in our application this implies to compute the values of the function $G_{L,\alpha}$ or $G_{L,\alpha}^{(e)}$ for a proper range of the variable x at the same time. We briefly review here the main steps of such techniques.

Fast and Fractional Fourier Transforms The FFT technique involves two steps:

- a numerical quadrature scheme to approximate through a N -point sum the integral

$$I(x) = \frac{1}{\pi} \int_0^{+\infty} \Re \left[e^{-iux} F(u) \right] du.$$

By using an equi-spaced grid $\{u_n\}_{n=1,\dots,N}$ of the line $\{z = u + iv \in \mathbb{C} : u \in \mathbb{R}^+, v = \nu\}$ with spacing Δ , we have

$$I(x) \approx \Sigma_N(x) = \frac{\Delta}{\pi} \sum_{n=0}^{N-1} \Re \left[e^{-iu_n x} F(u_n) w_n \right],$$

where w_n are the integration weights²;

- given a grid $x_m = x_1 + \gamma m, m = 0, \dots, N-1$, denoted by \mathbf{x} , the sum $\Sigma_N(x_m)$ is written as a discrete Fourier transform (DFT) when

$$\Delta \cdot \gamma = \frac{2\pi}{N} \quad (15)$$

that is

$$\Sigma_N(x_m) = \frac{\Delta}{\pi} \sum_{n=0}^{N-1} e^{-inm\Delta\gamma} e^{-in\Delta x_1} F(n\Delta) w_n = \frac{\Delta}{\pi} \sum_{n=0}^{N-1} e^{-inm\frac{2\pi}{N}} h_n$$

where

$$h_n = e^{-in\Delta x_1} F(n\Delta) w_n. \quad (16)$$

The integral $I(x)$ is therefore approximated over the grid \mathbf{x} as $I(x_m) \approx \Sigma_N(x_m)$ that can be efficiently computed by means of the Fast Fourier Transform algorithm, $I(\mathbf{x}) \approx FFT(\mathbf{x}, \mathbf{h})$. The required values are computed with $O(N \log(N))$ operations. A thorough discussion on sampling and truncation errors is found in Lee (2004).

The condition $\Delta \cdot \gamma = 2\pi/N$ imposes that if we refine the integration grid (Δ small), the range for the variable x becomes larger, thus including values which cannot be useful in our valuation procedure. Fractional Fourier transform (FRFT) permits on the contrary to decouple the two steps: it is therefore possible to choose properly the integration range and the x -spacing grid. The resulting algorithm, introduced in the financial literature in Chourdakis (2005), involves the use of standard FFT: in terms of the number of elementary operations, the computational cost of a FRFT procedure with N -point, N -FRFT, is about the same as a $4N$ -FFT. The advantage of running a FRFT with smaller N is that it may achieve the same accuracy than a FFT with much larger N .

The m -th component of the η -fractional discrete Fourier transform of the vector \mathbf{h} is defined as

$$FRFT(\mathbf{h}, \eta)_m = \sum_{n=0}^{N-1} e^{-i2\pi nm\eta} h_n, \quad k = 0, \dots, N - 1$$

with $\eta = \Delta\gamma/2\pi$. The algorithm works as follows: firstly define two $2N$ -point vectors

$$\mathbf{y} = (y_0, \dots, y_{n-1}, y_n, \dots, y_{2n}), \quad y_j = h_j e^{-i\pi j^2 \eta}, \quad 0 \leq j < n - 1, y_j = 0, \quad n \leq j < 2n,$$

$$\mathbf{z} = (z_0, \dots, z_{n-1}, \bar{z}_n, \dots, \bar{z}_{2n}), \quad z_j = e^{i\pi j^2 \eta}, \quad 0 \leq j < n - 1, \bar{z}_j = e^{i\pi(n-j)^2 \eta}, \quad 0 \leq j < n - 1.$$

The m -th component of $FRFT(\mathbf{h}, \eta)_k$ is then computed as

$$FRFT(\mathbf{h}, \eta)_m = e^{i\pi m^2 \eta} \odot FFT_m^{-1}(FFT(\mathbf{y})_m \odot FFT(\mathbf{z})_m), \quad m = 1, \dots, n,$$

where FFT^{-1} is the inverse fast Fourier transform and \odot is the component-wise vector multiplication. As before, the integral $I(x)$ can be approximated over the grid \mathbf{x} by means of the Fractional Fourier Transform algorithm, $I(\mathbf{x}) \approx FRFT(\mathbf{x}, \mathbf{h}, \eta)$

4 Implementation and Results

The numerical procedures outlined in Section 2 for the computation of $V@R$ and $CV@R$ both require to evaluate the distribution function and/or the stop-loss expectation of the random variable L . ALGORITHM 1 firstly calls for a zero-finding routine that needs to compute (a sequence of) values of $F_L(\cdot)$ and then $SL(L)$ must be evaluated. ALGORITHM 2 must solve iteratively a univariate minimization problem requiring at each step to compute an expectation. Efficient numerical quadratures for computing Fourier integrals suffice for implementing the two algorithms without the need of calling for fast Fourier transforms. Error bounds on the approximation obtained clearly depends on the computational methods chosen for numerical integration, zero-finding and minimization routines. A general result on the behavior of the error can be simply outlined by exploiting the properties of the

²Different spacing rules can be implemented, e.g. the midpoint rule.

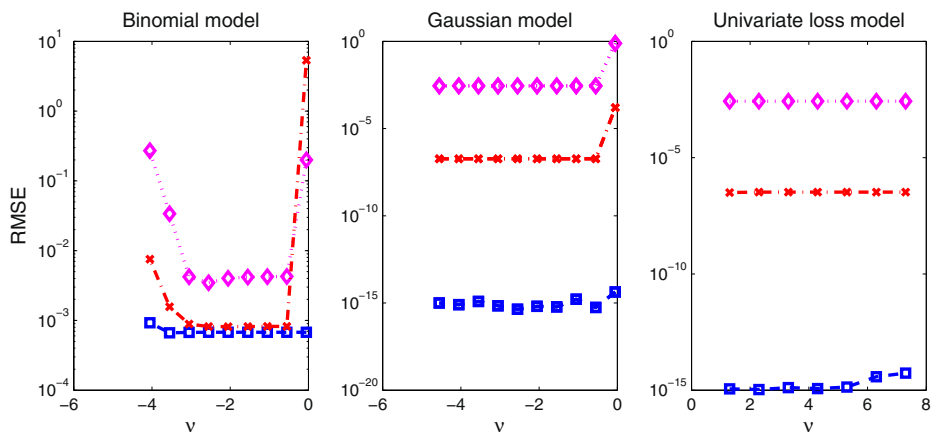


Fig. 1 RMSE between the functions $G_{L,\alpha}(x)$ and $G_{L,\alpha}^e(x)$ and their approximation evaluated on a finite grid, as a function of the damping parameter ν for the three toy models. The squared-blue lines refer to the adaptive Lobatto quadrature, the star-red and the diamond-magenta to the FRFT and FFT algorithms, respectively

convex function $G_{L,\alpha}$. As a matter of fact, let \bar{x} and x^* be given points: since it can be proved that (see Rockafellar and Uryasev 2002, Theorem 10)

$$\frac{\mathbb{E}[(L - \bar{x})^+ - (L - x^*)^+]}{\bar{x} - x^*} = -(1 - F_L(\bar{x})) - \rho(\bar{x}, x^*)(F_L(\bar{x}) - F_L(x^*))$$

Table 1 Comparison between the Fourier-based numerical procedures for the gaussian model (upper table), $X \sim N(\mu, \sigma^2)$ and the binomial model (lower table), $X \sim B(n, p)$, with $\alpha = 0.99$

	V@R Abs Err	CV@R Abs Err	Relative Time
ALG 1	0.53×10^{-14}	0	1
ALG 2	0.33×10^{-07}	0	0.3
2^{12} -FFT	0.15×10^{-02}	0.32×10^{-05}	0.7×10^{-05}
2^{10} -FRFT	0.14×10^{-03}	0.27×10^{-07}	0.2×10^{-05}
$n = 5, p = 0.1$	V@R Abs Err	CV@R Abs Err	Relative Time
ALG 2	0.0092	0.0027	1
2^{12} -FFT	0.0106	0.0025	0.0092
2^{10} -FRFT	0.0176	0.0047	0.0014

The second and third columns report absolute errors for the V@R and CV@R with respect to the true values, as obtained by applying zero-finding, univariate minimization (golden section search) and quadrature (adaptive Lobatto algorithm) build-in functions of MatLab. In view of the inversion results for discontinuous distribution functions, we did not apply Alg 1 to the binomial case. In these experiments, the zero-finding algorithm had starting point equal to mean of the r.v.; the univariate minimization requires a starting interval set to $[0, n]$ and $[0, \mu + 3\sigma]$, respectively. The FFT and FRFT algorithms have input vectors of length 2^N ; the integral was approximated between 0 and 100 in the gaussian case and 0 and 200 in the binomial case; the x -grid was started at $x_1 = 0$, with $\gamma = 0.004$ for FRFT. In the last column the relative CPU times are shown, normalized to the slowest algorithm (Alg 1)

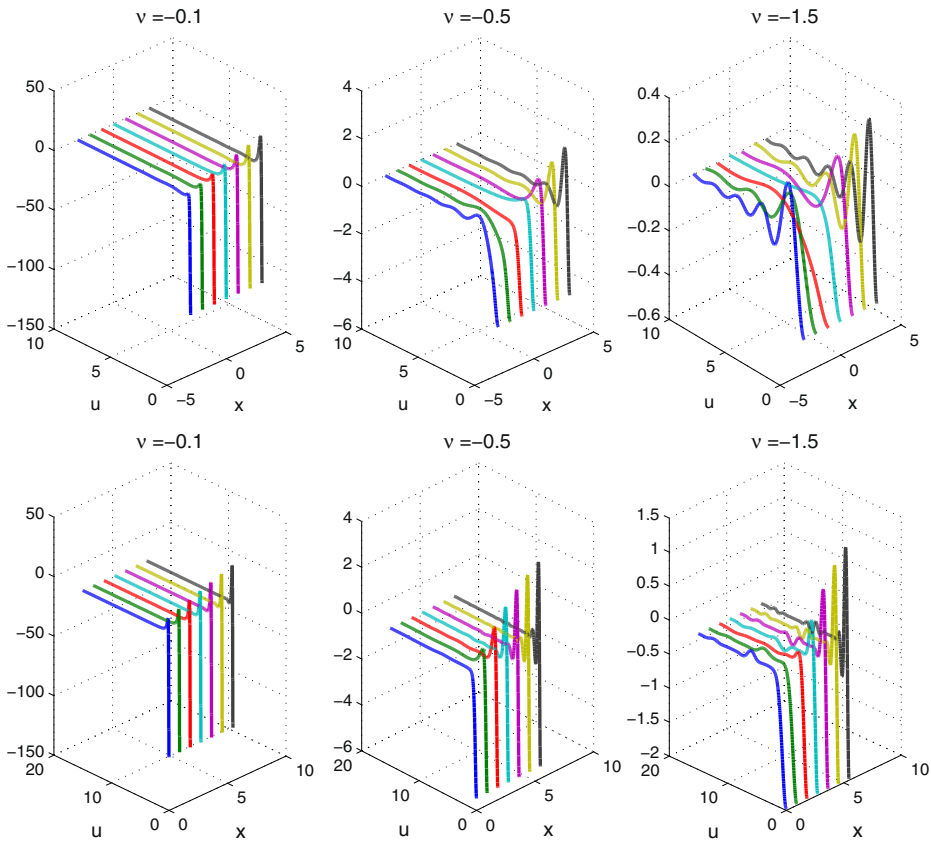


Fig. 2 Integrand of function (13) for the Gaussian and the binomial model

with $\rho(\bar{x}, x^*) \in (0, 1)$, then

$$|G_{L,\alpha}(\bar{x}) - G_{L,\alpha}(x^*)| = |\bar{x} - x^*| \frac{|(F_L(\bar{x}) - \alpha) - \rho(\bar{x}, x^*)(F_L(\bar{x}) - F_L(x^*))|}{1 - \alpha}.$$

If we choose $x^* = V@R_\alpha(L)$ (i.e. the true minimizer of $G_{L,\alpha}$), which implies that $F_L(x^*) = \alpha$ and $G_{L,\alpha}(x^*) = CV@R_\alpha(L)$, and \bar{x} as an approximated minimizer, then

$$|G_{L,\alpha}(\bar{x}) - CV@R_\alpha(L)| \leq |\bar{x} - V@R_\alpha(L)| \frac{|F_L(\bar{x}) - \alpha|}{1 - \alpha} < |\bar{x} - V@R_\alpha(L)|.$$

More generally the function $G_{L,\alpha}$ too is approximately evaluated through an N -point sum defined over a given grid, that is $G_{L,\alpha}(\bar{x}) \approx \tilde{G}_{L,\alpha}(\bar{x})$: therefore

$$\begin{aligned} |\tilde{G}_{L,\alpha}(\bar{x}) - CV@R_\alpha(L)| &\leq |\tilde{G}_{L,\alpha}(\bar{x}) - G_{L,\alpha}(\bar{x})| + |G_{L,\alpha}(\bar{x}) \\ &\quad - CV@R_\alpha(L)| \leq |\tilde{G}_{L,\alpha}(\bar{x}) - G_{L,\alpha}(\bar{x})| + |\bar{x} - V@R_\alpha(L)|. \end{aligned}$$

The error term $|\tilde{G}_{L,\alpha}(\bar{x}) - G_{L,\alpha}(\bar{x})|$ depends of course on the numerical quadrature scheme applied: as it has been discussed in details in Lee (2004) in the Fourier inversion framework, it may be further decomposed in sampling error (due to the discrete evaluation of the integrand in a finite grid) and truncation error (due the finiteness of the upper

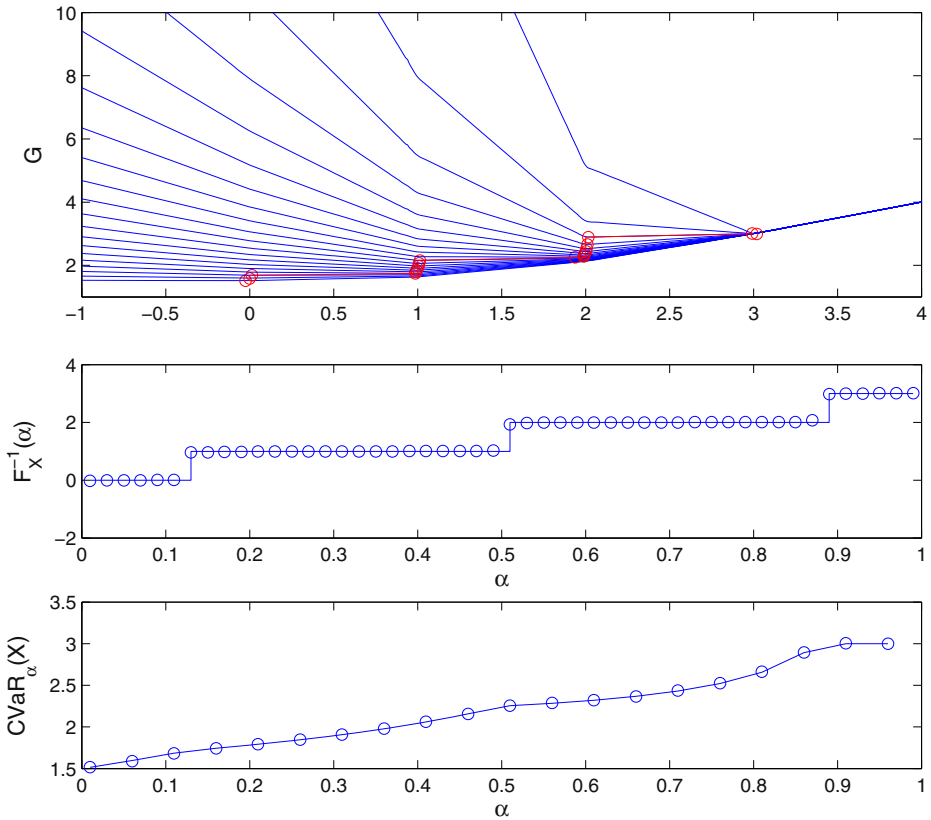


Fig. 3 Binomial model, $X \sim Binomial(N, p)$. In the upper plot, the functions G and the corresponding solutions of the minimization problem, for $\alpha \in [0.01, 0.99]$. In the lower plots, true $V@R$ and $CV@R$ (continuous lines) and the corresponding values ('o') computed by solving the minimization problem

integration limit). For both errors bounds are available depending on the payoff and the characteristic function of the random variable L .

A third algorithm can be outlined, providing a “quick-and-dirty” solution, based on FFT/FRFT. Since we want to minimize w.r.t. x the functions $G_{L,\alpha}(x)$ or $G_{L,\alpha}^{(e)}(x)$ we can approximate them for a whole range of values \mathbf{x} by applying once FFT/FRFT: from the integral representation (13) and (14), we get

$$G_{L,\alpha}(\mathbf{x}) \approx \hat{G}_\alpha(\mathbf{x}) \equiv \begin{cases} \mathbf{x} - \frac{e^{i\mathbf{v}\mathbf{x}}}{(1-\alpha)\pi} \odot FFT(\mathbf{x}, \mathbf{h}) \\ \mathbf{x} - \frac{e^{i\mathbf{v}\mathbf{x}}}{(1-\alpha)\pi} \odot FRFT(\mathbf{x}, \mathbf{h}, \eta) \end{cases}$$

and

$$G_{L,\alpha}^{(e)}(\mathbf{x}) \approx \hat{G}_\alpha^{(e)}(\mathbf{x}) \equiv \begin{cases} \mathbf{x} + \frac{e^{i\mathbf{v}\mathbf{x}}}{(1-\alpha)\pi} \odot FFT(\mathbf{x}, \mathbf{h}) \\ \mathbf{x} + \frac{e^{i\mathbf{v}\mathbf{x}}}{(1-\alpha)\pi} \odot FRFT(\mathbf{x}, \mathbf{h}, \eta) \end{cases}$$

where \mathbf{h} is the vector defined in (16), evaluated according to the functions in (13) or (14). Hence, the minimum and the corresponding minimizer of the vector $\hat{G}_{L,\alpha}$ will provide approximated values for $CV@R$ and $V@R$, respectively.

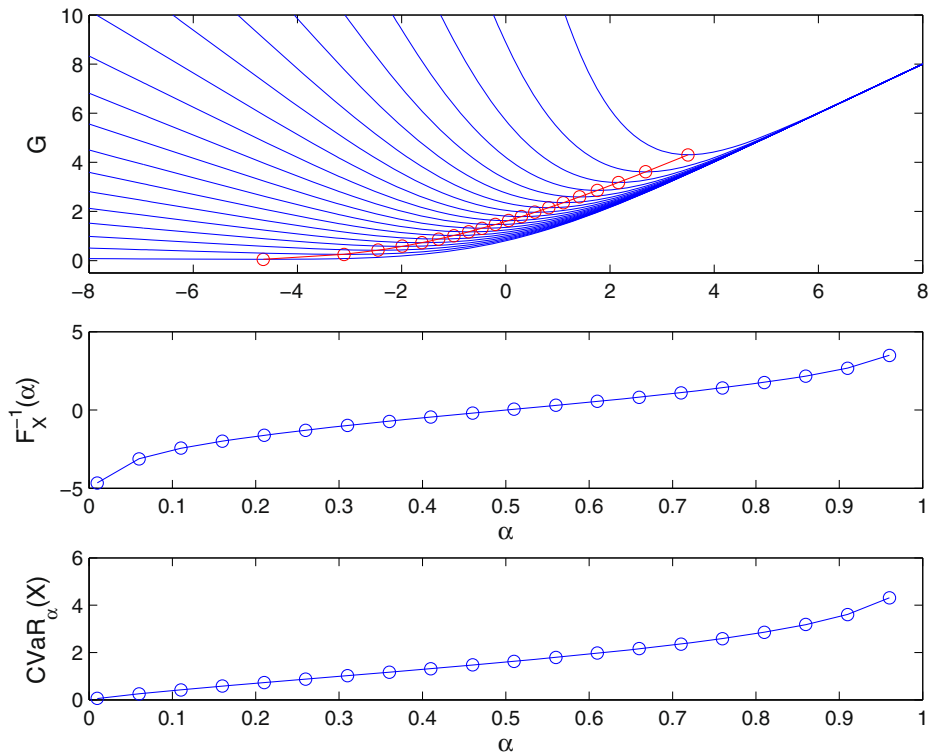


Fig. 4 Gaussian model, $X \sim N(\mu, \sigma^2)$. In the upper plot, the functions G and the corresponding solutions of the minimization problem, for $\alpha \in [0.01, 0.99]$. In the other plots, true V@R and CVaR (continuous lines) and the corresponding values ('o') computed by solving the minimization problem

ALGORITHM 3 FFT/FRFT

1. Compute the vector $\hat{G}_{L,\alpha}$ through FFT/FRFT algorithm for a proper grid x ;
2. Find the minimum and the corresponding minimizer of $\hat{G}_{L,\alpha}$

Efficiency and reliability of each procedure depend on an appropriate choice of the integration contour, i.e. the damping parameter ν : this is a well recognized need in the quantitative finance literature, since by changing ν the integrand can become highly peaked or strongly oscillatory. The hypothesis on the integral representation for the function G require the finiteness of certain moments which implies the existence of a range of possible values of the form (ν_-, ν_+) . Once these bounds have been determined for the considered model, an optimization problem can be setup taking into account for error bounds: see Lee (2004) and Lord and Kahl (2007) where the bounds for most of the models we used in the following experimental results have been derived. Furthermore, for our problem we need to select a “good value” of ν for every point x . We noticed in our numerical experiments that a preliminary study of the behavior of the functions $G_{L,\alpha}(x)$ and $G_{L,\alpha}^\ell(x)$ permits to select a proper range in which the minimum is located: hence the integrands can be analyzed for each value of x and a suitable value of ν may be selected according to a given criterion. In particular we

Table 2 Comparison between the numerical procedures for the benchmark univariate loss model - Example (21) - with $\alpha = 0.99$

$\mu = 0, \sigma = 0.2, T = \frac{1}{4}$	V@R Abs Err	CV@R Abs Err	Relative Time
ALG 1	0.11×10^{-15}	0.26×10^{-14}	1
ALG 2	0.37×10^{-08}	0.22×10^{-15}	0.4628
2^{12} -FFT	0.0011	0.0017	0.44×10^{-03}
2^{10} -FRFT	0.14×10^{-03}	0.22×10^{-05}	0.58×10^{-03}
$\mu = -0.8, \sigma = 0.35, T = \frac{1}{12}$	V@R Abs Err	CV@R Abs Err	Relative Time
ALG 1	0	0.55×10^{-15}	1
ALG 2	0.36×10^{-08}	0	0.3583
2^{12} -FFT	0.005	0.0004	0.0011
2^{10} -FRFT	0.88×10^{-04}	0.23×10^{-05}	0.0013

The second and third columns report absolute errors with respect to the true values for the V@R and CV@R. In these experiments, the zero-finding algorithm had starting point equal to the mid point of the interval $[0, V_0 e^{rT}]$, while for the univariate minimization the starting interval was set to $[0, e^{rT}]$. The FFT and FRFT algorithms have input vectors of length 2^N ; the integrals were approximated between 0 and 100; the x -grid has right end point at $\log(V_0 * \exp(rT))$, with $\gamma = 6.7 \times 10^{-04}$ for FRFT. In the last column the relative CPU times are shown, normalized to the slowest algorithm (Alg 1)

implemented the procedure suggested in Lord and Kahl (2007) consisting in the minimization of the damped integrand w.r.t. ν , but we didn't observe a substantial improvement in our minimization results over an empirical ad-hoc selection of ν , at least for the considered models (see Fig. 1). Of course precision and speed depend on the algorithms implemented, the programming language and on the computer available. In our experiment we used MatLab R2012a on a Intel Core i5 CPU with 2.40 GHz. The accuracy in double precision is $2.2204e-16$. The basic steps of the algorithms (quadrature, univariate minimization, zero-finding and FFT) are those available as MatLab build-in functions. In particular we used an adaptive Lobatto quadrature and a minimization routine based on golden section search and parabolic interpolation.

4.1 Toy Models Results

In this section we report some results obtained by applying the computational procedures outlined above, that is Algorithm 1, 2 and 3, in three simple models. We considered a Gaussian and a binomial variates (Table 1 and Figs. 1, 2, 3, 4) the univariate loss model framework of Example (2.1) (Table 2). In this case, due to the analytical form of the function to be minimized (4), we applied integral representation w.r.t. the scaled variable x/V_0 . In order to keep a sensible number of figures and tables, we show only few cases, but the behavior of the procedures is almost the same for a wide range of parameters and also for different choices of the random variables. Exact results are readily available in all these models. Pros and cons of each algorithm may be easily summarized: Algorithm 1 is the slowest and the most accurate but it cannot be safely applied in the discontinuous case, at least by using standard numerical inversion of the Fourier representation of the cdf. As a matter of fact the Gibbs phenomenon can seriously affect the computations (see Remark 3.1): more sophisticated techniques should be applied in this case, such that (the filtered-COS method) proposed in Ortiz-Gracia and Oosterlee (2014). Algorithm 2 proves to be a bit faster and accurate enough in all the cases while Algorithms 3 are certainly the fastest

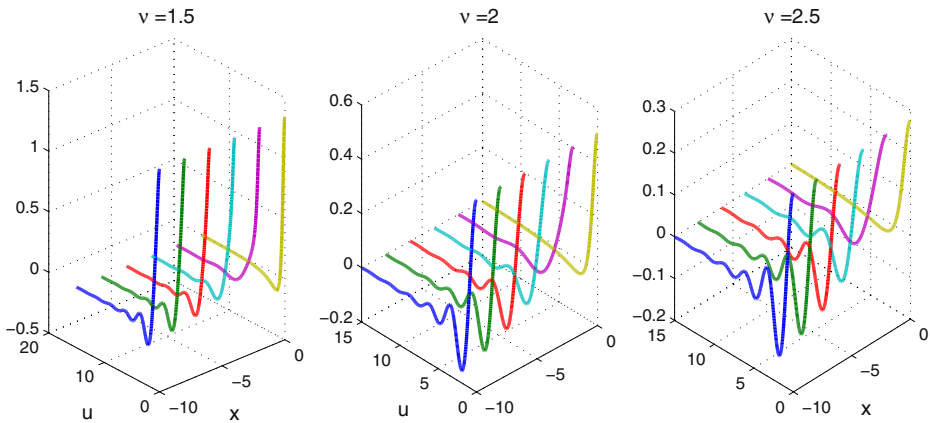


Fig. 5 Integrand of function (14) for the univariate loss model with $\mu = 0$ and $\sigma = 0.2$

but the less precise. In our implementation the slowest algorithm computes the two quantities in about 1.5 seconds. Finally, we observe that the choice of the damping parameter for these models is less critical in the case of continuous random variables, as it can be seen in Figs. 1, 2 and 5.

Remark 3.1 It is worth noting that the reconstruction of a discrete cdf (in particular its generalized inverse) in our numerical examples through the minimization of the function G does not suffer the well-known Gibbs phenomenon (see Fig. 3), which on the contrary affects the direct use of Fourier representation for discontinuous functions. In fact, in our approach, $F_X^{-1}(\alpha)$ is characterized as the minimum of a continuous function (see Theorem 1) and the Fourier representation is only used to represent such a function. This phenomenon in turn may have a serious impact on the estimation of quantiles: for this reason we didn't apply Algorithm 1 to the binomial random variable.

4.2 Loss Models with Jumps and Stochastic Volatility

In the second set of experiments, we show the effectiveness of the considered computational procedures in the univariate loss model by assuming different dynamics for X_T and evaluating the impact of the relevant parameters on the computation of $V@R$ and $CV@R$. The instances we consider are Lévy models with finite activity (Merton Jump-Diffusion) and infinite activity (Variance Gamma), and stochastic volatility models (Heston model) with jumps (Regime Switching Jump-Diffusion). But our procedure applies to all models characterized by having a computable (generalized) characteristic function. The bounds for the damping parameter ν can be found in Lord and Kahl (2007). In such a cases, we used a hybrid approach consisting in two steps:

1. FRFT approximation of the function $G_{L,\alpha}^{(e)}$ for determining a proper interval in which the minimum is located and for automatically finding a feasible starting point x_0 ;
2. refinement of the previous estimate by starting from x_0 a local minimization routine.

In the following we simply report the GCF of the considered dynamics. Details can be found e.g. in Cont and Tankov (2003).

Merton Jump-Diffusion Model We consider a jump-diffusion setting in which the jump process is described as a marked point process (MPP). Let $\mu : \mathcal{S} \rightarrow \mathbb{R}$, $\sigma : \mathcal{S} \rightarrow \mathbb{R}$ and $\gamma : E \times \mathcal{S} \rightarrow \mathbb{R}$ be given functions, (E, \mathcal{E}) being the measurable mark space. Without loss of generality, we can assume in the following $E \subseteq \mathbb{R}$. In the given interval $0 \leq t \leq T$, we consider therefore the dynamic

$$X(t) = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) + \int_0^t \int_E \gamma(y) p(dy, ds), \tag{17}$$

where $W(t)$ is a standard Brownian motion and $p(dy, dt)$ is a MPP characterized by the intensity

$$\lambda_t(dy) \equiv \lambda m(dy).$$

Here λ represents the intensity of the Poisson process N_t , while $m(dy)$ is a probability measures on E which specifies the jump variable Y . We assume that $W(\cdot)$ and $p(dy, dt)$ are independent and that $\mathbb{E}[e^{\gamma(Y)}] = \int_E e^{\gamma(y)} m(dy)$ is finite. The function $\gamma(y)$ represents the jump amplitude relative to the mark y : without loss of generality, we set $\gamma(y) = y$ in the following. The GCF for X_T is then given by

$$\phi_{X_T}(z) = e^{i(\mu - \sigma^2/2)Tz - \sigma^2 T z^2/2 + \lambda T(\phi_Y(z) - 1)}$$

where $\phi_Y(z) = \mathbb{E}[e^{izY}]$. In the numerical example, we consider jumps characterized by a Normal distribution, $Y \sim N(a, b)$ so that

$$\phi_Y(z) = e^{iaz - b^2 z^2/2}.$$

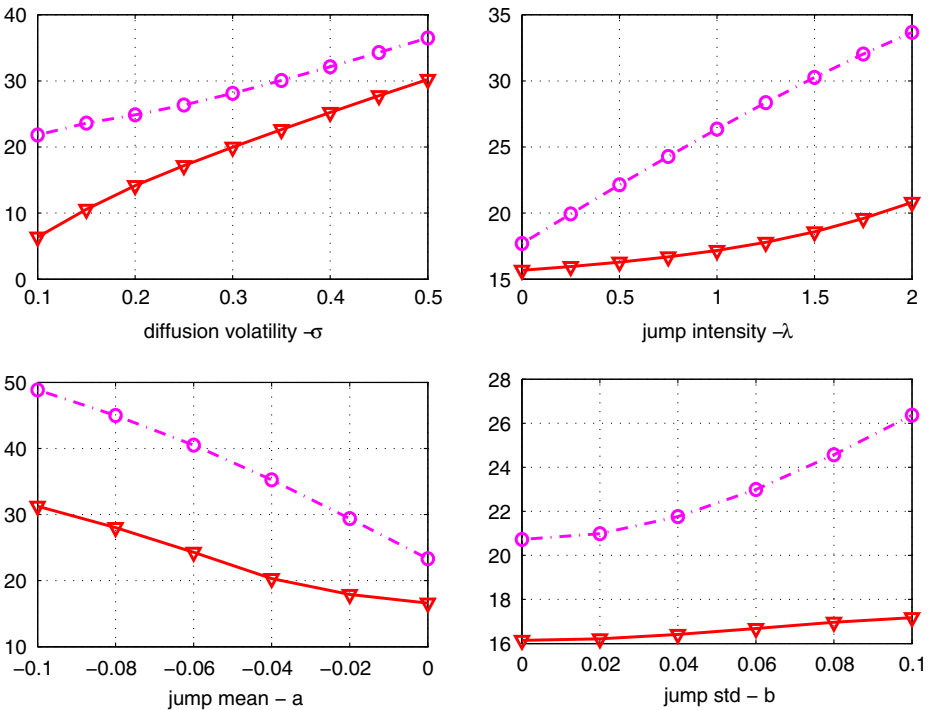


Fig. 6 V@R (triangle) and CV@R (circle) for varying parameters of the Merton jump-diffusion model with $V_0 = 100$, $r = 0$, $T = 1/12$, $\mu = 0$, $\sigma = 0.25$, $a = -0.01$, $b = 0.1$ and $\lambda = 1$

V@R and CV@R obtained by varying the diffusion volatility σ , the jump intensity λ and the jump parameters a and b are reported in Fig. 6.

VG Model The Variance Gamma model was introduced in Madan and Seneta (1990) and represents one of the simpler example of infinite activity Lévy model for describing an asset value dynamic. It can be defined as a Brownian motion with drift, where time is changed by an independent gamma process with mean rate unity and variance rate ν , $G(t; 1, \nu)$:

$$X_t = \theta G(t; 1, \nu) + \sigma W_{G(t;1,\nu)}.$$

It has three parameters θ, σ, ν and the characteristic function is given by

$$\phi_{X_T}(z) = \left(\frac{1}{1 - i\theta\nu z + \sigma^2\nu z^2/2} \right)^{T/\nu}.$$

In Fig. 7 the behavior of V@R and CV@R are compared for different values of the parameters.

Regime-Switching Jump-Diffusion Model We consider a jump-diffusion model the parameters of which are driven by a finite state and continuous time Markov chain. To be definite, let $\omega(t)$ be a continuous time, homogeneous and stationary Markov chain on the state space $\mathcal{S} = \{1, 2, \dots, M\}$ with a generator $Q \in \mathbb{R}^{M \times M}$: the elements q_{ij} of the matrix Q are positive numbers such that $\sum_{j \neq i, j=1}^M q_{ij} = -q_{ii}$, for $i = 1, \dots, M$. The jump-diffusion dynamic is then modified as

$$X(t) = \int_0^t (\mu(\omega(s)) - \frac{1}{2}\sigma^2(\omega(s)))ds + \int_0^t \sigma(\omega(s))dW(s) + \int_0^t \int_E \gamma(y, \omega(t^-))p^\omega(dy, ds), \tag{18}$$

where $p^\omega(dy, dt)$ is a MPP characterized by the regime-switching intensity $\lambda_i^\omega(dy) \equiv \lambda(\omega)m(\omega, dy)$, $m(\cdot, dy)$ being a set of probability measures on E , one for each state (regime) $i \in \mathcal{S}$. The function $\gamma(y, \omega)$ represents the jump amplitude relative to the mark y in regime ω . We assume that the processes $\omega(\cdot)$ and $W(\cdot)$ are independent, $W(\cdot)$ and $p^\omega(dy, dt)$ are conditionally independent given $\omega(t)$ and that $\mathbb{E}[e^{\gamma(Y,\omega)}] = \int_E e^{\gamma(y,\omega)}m(\omega, dy)$ is finite for each regime ω .

In Ramponi (2012) (see also Chourdakis 2005) it was proved the following

Proposition 3 Let $\phi_j(z) = \mathbb{E}[e^{iz\gamma(Y(j),j)}]$ be the generalized Fourier transform of the jump magnitude under the historical measure. Then, by letting

$$\vartheta_j(z) = z(\mu(j) - \frac{1}{2}\sigma^2(j)) + \frac{1}{2}iz^2\sigma^2(j) - i\lambda(j)(\phi_j(z) - 1) \tag{19}$$

and $\tilde{\vartheta}_i(z) = \vartheta_j(z) - \vartheta_M(z)$, we have

$$\begin{aligned} \phi_{X_T}(z) &= e^{i\vartheta_M(z)T} \left(\mathbf{1}' \cdot e^{(Q' + i \text{diag}(\tilde{\vartheta}_1(z), \dots, \tilde{\vartheta}_{M-1}(z), 0))T} \cdot \mathbb{I}(0) \right) \\ &= \mathbf{1}' \cdot e^{(Q' + i \text{diag}(\vartheta_1(z), \dots, \vartheta_M(z)))T} \cdot \mathbb{I}(0), \end{aligned} \tag{20}$$

where $\mathbf{1} = (1, \dots, 1)' \in \mathbb{R}^{M \times 1}$, $\mathbb{I}(0) = (\mathbb{I}_{\omega(0)=1}, \dots, \mathbb{I}_{\omega(0)=M})' \in \mathbb{R}^{M \times 1}$ and Q is the transpose of Q .

Simple linear constraints on the full parameter set of RSJD dynamic (18) permit to specify different models: from a regime-switching without jumps (the Naik model (Naik 1993)

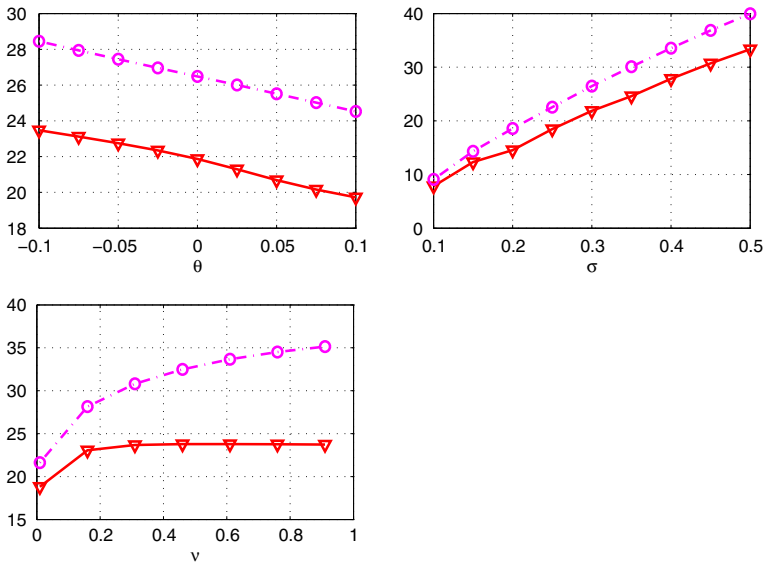


Fig. 7 V@R (triangle) and CV@R (circle) for varying parameters of the *Variance Gamma* model with $V_0 = 100$, $r = 0$, $T = 1/12$, $\theta = 0$, $\sigma = 0.3$, $\nu = 0.1$

- RSGBM) to a unique regime jump-diffusion model (JDM), which includes the standard geometrical Brownian motion (GBM).

The evaluation of the characteristic function requires to compute matrix exponentials for which efficient numerical techniques are available (Higham 2009); conversely, the case $M = 2$ can be considered explicitly. The following can be proved (see Ramponi (2012) and the references therein).

Proposition 4 Let $y_{1,2}$ be the solutions of the quadratic equation $y^2 + (q_1 + q_2 - i\theta)y - i\theta q_2 = 0$ and

$$q_1^T(\theta) = \frac{1}{y_1 - y_2} (e^{y_1 T} (y_1 + q_1 + q_2) - e^{y_2 T} (y_2 + q_1 + q_2))$$

$$q_2^T(\theta) = \frac{1}{y_1 - y_2} (e^{y_1 T} (y_1 + q_1 + q_2 - i\theta) - e^{y_2 T} (y_2 + q_1 + q_2 - i\theta)).$$

Then

$$e_t[e^{i\theta T_1}] = \mathbb{I}_{\omega(t)=1} q_1^T(\theta) + \mathbb{I}_{\omega(t)=2} q_2^T(\theta)$$

and therefore

$$\varphi_{X_T}(z) = e^{i\theta_2(z)T} \left(\mathbb{I}_{\omega(t)=1} q_1^T(\theta(z)) + \mathbb{I}_{\omega(t)=2} q_2^T(\theta(z)) \right)$$

Numerical tests are reported in Figs. 8, 9 for a two-state model. In order to single out the effects of the switching parameters, we firstly consider the RSGMB model, thus discarding the jump component ($\lambda_1 = \lambda_2 = 0$) - Fig. 8; then we fix the diffusive dynamic ($\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$) and vary the jump parameters according to the switching model - Fig. 9.

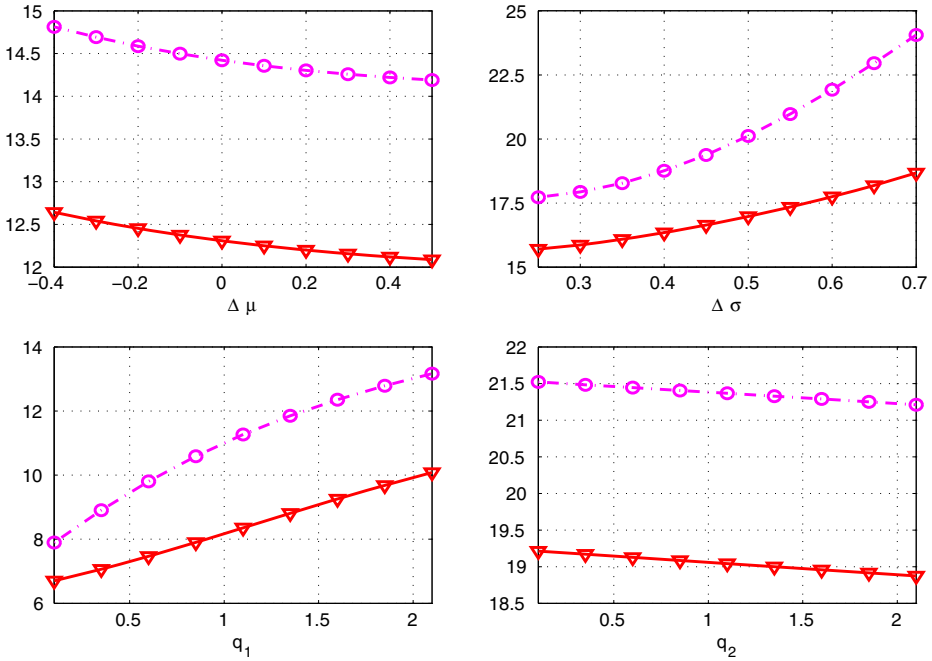


Fig. 8 V@R (triangle) and CV@R (circle) for varying parameters of the *RSGBM model* with $V_0 = 100$, $r = 0$, $T = 1/12$. In this model we consider a varying gap for the drift and volatility, $\mu_2 = 0.5 + \Delta\mu$, $\sigma_2 = 0.1 + \Delta\sigma$; when fixed the parameters were set to $\mu_1 = 0$, $\mu_2 = -0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.3$, $q_1 = 0.5$, $q_2 = 0.5$

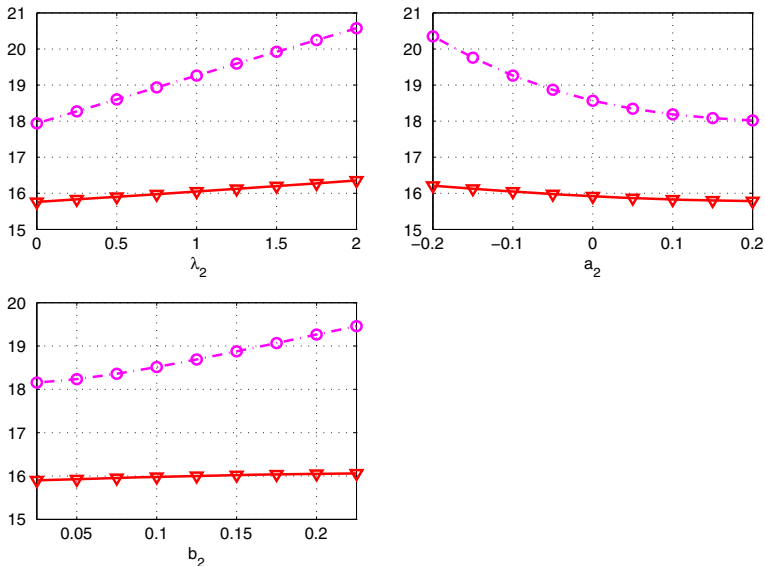


Fig. 9 V@R (triangle) and CV@R (circle) for varying parameters of the *RSJD model* with $V_0 = 100$, $r = 0$, $T = 1/12$. In this model we consider a fixed drift and volatility, $\mu = 0$, $\sigma = 0.25$ and vary the jump parameters with $\lambda_1 = 1$, $a_1 = 0.1$, $b_1 = 0.1$, and $q_1 = 0.5$, $q_2 = 0.5$

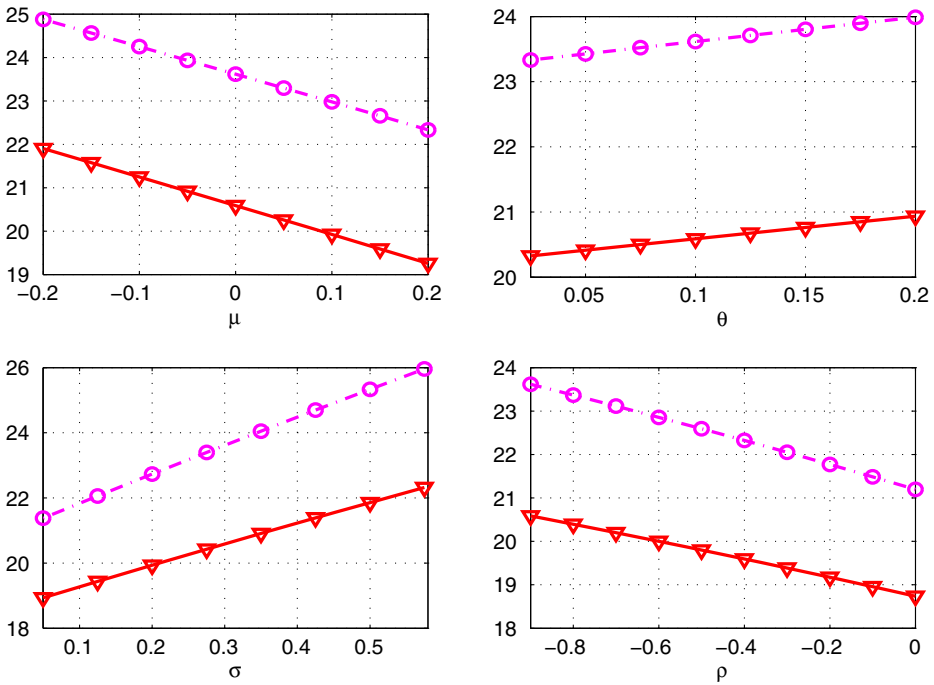


Fig. 10 V@R (triangle) and CV@R (circle) for varying parameters of the Heston model with $V_0 = 100$, $r = 0$, $T = 1/12$, $\mu = 0$, $\theta = 0.1$, $\sigma = 0.3$, $\kappa = 1$ and $\rho = -0.9$

Heston Stochastic volatility model Heston model (Heston 1993) is certainly one of the most famous stochastic volatility dynamic for an asset price: it is defined as

$$V_t = V_0 + \int_0^t V_s \mu ds + \int_0^t \sqrt{v_s} dW_s^1 \tag{21}$$

$$v_t = v_0 + \int_0^t \kappa(\theta - v_s) ds + \sigma \int_0^t \sqrt{v_s} (\rho dW_s^1 + \sqrt{1 - \rho^2} dW_s^2). \tag{22}$$

where $V_0 > 0$, μ is the rate of return and v_t , the volatility process, satisfies a CIR mean reverting dynamic with parameters κ (the mean reversion speed), θ (the long term volatility) and σ (the vol-vol). Furthermore, the two process are ρ -correlated, with $-1 < \rho \leq 0$. The Feller condition $2\kappa\theta > \sigma^2$ ensures the strict positivity of v_t . The (generalized) characteristic function of the log-price is

$$\phi_{X_T}(z) = e^{C(T,z) + D(T,z)v_0 + iz(\mu + \log(V_0))}$$

where

$$C(T, z) = \frac{\kappa\theta}{\sigma^2} \left((\kappa - \rho\sigma zi + d(z))T - 2 \log \left(\frac{c(z)e^{d(z)T} - 1}{c(z) - 1} \right) \right)$$

$$D(T, z) = \frac{\kappa - \rho\sigma zi + d(z)}{\sigma^2} \left(\frac{e^{d(z)T} - 1}{c(z)e^{d(z)T} - 1} \right)$$

and

$$c(z) = \frac{\kappa - \rho\sigma zi + d(z)}{\kappa - \rho\sigma zi - d(z)}, \quad d(z) = \sqrt{(\rho\sigma zi - \kappa)^2 + i\sigma^2 z + \sigma^2 z^2}.$$

For the numerical implementation, we used the procedure outlined in Lord and Kahl (2010). The corresponding results are plotted in Fig. 10.

5 Conclusions

In this paper we consider the problem of efficiently computing CV@R and V@R of an arbitrary loss function, characterized by having a computable generalized characteristic function. We compare different numerical procedures to compute the risk measures based on the integral representation of the distribution function and the stop/loss expectation of the target loss random variable. In particular, the main contribution of the paper is to exploit the characterization of CV@R and V@R as the solution of an univariate minimization problem, as obtained by Rockafellar and Uryasev in Rockafellar and Uryasev (2002). The function to be minimized admits an integral representation as an inverse Fourier transform, under some hypothesis on the finiteness of the exponential moments of the loss distribution. Fast and reliable numerical procedures can be designed to compute the quantities of interest based on the fast Fourier transform algorithm. We finally notice that the basic characterization by Rockafellar and Uryasev is more general, since decision variables can be considered in the minimization problem: our procedure can therefore be included as part of more general optimization problem, like portfolio risk management, where the computation of V@R and CV@R plays a central role as objective functionals and/or constraints to be satisfied.

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