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ERRATUM TO "ALGEBRAIC SUPERGROUPS OF CARTAN TYPE"

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Abstract

[1] In this note I fix a mistake in my previous paper [5]: namely, the result concerning the uniqueness (up to isomorphisms) of such supergroups needs a new formulation and proof. By the same occasion, I explain more in detail the existence result which comes out of the construction of Chevalley supergroups.

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1 Algebraic supergroups of Cartan type: existence

All notation and terminology throughout this note will be as in [5]. In particular, every supergroup $\mathbf{G}$ we shall consider will be *fine*, which means that its tangent Lie algebra functor $Lie(\mathbf{G})$ is of the form $A \mapsto Lie(\mathbf{G})(A) = \big(A \otimes_{\Bbbk} \mathfrak{g}\big)_{\bar{0}}$ (where A ranges among all commutative superalgebras) for some Lie superalgebra $\mathfrak{g}$ over $\Bbbk$, with the additional requirement that $\mathfrak{g}_{\bar{1}}$ (as $\Bbbk$–module) be free of finite rank.

The main result in [5] was the construction of the "Chevalley supergroups" of Cartan type, denoted $\mathbf{G}_V$ as their construction depends on some suitable $\mathfrak{g}$–module V: these are connected algebraic $\Bbbk$–supergroups, defined over $\mathbb{Z}$, such that the complexification of their tangent Lie superalgebra be (finite dimensional) simple of Cartan type. In particular, this proves that supergroups with such properties do exist. However, the presentation in [5] might be obscure on this point, since the construction of $\mathbf{G}_V$ is based upon the choice of V and of a suitable lattice inside it, and the existence of such data might be unclear. This point deserves to be made clear, which is what I am doing in this section.

Let $\mathfrak{g}$ be a complex Lie superalgebra which is simple of Cartan type; let then L_r and L_w be its root lattice and (integral) weight lattice, respectively. As explained in [5], Subsec. 4.5, for any Chevalley supergroup $\mathbf{G}_V$ associated with $\mathfrak{g}$ and a suitable $\mathfrak{g}$–module V its group of characters is a lattice $\Lambda := \Lambda_V$ that lies between L_r and L_w; indeed, it is the lattice of weights spanned by the weights (for the action of a Cartan subalgebra of $\mathfrak{g}$) of V itself. In addition, this V has to be finite dimensional, faithful and rational, and also has to contain an admissible lattice, say M. Thus what we need to show is the following:

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Lemma 1.1. *For any choice of an intermediate lattice Λ lying between L_r and L_w, there exists a finite dimensional, rational, faithful $\mathfrak{g}$–module which contains an admissible lattice M and whose set of weights spans Λ.*

Proof. We start recalling that the even part of $\mathfrak{g}$ is of the form $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{0}^\uparrow}$ where $\mathfrak{g}_0$ is a reductive Lie subalgebra and $\mathfrak{g}_{\bar{0}^\uparrow}$ is a nilpotent ideal (cf. [5], Subsec. 2.3).

First, by classical theory of reductive Lie algebras, for any Λ as before there is a faithful, finite dimensional, rational $\mathfrak{g}_0$–module W whose weights span Λ; moreover, such a W contains a lattice N that is "admissible", which means stable for the Kostant form $K_{\mathbb{Z}}(\mathfrak{g}_0)$ of $U(\mathfrak{g}_0)$ — cf. [1], Ch. VIII, §12.7 (taking into account that $\mathfrak{g}_0$ is always *simple* but when $\mathfrak{g}$ is of type $W(n)$, for then it has a one-dimensional center, and Theorem 2 in [*loc. cit.*] applies again). Since $\mathfrak{g}_{\bar{0}^\uparrow}$ is an ideal in $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{0}} \big/ \mathfrak{g}_{\bar{0}^\uparrow} \cong \mathfrak{g}_0$, the same W is also a $\mathfrak{g}_{\bar{0}}$–module (by scalar extension), the lattice N being again "admissible", i.e. stable for $K_{\mathbb{Z}}(\mathfrak{g}_{\bar{0}^\uparrow})$; of course W is still rational and finite dimensional.

Second, consider $V := Ind_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}(W) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} W$: this is a $\mathfrak{g}$–module which is still finite dimensional and rational; moreover, it contains the admissible lattice $M := K_{\mathbb{Z}}(\mathfrak{g}) \otimes_{K_{\mathbb{Z}}(\mathfrak{g}_{\bar{0}})} N$, and by construction the lattice Λ_V spanned by the weights of V is exactly Λ (since we initially assumed that $\Lambda \supseteq L_r$, i.e. Λ contains all roots of $\mathfrak{g}$). Finally, as $\mathfrak{g}$ is simple and its action on V is non-trivial, V itself is also faithful, as required. □

Remarks 1.2. *(a)* Every Cartan type simple Lie superalgebra $\mathfrak{g}$ can be realized (or defined, if you wish) as a Lie subsuperalgebra of some $\mathfrak{gl}(V)$ for a suitable V — more precisely, as a suitable Lie superalgebra of superderivations of a Grassmann algebra, say $\Bbbk\big[\xi_1, \dots, \xi_n\big]$, which stands for V (the "standard representation"). It so happens that such a $\mathfrak{g}$–module $V := \Bbbk\big[\xi_1, \dots, \xi_n\big]$ is finite dimensional, faithful and rational, and in addition it contains an admissible lattice, namely $M := \mathbb{Z}\big[\xi_1, \dots, \xi_n\big]$: so everything is in place to construct the corresponding Chevalley supergroup $\mathbf{G}_V$. In Sec. 5 of [5] this construction is explicitly carried out for type $W(n)$; in this case, the associated lattice of weights is L_w, the *full* lattice of weights of $\mathfrak{g}$. One can clearly do the same, along the same lines, for types S, $\widetilde{S}$ and H still using the standard representation $V := \Bbbk\big[\xi_1, \dots, \xi_n\big]$.

More generally, the weight lattices Λ between L_r and L_w are in bijections with the sublattices of the quotient $L_w \big/ L_r$: but then — see §4.27 in [5] — there are very few possibilities for Λ, namely four cases for type $H(2r)$, two cases for type $H(2r+1)$, and just one case for types W, S and $\widetilde{S}$. In particular, for the last three cases the construction of Chevalley supergroups $\mathbf{G}_V$ with V the standard representation exhausts all possibilities.

(b) The arguments used to prove Lemma 1.1 above also apply to give a similar result for the case when $\mathfrak{g}$ is simple of *classical* type: one only needs minimal adaptations, actually simplifications, because $\mathfrak{g}_{\bar{0}}$ is reductive (there is no "extra nilpotent part" such as $\mathfrak{g}_{\bar{0}^\uparrow}$, say). As a consequence, one has a proof of the fact that "Chevalley supergroups of classical type" as considered in [2], [3], [4] and [6] actually do exist.

(c) It is proved in [5], Proposition 4.26, that, under mild assumptions, every Chevalley supergroup of Cartan type $\mathbf{G}_V$ is a *closed* supersubgroup of $\mathrm{GL}(V)$. Actually, these conditions are slightly ill settled in the statement of that Proposition: indeed, instead of

"Assume that $\mathfrak{g}_{\bar{1}}$ as a $\Bbbk$–submodule of $\mathfrak{gl}(V)_{\bar{1}}$ is a direct summand"

one should read

"Assume that $\mathfrak{g}_{\bar{1}}$ as a $\Bbbk$–submodule of $\mathfrak{gl}(V)_{\bar{1}}$
is a direct summand with a $\Bbbk$–free complement"

or (what amounts to be the same)

$$\text{“}\textit{Assume that the } \Bbbk\text{–}\textit{module } \mathfrak{gl}(V)_{\bar{1}} \Big/ \mathfrak{g}_{\bar{1}} \textit{ is free”.}$$

In fact, the extra condition of "having a $\Bbbk$–*free* complement" was actually used in the proof of the Proposition, but it was not mentioned in the statement itself. By the way, when $\Bbbk$ is *local* this extra condition automatically holds true, by Kaplansky's theorem.

2 Splittings for supergroups and Hopf superalgebras

In what follows we need the notion of "splitting" for both supergroups and Hopf superalgebras. We take it from [7], where further details may be found. Hereafter, we will think of $\Bbbk$ as being a totally even superalgebra.

2.1. Strongly split Hopf superalgebras. Let H be any commutative Hopf $\Bbbk$–superalgebra. Then $J_H := H_{\bar{1}}^{\,2} \oplus H_{\bar{1}}$ is in fact a *Hopf ideal* of H, hence $\overline{H} := H \big/ J_H$ is a classical (i.e. super but with trivial odd component) commutative Hopf algebra. Moreover, the coproduct of H induces a structure of super left $\overline{H}$–comodule on H (via the projection $H \longrightarrow\!\!\!\!\!\rightarrow \overline{H}$), such that H is a counital super left $\overline{H}$–comodule $\Bbbk$–algebra.

Let $\epsilon : H \longrightarrow \Bbbk$ be the counit map, let $H^+ := Ker(\epsilon)$, $H_{\bar{0}}^+ := H_{\bar{0}} \cap H^+$, $W^H := H_{\bar{1}} \big/ H_{\bar{0}}^+ H_{\bar{1}}$ and consider $\bigwedge W^H$. Then $\overline{H} \otimes \bigwedge W^H$ has a natural structure of a commutative superalgebra, endowed with a natural "augmentation" map (i.e. a $\Bbbk$–valued morphism of $\Bbbk$–superalgebras); moreover, the coproduct of $\overline{H}$ induces on $\overline{H} \otimes \bigwedge W^H$ a super left $\overline{H}$–comodule structure, so $\overline{H} \otimes \bigwedge W^H$ *is a super counital left* $\overline{H}$*–comodule* $\Bbbk$*–algebra.*

The notion of "strongly split" (commutative) Hopf superalgebra, essentially due to Masuoka — as the core idea was already in his papers [8] and [9], but the present terminology is borrowed from [7] — reads as follows: a commutative Hopf superalgebra H as above is said to be *strongly split* if W^H is $\Bbbk$–free and there is an isomorphism $\zeta : H \xrightarrow{\cong} \overline{H} \otimes_{\Bbbk} \bigwedge W^H$ of super counital left $\overline{H}$–comodule $\Bbbk$–algebras. In particular, Masuoka proved that any commutative Hopf superalgebra over $\Bbbk$ is automatically strongly split when $\Bbbk$ is a field whose characteristic is not 2: cf. [8], Theorem 4.5.

2.2. Global splittings for supergroups. Let $\mathbf{G}$ be an (affine) *supergroup* over $\Bbbk$, $H := \mathcal{O}(\mathbf{G})$ the Hopf $\Bbbk$–superalgebra representing it, and $\overline{H} := H \Big/ J_H = H_{\bar{0}} \Big/ H_{\bar{1}}^{\,2}$, which is a (classical) commutative Hopf $\Bbbk$–algebra. The affine group-scheme $\mathbf{G}_{ev}$ represented by $\overline{H} = \overline{\mathcal{O}(\mathbf{G})}$ — so that $\mathcal{O}\big(\mathbf{G}_{ev}\big) = \overline{\mathcal{O}(\mathbf{G})}$ — is called *the classical supersubgroup(-scheme) associated with* $\mathbf{G}$. The projection $\pi : H := \mathcal{O}(\mathbf{G}) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}\big(\mathbf{G}_{ev}\big) = \overline{H}$ yields an embedding $j : \mathbf{G}_{ev} \hookrightarrow \mathbf{G}$, so $\mathbf{G}_{ev}$ identifies with a closed (super)subgroup of $\mathbf{G}$. Moreover, every closed supersubgroup of $\mathbf{G}$ which is *classical* is a closed subgroup of $\mathbf{G}_{ev}$.

Here is now the definition of "globally split supergroups". Let $\mathbf{G}$ be an affine $\Bbbk$–supergroup for which there exists a closed subsupercheme $\mathbf{G}_-$ of $\mathbf{G}$, stable by the adjoint $\mathbf{G}_{ev}$–action, such that

(a) $1_{\mathbf{G}} \in \mathbf{G}_-$, hence we look at $\mathbf{G}_-$ as a *pointed superscheme*,

(b) the product in $\mathbf{G}$ restricts to an isomorphism $\mathbf{G}_{ev} \times \mathbf{G}_- \xrightarrow{\cong} \mathbf{G}$ of pointed left $\mathbf{G}_{ev}$–superschemes (which will be called a *(global) splitting* of $\mathbf{G}$),

(c) $\mathbf{G}_-$ is isomorphic to a totally odd affine superscheme $\mathbb{A}_{\Bbbk}^{0|d_-}$, as a pointed superscheme.

When all this holds, we say that $\mathbf{G}$ is *globally strongly split*, or in short that it is *gs-split*.

As the referee kindly suggested, a very inspiring (and suggestive) alternative terminology might be that of *"equivariantly split"* supergroup, which stresses the fact the splitting of such a supergroup G is $\mathbf{G}_{ev}$–equivariant; nevertheless, we have adopted here the terminology of [7] as we quote results from there.

The link with "splittability" of (commutative) Hopf superalgebras is the one we could expect (cf. Theorem 4.5 in [8], as well as Theorem 3.2.8 and Corollary 3.2.9 in [7]):

Theorem 2.3. *Let $\mathbf{G}$ be an affine supergroup, defined over a ring $\Bbbk$, and let $H := \mathcal{O}(\mathbf{G})$ be its representing (commutative Hopf) $\Bbbk$–superalgebra. Then $\mathbf{G}$ is globally strongly split if and only if the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ is strongly split. In particular, if $\Bbbk$ is a field whose characteristic is not 2, then $\mathbf{G}$ is automatically globally strongly split.*

Finally, by Corollary 4.22*(c)* and Proposition 4.23 in [5] one gets the following:

Theorem 2.4. *All Chevalley supergroups of Cartan type $\mathbf{G}_V$ as in [5] are gs-split.*

3 Algebraic supergroups of Cartan type: uniqueness

As recalled in Sec. 1, the main result in [5] was the construction of the "Chevalley supergroups" of Cartan type: this proved the *existence* of any possible type of connected algebraic $\mathbb{Z}$–supergroup whose complexified tangent Lie superalgebra is (finite dimensional) simple of Cartan type. On the other hand, the *uniqueness* question was addressed in Subsection 4.8 of [5], devoted to proving that any algebraic supergroup with the above mentioned properties (in particular for its tangent Lie superalgebra) is necessarily isomorphic to some Chevalley supergroup of Cartan type. However, the result and proof presented there were wrong: hereafter I provide a correct (modified) statement and proof, with changes that affect everything from §4.38 through Theorem 4.42 in Sec. 4.8 of [5].

3.1. Gs-split, $\Bbbk$–split supergroups of Cartan type and the Uniqueness Theorem. Let $\mathbf{G}$ be a connected, gs-split $\Bbbk$–supergroup; we assume for it that its tangent Lie superalgebra $\mathfrak{g} := \mathit{Lie}(\mathbf{G})$ be a $\Bbbk$–form of a complex Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$ — i.e., there exists a Lie superalgebra $\mathfrak{g}_{\mathbb{Z}}$ over $\mathbb{Z}$ such that $\mathfrak{g} = \Bbbk \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ and $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{g}_{\mathbb{Z}}$ — and this $\mathfrak{g}_{\mathbb{C}}$ is simple of Cartan type. Moreover, we assume that the classical subgroup $\mathbf{G}_{ev}$ of $\mathbf{G}$ has a $\Bbbk$–split maximal torus. In short, we say that $\mathbf{G}$ *is a gs-split, $\Bbbk$–split supergroup of Cartan type*. The group of characters of any $\Bbbk$–split maximal torus in $\mathbf{G}_{ev}$, call it Λ, contains the root lattice, since $\mathbf{G}_{ev}$ acts on $\mathfrak{g}$ by the adjoint action: hence Λ is an intermediate lattice lying between L_r and L_w (notation of Sec. 1).

Now, for any pair $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$ as above there exists a Chevalley $\Bbbk$–supergroup $\mathbf{G}_V$ of Cartan type whose associated pair is exactly $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$ — cf. Sec. 1 above. This yields an "Existence Theorem" for gs-split, $\Bbbk$–split supergroups of Cartan type.

A related "Uniqueness Theorem" was presented in Subsec. 4.8 of [5]. However, it was based on a wrong analysis, so it requires crucial amendments. In fact, I shall present a double version of such a result, one holding true for any ring — but requiring a stronger assumption than that concerning $\mathit{Lie}(\mathbf{G})$ — and one that applies only for fields of zero characteristic – for which the above requirement on $\mathit{Lie}(\mathbf{G})$ is enough.

Here comes the first result:

Theorem 3.2. *Let* $\mathbf{G}$ *be a connected gs-split,* $\Bbbk$*–split supergroup of Cartan type, let* $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$ *be its associated pair and let* $\mathbf{G}_V$ *be the Chevalley supergroup of Cartan type whose group is associated with the same pair* $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$*. Assume in addition that* $\mathbf{G}_{ev}$ *is isomorphic to* $\big(\mathbf{G}_V\big)_{ev}$*. Then* $\mathbf{G}$ *is isomorphic to* $\mathbf{G}_V$*.*

Proof. To begin with, recall that the classical subgroup $(\mathbf{G}_V)_{ev}$ of the supergroup $\mathbf{G}_V$ splits into a semidirect product $\mathbf{G}_V = \mathbf{G}_0^V \ltimes \mathbf{G}_{\bar{0}\uparrow}^V$, where $\mathbf{G}_0^V$ is a connected $\Bbbk$–split subgroup with $Lie\big(\mathbf{G}_0^V\big) = \mathfrak{g}_0$ and group of characters Λ, and $\mathbf{G}_{\bar{0}\uparrow}^V$ is a connected, unipotent normal subgroup with $Lie\big(\mathbf{G}_{\bar{0}\uparrow}^V\big) = \mathfrak{g}_{\bar{0}\uparrow}$. In addition, by construction $\mathbf{G}_0^V$ is obtained via a classical procedure "à la Chevalley" — cf. Proposition 4.9 in [5] — based on V thought of as a $\mathfrak{g}_0$–module, the Kostant form of $U(\mathfrak{g}_0)$, etc. Similarly, $\mathbf{G}_{\bar{0}\uparrow}^V$ too is realized via a construction "à la Chevalley", which roughly speaking "integrates" the nilpotent Lie algebra $\mathfrak{g}_{\bar{0}\uparrow}$ (linearized through V). These Chevalley constructions for $\mathbf{G}_0^V$ and $\mathbf{G}_{\bar{0}\uparrow}^V$ are realized simultaneously as parts of the overarching procedure which constructs *all* of $\mathbf{G}_V$. In addition (as a consequence), the adjoint action of $(\mathbf{G}_V)_{ev}$ onto $Lie\big(\mathbf{G}_V\big) = \mathfrak{g}$ is uniquely determined by the adjoint action of $Lie\big((\mathbf{G}_V)_{ev}\big) = \mathfrak{g}_{\bar{0}}$ onto $Lie\big(\mathbf{G}_V\big) = \mathfrak{g}$ as well as by Λ.

From the isomorphism $\phi : \mathbf{G}_{ev} \cong \big(\mathbf{G}_V\big)_{ev}$ we get a corresponding decomposition of $\mathbf{G}_{ev}$ as a semidirect product, and also that the adjoint action of $\mathbf{G}_{ev}$ onto $Lie\big(\mathbf{G}\big) = \mathfrak{g}$ is uniquely determined by the action of $Lie\big(\mathbf{G}_{ev}\big) = \mathfrak{g}_{\bar{0}}$ onto $Lie\big(\mathbf{G}\big) = \mathfrak{g}$ and by Λ.

Putting all this together we get, in the language of [7], that one can express this by saying that the "super Harish-Chandra pairs" of $\mathbf{G}_V$ and $\mathbf{G}$, namely $\big((\mathbf{G}_V)_{ev}, \mathfrak{g}\big)$ and $\big(\mathbf{G}_{ev}, \mathfrak{g}\big)$, are isomorphic. Now, the main result in [7] is exactly — cf. Theorem 4.3.14 therein — that the (suitably defined) category of "super Harish-Chandra pairs" is equivalent to the category of (fine) gs-split supergroups. But the latter category contains both our supergroups $\mathbf{G}$ and $\mathbf{G}_V$ (by assumption for the former, and by the remark at the beginning of the proof for the latter): therefore, we can conclude that $\mathbf{G}$ is isomorphic to $\mathbf{G}_V$. □

The second result is a direct consequence:

Theorem 3.3. *Let* $\mathbf{G}$ *be a connected gs-split,* $\Bbbk$*–split supergroup of Cartan type, let* $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$ *be its associated pair and let* $\mathbf{G}_V$ *be the Chevalley supergroup of Cartan type associated with* $\big(\mathfrak{g}_{\mathbb{C}}\,,\Lambda\big)$*. Moreover, assume that* $\Bbbk$ *is a field of characteristic zero.*

Then $\mathbf{G}$ *is isomorphic to* $\mathbf{G}_V$*.*

Proof. From the proof of Theorem 3.2 above we know that the subgroup $(\mathbf{G}_V)_{ev}$ of $\mathbf{G}_V$ splits as $\mathbf{G}_V = \mathbf{G}_0^V \ltimes \mathbf{G}_{\bar{0}\uparrow}^V$, where $\mathbf{G}_0^V$ is connected, $\Bbbk$–split reductive with $Lie\big(\mathbf{G}_0^V\big) = \mathfrak{g}_0$ and group of characters Λ, and $\mathbf{G}_{\bar{0}\uparrow}^V$ is connected, unipotent, normal with $Lie\big(\mathbf{G}_{\bar{0}\uparrow}^V\big) = \mathfrak{g}_{\bar{0}\uparrow}$. Moreover, the (conjugacy) action of $\mathbf{G}_0^V$ onto $\mathbf{G}_{\bar{0}\uparrow}^V$ is entirely encoded by the adjoint action of $\mathfrak{g}_0$ onto $\mathfrak{g}_{\bar{0}\uparrow}$ and by the lattice Λ.

On the other hand, the classical subgroup $\mathbf{G}_{ev}$ of $\mathbf{G}$ has tangent Lie algebra $Lie\big(\mathbf{G}_{ev}\big) = \mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\bar{0}\uparrow}$, where $\mathfrak{g}_0$ is a reductive Lie subalgebra and $\mathfrak{g}_{\bar{0}\uparrow}$ is a nilpotent ideal of $\mathfrak{g}_{\bar{0}}$ (see the proof of Lemma 1.1). Since $\Bbbk$ is a field of characteristic zero, we have a Chevalley decomposition of $\mathbf{G}_{ev}$ into a semidirect product $\mathbf{G}_{ev} = \mathbf{G}_0 \ltimes R_u\big(\mathbf{G}_{\bar{0}}\big)$, where $R_u\big(\mathbf{G}_{\bar{0}}\big)$ is the unipotent radical of $\mathbf{G}_{\bar{0}}$; then this $R_u\big(\mathbf{G}_{\bar{0}}\big)$ is a connected, unipotent normal subgroup with $Lie\big(R_u\big(\mathbf{G}_{\bar{0}}\big)\big) = \mathfrak{g}_{\bar{0}\uparrow}$, while $\mathbf{G}_0\ \big(\cong \mathbf{G}\big/R_u\big(\mathbf{G}_{\bar{0}}\big)\big)$ is a connected, $\Bbbk$–split reductive subgroup with $Lie\big(\mathbf{G}_0\big) = \mathfrak{g}_0$ and group of characters the lattice Λ.

By classification theory of $\Bbbk$–split reductive groups, $\mathbf{G}_0$ corresponds to the pair $\big(\mathfrak{g}_0\,,\Lambda\big)$; but $\mathbf{G}_0^V$, which also is $\Bbbk$–split reductive, corresponds to the same pair as well, whence there exists an isomorphism $\mathbf{G}_0 \cong \mathbf{G}_0^V$. Similarly, in the classification of connected unipotent (algebraic) group-schemes over fields of characteristic zero both $\mathbf{G}_{\bar{0}\uparrow}^V$ and $R_u\big(\mathbf{G}_{\bar{0}}\big)$ correspond to the same nilpotent Lie algebra (namely $\mathfrak{g}_{\bar{0}\uparrow}$), hence there exists an isomorphism between them. In addition, the (conjugacy) action of $\mathbf{G}_0$ onto $R_u\big(\mathbf{G}_{\bar{0}}\big)$ is again entirely encoded by the adjoint action of $\mathfrak{g}_0$ onto $\mathfrak{g}_{\bar{0}\uparrow}$ and by the lattice Λ.

Comparing now $(\mathbf{G}_V)_{ev}$ and $\mathbf{G}_{ev}$ we conclude that both are semidirect products, with pairwise isomorphic factors, and the action of the reductive factor onto the unipotent (normal) one is ruled in the same way; so these semidirect products are isomorphic, i.e. $\mathbf{G}_{ev} \cong (\mathbf{G}_V)_{ev}$. Then Theorem 3.2 applies, and we find that $\mathbf{G}$ is isomorphic to $\mathbf{G}_V$. □

Remark 3.4. The same statements as in Theorems 3.2 and 3.3 above also hold true in the case when $\mathfrak{g}$ is simple of *classical* type and $\mathbf{G}_V$ is a "Chevalley supergroup" as in [3]; indeed, one proves them via the same arguments, and in the second case the proof is even simpler, as $\mathfrak{g}_{\bar{0}}$ is reductive. This yields another, more general proof of the fact that "Chevalley supergroups of *classical* type" are unique up to isomorphism (cf. [4]).

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