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The equivariant ring of conditions of conics

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ABSTRACT

In this paper we are going to give an explicit description of the so-called ring of conditions of conics, i.e. of the homogenous space $PGL(3)/PSO(3)$. This is achieved by first describing its equivariant version.

1. Introduction

A classical result of enumerative geometry is the computation by Halphen of the number 3264 of conics in the projective plane simultaneously tangent to 5 given conics in general position. One way to perform this computation is via the study of the cohomology ring of the variety of complete conics (see for example [DP,CX]). Indeed in [CX] the authors make a very detailed study of any type of conditions on conics giving a basis of the so-called ring of conditions. This ring has been introduced in [DP2] exactly to put in a general framework the study of such enumerative questions for a large class of homogeneous spaces, including the quotient of a semisimple algebraic group modulo the subgroup of fixpoints of an involution.

Our aim in this paper is different from the one in [CX], since we are going to give a set of generators and relations for the ring of conditions of conics tensored with the rational numbers. This will be achieved by first computing the equivariant (with respect to $PGL(3)$) cohomology of this ring.

The methods and techniques we are going to employ are similar to those we have used in our previous papers [St2,St3,St4], in which analogous results have been obtained for the ring of conditions of semisimple groups G considered as the quotient of $G \times G$ modulo the diagonal, i.e. the fixpoints of the involution flipping the two copies of G. A key ingredient of this computation has been the fact that a $G \times G$-equivariant completion of this homogeneous space has all of its $T \times T$-fixpoints

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contained in a closed orbit, so the equivariant cohomology is detected by the equivariant cohomology of the closed orbits (this fact has been also used by Brion and Joshua in [BJ]).

The case treated in this paper is the first not completely straightforward in which there are non-closed orbits containing fixpoints under the action of a maximal torus and it might be of inspiration for more general results.

The paper is organized as follows. In Section 2 we recall a few known facts and introduce the equivariant ring of conditions for conics. We are careful to use only the minimum amount of algebraic groups machinery at the cost of being somewhat cumbersome. In Section 3 we perform the computation of the equivariant cohomology of complete conics. It turns out that this ring has a remarkably simple presentation, while for example, even in simple cases for the ring of equivariant cohomology of the wonderful group compactification (see [St3] and [U]) this is not true. In Section 4 we compute the equivariant cohomology of a general smooth equivariant compactification of the space of nonsingular conics. Finally in the last section we use these results to get our description of the equivariant ring of conditions.

In this paper the cohomology is going to be with coefficients in $\mathbb{Q}$.

2. Recollections

Let us recall the classical construction of the variety of complete conics (see for example [CX] and references in it). We shall work over the field of complex number $\mathbb{C}$. Consider a three dimensional vector space V and its second symmetric power S^2V: this is a six dimensional space which we can think as the space of 3×3 symmetric matrices. The projective space $\mathbb{P}(S^2V)$ is acted upon by the group $G := PGL(3)$ and contains 3 orbits, respectively given by rank 1, rank 2, rank 3 matrices. If we consider the closed orbits of rank 1 matrices, this is isomorphic to the projective plane $\mathbb{P}^2$ via the Veronese embedding.

The variety X of complete conics is by definition the variety obtained from $\mathbb{P}(S^2V)$ by blowing up the closed orbit, i.e. the Veronese surface.

Clearly G acts on X, and X contains 4 orbits. They are described as follows:

(1) The open orbit X_0, which is nothing else than the variety of nondegenerate conics.
(2) Two codimension one orbits $\mathcal{O}_1$, $\mathcal{O}_2$, which we can think respectively $\mathcal{O}_1$ as the variety of rank 2 conics, $\mathcal{O}_2$ as the variety of rank 1 conics (double lines) together with a pair of distinct points in the line.
(3) The closed orbit $\mathcal{O}_{1,2} = \mathcal{F}$, which is isomorphic to the variety of flags $(p \subset \ell)$ with p a point in $\mathbb{P}^2$ and ℓ a line containing p.

The closure of each orbit is smooth and the closed orbit is the transversal intersection of the two divisors $\overline{\mathcal{O}}_1$ and $\overline{\mathcal{O}}_2$.

Let us now see how to define the equivariant ring of conditions of conics.

Given a sequence $\sigma = (v_1, \dots, v_m)$ of vectors $v_i = (a_i, b_i) \in \mathbb{Z}^2$ with nonnegative entries, we say that σ is obtained from the sequence $\tau = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m)$ by an elementary move if $v_i = v_{i-1} + v_{i+1}$.

A sequence σ is admissible, if it is obtained from the sequence $((1,0),(0,1))$ by a finite succession of elementary moves.

The set Σ of admissible sequences is partially ordered by inclusion.

We now construct a G-variety X_σ for each admissible $\sigma = (v_1, \dots, v_m)$, having the following properties:

(1) X_σ is a compactification of the variety X_0 of nondegenerate conics.
(2) X_σ contains m codimension one orbits, $\mathcal{O}_{v_1}, \dots, \mathcal{O}_{v_m}$.
(3) X_σ contains $m-1$ closed orbits $\mathcal{O}_{v_i,v_{i+1}}$, $i = 1, \dots, m-1$, each isomorphic to the flag variety $\mathcal{F}$ and furthermore $\mathcal{O}_{v_i,v_{i+1}}$ is the transversal intersection $\overline{\mathcal{O}}_{v_i} \cap \overline{\mathcal{O}}_{v_{i+1}}$.

Notice that by construction the G-orbits of X_σ are indexed by the set B_σ consisting of those subsets in σ which have cardinality at most two and, if they have cardinality 2, they are of the form $\{v_i, v_{i+1})$ for some $i = 1, \dots, m-1$ (of course the open orbit X_0 corresponds to the empty set).

If $\sigma = ((1,0),(0,1))$, we set $X_\sigma = X$, $\mathcal{O}_{(0,1)} = \mathcal{O}_1$ and $\mathcal{O}_{(1,0)} = \mathcal{O}_2$. We then proceed by induction on the cardinality of σ, which we assume to have at least 3 elements. Assume that σ is obtained from the sequence $\tau = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m)$ by an elementary move as above. Then X_σ is obtained from X_τ blowing up the closed orbit $\mathcal{O}_{v_{i-1},v_{i+1}}$.

It is immediate to see that, if $\tau \subset \sigma$, we get a (unique) G-equivariant morphism $\gamma_{\sigma,\tau} : X_\sigma \to X_\tau$, extending the identity on X_0, and thus a ring homomorphism $\gamma^*_{\sigma,\tau} : H^*_G(X_\tau) \to H^*_G(X_\sigma)$. Under this the rings $H^*_G(X_\sigma)$ form a directed system with respect to the partially ordered set Σ.

From the general theory developed in [DP2], we then get that the equivariant ring of conditions R_G of conics is just the direct limit

$$R_G = \lim_{\sigma} H^*_G(X_\sigma).$$

Thus our first step will be to compute the equivariant cohomology of X, then compute the one of a general X_σ and finally take the limit.

The variety X_σ is what is called a regular G-embedding and a strategy to compute its G-equivariant cohomology is developed in [BDP].

Let us recall this procedure in our special case.

Definition 1. Given an admissible sequence σ, a Stanley–Reisner system on σ is given by

(1) For each $b \in B$, a graded commutative algebra with 1, A_b.
(2) For any pair b, b' with $b' = b \setminus v_j$, a graded commutative algebra with 1, $B^b_{b'}$ together with two graded homomorphisms

$$\phi^{b'}_b : A_b \to B^b_{b'} \quad \text{and} \quad \psi^b_{b'} : A_{b'} \to B^b_{b'}$$

with $\phi^{b'}_b$ surjective.

Let us now see how to define a Stanley–Reisner system associated to X_σ. Given a point $p \in X$, we shall denote by G_p its stabilizer in G.

Take $b \in B$. Consider the corresponding orbit $\mathcal{O}_b$ in X_σ. Set

$$A_b := H^*_G(\mathcal{O}_b).$$

Given a pair b, b' with $b' = b \setminus \{v_j\}$, $\mathcal{O}_b$ is a codimension one orbit in $\overline{\mathcal{O}_{b'}}$. Consider the union $\mathcal{M}^b_{b'} := \mathcal{O}_{b'} \cup \mathcal{O}_b$.

Take the normal line bundle $\mathcal{L}$ to $\mathcal{O}_b$ in $\mathcal{M}^b_{b'}$. $\mathcal{L}$ is G-linearized, so G acts on the complement $\mathcal{L}'$ of the zero section of $\mathcal{L}$. We now set

$$B^b_{b'} = H^*_G(\mathcal{L}').$$

One then observes that $\mathcal{L}'$ is itself a G-homogeneous space and for any point $p \in \mathcal{O}_b$ the stabilizer of a point above p is the kernel of a character $\chi^b_{b'}$ of G_p.

It turns out that we can choose a point p_b in each orbit $\mathcal{O}_b$ in such a way that the stabilizers G_{p_b} have the following property.

For any two orbits $\mathcal{O}_b, \mathcal{O}_{b'}$ with $\mathcal{O}_b$ of codimension one in $\overline{\mathcal{O}}_{b'}$, there are Levi factors L^b, $L^{b'}$ of G_{p_b} and respectively $G_{p_{b'}}$ such that, if we denote by $L^b_{b'}$ the intersection of the kernel of $\chi^b_{b'}$ with L^b, then $L^b_{b'} \subset L^{b'}$.

Notice that A_b is also isomorphic to $H^*_{L^b}(pt)$ and $B^b_{b'}$ is isomorphic to $H^*_{L^b_{b'}}(pt)$.

The projection $L' \to \mathcal{O}_b$ induces a homomorphism

$$\phi^{b'}_b : A_b \to B^b_{b'}.$$

Under the identification of A_b with $H^*_{L^b}(pt)$ and of $B^b_{b'}$ with $H^*_{L^b_{b'}}(pt)$, this is the homomorphism induced by the inclusion $L^b_{b'} \subset L^b$. It follows that $\phi^{b'}_b$ is surjective with kernel the ideal generated by the equivariant Chern class of the line bundle $\mathcal{L}$.

On the other hand the inclusion $L^b_{b'} \subset L^{b'}$ induces the homomorphism

$$\psi^b_{b'} : A_{b'} \to B^b_{b'}.$$

At this point one defines the graded algebra

$$A_\sigma \subset \bigoplus_{b \in B} A_b$$

consisting of the sequences $(a_b)_{b\in B}$ such that for every pair b, b' as before,

$$\psi^b_{b'}(a_{b'}) = \phi^{b'}_b(a_b).$$

Usually in what follows we shall write A_i for A_{v_i}, $A_{i,j}$ for $A_{\{v_i,v_j\}}$ and so on and similarly for the corresponding orbits, the rings B and the maps ϕ and ψ.

The main result in [BDP] then gives

Theorem 2.1. *The algebra A_σ is isomorphic to the cohomology ring $H^*_G(X_\sigma)$.*

3. The equivariant cohomology of complete conics

We are now going to explain how to use Theorem 2.1 to compute $H^*_G(X)$. In order to obtain the Stanley–Reisner system in this case, we may use the results in [Sc], but since our situation is quite simple, we do it directly. Recall that for any homogeneous space G/H, $H^*_G(G/H) \simeq H^*_H(pt)$. Now X has 4 orbits.

For X_0 we have that $X_0 = G/PSO(3)$. Since $PSO(3)$ is isomorphic to $PGL(2)$, we deduce immediately that

$$A_\emptyset = \mathbb{Q}[t]$$

with $\deg t = 4$.

As we have already mentioned, the closed orbit $\mathcal{O}_{1,2}$ is isomorphic to the flag variety G/B, B the Borel subgroup of G which is the image of the subgroup of upper triangular matrices in $SL(3)$. Thus we get that $L^{1,2} = T$ is a maximal torus in $PGL(3)$. As usual we consider T as the group of triples (t_1, t_2, t_3) of nonzero complex numbers such that $t_1t_2t_3 = 1$, modulo the subgroup of cubic roots of 1. It follows that we have

$$A_{1,2} = \mathbb{Q}[x_1, x_2]$$

with $\deg x_1 = \deg x_2 = 2$. We may as well assume that x_1 (resp. x_2) is the equivariant Chern class of the normal line bundle to $\mathcal{O}_{1,2}$ in $\overline{\mathcal{O}}_1$ (resp. $\overline{\mathcal{O}}_2$).

In the previous section we have seen that $\mathcal{O}_1$ is the variety of (unordered) pairs of distinct lines in $\mathbb{P}^2$. Consider the G-equivariant projection $\rho : \mathcal{O}_1 \to \mathbb{P}^2$ mapping such a pair of lines ℓ_1, ℓ_2 to the point $\ell_1 \cap \ell_2$. Let P denote the parabolic subgroup fixing a given point in $\mathbb{P}^2$. We have a group homomorphism $P \to PGL(2)$ whose kernel is the solvable radical in P. $PGL(2)$ acts on the pencil of lines through such a point, hence on the pairs of such lines and the stabilizer of a given unordered pair ℓ_1, ℓ_2 is just the normalizer of a maximal torus in $PGL(2)$. Thus the stabilizer of ℓ_1, ℓ_2 in G is just the pre-image of such a normalizer in P.

It is then immediate to see that, in this case, L^1 is obtained as follows. Consider the normalizer $N(T)$. Then $N(T)/T$ is the symmetric group S_3 and L^1 is the pre-image of the order 2 subgroup in S_3 generated by the transposition $(1, 2)$. We deduce that

$$A_1 = \mathbb{Q}[y, u]$$

with $\deg y = 2$ and $\deg u = 4$. We can assume that y is the Chern class of the normal bundle to $\mathcal{O}_1$.

The case of $\mathcal{O}_2$ is completely analogous and we have that L^2 is the pre-image of the order 2 subgroup in S_3 generated by the transposition $(2, 3)$. So

$$A_2 = \mathbb{Q}[z, v]$$

with $\deg z = 2$ and $\deg v = 4$. Again we can assume that z is the Chern class of the normal bundle to $\mathcal{O}_2$.

At this point it is very easy to compute the groups $L^b_{b'}$. As a matter of fact by their definition one has the following:

$$\begin{gathered}
L_1^{1,2} \subset L^{1,2} \text{ is the kernel of the root } e^{\alpha_1} : (t_1, t_2, t_3) \to t_1 t_2^{-1}, \\
L_2^{1,2} \subset L^{1,2} \text{ is the kernel of the root } e^{\alpha_2} : (t_1, t_2, t_3) \to t_2 t_3^{-1}, \\
L_\emptyset^1 \subset L^1 \text{ is the kernel of the root } e^{\alpha_2} \text{ which is still defined on } L^1, \\
L_\emptyset^2 \subset L^2 \text{ is the kernel of the root } e^{\alpha_2} \text{ which is still defined on } L^2.
\end{gathered}$$

From this one sees that

$$\begin{gathered}
B_1^{1,2} = \mathbb{Q}[x_1, x_2]/(x_1) = \mathbb{Q}[x_2]; \qquad B_2^{1,2} = \mathbb{Q}[x_1, x_2]/(x_2) = \mathbb{Q}[x_1]; \\
B_\emptyset^1 = \mathbb{Q}[y, u]/(y) = \mathbb{Q}[u]; \qquad B_\emptyset^2 = \mathbb{Q}[z, v]/(z) = \mathbb{Q}[v],
\end{gathered}$$

and the homomorphisms $\phi_b^{b'}$ are the quotient homomorphisms.

On the other hand one readily sees that the maps $\psi^b_{b'}$ are given by

$$\begin{gathered}
\psi_1^{1,2}(u) = 0, \qquad \psi_1^{1,2}(y) = x_2; \qquad \psi_2^{1,2}(v) = 0, \qquad \psi_1^{1,2}(z) = x_1, \\
\psi_\emptyset^1(t) = u; \qquad \psi_\emptyset^2(t) = v.
\end{gathered}$$

Using these facts, we deduce the following

Theorem 3.1. *The equivariant cohomology ring $H^*_G(X)$ is isomorphic as a graded ring to $\mathbb{Q}[q_1, q_2, q]/(q_1 q_2 q)$ with $\deg q_1 = \deg q_2 = 2$ and $\deg q = 4$.*

Proof. By our previous computations, $H^*_G(X)$ is the subring in $A_\emptyset \oplus A_1 \oplus A_2 \oplus A_{1,2}$ consisting of the elements of the form

$$\begin{aligned}&\big(a+tf(t), a+uf(u)+yg(y)+uyh(y,u), a+vf(v)+zm(z)+vzk(z,v),\\&a+x_2g(x_2)+x_1m(x_1)+x_1x_2p(x_1,x_2)\big)\end{aligned}\tag{1}$$

where $a \in \mathbb{Q}$ and f, g, m, h, k, p are polynomials. Now consider $(t, u, v, 0) = \tilde{q}$, $(0, 0, z, x_1) = \tilde{q}_1$, $(0, y, 0, x_2) = \tilde{q}_2$. Notice that these elements lie in $H^*_G(X)$ and that

$$\tilde{q}\tilde{q}_1 = (0, 0, vz, 0), \qquad \tilde{q}\tilde{q}_2 = (0, uy, 0, 0), \qquad \tilde{q}_1\tilde{q}_2 = (0, 0, 0, x_1x_2).$$

Furthermore $\tilde{q}\tilde{q}_1\tilde{q}_2 = (0, 0, 0, 0)$. These elements generate $H^*_G(X)$, so we have a degree preserving surjective homomorphism

$$\gamma : \mathbb{Q}[q_1, q_2, q]/(q_1q_2q) \to H^*_G(X).$$

By [DS] we get the Poincaré polynomial

$$\sum_n \dim H^{n/2}(X)t^n = 1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5. \tag{2}$$

Thus

$$\sum_n \dim H^{n/2}_G(X)t^n = \frac{1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5}{(1-t^2)(1-t^3)} = \frac{1-t^4}{(1-t)^2(1-t^2)} \tag{3}$$

which is also the Poincaré series of the ring $\mathbb{Q}[q_1, q_2, q]/(q_1q_2q)$ (with halved degrees). So the surjective homomorphism γ must be an isomorphism. $\square$

Remark 3.2. A different approach to the computation of the cohomology of complete conics and indeed complete quadrics, has been developed in [DGMP]: we leave to the reader the comparison between the two approaches as well as with the approach developed in [LP].

4. The equivariant cohomology of X_σ

We now fix an admissible $\sigma = (v_1, \dots, v_m)$, $m \geqslant 3$, and consider the corresponding variety X_σ. Recall that

(i) X_σ contains m codimension one orbits, $\mathcal{O}_{v_1}, \dots, \mathcal{O}_{v_m}$.
(ii) X_σ contains $m-1$ closed orbits $\mathcal{O}_{v_i,v_{i+1}}$, $i = 1, \dots, m-1$, each isomorphic to the flag variety $\mathcal{F}$ and furthermore $\mathcal{O}_{v_i,v_{i+1}}$ is the transversal intersection $\overline{\mathcal{O}}_{v_i} \cap \overline{\mathcal{O}}_{v_{i+1}}$.

We begin by computing the Poincaré polynomial of X_σ. We have the following

Proposition 4.1. *The Poincaré polynomial of X_σ is*

$$\sum_n \dim H^{n/2}(X_\sigma)t^n = 1 + mt + (2m-1)t^2 + (2m-1)t^3 + mt^4 + t^5.$$

Proof. In the case $m = 2$ i.e. $X_\sigma = X$, this is just formula (2). So we proceed by induction on m.

Assume that σ is obtained from the sequence $\tau = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m)$ by the elementary move $v_i = v_{i-1} + v_{i+1}$. Then by induction we have

$$\sum_n \dim H^{n/2}(X_\tau)t^n = 1 + (m-1)t + (2m-3)t^2 + (2m-3)t^3 + (m-1)t^4 + t^5.$$

We pass from X_τ to X_σ blowing up a closed orbit $\mathcal{O}$, whose Poincaré polynomial equals $1 + 2t + 2t^2 + t^3$. We claim that the restriction map $H^*(X_\tau) \to H^*(\mathcal{O})$ is surjective. Indeed it is easily seen that the restriction map $H^*(X) \to H^*(\mathcal{O}_{1,2})$ is surjective, since $H^*(\mathcal{O}_{1,2})$ is generated by the image of the two equivariant classes $\tilde{q}_1$, $\tilde{q}_2$ considered in Theorem 3.1. The claim now follows for X_τ from the commutativity of the diagram

$$\begin{tikzcd} \mathcal{O} \arrow[r, "i_\tau"] \arrow[d, "\pi"'] & X_\tau \arrow[d, "\pi"'] \\ \mathcal{O}_{1,2} \arrow[r, "i"] & X \end{tikzcd}$$

where π is the G-equivariant projection and i, i_τ are the inclusions, once we remark that the restriction of π to $\mathcal{O}$ is an isomorphism.

It then follows from the formula of the cohomology of a blow up that

$$\begin{aligned} \sum_n \dim H^{n/2}(X_\sigma)t^n &= 1 + (m-1)t + (2m-3)t^2 + (2m-3)t^3 + (m-1)t^4 + t^5 \\ &\quad + t\bigl(1 + 2t + 2t^2 + t^3\bigr) \\ &= 1 + mt + (2m-1)t^2 + (2m-1)t^3 + mt^4 + t^5 \end{aligned}$$

as desired. □

Corollary 4.2. *For the equivariant cohomology ring we have*

$$\sum_n \dim H_G^{n/2}(X_\sigma)t^n = \frac{1 + mt + (2m-1)t^2 + (2m-1)t^3 + mt^4 + t^5}{(1-t^2)(1-t^3)}.$$

We are now ready to determine the ring $H_G^*(X\sigma)$. Since each closed orbit in X_σ is isomorphic to the flag variety $\mathcal{F}$, $A_{i,i+1} = H_G^*(\mathcal{O}_{v_i,v_{i+1}}) \simeq \mathbb{Q}[x_1^{(i)}, x_2^{(i)}]$ and we can assume that $x_1^{(i)}$ (resp. $x_2^{(i)}$) is the Chern class of the normal line bundle to $\mathcal{O}_{v_i,v_{i+1}}$ in $\overline{\mathcal{O}}_{v_i}$ (resp. $\overline{\mathcal{O}}_{v_{i+1}}$).

We have already computed

$$A_1 = H_G^*(\mathcal{O}_{v_1}) = \mathbb{Q}[y,u], \qquad A_m = H_G^*(\mathcal{O}_{v_m}) = \mathbb{Q}[z,v]$$

and

$$A_\emptyset = H_G^*(X_0) = \mathbb{Q}[t].$$

As for the remaining codimension one orbits, it is straightforward to see that

$$A_i = H_G^*(\mathcal{O}_{v_i}) = \mathbb{Q}\bigl[y^{(i)}\bigr]$$

with $y^{(i)}$ equal to the Chern class of the normal line bundle to $\mathcal{O}_{v_i}$ in X. This follows from the fact that $\mathcal{O}_{v_i}$ is the open orbit in an exceptional divisor obtained by blowing up a closed orbit. Thus $\mathcal{O}_{v_i}$ is the complement of the zero section in a suitable line bundle on the flag variety.

We now have to determine the rings $B_{b'}^b$ and the homomorphisms $\phi_b^{b'}$ and $\psi_{b'}^b$.

Assume first $b' = \emptyset$, $b = \{v_i\}$. If $i = 1$ or $i = m$, the computations have already been done above. Otherwise we see that

$$B^i_\emptyset = \mathbb{Q} \quad \text{and} \quad \phi^\emptyset_i\big(y^{(i)}\big) = 0 = \psi^i_\emptyset(t).$$

Let $b = \{v_i\}$. Again, if $i = 1$ or $i = m$, the computations have already been done above. Otherwise reasoning in a similar way we have that

$$B^i_{i,i+1} = \mathbb{Q}\big[x^{(i)}_1, x^{(i)}_2\big]/\big(x^{(i)}_1\big); \qquad B^i_{i-1,i} = \mathbb{Q}\big[x^{(i)}_1, x^{(i)}_2\big]/\big(x^{(i)}_2\big)$$

the homomorphism $\phi^i_{i,i+1}$ and $\phi^{i-1}_{i-1,i}$ are the quotient homomorphisms, while $\psi^{i,i+1}_i(y^{(i)}) \equiv x^{(i)}_2 \bmod x^{(i)}_1$ and $\psi^{i-1,i}_{i-1}(y^{(i-1)}) \cong x^{(i-1)}_1 \bmod x^{(i-1)}_2$.

We deduce

Theorem 4.3. *The equivariant cohomology ring $H^*_G(X_\sigma)$ is isomorphic, as a graded ring, to $\mathbb{Q}[z_1, \dots, z_m, \zeta]/I$, where I is the ideal generated by the relations*

$$z_i z_j, \quad \text{if } |i-j| > 1; \qquad \zeta z_i, \quad \text{if } 1 < i < m. \tag{4}$$

and $\deg z_i = 2$, $\deg \zeta = 4$.

Proof. We know that

$$H^*_G(X_\sigma) \subset \bigoplus_{i=1}^{m-1} A_{i,i+1} \oplus \bigoplus_{i=0}^{m} A_i \oplus A_\emptyset.$$

However from Theorem 2.1 and our computations, we deduce that, if we project onto the ring $A_1 \oplus \bigoplus_{i=1}^{m-1} A_{i,i+1} \oplus A_m$, we still get an inclusion (this also follows using the localization theorem for equivariant cohomology, since the only orbits containing fixpoints under a maximal torus are exactly the closed orbits, $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_m}$).

We then get that $H^*_G(X_\sigma)$ consists of the sequences $(g_1, f_{1,2}, \dots, f_{m-1,m}, g_m)$ with $g_1(y,u) \in A_1$, $g_m(v,z) \in A_m$, $f_{i,i+1}(x^{(i)}_1, x^{(i)}_2) \in A_{i,i+1}$ and

$$g_1(x,0) = f_{1,2}(0,x); \qquad f_{i-1,i}(x,0) = f_{i,i+1}(0,x), \quad \forall 1 < i < m-1;$$
$$f_{m-1,m}(x,0) = g_m(0,x); \qquad g_1(0,x) = g_m(x,0)$$

x being an auxiliary variable. Now set

$$d^\sigma_i = \begin{cases} (y, x^1_2, 0, \dots, 0), & \text{if } i = 1; \\ (0, \dots, x^{i-1}_1, x^i_2, 0, \dots, 0), & \text{if } 1 < i < m; \\ (0, \dots, 0, x^m_1, z), & \text{if } i = m. \end{cases}$$

Also set

$$q = (u, 0, \dots, 0, v).$$

The elements d^σ_i and q lie in $H^*_G(X_\sigma)$ and generate it. Furthermore they satisfy the relations

$$d^\sigma_i d^\sigma_j = 0, \quad \text{if } |i-j| > 1; \qquad q d^\sigma_i = 0, \quad \text{if } 1 < i < m. \tag{5}$$

Thus, defining $\tilde{A}_\sigma = \mathbb{Q}[z_1, \dots, z_m, \zeta]/I$, we get a surjection of graded rings $\Psi : \tilde{A}_\sigma \to H^*_G(X_\sigma)$.

As before let $\tau = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m)$ and assume that σ is obtained from τ by the elementary move $v_i = v_{i-1} + v_{i+1}$. Consider the subring $C \subset \tilde{A}_\sigma$ generated by the elements ζ, z_j for $j \neq i-1, i+1$, $z_{i-1} + z_i$, $z_i + z_{i+1}$. It is immediate to see that C is a quotient of $\tilde{A}_\tau$. So using Corollary 4.2 and induction we get that

$$\sum_n \dim C_{n/2} t^n \leqslant \frac{1 + (m-1)t + (2m-3)t^2 + (2m-3)t^3 + (m-1)t^4 + t^5}{(1-t^2)(1-t^3)}.$$

On the other hand, using the relations (4), one immediately sees that any element in $a \in \tilde{A}_\sigma$ can be written as $a = z_i f(z_i, z_{i-1} + z_i) + c$, with $c \in C$ and f a polynomial. It follows that

$$\begin{aligned}
&\sum_n \dim \tilde{A}_{\sigma, n/2} t^n \\
&\quad \leqslant \frac{t}{(1-t)^2} + \frac{1 + (m-1)t + (2m-3)t^2 + (2m-3)t^3 + (m-1)t^4 + t^5}{(1-t^2)(1-t^3)} \\
&\quad = \frac{t(1 + 2t + 2t^2 + t^3) + 1 + (m-1)t + (2m-3)t^2 + (2m-3)t^3 + (m-1)t^4 + t^5}{(1-t^2)(1-t^3)} \\
&\quad = \frac{1 + mt + (2m-1)t^2 + (2m-1)t^3 + mt^4 + t^5}{(1-t^2)(1-t^3)} = \sum_n \dim H_G^{n/2}(X_\sigma) t^n.
\end{aligned}$$

Thus the Ψ is necessarily an isomorphism, proving our claim. $\square$

We want to give a reformulation of Theorem 4.3, which will be useful in the sequel. Fix $\sigma = (v_1, \dots, v_m)$. Consider the first quadrant $C = \{(a,b) \in \mathbb{R}^2 \mid a, b \geqslant 0\}$. Then σ gives a decomposition of C into the cones $C_1, \dots, C_{m-1}$ with

$$C_i = \{v = \alpha v_i + \beta v_{i+1} \mid \alpha, \beta \geqslant 0\}.$$

Define K_σ as the space of continuous functions on C which take rational values on integer points and are such that their restriction to each cone C_i is a polynomial. K_σ is clearly a graded $\mathbb{Q}$-algebra. We can give generators $\bar{d}_1^\sigma, \dots, \bar{d}_m^\sigma$, for K_σ as follows. Given a vector $v \in C$, it will lie in one of the C_i's (and in only one unless it is a multiple of one of the v_j's). We can then write $v = \alpha_i(v) v_i + \beta_i(v) v_{i+1}$. Notice that $\beta_i(v) = \alpha_{i+1}(v)$ on the multiples of v_{i+1}. Let us then set

$$\bar{d}_1^\sigma = \begin{cases} \alpha_1(v) & \text{if } v \in C_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\bar{d}_m^\sigma = \begin{cases} \beta_{m-1}(v) & \text{if } v \in C_{m-1}, \\ 0 & \text{otherwise} \end{cases}$$

and for $1 < i < m$

$$\bar{d}_i^\sigma = \begin{cases} \beta_i(v) & \text{if } v \in C_i, \\ \alpha_{i+1}(v) & \text{if } v \in C_{i+1}, \\ 0 & \text{otherwise}. \end{cases}$$

So that if we consider the subring $D \subset H^*_G(X_\sigma)$ generated by the d_i^σ, we get a surjective homomorphism

$$\mu : D \to K_\sigma.$$

It is clear by their definition that the $\bar{d}_i^\sigma$ satisfy the same relations (5) as the d_i^σ. Furthermore the restriction to each of the cones C_i gives a homomorphism $\gamma_i : K_\sigma \to \mathbb{Q}[x_1^{(i)}, x_2^{(i)}] = A_{i,i+1}$ such that $\gamma_i(d_i^\sigma) = x_1^{(i)}$, $\gamma_i(d_{i+1}^\sigma) = x_2^{(i)}$ and $\gamma_i(d_j^\sigma) = 0$ for $j \neq i, i+1$. Thus we get a homomorphism

$$\gamma : K_\sigma \to \bigoplus_{i=1}^{m-1} A_{i,i+1}.$$

Composing this with μ, we get a homomorphism

$$\gamma\mu : D \to \bigoplus_{i=1}^{m-1} A_{i,i+1}$$

which clearly coincides with the restriction to $D \subset A_1 \oplus \bigoplus_{i=1}^{m-1} A_{i,i+1} \oplus A_m$ of the projection onto $\bigoplus_{i=1}^{m-1} A_{i,i+1}$. Since this is clearly injective, we have shown the following

Theorem 4.4. *The ring $H_G^*(X\sigma)$ is isomorphic as a graded ring to the ring $K_\sigma[\zeta]/\tilde{I}$, where $\tilde{I}$ is the ideal generated by the relations ζd_j^σ, $j + 2, \dots, m-1$.*

Remark 4.5. (1) By our description it follows that $K_\sigma \simeq H_G^*(X\sigma)/(\zeta)$.

(2) Notice that the ideal J_σ of functions $f \in K_\sigma$ such that $\zeta f \in \tilde{I}$ coincides with the ideal of functions whose support is contained in the open quadrant $C_0 = \{(a,b) \mid ab \neq 0\}$. So $H_G^*(X_\sigma) \equiv K_\sigma[\zeta]/\zeta J_\sigma$.

5. The equivariant ring of conditions

Theorem 4.4 allows us to determine the equivariant ring of conditions. In order to do this let us give the following

Definition 2. A function on the quadrant C is called admissible if it lies in K_σ, for some admissible sequence σ.

The following lemma is well known, but we give its proof for completeness.

Lemma 5.1. *Let σ_1, σ_2 be two admissible sequences. Then there exist an admissible sequence τ such that $\tau \geqslant \sigma_1$ and $\tau \geqslant \sigma_2$.*

Proof. We proceed by induction on the cardinality of σ, the case in which $\sigma = ((0,1),(1,0))$ being trivial.

Remark that if $v, w \in \mathbb{Q}^2$ are such that $\det(v,w) = 1$ and u is such that $\det(v,u) = \det(u,w) = 1$, then $u = v + w$.

Now assume that $\sigma_1 = (v_1, \dots, v_m)$ and $v_i = v_{i-1} + v_{i+1}$ so that $\sigma_1' = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m)$ is also admissible. By induction there is $\tau' = (w_1, \dots, w_n)$ so that $\tau' \geqslant \sigma_1'$ and $\tau' \geqslant \sigma_2$. Now we must have $v_{i-1} = w_j$ and $v_{i+1} = w_k$ for some $j < k$. If $k = j+1$, we are done taking $\tau = (w_1, \dots, w_{j-1}, v_i, w_{j+1}, w_n)$. Otherwise the sequence $(w_j, \dots, w_k)$ is obtained from the sequence (v_{i-1}, v_{i+1}) by a succession of elementary moves, so, by our previous remark, it has to contain v_i and hence $\tau' \geqslant \sigma_1$. $\square$

Since clearly, if $\tau \geqslant \sigma$, K_σ is naturally contained in K_τ, Lemma 5.1 implies that the space K of admissible functions is a ring and

$$K = \lim_{\sigma \in \Sigma} K_\sigma.$$

We are now ready to state

Theorem 5.2. *There is an isomorphism of graded rings*

$$R_G = K[\tau]/\tau J$$

where $J \subset K$ is the ideal of functions whose support is contained in the open quadrant C_0.

Proof. Let $\tau \geqslant \sigma$ be two admissible sequences. Denote by $j_\sigma^\tau : R_\sigma \to R_\tau$ the inclusion. We have that $j_\sigma^\tau(J_\sigma) \subset J_\tau$, so we get an inclusion, which we also denote by j_σ^τ, of $R_\sigma[\zeta]/\zeta J_\sigma$ into $R_\tau[\zeta]/\zeta J_\tau$.

On the other hand, let us consider the G-equivariant projection

$$\pi_\sigma^\tau : X_\tau \to X_\sigma$$

and the corresponding homomorphism

$$\left(\pi_\sigma^\tau\right)^* : H_G^*(X_\sigma) \to H_G^*(X_\tau).$$

Putting together all our previous considerations, in order to prove the theorem, the only thing we need to show is that the diagram

$$\begin{tikzcd}
H_G^*(X_\sigma) \arrow[r, "i_\sigma"] \arrow[d, "(\pi_\sigma^\tau)^*"'] & R_\sigma[\zeta]/\zeta J_\sigma \arrow[d, "j_\sigma^\tau"'] \\
H_G^*(X_\tau) \arrow[r, "i_\tau"] & R_\tau[\zeta]/\zeta J_\tau
\end{tikzcd}$$

commutes.

For this we can assume that $\tau = (v_1, \ldots, v_m)$, $v_i = v_{i-1} + v_{i+1}$ and $\tau = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m)$, so that X_τ is obtained from X_σ blowing up the closed orbit $\mathcal{O}_{v_{i-1}, v_{i+1}}$.

But then, since π_σ^τ is a blow up we have

$$\left(\pi_\sigma^\tau\right)^* d_j^\sigma = \begin{cases} d_j^\tau & \text{if } j \neq i-1, i+i, \\ d_{i-1}^\tau + d_i^\tau & \text{if } j = i-1, \\ d_{i+1}^\tau + d_i^\tau & \text{if } j = i+1 \end{cases}$$

and on the other hand by definition we also have

$$\left(j_\sigma^\tau\right)^* \bar{d}_j^\sigma = \begin{cases} \bar{d}_j^\tau & \text{if } j \neq i-1, i+i, \\ \bar{d}_{i-1}^\tau + \bar{d}_i^\tau & \text{if } j = i-1, \\ \bar{d}_{i+1}^\tau + \bar{d}_i^\tau & \text{if } j = i+1. \end{cases} \qquad \square$$

6. The ring of conditions

It remains to determine the ring of conditions of conics as a quotient of R_G. For this one needs to determine the image in R_G of the ring $H^*_G(pt)$, which is a polynomial ring on two generators, one of degree 4 and one of degree 6. These are the two generating invariants of $H^*_T(pt)^{S_3}$.

A simple computation, which we leave to the reader, shows that, suitably normalizing the various elements and setting p_1 and p_2 equal to the linear functions giving the dual basis to $(1,0),(0,1)$, the images of the two polynomial generators of $H^*_G(pt)$ are the two elements

$$H_1 = p_1^2 + p_1 p_2 + p_2^2 + \zeta$$

and

$$H_2 = 2p_1^3 + 3p_1^2 p_2 - 3p_1 p_2^2 - 2p_2^3 - 6(p_1 - p_2)\zeta$$

of R_G. Thus finally we get

Theorem 6.1. *The ring of conditions for conics is the ring*

$$R = R_G/(H_1, H_2).$$

Remark 6.2. Notice that Theorem 6.1 clearly implies that R is generated by elements of degree two.

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