Endogenous Credit Cycles

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This paper studies models of credit with limited commitment and, therefore, endogenous debt limits. There are multiple stationary equilibria plus nonstationary equilibria in which credit conditions change simply because of beliefs. There can be equilibria in which debt limits display deterministic cyclic or chaotic dynamics, as well as stochastic (sunspot) equilibria in which they fluctuate randomly, even though fundamentals are deterministic and time invariant. Examples and applications are discussed. We also consider different mechanisms for determining the terms of trade and compare the setup to other credit models in the literature.

> As we have seen, all financial institutions are at the mercy of our innate inclination to veer from euphoria to despondency. (Niall Ferguson, *The Ascent of Money* [2008])

I. Introduction

Are credit markets susceptible to animal spirits, self-fulfilling prophecies, and endogenous fluctuations? This is different from asking if there are channels in credit markets that amplify or propagate fluctuations driven by fundamental factors, as emphasized in the literature surveyed by Gertler and Kiyotaki (2010). In this paper, we show that even if fundamentals are deterministic and time invariant, economies with credit market frictions can display cyclic, chaotic, and stochastic dynamics driven purely by beliefs. The key friction is limited commitment, which leads to endogenous borrowing constraints.[1] While most studies of these kinds of models focus on stationary equilibria, we show that economies with commitment frictions not only generate multiple steady states, with different credit conditions and allocations, but also display equilibria in which credit conditions and allocations vary over time, even when fundamentals do not. This is true when the terms of trade are determined by Walrasian pricing or by generalized Nash bargaining, although the economic forces generating dynamics in economies with the two pricing mechanisms are very different.

With limited commitment, agents honor their debt obligations to avoid punishment by exclusion from future credit. If one believes that one's debt limit in the future will be 0, one has nothing to lose by reneging on current obligations, which makes the equilibrium debt limit 0 today. Hence there is always a no-credit equilibrium. Generally, there is also a steady-state equilibrium with a positive debt limit. To explain how we get cycles, we find it useful to first build some intuition by considering a competitive labor market. Let the unconstrained equilibrium hours and wages be (ℓ^*, w^*). Suppose that we impose an exogenous restriction $\ell \leq \phi$, where $\phi = \ell^* - \varepsilon$ with $\varepsilon > 0$. There are two effects on workers: ℓ goes down and w goes up. The first effect has a second-order impact on their utility, by the envelope theorem, if ε is not too big. The increase in

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[1] By modeling debt limits endogenously using limited commitment, we follow Kehoe and Levine (1993, 2001) and Alvarez and Jermann (2000). See Azariadis and Kass (2007, 2013), Lorenzoni (2008), Hellwig and Lorenzoni (2009), and Sanches and Williamson (2010) for related work. As discussed in more detail below, our setup differs somewhat from the usual limited-commitment model and is based on a framework we use elsewhere to study banking (Gu et al. 2013). While there are no banks here, the setup is used because it is tractable and flexible. In particular, we have production and investment because we think that it may be interesting to see how these vary over the credit cycle; but as we show, the model can be reinterpreted as a Kehoe-Levine pure-endowment economy or as some other standard credit models.

w, however, generates a first-order increase in utility. This is why unions want to increase *w* and lower ℓ, compared to a competitive market. Of course, no individual worker can affect *w* in a competitive market, but an outside (e.g., union or policy) restriction on ℓ can generate a wage-increasing effect.[2]

Now consider a competitive credit market. No individual borrower can affect the terms of his loan by taking on less debt than his unconstrained equilibrium choice, say y^*. But if all borrowers face a constraint $y \leq \phi = y^* - \varepsilon$, as long as ε is not too big, they are all better off because the improvement in the terms of trade implies a first-order increase in utility, while the impact of the reduction in *y* is second order, again by the envelope theorem. Once this is understood, credit cycles emerge as follows. If you believe that the debt limit tomorrow will decrease somewhat, but not too much, your utility will increase tomorrow. This makes you more reluctant to renege on current debt because you do not want to be shut out of tomorrow's market. The reluctance to renege makes today's endogenous debt limit high. Tomorrow, you believe that the debt limit the next day will go back up. This makes you worse off the day after tomorrow, again by the envelope theorem. So tomorrow you are more inclined to renege, and this makes tomorrow's debt limit low. There is thus a natural tendency for credit conditions to oscillate in a two-period cycle. Higher-order cycles, chaotic dynamics, and sunspot equilibria are more complicated versions of the same idea.

A bilateral bargaining model can generate similar results for completely different reasons. Generalized Nash bargaining has the property that the surplus of both parties need not increase with the total surplus: one agent's surplus must rise but the other's may fall (see Aruoba, Rocheteau, and Waller [2007] for an extended discussion). When a borrower is negotiating over the terms of a loan, he can therefore be worse off if his debt limit goes up. This is different from the envelope-theoretic reasoning in Walrasian markets but still implies credit cycles: if your debt limit goes up, you can be worse off when negotiating tomorrow's credit contract, justifying a low limit today, and so on. Note that there cannot be cycles in the model with take-it-or-leave-it offers by the borrower or with Kalai's proportional bargaining solution, both of which imply that the borrower's surplus must increase with his debt limit. But with generalized Nash bargaining, as is the case with Walrasian pricing, borrowers can be better off when debt limits tighten. In either case, in contrast to the labor market example, in credit markets we do not need outside forces such as unions or government to impose the relevant quantity restriction: limited commitment delivers this as an equilibrium outcome.

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[2] As suggested by a referee, one can also provide related examples using international trade theory with tariffs; we think that the labor example should suffice.

The rest of the paper involves making this precise and providing the proofs. Before beginning, we mention that, as is well known, similar dynamics emerge in monetary models. In particular, the no-credit equilibrium is reminiscent of the nonmonetary equilibrium that always exists with fiat currency. The economic forces are different here. At the risk of oversimplifying, the basic mechanism at work in those models is a backward-bending savings function. The key economic mechanism at work in our model—payoffs decreasing as quantity constraints are relaxed—is different.[3]

II. The Model

Time is discrete and continues forever. Each period has two subperiods. There are equal measures of two types of agents. Type 1 agents consume and type 2 agents produce a good x in the first subperiod. Type 1 agents produce an intermediate good in the first subperiod and invest it, to generate a gross return y that type 2 agents like to consume in the second subperiod. There are gains from trade using the following credit relationship: type 1 agents (borrowers) get x now from type 2 agents (lenders) and promise to deliver y later, when their investments come to fruition. Only type 1 agents have access to the investment opportunity. The utility from this arrangement is $U^1(x, y)$ for type 1 and $U^2(y, x)$ for type 2, where U^j is strictly increasing in the first argument, which is j's consumption, and strictly decreasing in the second, which is j's production. Utility is also concave and twice differentiable and satisfies $U^j(0, 0) = 0$ (which can be relaxed). We also assume normal goods for some results.

The key friction in the model is limited commitment: rather than honor his obligation to deliver y in the second subperiod, type 1 may abscond with or otherwise divert the investment returns for his own purposes, as in Biais et al. (2007) or DeMarzo and Fishman (2007). If type 1 behaves opportunistically in this way, he gets a payoff λy over and above $U^1(x, y)$. Therefore, λ parameterizes the temptation to renege. We im-

pose $U^1(x, y + y') + \lambda y' \leq U^1(x, y)$ for all $x, y, y' \geq 0$. This says that it is never efficient ex ante for type 1 to divert y' because he is better off not producing/investing in the first place. However, he may be tempted to divert resources ex post, after the production cost is sunk. Incentive compatibility requires trade to be self-enforcing: U^j must be nonnegative for both agents, and we have to ensure compliance by type 1 in the second subperiod. The incentive to honor his obligation comes from the threat to exclude type 1 from future credit, which gives him an autarky payoff normalized to zero. Because it is useful in applications and examples, we allow imperfect monitoring: a defaulting type 1 agent gets caught only with probability π.[4]

Let V_t^j be type j's value function at t given an allocation (x_t, y_t), which we call a *credit contract*, since it specifies that type 1 gets (type 2 gives) x now and type 1 repays (type 2 receives) y later. Let $\beta \in (0, 1)$ be the discount factor across periods. Discounting across subperiods is subsumed in the U^j notation. Then[5]

$$V_t^1 = U^1(x_t, y_t) + \beta V_{t+1}^1, \tag{1}$$

$$V_t^2 = U^2(y_t, x_t) + \beta V_{t+1}^2. \tag{2}$$

A feasible contract must satisfy the *participation constraints* in the first subperiod,

$$U^1(x_t, y_t) \geq 0 \quad \text{and} \quad U^2(y_t, x_t) \geq 0. \tag{3}$$

The critical condition is the *repayment constraint* for type 1 in the second subperiod,

$$\lambda y_t + (1 - \pi)\beta V_{t+1}^1 \leq \beta V_{t+1}^1. \tag{4}$$

The left-hand side of (4) is type 1's instantaneous deviation payoff λy_t plus the continuation value: with probability π, he is caught and excluded from future markets; with probability $1 - \pi$, he is not caught and continues in good standing. It is convenient to rewrite (4) as

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[4] As mentioned, this environment is taken from Gu et al. (2013), where we discuss imperfect monitoring in more detail. But there we study only the incentive-feasible set in the spirit of mechanism design, while here we study price-taking and bargaining equilibria. To be clear, we do not need $\pi < 1$ for any general results, but it can be useful in applications and examples.

[5] These equations are sometimes referred to as "promise-keeping constraints" in the related literature.

$$y_t \leq \phi_t, \tag{5}$$

where $\phi_t \equiv (\beta\pi/\lambda)V_{t+1}^1$ is the *debt limit*. Using (1), we can express this recursively to make it clear that the debt limit in one period depends on the debt limit in the next period:

$$\phi_{t-1} = \frac{\beta\pi}{\lambda}U^1(x_t, y_t) + \beta\phi_t. \tag{6}$$

We consider both competitive markets and search-and-bargaining markets, but to ease the presentation, we present only the former in the text and relegate the latter to Appendix B. Thus, each period, everyone is assigned to one of a large number of spatially distinct Walrasian markets, where they trade short-term (across subperiod) credit contracts taking prices as given.[6] Let y be the numeraire. Then type 1 maximizes utility given his budget and credit constraints. We drop the participation constraint because autarky is always feasible:

$$\max_{x_t, y_t} U^1(x_t, y_t) \text{ subject to } p_t x_t = y_t \text{ and (5)}. \tag{7}$$

Meanwhile, type 2, with no repayment issues, solves

$$\max_{x_t, y_t} U^2(y_t, x_t) \text{ subject to } p_t x_t = y_t. \tag{8}$$

Let (x^*, y^*) denote equilibrium ignoring the repayment constraint, the solution to

$$U_x^1(x_t, y_t)x_t + U_y^1(x_t, y_t)y_t = 0, \tag{9}$$

$$U_x^2(y_t, x_t)x_t + U_y^2(y_t, x_t)y_t = 0. \tag{10}$$

If $y^* \leq \phi_t$, we can implement the unconstrained allocation (x^*, y^*). Otherwise equilibrium is constrained: $y_t = \phi_t$ and $x_t = h(\phi_t)$ is the solution to (10) with $y_t = \phi_t$.

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[6] There are no long-term trades. One way to motivate this is to say that agents meeting in one market never meet again, as formalized in Aliprantis, Camera, and Puzzello (2006, 2007), which is why we assume a large number of segmented markets. It is even easier to motivate the lack of long-term credit contracts in the search-and-bargaining version of the model, where agents trade bilaterally and never meet again, with probability one. Also, to avoid issues concerning renegotiation or the incentive compatibility of punishments, when we say that deviators are excluded from future credit markets, we mean that they literally lose access to these markets and are simply not allowed in. We also emphasize that we do not allow entry: the measures of borrowers and lenders (types 1 and 2) are fixed.

Since $x^* = h(y^*)$, equilibrium can be represented as follows:

$$\text{if } \phi_t < y^*, \text{ then } x_t = h(\phi_t) \text{ and } y_t = \phi_t;$$
$$\text{if } \phi_t \geq y^*, \text{ then } x_t = h(y^*) \text{ and } y_t = y^*. \tag{11}$$

It will be useful below to note the following: when $y_t \leq \phi_t$ is binding, x_t is not necessarily increasing in ϕ, since the sign of

$$\frac{\partial x}{\partial \phi} = h'(\phi) = -\frac{U_y^2 + y\left(U_{yy}^2 - \dfrac{U_y^2}{U_x^2}U_{xy}^2\right)}{U_x^2 + x\left(U_{xx}^2 - \dfrac{U_x^2}{U_y^2}U_{xy}^2\right)} \tag{12}$$

is ambiguous. More importantly, the sign of

$$\frac{\partial U^1[h(\phi), \phi]}{\partial \phi}$$

$$= \frac{U_y^1 U_x^2 - U_x^1 U_y^2 - y U_x^1\left(U_{yy}^2 - \dfrac{U_y^2}{U_x^2}U_{xy}^2\right) + x U_y^1\left(U_{xx}^2 - \dfrac{U_x^2}{U_y^2}U_{xy}^2\right)}{U_x^2 + x\left(U_{xx}^2 - \dfrac{U_x^2}{U_y^2}U_{xy}^2\right)} \tag{13}$$

is ambiguous. As we show below, $\partial U^1/\partial \phi < 0$ is not only possible but inevitable for some ϕ. Hence, a borrower's payoff can decrease with his credit limit.

Following Alvarez and Jermann (2000), for all t, the equilibrium debt limit ϕ_t is defined as follows: type 1 is indifferent between repaying ϕ_t and defaulting. In any feasible allocation, payoffs, and hence ϕ_t, must be bounded (so, as in many other models, we rule out explosive bubbles). We can also bound x_t and y_t without loss in generality. Hence we have the following definition.

DEFINITION 1.   An equilibrium is given by nonnegative and bounded sequences of credit limits $\{\phi_t\}_{t=1}^{\infty}$ and credit contracts $\{x_t, y_t\}_{t=1}^{\infty}$ such that (i) for all t, (x_t, y_t) solves (11) given ϕ_t and (ii) ϕ_t solves the difference equation (6) given $\{x_t, y_t\}_{t=1}^{\infty}$.

We can collapse the two conditions in definition 1 into one:

$$\phi_{t-1} = f(\phi_t) \equiv \begin{cases} (\beta\pi/\lambda)U^1[h(\phi_t), \phi_t] + \beta\phi_t & \text{if } \phi_t < y^* \\ (\beta\pi/\lambda)U^1(x^*, y^*) + \beta\phi_t & \text{otherwise.} \end{cases} \tag{14}$$

The dynamical system (14) describes the evolution of the debt limit in terms of itself. This system is forward looking, naturally, in the sense that the debt limit in one period depends on the debt limit in the next period. Equilibria are characterized by nonnegative bounded solutions to (14), from which one can solve for the allocation using (11). This completes our description of the baseline model.

III.   Results

A stationary equilibrium, or steady state, is a fixed point $f(\phi) = \phi$. Obviously $\phi = 0$ is one such point, associated with $(x, y) = (0, 0)$. Intuitively, if there is to be no credit in the future, you have nothing to lose by reneging, so no one will extend you credit today. A nondegenerate steady state is a solution to $f(\phi^s) = \phi^s > 0$. The mild assumption $U_x^1(0,0)h'(0) + U_y^1(0,0) > \lambda(1 - \beta)/\beta\pi$ guarantees the following.

PROPOSITION 1.   There exists at least one positive steady state, $\phi^s = f(\phi^s) > 0$. If $f(y^*) < y^*$, the repayment constraint is binding at any steady state.

All proofs are in Appendix A, but the existence result is easy to understand from figures 1A and B. In figure 1A, the debt constraint in steady state is slack, $\phi^s \geq y^*$, and in figure 1B, it is binding, $\phi^s < y^*$. Note that $f(\phi)$ is not necessarily monotone for $\phi \in (0, y^*)$, so we cannot guarantee uniqueness of a positive steady state ϕ^s, in general, although it was unique in all the examples we tried. Some statements of results below proceed as if ϕ^s were unique, but this is only to ease the presentation, and the results hold more generally if one replaces the clause "the unique positive" with "the smallest positive" steady state.[7]

It is clear that there exist equilibria in which $\phi_t \rightarrow 0$, as shown in figures 2A and B. Note that with ϕ_{t-1} on the horizontal and ϕ_t on the vertical axis, in these diagrams the curve is f^{-1}. We draw it this way even though the dynamics are forward looking, with causality running from future to previous periods. In figure 2A, f is monotone, so f^{-1} is a function; in figure 2B, it is not monotone, so f^{-1} is a correspondence. When f is monotone, once we pick an initial $\phi_0 \in (0, \phi^s)$, the sequence $\{\phi_t\}$ is pinned down. When f is not monotone, over some range for ϕ_0 there are multiple equilibrium paths for $\{\phi_t\}$. There are even perfect-foresight equilibria in which we start at ϕ^s and stay there for any number of periods, until dropping to the lower branch of f^{-1}, and then $\phi_t \rightarrow 0$. This is a serious and escalating credit crunch caused by nothing more than beliefs—a self-fulfilling prophecy.

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[7] One can make a few observations about the case with multiple positive steady states; e.g., they alternate between stable and unstable. We omit this in the interest of space.
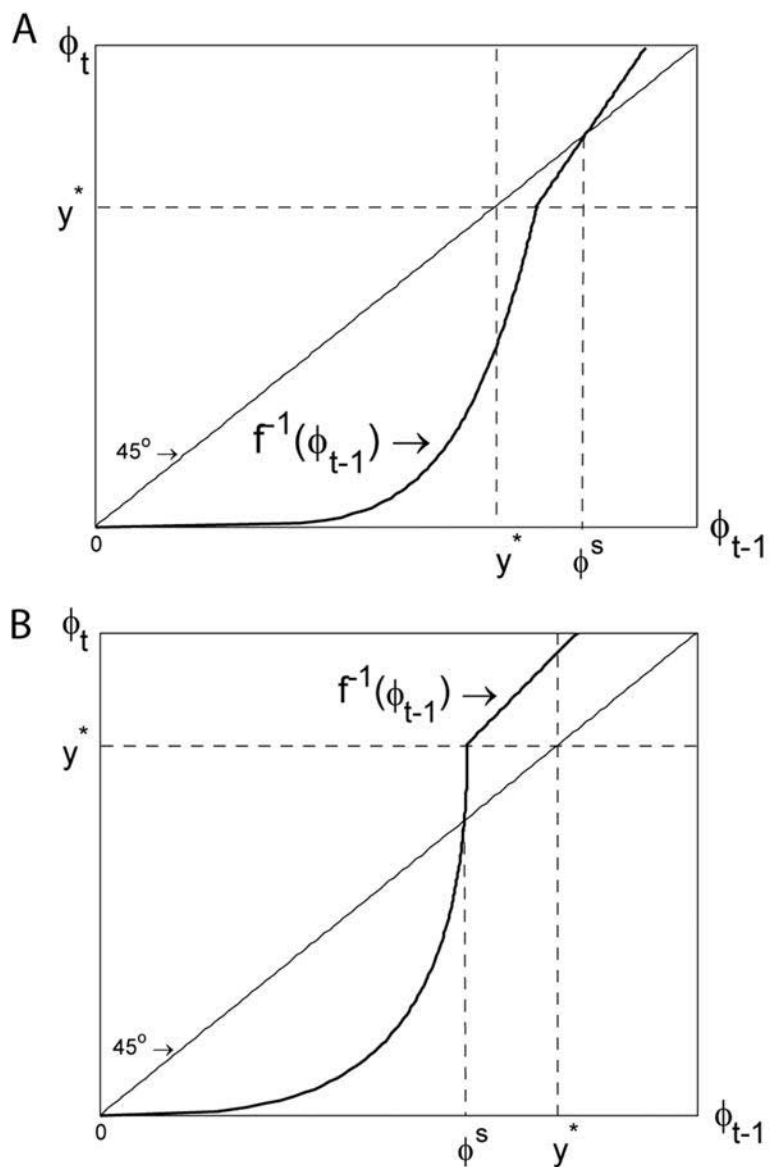
FIG. 1.—Steady state with (A) $\phi^s > y^*$ and (B) $\phi^s < y^*$

Summarizing this discussion, we have established the following proposition.

PROPOSITION 2. Suppose that there is a unique positive steady state ϕ^s, and let $\tilde{\phi} = \arg\max f(\phi_t)$ subject to $\phi_t \in [0, \phi^s]$. Starting from any $\phi_0 < \tilde{\phi}$, there is a nonstationary equilibrium, and possibly more than one, with $\phi_t \to 0$.

A

ϕ_t

$f^{-1}(\phi_{t-1}) \rightarrow$

y^*

$45° \rightarrow$

0

ϕ^s y^* ϕ_{t-1}

B

ϕ_t

$f^{-1}(\phi_{t-1}) \rightarrow$

y^*

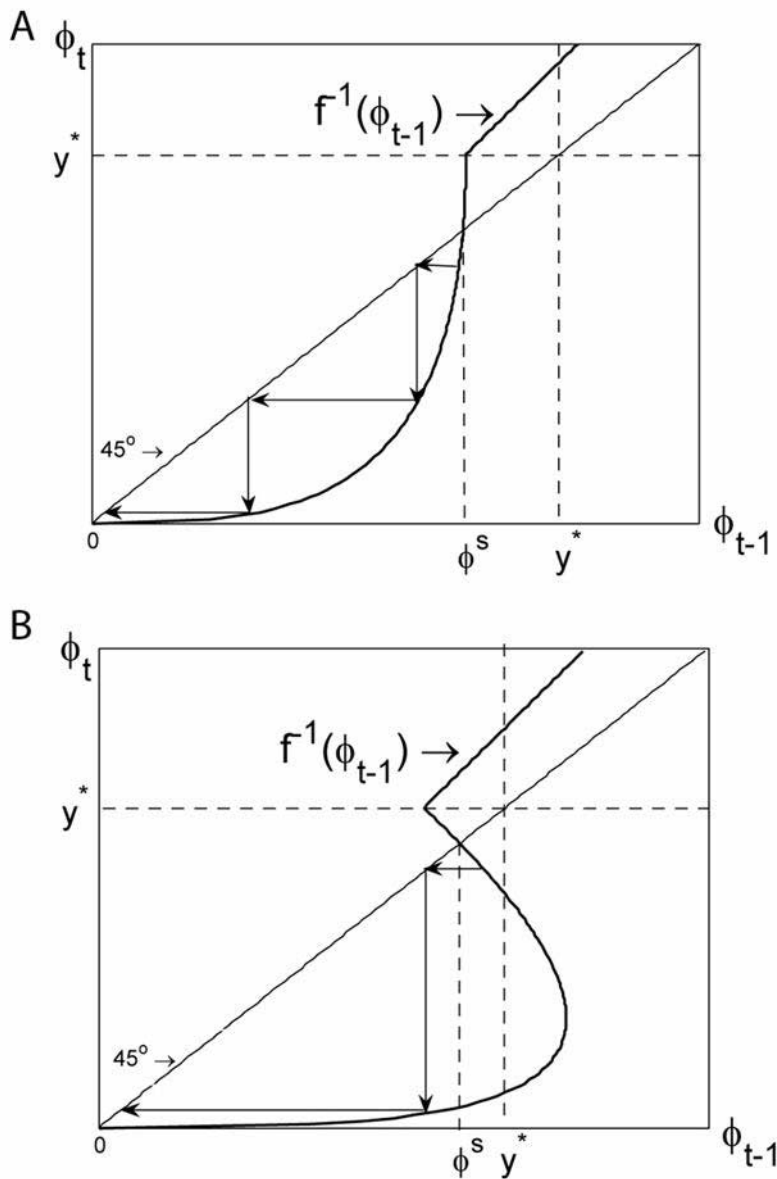$45° \rightarrow$

0

ϕ^s y^* ϕ_{t-1}

Fig. 2.—Nonstationary equilibria

Now consider two-period cycles, with ϕ^1 and $\phi^2 > \phi^1$ denoting the periodic points. Following standard methods (see Azariadis [1993] for a textbook treatment), we have the following proposition.

PROPOSITION 3.   Suppose that there is a unique positive steady state ϕ^s. If $f'(\phi^s) < -1$, there is a two-period cycle with $\phi^1 < \phi^s < \phi^2$.

We illustrate the previous result with examples using the following functional forms:

$$U^1(x, y) = \frac{x^{1-\alpha}}{1-\alpha} - y \quad \text{and} \quad U^2(y, x) = y - \frac{x^{1+\gamma}}{1+\gamma}. \qquad (15)$$

For simplicity, for now we set $\alpha = 0$.[8] Thus, U^1 is linear, but since U^2 is nonlinear, interesting dynamics still obtain via $x = h(\phi)$. The parameters are not calibrated to be empirically reasonable, only to illustrate some mathematical possibilities (this is not to say that the model could not be calibrated more realistically in future work). Examples 1 and 2 show cycles in which $\phi^1 < \phi^2 < y^*$ and $\phi^1 < y^* < \phi^2$, respectively.

EXAMPLE 1. Let $\gamma = 2.1$, $\beta = 0.4$, and $\pi/\lambda = 6$. Then $\phi^s = 0.7194$ and there is a two-cycle with $\phi^1 = 0.4777 < y^*$ and $\phi^2 = 0.9357 < y^*$. See figure 3A.

EXAMPLE 2. Let $\gamma = 0.5$, $\beta = 0.9$, and $\pi/\lambda = 10$. Now $\phi^s = 0.9674$ and there is a two-cycle with $\phi^1 = 0.9328 < y^*$ and $\phi^2 = 1.0365 > y^*$. See figure 3B.

The next example has a three-period cycle, shown in figure 4A. The existence of a three-period cycle implies the existence of cycles of all orders by the Sarkovskii theorem, as well as chaotic dynamics by the Li-Yorke theorem (again see Azariadis 1993). Chaos is observationally equivalent to a stochastic process for ϕ_t. Hence, debt limits, investment, and consumption can appear random when they are not (see fig. 4B). Note that credit limits are marked by x's in the figure. As one can see, there are recurrent episodes with three or four periods of unconstrained credit followed by a crunch, as well as episodes with three periods of tight credit followed by easy credit. Hence, compared to equilibria with two-period cycles, these chaotic equilibria can have much more intricate dynamics.

EXAMPLE 3. Let $\gamma = 0.8$, $\beta = 0.9$, and $\pi/\lambda = 15$. Now $\phi^s = 0.9835$, and there is a three-cycle with $\phi^1 = 0.9464 < y^*$, $\phi^2 = 1.0516 > y^*$, and $\phi^3 = 1.1684 > y^*$.

The next result says that in any cycle we must have $\phi_t < y^*$ at some point over the cycle, but not necessarily at all points. In other words, the debt limit binds infinitely often.

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[8] It is not merely a normalization to set $\alpha = 0$, although that would be the case in the model with bargaining. In the bargaining version, it is equivalent to bargain over either x and y, on the one hand, or $x^{1-\alpha}/(1-\alpha)$ and y, on the other. In the Walrasian model, however, we have already bought into linear pricing. If this is not obvious, consider a competitive profit-maximizing firm with cost function $c(y)$, for which it obviously matters, for profit, if $c(y)$ is linear or strictly convex. We return to $\alpha > 0$ below. For now, $\alpha = 0$ provides a stark illustration of the economic mechanism at work: in this case, type 1 agents realize no gains from trade, like a competitive profit-maximizing firm with linear cost, if the constraint is slack.
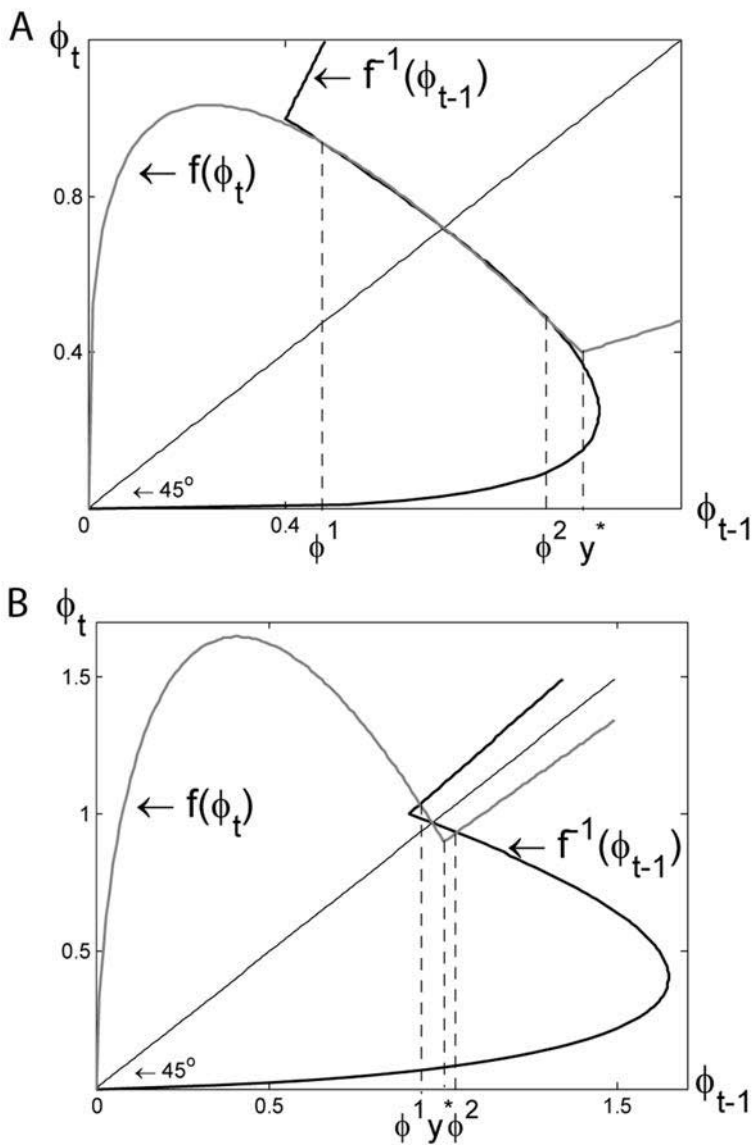
FIG. 3.—Cycle with (A) $\phi^1 < \phi^2 < y^*$ and (B) $\phi^1 < y^* < \phi^2$

PROPOSITION 4.   In any n-period cycle, at least one periodic point has the repayment constraint binding, $\phi_t < y^*$.

The model can also generate stochastic (sunspot) cycles, where ϕ_t and (x_t, y_t) fluctuate randomly. Consider a Markov sunspot variable $s_t \in \{1, 2\}$, which does not affect fundamentals but may affect equilibrium. Let
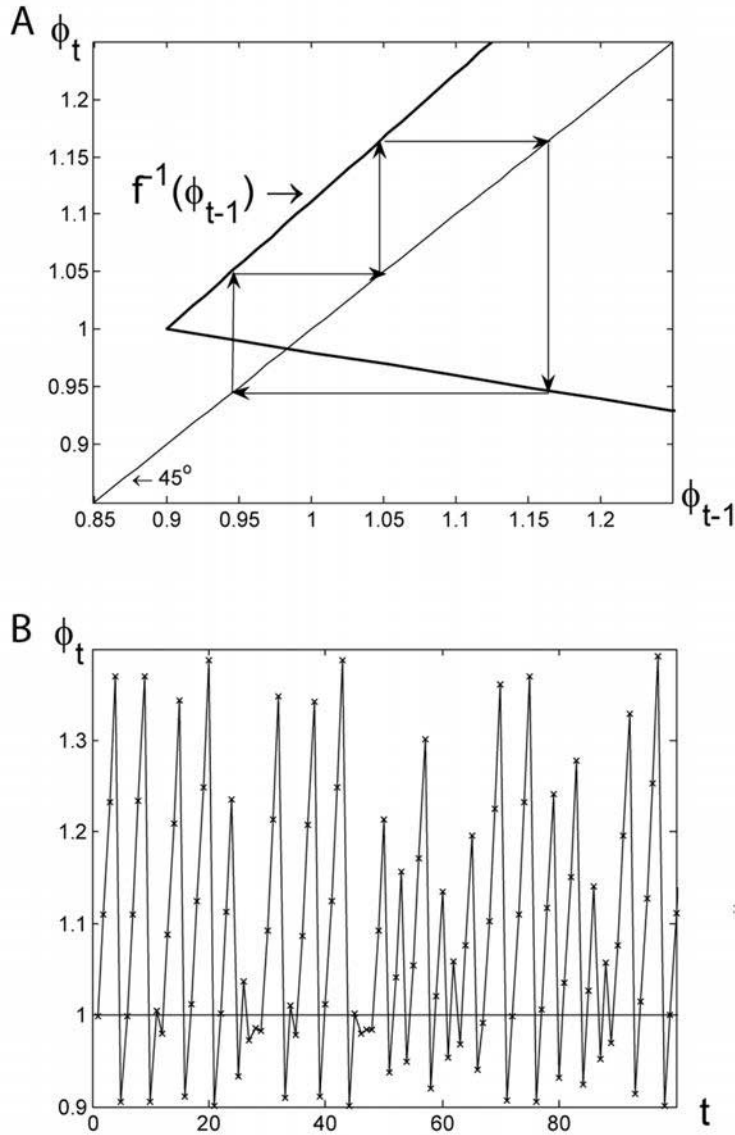
FIG. 4.—*A*, A three-period cycle; *B*, chaotic dynamics

$\Pr(s_{t+1} = 1 | s_t = 1) = \sigma_1$ and $\Pr(s_{t+1} = 2 | s_t = 2) = \sigma_2$. The economy is in state s at date t if $s_t = s$. Let $V^j_{s,t}$ be type j's value function in state s at date t, and let $(x_{s,t}, y_{s,t})$ be the state-contingent credit allocation. Then

$$V^1_{s,t} = U^1(x_{s,t}, y_{s,t}) + \beta[\sigma_s V^1_{s,t+1} + (1 - \sigma_s) V^1_{-s,t+1}], \qquad (16)$$

$$V_{s,t}^2 = U^2(y_{s,t}, x_{s,t}) + \beta[\sigma_s V_{s,t+1}^2 + (1 - \sigma_s) V_{-s,t+1}^2]. \tag{17}$$

The generalized repayment constraint is

$$\lambda y_{s,t} \leq \phi_{s,t} \equiv (\beta\pi/\lambda)[\sigma_s V_{s,t+1}^1 + (1 - \sigma_s) V_{-s,t+1}^1]. \tag{18}$$

Equilibrium in state s at date t is given as follows:

$$\begin{aligned} &\text{if } \phi_{s,t} < y^*, \text{ then } x_{s,t} = h(\phi_{s,t}) \text{ and } y_{s,t} = \phi_{s,t}; \\ &\text{if } \phi_{s,t} \geq y^*, \text{ then } x_{s,t} = h(y^*) \text{ and } y_{s,t} = y^*. \end{aligned} \tag{19}$$

The analogue to (6) is

$$\begin{aligned} \phi_{s,t-1} = {}& \sigma_s \beta[(\pi/\lambda) U^1(x_{s,t}, y_{s,t}) + \phi_{s,t}] \\ &+ (1 - \sigma_s)\beta[(\pi/\lambda) U^1(x_{-s,t}, y_{-s,t}) + \phi_{-s,t}]. \end{aligned}$$

This defines a bivariate dynamical system:

$$\begin{bmatrix} \phi_{1,t-1} \\ \phi_{2,t-1} \end{bmatrix} = \begin{bmatrix} \sigma_1 f(\phi_{1,t}) + (1 - \sigma_1)f(\phi_{2,t}) \\ \sigma_2 f(\phi_{2,t}) + (1 - \sigma_2)f(\phi_{1,t}) \end{bmatrix}. \tag{20}$$

DEFINITION 2.   A sunspot equilibrium is given by nonnegative and bounded sequences of credit limits $\{\phi_{s,t}\}_{t=1,s=1,2}^{\infty}$ and contingent contracts $\{x_{s,t}, y_{s,t}\}_{t=1,s=1,2}^{\infty}$ such that (i) $(x_{s,t}, y_{s,t})$ solves (19) given $\phi_{s,t}$ for all t and s and (ii) $(\phi_{1,t}, \phi_{2,t})$ solves the difference equation (20) given $\{x_{s,t}, y_{s,t}\}_{t=1,s=1,2}^{\infty}$.

A proper sunspot equilibrium has $\phi_{1,t} \neq \phi_{2,t}$ for some t. Consider proper sunspot equilibria that depend only on the state, not on the date, given by (ϕ_1, ϕ_2) with $\phi_2 > \phi_1$. Then the repayment constraint is binding in state 1; otherwise, we have $x^s = x^*$ and $y^s = y^*$ in both states, which implies $\phi_1 = \phi_2$. Following standard methods (again see Azariadis 1993), one can show that proper sunspot equilibria exist when $f'(\phi^s) < -1$, the same condition for two-period cycles, and so the examples with cycles also have sunspot equilibria. Formally, we have the following proposition.

PROPOSITION 5.   Suppose that there is a unique positive steady state ϕ^s. If $f'(\phi^s) < -1$, then there exist (σ_1, σ_2), $\sigma_1 + \sigma_2 < 1$, such that the economy has a proper sunspot equilibrium with ϕ_1 and ϕ_2 in the neighborhood of ϕ^s and Markov transitions given by $\Pr(s_{t+1} = 1|s_t = 1) = \sigma_1$ and $\Pr(s_{t+1} = 2|s_t = 2) = \sigma_2$.

If the conditions in proposition 5 are satisfied, one of the credit regimes can be highly persistent: when σ_1 is big, for example, the economy can stay in a regime in which debt limits are tight for a long time before switching to the other regime. The sunspot equilibria guaranteed by the proposition satisfy $\sigma_1 + \sigma_2 < 1$, however, so they cannot have both regimes highly persistent. This does not mean that it is impossible to generate sunspot equilibria in which σ_1 and σ_2 are both big; they are just not guaranteed by the proposition.

IV. Discussion

The existence of equilibria with deterministic or stochastic cycles relies on the nonmonotonicity of $f(\phi_t)$. To understand this, recall that

$$\phi_t = f(\phi_{t+1}) = \frac{\beta\pi}{\lambda} U^1(x_{t+1}, y_{t+1}) + \beta\phi_{t+1}. \tag{21}$$

An increase in the debt limit at $t+1$ influences the economy at t in two ways. First, it directly raises ϕ_t through the linear term $\beta\phi_{t+1}$. This effect in isolation works against cycles; for example, for $\phi_{t+1} > y^*$, the allocation (x_{t+1}, y_{t+1}) does not depend on ϕ_{t+1}, so only the linear term in (21) is operative, and the system $\phi_{t+1} = (1/\beta)\phi_t$ is explosive. But there is a second, nonlinear, effect coming from $U^1(x_{t+1}, y_{t+1})$, which is ambiguous in general and negative when ϕ_{t+1} is near y^*.[9]

Proposition 6.   With Walrasian pricing, $\partial U^1[h(\phi), \phi]/\partial\phi < 0$ at $\phi = y^* - \varepsilon$ for some $\varepsilon > 0$ if y is a normal good for type 2.

Heuristically, in Walrasian equilibrium, a buyer of good y is always better off under the restriction $y \leq y^* - \varepsilon$ for some $\varepsilon > 0$, for the same reason that monopolists produce less than competitive suppliers. Pursuing the monopolist analogy, we do not suggest that ε has to be small—that is merely a sufficient condition for the envelope theorem—and the borrowers may be better off for relatively big ε, just not too big. While debt limits can make borrowers better off, they cannot make everyone better off. For small ε, lenders lose, and for sufficiently tight debt limits, everyone loses (consider $\phi = 0$).

When the nonlinear term in (21) is negative and big, $f(\phi_{t+1})$ can be decreasing. We now present conditions guaranteeing that the system satisfies the condition for cycles, $f'(\phi^s) < -1$. We start by defining the elasticity

$$\eta(\phi) = \frac{\phi}{U^1[h(\phi),\phi]} \frac{\partial U^1[h(\phi),\phi]}{\partial \phi}.$$

COROLLARY 1. If $\eta(y^*) < -1$, then we can always pick β and π/λ to generate cycles.

It is not hard to satisfy this elasticity condition.[10] If $U^1 = \log(1+x) - A\log(1-y)$ and $U^2 = \log(1+y) - A\log(1-x)$, for $A \in (0,1)$, $\eta(y^*) < -1$ iff $A > \hat{A}$, where $\hat{A} \simeq 0.132$. Basically, $\eta(y^*) < -1$ allows us to choose $\beta\pi/(1-\beta)\lambda$ close to $y^*/U^1(x^*,y^*)$ so that ϕ^s is near y^* and then pick β and π/λ to guarantee $f'(\phi^s) < -1$. Although this is not a quantitative paper, we mention that having π/λ and β makes it easier to construct interesting examples. This is not surprising but is relevant because in future quantitative work it may be useful to allow the monitoring probability π, which is absent in most related models, to play a role.

To say more about conditions to guarantee cycles, reconsider the functional forms in (15), where now $\alpha \in [0,1)$. Also assume $(1-\alpha)\gamma > (1+\gamma)\alpha$. Then we have the following corollary.

COROLLARY 2. Given the parameter restrictions listed above, there is a unique positive steady state $\phi^s > 0$, and there are cycles around ϕ^s if

$$\beta < \frac{\gamma(1-\alpha) - \alpha(1+\gamma)}{\alpha + \gamma} \quad \text{and}$$

$$1 + \frac{2 + \gamma - \alpha}{\beta(\alpha+\gamma)} < \frac{\pi}{\lambda} < \frac{1-\alpha}{\alpha}\frac{1-\beta}{\beta}. \tag{22}$$

COROLLARY 3. As a special case, $\alpha = 0$ implies that cycles exist if $\pi/\lambda > 1 + (2+\gamma)/\beta\gamma$.

While it is hard to identify parameter conditions for a local bifurcation around ϕ^s, in general, given these functional forms, we get $f'(\phi^s) < -1$ when π/λ is in the interval defined in corollary 2. When $\alpha = 0$, so that U^1 is linear, all we need is $\pi/\lambda > 1 + (2+\gamma)/\beta\gamma$. In this case, credit cycles are more likely to emerge when β and π/λ are big. It may be surprising that cycles are more likely when agents are patient, the temptation to default is low, and the monitoring probability is high, but that is what we find in this specification.

The last item to discuss is the economic interpretation. In the baseline model, type 2 produces 1's consumption good, while type 1 produces intermediate goods, invests, and repays 2 from the proceeds. For concreteness, consider an example suggested by the editor. Agent 1 is a contractor whom agent 2 wants to do some work on his house. Agent 1 needs

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[10] At the suggestion of a referee and the editor, an alternative elasticity condition is provided in App. C.

to be paid x in advance; then he invests in supplies, or working capital, generally. There is a cost to agent 2 of providing x, which can be a production cost or an opportunity cost of consuming less himself. Agent 1 promises to deliver home improvements/repairs using his labor y, but as usual, he is tempted to renege for the opportunistic payoff λy. So we impose the constraint $\lambda y + (1 - \pi)\beta V_{t+1}^1 \leq \beta V_{t+1}^1$, saying that the contractor prefers to do the work rather than renege and risk ruining his reputation, which occurs with probability π, say because homeowners report bad behavior only probabilistically. Hence, there is a maximum $y = \phi$ that agent 1 can credibly commit to deliver. Our analysis implies that this example has equilibria in which ϕ_t displays complicated dynamics.[11]

We like having production and investment in the model because it is interesting to know that there can be endogenous dynamics in these variables, and in aggregate output, that cannot arise in pure-endowment specifications. However, similar results apply to endowment economies. Consider a simple Kehoe-Levine economy, where type 1 agents are endowed with one unit of a generic good in the second subperiod and type 2 with a unit in the first. There is no production or investment. Agents all want to consume in both subperiods. If 1 gets x_t from 2 in the first subperiod and gives him y_t in the second, payoffs are $U^1 = u(x_t) + v(1 - y_t)$ and $U^2 = u(1 - x_t) + v(y_t)$, which is a special case of our general setup. The value functions still satisfy (1) and (2), while the repayment constraint becomes $v(1) - v(1 - y_t) \leq \beta \pi V_{1t+1} \equiv \phi_t$, or $y_t \leq g(\phi_t)$ with $g(\phi_t) = 1 - v^{-1}[v(1) - \phi_t]$. The interpretation of ϕ_t here is slightly different, compared to the baseline model, where ϕ_t was the maximum credible promise, because the deviation payoff is now nonlinear. Still, following the steps taken above, we get

$$\phi_{t-1} = f(\phi_t)$$
$$\equiv \begin{cases} \beta \pi u[h \circ g(\phi_t)] + \beta \pi v[1 - g(\phi_t)] + \beta \phi_t & \text{if } g(\phi_t) < y^* \\ \beta \pi u(x^*) + \beta \pi v(1 - y^*) + \beta \phi_t & \text{otherwise.} \end{cases} \quad (23)$$

System (23) is similar to (14), except now the debt constraint is $y \leq g(\phi)$, while in the baseline model it is $y \leq \phi$. To get cycles, we need $f'(\phi^s) < -1$, which in this version means that $g'(\phi)$ is big. Since $g'(y^*)$

$= 1/v'(1 - y^*)$, we need $v'(1 - y^*)$ small. Given this, all the results hold, although the algebra was easier when deviation payoffs and hence the repayment constraints were linear. To be clear, usually pure-credit models have two types, one endowed with 1 in even periods and 0 in odd periods and vice versa for the other, while we have them endowed this way across subperiods. That is not crucial: Kehoe and Levine use $U = u(c_t) + \beta u(c_{t+1}) + \cdots$, while we use $U(x_t, y_t) + \beta U(x_{t+1}, y_{t+1}) + \cdots$ in general and $U(x_t, y_t) = u(x_t) + v(y_t)$ as a special case, but one can simply reinterpret two periods in their model as one period in ours, with a different β.

We can also map our setup into a model of secured lending, as in Kiyotaki and Moore (1997). In the first subperiod, type 1 is endowed with C units of a good he likes but type 2 does not, while type 2 is endowed with K units of a good that can be consumed by both types or used as capital to produce more goods in the second subperiod via technology $F(K)$. For simplicity, K and goods fully depreciate across periods. Type 1 considers both goods perfect substitutes but consumes only in the second subperiod, while type 2 values goods in both subperiods. If type 1 borrows x from 2 and promises to repay y, $U^1 = F(x) - y + c - C$, where c is the consumption of his endowed goods. We subtract C from his utility function, so his utility in autarky is normalized to zero. Alternatively, $c - C$ is his consumption over and above his endowment (think of him as a profit-maximizing producer/investor). Type 2's payoff is $U^2 = U^2(K - x, y)$ (think of him as a consumer/saver). To secure a loan, type 1 pledges C as collateral: if he delivers y to the lender, he gets C back; otherwise he gets $f(x) - (1 - \lambda)y - C$, as the lender gets a fraction of what he was promised, while he takes the rest and forfeits his collateral plus future credit with probability π.

The relevant constraint becomes

$$F(x_t) - y_t + \beta V_{t+1}^1 \geq F(x_t) - (1 - \lambda)y_t - C + (1 - \pi)\beta V_{t+1}^1,$$

or the following linear transformation of our baseline constraint,

$$y_t \leq C/\lambda + (\beta\pi/\lambda)V_{t+1}^1 \equiv \phi_t. \tag{24}$$

If $\pi = 0$, this is a simple model of collateralized lending à la Kiyotaki-Moore; if $C = 0$, this is a special case of our baseline model. Following the same steps, we again get a system similar to (14):

$$\phi_{t-1} = f(\phi_t)$$
$$\equiv \begin{cases} (\beta\pi/\lambda)F[h(\phi_t)] - (\beta\pi/\lambda)\phi_t + \beta\phi_t + (1 - \beta)C/\lambda & \text{if } \phi_t < y^* \\ (\beta\pi/\lambda)F(x^*) - (\beta\pi/\lambda)y^* + \beta\phi_t + (1 - \beta)C/\lambda & \text{otherwise.} \end{cases}$$
$$\tag{25}$$

By virtue of (24), the equilibrium credit limit is at least C/λ. Hence, we lose the no-credit equilibrium (as in the original Kehoe-Levine model, where they endowed agents with assets that can be confiscated in the event of default). Intuitively, even if there were no future credit, you would repay a debt of $y < C/\lambda$ today to avoid forfeiting valuable collateral. We also lose dynamic equilibria converging to a no-credit equilibrium. But that does not affect our main results. As long as $\pi > 0$, all the results on cycles continue to hold as long as C is not too big, simply by continuity (payoffs and behavior over a cycle around the positive steady state do not depend critically on what happens near the origin). We mention this not only to show the flexibility of the framework but to make the following point: What generates interesting dynamics is the forward-looking nature of credit markets. If $\pi = 0$, there is only secured lending, as in Kiyotaki-Moore, and hence no endogenous dynamics. However, that is particular to the simplicity of this example, as Kiyotaki-Moore models more generally can also generate endogenous cycles (see, e.g., He, Wright, and Zhu [2013] or Rocheteau and Wright [2013] for recent examples).

V.   Conclusion

This paper has developed a framework to study dynamics in lending, production, and investment. There always exist multiple steady states and multiple dynamic equilibria in the baseline model. There can exist deterministic, chaotic, and stochastic cycles, where credit conditions fluctuate even though fundamentals are deterministic and time invariant. Even if we add features, such as collateral, that rule out the no-credit equilibrium and paths that converge to the no-credit equilibrium, cycles still exist. The key friction in the theory is limited commitment, although there are other ingredients, including imperfect monitoring. Still, the model is simple to use. One reason is that equilibrium allocations reduce to a sequence of two-period (or two-subperiod) credit arrangements, although the economy goes on forever. The setup is also quite flexible and can easily accommodate both price taking and bargaining. We provided several interpretations, including pure-credit economies, as well as those with production and investment.

Of course, we made some strong assumptions, including our assumption of perfect-foresight or rational expectations. Obviously, if one relaxes these assumptions, even more exotic equilibria could arise; the goal here was to show that even with the discipline of rational expectations, one can get complicated dynamics. It would be interesting to see if combinations of fundamental shocks and the endogenous dynamics emphasized here might generate empirically plausible cycles. This and other

quantitative work is left to future research. As a final remark, we want to emphasize that the mathematics used here is neither new nor difficult and has been applied in many earlier papers to monetary and other models (recall n. 3). Our contribution was not meant to be technical but to present a novel economic application to credit markets. Some further progress has been made along these lines in recent work by He et al. (2013) and Rocheteau and Wright (2013), where similar methods are applied to markets for financial assets and markets for housing.

Appendix A

Proofs

Proof of Proposition 1

Define $T(\phi) = f(\phi) - \phi$. Our assumptions imply $T'(0) > 0$. Also, $T'(\phi) = \beta - 1 < 0$ for $\phi > y^*$. By the continuity and monotonicity of $T(\phi)$ for $\phi > y^*$, it is easy to see the following: if $T(y^*) \geq 0$, there exists $\phi^s > y^*$ such that $T(\phi^s) = 0$; and if $T(y^*) < 0$, there exists ϕ^s in $(0, y^*)$ such that $T(\phi^s) = 0$. In the latter case, there are no stationary equilibria in which $\phi^s > y^*$ because $T(\phi)$ is strictly decreasing for $\phi > y^*$. QED

Proof of Proposition 2

Because $f(\phi_t)$ is continuous, ϕ_{t-1} covers the interval $[0, \tilde{\phi}]$ for $\phi_t \in [0, \phi^s]$. Since there is a unique positive steady state, $f(\phi_t) > \phi_t$ for $\phi_t \in (0, \phi^s)$ and $f(\phi_t) < \phi_t$ for $\phi_t \in (\phi^s, \infty)$. That is, $\phi_{t-1} > \phi_t$ for $\phi_t \in (0, \phi^s)$ and $\phi_{t-1} < \phi_t$ for $\phi_t \in (\phi^s, \infty)$. Given $\phi_0 < \tilde{\phi}$, there is a ϕ_1 such that $\phi_1 \in (0, \phi^s)$ and $\phi_1 < \phi_0$, which implies a $\phi_2 \in (0, \phi^s)$ with $\phi_2 < \phi_1$, and so on. This sequence $\{\phi_t\}_0^\infty$ converges to zero. QED

Proof of Proposition 3

Let $f^2(\phi) = f \circ f(\phi)$. Because ϕ^s is the unique positive steady state, $f(\phi) > \phi$ for $\phi < \phi^s$ and $f(\phi) < \phi$ for $\phi > \phi^s$. Because $f(\phi)$ is linearly increasing for $\phi > y^*$, there exists a $\tilde{\phi} > y^*$ such that $f(\tilde{\phi}) > y^*$. By the uniqueness of the positive stationary equilibrium, $f^2(\tilde{\phi}) < f(\tilde{\phi}) < \tilde{\phi}$. Note that 0 and ϕ^s are two fixed points of f^2. The slope of $f^2(\phi^s)$ is

$$\frac{df^2(\phi^s)}{d\phi^s} = f'[f(\phi^s)]f'(\phi^s) = f'(\phi^s)f'(\phi^s) = [f'(\phi^s)]^2 > 1.$$

The last inequality uses $f'(\phi^s) < -1$. Similarly, $df^2(0)/d\phi > 1$. By continuity, f^2 must cross the 45-degree line in $(0, \phi^s)$. Because f^2 lies below the diagonal at $\tilde{\phi}$, it crosses it at least once in $(\phi^s, \tilde{\phi})$. So there are fixed points ϕ^1 and ϕ^2 such that $0 < \phi^1 < \phi^s < \phi^2$ for $f^2(\phi)$. QED

Proof of Proposition 4

Let $\phi^1, \phi^2, \ldots, \phi^n$ be the periodic points of an n-cycle. Suppose by way of contradiction that $\phi^j > y^*$ for all $j = 1, 2, \ldots, n$. As $\phi^j > y^*$ for all j, the payoff in each period of the cycle is $U^1(x^*, y^*)$, and $V_t^1 = U^1(x^*, y^*)/(1 - \beta)$ for all t. By the definition of ϕ_t, we have $\phi_t = \beta\pi/[(1 - \beta)\lambda]U^1(x^*, y^*)$ for all t. This is a contradiction. QED

Proof of Proposition 5

Since $f'(\phi^s) < 0$, there is an interval $[\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]$, with $\varepsilon_1, \varepsilon_2 > 0$, such that $f(\phi_1) > f(\phi_2)$ for $\phi_1 \in [\phi^s - \varepsilon_1, \phi^s)$ and $\phi_2 \in (\phi^s, \phi^s + \varepsilon_2]$. By definition, (ϕ_1, ϕ_2) is a proper sunspot equilibrium if there exists (σ_1, σ_2), with $\sigma_1, \sigma_2 < 1$ such that

$$\phi_1 = \sigma_1 f(\phi_1) + (1 - \sigma_1)f(\phi_2), \tag{A1}$$

$$\phi_2 = (1 - \sigma_2)f(\phi_1) + \sigma_2 f(\phi_2). \tag{A2}$$

Because ϕ_1 and ϕ_2 are weighted averages of $f(\phi_1)$ and $f(\phi_2)$, where $f(\phi_1) > \phi_1$ and $f(\phi_2) < \phi_2$, by the uniqueness of the positive steady state, necessary and sufficient conditions for (A1) and (A2) are

$$f(\phi_2) < \phi_1 < f(\phi_1) \quad \text{and} \quad f(\phi_2) < \phi_2 < f(\phi_1). \tag{A3}$$

Now, because $\phi_1 < \phi_2$, we can reduce this to

$$\phi_2 < f(\phi_1) \quad \text{and} \quad \phi_1 > f(\phi_2). \tag{A4}$$

When we expand $f(\phi_1)$ and $f(\phi_2)$ around ϕ^s and use $f(\phi^s) = \phi^s$, the above inequalities are equivalent to

$$\frac{\phi_2 - \phi^s}{\phi^s - \phi_1} < -f'(\phi^s) < \frac{\phi^s - \phi_1}{\phi_2 - \phi^s}.$$

Because $-f'(\phi^s) > 1$, $(\phi_2 - \phi^s)/(\phi^s - \phi_1) < -f'(\phi^s)$ is redundant if $-f'(\phi^s) < (\phi^s - \phi_1)/(\phi_2 - \phi^s)$. Now we have two unknowns (ϕ_1, ϕ_2) and only one inequality, $-f'(\phi^s) < (\phi^s - \phi_1)/(\phi_2 - \phi^s)$, to solve. It is straightforward that multiple solutions exist on $[\phi^s - \varepsilon_1, \phi^s + \varepsilon_2]$. To show $\sigma_1 + \sigma_2 < 1$, rewrite (A1) and (A2) as

$$\sigma_1 + \sigma_2 = \frac{\phi_1 - f(\phi_2) - \phi_2 + f(\phi_1)}{f(\phi_1) - f(\phi_2)} = \frac{\phi_1 - \phi_2}{f(\phi_1) - f(\phi_2)} + 1 < 1,$$

which holds because $(\phi_1 - \phi_2)/[f(\phi_1) - f(\phi_2)]$ is negative. QED

Proof of Proposition 6

If $\phi = y^*$, equilibrium is on the contract curve and $-U_x^1/U_y^1 = -U_x^2/U_y^2 = y/x$. A calculation implies

$$\left.\frac{\partial U^1[h(\phi),\phi]x}{\partial\phi}\right|_{\phi\to y_-^*} = \frac{U_y^1}{x}\left[\frac{x^2U_{xx}^2 + 2xyU_{xy}^2 + y^2U_{yy}^2}{U_x^2 + x\left(U_{xx}^2 - \dfrac{U_x^2}{U_y^2}U_{xy}^2\right)}\right].$$

The term outside the brackets is negative. The term in brackets is positive as long as y is normal for type 2. QED

Proof of Corollary 1

To construct cycles, it suffices to show that we can choose β and π/λ such that ϕ^s is close to y^* and $f'(\phi^s) < -1$. A steady state exists at $\phi = y_-^*$ if $y_-^* = (\beta\pi/\lambda)U^1(x_-^*,y_-^*) + \beta y_-^*$, or

$$\frac{\pi}{\lambda} = \frac{1-\beta}{\beta}\left.\frac{\phi}{U^1[h(\phi),\phi]}\right|_{\phi\to y_-^*}. \tag{A5}$$

The derivative of $f(\phi)$ at $\phi = y_-^*$ is $f'(\phi) = (\beta\pi/\lambda)\partial U^1[h(\phi),\phi]/\partial\phi|_{\phi\to y_-^*} + \beta$. Hence, for $f'(\phi) < -1$, we need

$$\frac{\beta\pi}{\lambda}\left.\frac{\partial U^1[h(\phi),\phi]}{\partial\phi}\right|_{\phi\to y_-^*} + \beta < -1. \tag{A6}$$

Combining (A5) and (A6), after some algebra, we get

$$\beta < \left\{-1 - \frac{\phi}{U^1[h(\phi),\phi]}\left.\frac{\partial U^1[h(\phi),\phi]}{\partial\phi}\right|_{\phi\to y_-^*}\right\}$$
$$\div\left\{1 - \frac{\phi}{U^1[h(\phi),\phi]}\left.\frac{\partial U^1[h(\phi),\phi]}{\partial\phi}\right|_{\phi\to y_-^*}\right\}. \tag{A7}$$

If the elasticity condition holds, we can pick β and π/λ to satisfy (A7). QED

Proof of Corollary 2

Given these utility functions, unconstrained and constrained equilibria satisfy $(x,y) = (1,1)$ and $(x,y) = (\phi^{1/1+\gamma},\phi)$. The unique positive steady state solves

$$\phi^{-(\alpha+\gamma)/(1+\gamma)} = (1-\alpha)\left(1 + \frac{1-\beta}{\beta}\frac{\lambda}{\pi}\right).$$

The steady state is constrained if and only if $\pi/\lambda < (1-\alpha)(1-\beta)/\alpha\beta$. A calculation implies $f'(\phi^s) < -1$ if and only if $\pi/\lambda > 1 + [(1+\gamma) + (1-\alpha)]/\beta(\alpha+\gamma)$. For these to hold simultaneously, we need $\beta < [\gamma(1-\alpha) - \alpha(1+\gamma)]/(\alpha+\gamma)$ with $\beta > 0$ as $(1-\alpha)/\alpha > (1+\gamma)/\gamma$. QED

Appendix B

Bargaining

Here we show that a model with bargaining can generate the same type of dynamics. For ease of presentation, assume that each type 1 meets a random type 2 at each date, and they negotiate a contract (x_t, y_t), taking as given what happens in all other meetings. Generalized Nash bargaining determines (x_t, y_t). Note that the strategic foundations of Nash bargaining are not straightforward in nonstationary situations (see, e.g., Coles and Wright 1998; Ennis 2001, 2004; Coles and Muthoo 2003). Thus, we are taking the Nash solution as a primitive, and there is no claim here that it is derived from a strategic bargaining game as in standard stationary models. That said, there is no presumption that one could not generate interesting dynamics with some strategic bargaining model, such as the ones used in the above-mentioned papers, but that is not the goal here.

Let type 1's bargaining power be θ, and let threat points be given by continuation values. Since the continuation values and threat points cancel, the bargaining outcome solves the simple problem

$$\max_{(x_t, y_t)} U^1(x_t, y_t)^\theta U^2(y_t, x_t)^{1-\theta} \text{ subject to (5).} \tag{B1}$$

If we ignore the repayment constraint for a moment, the solution (x^*, y^*) satisfies

$$\theta U_x^1(x_t, y_t) U^2(y_t, x_t) + (1-\theta) U^1(x_t, y_t) U_x^2(y_t, x_t) = 0, \tag{B2}$$

$$\theta U_y^1(x_t, y_t) U^2(y_t, x_t) + (1-\theta) U^1(x_t, y_t) U_y^2(y_t, x_t) = 0. \tag{B3}$$

If $\phi_t \geq y^*$, we can implement the unconstrained contract; if $\phi_t < y^*$, let $x_t = h(\phi_t)$ solve (B2) with $y_t = \phi_t$. Equilibrium satisfies (11), exactly as in the Walrasian model. When $y \leq \phi$ binds, we have

$$\frac{\partial x}{\partial \phi} = \frac{-\theta(U_{xy}^1 U^2 + U_x^1 U_y^2) - (1-\theta)(U_y^1 U_x^2 + U^1 U_{xy}^2)}{\theta(U_{xx}^1 U^2 + U_x^1 U_x^2) + (1-\theta)(U_x^1 U_x^2 + U^1 U_{xx}^2)}, \tag{B4}$$

which is ambiguous, in general. So is

$$\frac{\partial U^1[h(\phi), \phi]}{\partial \phi} = \frac{\theta U^2(U_{xx}^1 U_y^1 - U_{xy}^1 U_x^1) + (1-\theta) U^1(U_y^1 U_{xx}^2 - U_x^1 U_{xy}^2)}{\theta(U_{xx}^1 U^2 + U_x^1 U_x^2) + (1-\theta)(U_x^1 U_x^2 + U^1 U_{xx}^2)}, \tag{B5}$$

but it is unambiguous that $\partial U / \partial \phi < 0$ for ϕ close to y^* if $\theta < 1$ and y is a normal good for both types.

It is worth mentioning that bargaining with $\theta = 1$ implies $\partial U^1 / \partial \phi > 0$, in which case we cannot get endogenous cycles; the same is true for bargaining with Kalai's proportional solution. With generalized Nash and $\theta < 1$, however, since $\partial U / \partial \phi < 0$ for ϕ close to y^*, we get results similar to what we found with price taking. In particular, it is a matter of writing down the appropriate dynamical system and using the same methods to generate endogenous cycles, even though the economic forces are different, as discussed in the introduction.

Appendix C

Alternative Elasticity Conditions

Here, at the request of the editor and referee, we present an alternative condition to guarantee the existence of cycles, in terms of the elasticity of credit supply. From (13), we have

$$\frac{\partial U^1[h(\phi), \phi]}{\partial \phi} = \frac{U_y^1 U_x^2 - U_x^1 U_y^2 - y U_x^1 \left(U_{yy}^2 - \frac{U_y^2}{U_x^2} U_{xy}^2 \right) + x U_y^1 \left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2 \right)}{U_x^2 + x \left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2 \right)}.$$

The left-hand side is $\partial U^1[h(\phi), \phi] / \partial \phi = U_x^1 \, dh/d\phi + U_y^1$. Around (y^*, x^*), we have

$$\left. \frac{\partial U^1[h(\phi), \phi]}{\partial \phi} \right|_{\phi \to y_-^*} = U_x^1 \frac{dh}{d\phi} + U_y^1$$

$$= \frac{U_y^1}{x} \left[\frac{x^2 U_{xx}^2 + 2xy U_{xy}^2 + y^2 U_{yy}^2}{U_x^2 + x \left(U_{xx}^2 - \frac{U_x^2}{U_y^2} U_{xy}^2 \right)} \right].$$

As the last term is negative, $U_x^1 \, dh/d\phi + U_y^1 < 0$. Around (y^*, x^*), $U_x^1 / U_y^1 = -\phi/h$, and the inequality implies

$$\left. \frac{dh}{d\phi} \frac{\phi}{h} \right|_{\phi \to y_-^*} < 1.$$

That is, around the unconstrained steady state, the elasticity of credit supply is less than one.

The (utility) elasticity $\eta(\phi)$ defined just before corollary 1 can be rewritten

$$\frac{\phi}{U^1(h(\phi), \phi)} \frac{\partial U^1(h(\phi), \phi)}{\partial \phi} = \frac{\phi}{U^1(h(\phi), \phi)} \left[U_x^1 \frac{dh}{d\phi} + U_y^1 \right]$$

$$= \frac{dh}{d\phi} \frac{\phi}{h} \frac{U_x^1 h}{U^1} + \frac{U_y^1 \phi}{U^1}.$$

Around y^*,

$$\frac{dh}{d\phi}\frac{\phi}{h}\frac{U_x^1 h}{U^1} + \frac{U_y^1 \phi}{U^1}\bigg|_{\phi \to y_-^*} = \frac{U_x^1 h}{U^1}\left(\frac{dh}{d\phi}\frac{\phi}{h} - 1\right).$$

The condition in corollary 1 is equivalent to

$$\frac{dh}{d\phi}\frac{\phi}{h} < 1 - \frac{U^1(x^*, y^*)}{U_x^1(x^*, y^*)x^*}.$$

We conclude that if the elasticity of credit supply is small, cycles exist. QED

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