COMBINATORIAL ASPECTS OF CONNES'S EMBEDDING CONJECTURE AND ASYMPTOTIC DISTRIBUTION OF TRACES OF PRODUCTS OF UNITARIES

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ABSTRACT. In this paper we study the asymptotic distribution of the moments of (non-normalized) traces $\mathrm{Tr}(w_1), \mathrm{Tr}(w_2), \ldots, \mathrm{Tr}(w_r)$, where $w_1, w_2, \ldots, w_r$ are reduced words in unitaries in the group $\mathcal{U}(N)$. We prove that as $N \to \infty$ these variables are distributed as normal gaussian variables $\sqrt{j_1}Z_1, \ldots, \sqrt{Z_r}$, where $j_1, \ldots, j_r$ are the number of cyclic rotations of the words $w_1, \ldots, w_s$ leaving them invariant. This extends a previous result by Diaconis ([4]), where this it was proved, that $\mathrm{Tr}(U), \mathrm{Tr}(U^2), \ldots, \mathrm{Tr}(U^p)$ are asymptotically distributed as $Z_1, \sqrt{2}Z_2, \ldots, \sqrt{p}Z_p$.

We establish a combinatorial formula for $\int |\mathrm{Tr}(w_1)|^2 \cdots |\mathrm{Tr}(w_p)|^2$. In our computation we reprove some results from [1].

1. INTRODUCTION

Connes's embedding conjecture ([3]) for the case of discrete groups states that every discrete group Γ can be asymptotically embedded in the algebra of N by N matrices, when N tends to infinity. As observed in [10] (see also [7] and [5]) it amounts to prove that for every finite subset F of Γ, for every $\varepsilon > 0$, there exist N and unitaries $\{a_f \mid f \in F\}$ in $\mathcal{U}(N)$ such that $\|a_{f_1}a_{f_2} - a_{f_1 f_2}\|_{\mathrm{HS}} \leqslant \varepsilon \|\mathrm{Id}\|_{\mathrm{HS}}$ and $f_1 f_2 \in F$ for all f_1, f_2 in F. Here by $\|\cdot\|_{\mathrm{HS}}$ we denote the Hilbert-Schmidt norm

$$\|A\|_{\mathrm{HS}} = \mathrm{Tr}(A^*A)^{1/2}, \quad A \in M_N(\mathbb{C}),$$

Tr being the (non-normalized) trace on $M_N(\mathbb{C})$. If Γ is a group with presentation $\langle F_\infty \mid R\rangle$, where R are the relators, it can be proved (see [10]) that the Connes's embedding conjecture is equivalent to show that for any $\varepsilon > 0$, $w_1, w_2, \ldots, w_s \in R$, and for any $w_0 \notin R$, assuming that $w_0, w_1, \ldots, w_s$ are the words on the letters $a_1, \ldots, a_M$, there exist N and unitaries $U_1, U_2, \ldots, U_P$ in $\mathcal{U}(N)$ such that if $W_0, \ldots, W_s$ are the corresponding words obtained by substituting $a_1, \ldots, a_M$ with $U_1, \ldots, U_p$ we have (with $\mathrm{Tr} = \frac{1}{n}\,\mathrm{Tr}$)

$$(1) \qquad |\operatorname{tr}(W_0)| < \varepsilon, \quad |\operatorname{tr}(W_1)| > 1 - \varepsilon, \quad \ldots, \quad |\operatorname{tr}(W_s)| > 1 - \varepsilon.$$

Consequently, a natural object to study is the following: Let F_M be the free group with M generators $a_1, a_2, \ldots, a_M$. Let $w_0, w_1, \ldots, w_s$ be the reduced words in F_M and let $f_{w_0}, \ldots, f_{w_s}$ be the functions on $(\mathcal{U}(N))^M$ obtained by evaluating the traces $\mathrm{Tr}(W_0), \ldots, \mathrm{Tr}(W_s)$ of the words $W_0, \ldots, W_s$ at an M-uple $(U_1, \ldots, U_M)$. Then one

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has to determine the joint moments of these functions, i.e. the quantities (for all $\alpha_0, \dots, \alpha_s$ in $\mathbb{N}$)

$$\int_{(\mathcal{U}(N))^M} |f_{w_0}|^{\alpha_1} \cdots |f_{w_s}|^{\alpha_s} \, \mathrm{d}U_1 \cdots \mathrm{d}U_M,$$

with respect to the Haar measure.

In particular, after normalizing with the factor $\frac{1}{N}$, if one determines the measure for these moments, then one could solve the inequality (1).

2. Computation of $\int_{\mathcal{U}(N)} u_{i_1 j_1} \cdots u_{i_p j_p} u^*_{r_1 s_1} \cdots u^*_{r_p s_p} \, \mathrm{d}U$

The following computation was first performed by D. Weingarten, F. Xu, and B. Collins. At the time of writting the paper we were not aware of the previous literature, so we include our own proof for this computation.

Let S_n be the group of n-permutations and let $\mathbb{C}[S_n]$ be the group algebra. As in [2], we denote by W^N_σ the coefficient of $\sigma \in S_n$ in the inverse of the element $\Phi^N = \sum\limits_{\sigma \in S_n} N^{\sharp\sigma}\sigma \in \mathbb{C}[S_n]$ ($\sharp\sigma$ is the number of cycles in σ; the element Φ^N is invertible as it will be proven bellow for $N > n$). Thus we take

$$(\Phi^N)^{-1} = \sum_{\sigma \in S_n} W^N_\sigma \cdot \sigma.$$

Note that Φ^N is a central element and hence so is $\sum\limits_{\sigma \in S_n} W^N_\sigma \cdot \sigma$. With these notations we have:

Theorem 2.1. *For N, n in $\mathbb{N}$, $N > n$ and $\mathrm{d}U$ the Haar measure on $\mathcal{U}(N)$, let $i_1, \dots, i_n$, $j_1, \dots, j_n, r_1, \dots, j_n, s_1, \dots, s_n$ be indices from 1 to N. Denote the entries of a unitary by u_{ij} and the entries of its adjoint by $u^*_{ij} = \overline{u_{ij}}$. Then*

$$\int_{\mathcal{U}(N)} u_{i_1 r_1} \cdots u_{i_n r_n} u^*_{s_1 j_1} \cdots u^*_{s_n j_n} \, \mathrm{d}U = \sum W^N_{\sigma\theta^{-1}}$$

with the sum in the right hand side running over all σ, θ in S_n such that $j_a = i_{\sigma(a)}$, $a = 1, 2, \dots, n$ and $s_b = r_{\theta(b)}$, $B = 1, 2, \dots, n$.

Proof. Let $L^2(M_N(\mathbb{C})^n, \mu^n_M)$ be the Hilbert space obtained by endowing $M_N(\mathbb{C})^n$ with the measure $C\mathrm{e}^{-\operatorname{Tr}(A_1^* A_1) - \cdots - \operatorname{Tr}(A_n^* A_n)}$, $(A_1, \dots, A_n) \in M_N(\mathbb{C})^n$, where C is a constant, so that the entries functions $(A_1, A_2, \dots, A_n) \mapsto a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}$, have norm 1. Here $a^{(t)}_{ij}$ are the ij-entries of the matrix $A^{(t)}$ on the t-th component of the product $(M_N(\mathbb{C}))^n$.

Denote, for σ in S_n, by χ_σ the function

$$\chi_\sigma = \sum_{i_1, \dots, i_n = 1}^{N} a^{(1)}_{i_1 \sigma(i_1)} a^{(2)}_{i_2 \sigma(i_2)} \cdots a^{(n)}_{i_n \sigma(i_n)}. \qquad (?)$$

Then, from the theory of symmetric functions ([6], [9]), the functions χ_σ generate the subspace functions on $(M_N(\mathbb{C}))^n$ that are invariant to the diagonal action of $\mathcal{U}(n)$ on $(M_N(\mathbb{C}))^n$: $(A_1, \dots, A_n) \mapsto (UA_1U^*, \dots, UA_nU^*)$, $U \in \mathcal{U}(N)$. Moreover, for $n < N$ the functions $\{\chi_\sigma \mid \sigma \in S_n\}$ are independent ([9]) and the scalar product $\langle \chi_\sigma, \chi_\mu \rangle$ depends only on $\sigma^{-1}\mu$ and it is equal to $N^{\sharp(\sigma^{-1}\mu)}$.

Consequently, $\langle \chi_\sigma, \chi_\mu \rangle_{\sigma,\mu\in S_n}$ represents the matrix of the convolution with Φ_N on $L^2(S_n)$. Consequently, the inverse of Φ_N (which exists since the functions are independent) is the matrix $(W_{\sigma^{-1}\mu})_{\sigma,\mu\in S_n}$. Let P be the projection from $L^2((M_N(\mathbb{C}))^n),\mu)$ onto the space of $\mathcal{U}(N)$ invariant functions. Then on one hand, since μ is an invariant measure, it follows that P is the average over $\mathcal{U}(N)$ by integration. Hence

$$P(a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}) = \int_{\mathcal{U}(N)} (ua^{(1)}u^*)_{i_1 j_1} \cdots (ua^{(n)}u^*)_{i_n j_n} \,\mathrm{d}U. \tag{2}$$

On the other hand, assume $P(a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}) = \sum_{\sigma\in S_n} c_\sigma \chi_\sigma$, where c_σ depends on $i_1,\dots,i_n$, $j_1,\dots,j_n$. Then, for all $\mu \in S_n$,

$$\Big\langle a^{(1)}_{i_1,j_1} \cdots a^{(n)}_{i_n j_n} - \sum_\sigma c_\sigma \chi_\sigma, \chi_\mu \Big\rangle = 0,$$

and hence

$$\langle a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}, \chi_\mu \rangle = \sum_\sigma c_\sigma \langle \chi_\sigma, \chi_\mu \rangle, \quad \forall \mu \in S_n \tag{3}$$

But $\langle a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}, \chi_\mu \rangle_\mu$ is the vector (indexed) by $\mu \in S_n$) with the property that the μ-th component is equal to 1 if and only if $j_a = i_{\mu(a)}$, $a = 1,2,\dots,n$.

Let $R_{\sigma,\mu}$ be the inverse of the matrix $(\langle \sigma, \mu \rangle)_{\sigma,\mu\in S_n}$. We have noted before that $R_{\sigma,\mu} = W^N_{\sigma^{-1}\mu}$. From (3), by inversion, we deduce that $c_\sigma = \sum_\mu R_{\sigma,\mu}$, where the sum runs over all μ such that $j_a = i_{\mu(a)}$, $a = 1,2,\dots n$. Thus

$$P(a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}) = \sum_{\substack{\sigma\in S_n\\ \mu\in S'}} R_{\sigma,\mu}\chi_\sigma, \quad S' = \{\mu \in S_n \mid j_a = i_{\mu(a)}, a = 1,2,\dots,n\}. \tag{4}$$

From (2) we obtain that

$$\begin{aligned} P(a^{(1)}_{i_1 j_1} \cdots a^{(n)}_{i_n j_n}) &= \sum_{\substack{r_1,\dots,r_n=1\\ s_1,\dots,s_n=1}} \int_{\mathcal{U}(N)} u_{i_1 r_1} \cdots u_{i_n r_n} a^{(1)}_{r_1 s_1} \cdots a^{(n)}_{r_n s_n} u^*_{s_1 j_1} \cdots u^*_{s_n j_n} \,\mathrm{d}U \\ &= \sum_{\substack{r_1,\dots,r_n=1\\ s_1,\dots,s_n=1}} a^{(1)}_{r_1 s_1} \cdots a^{(n)}_{r_n s_n} \int_{\mathcal{U}(N)} u_{i_1 r_1} \cdots u_{i_n r_n} u^*_{s_1 j_1} \cdots u^*_{s_n j_n} \,\mathrm{d}U. \end{aligned}$$

Identifying the coefficients from the last formula with (4) we obtain our statement. □

3. FORMULA FOR $\iint_{\mathcal{U}(N)^2} \mathrm{Tr}(W_1)\cdots \mathrm{Tr}(W_s)\,\mathrm{d}U\,\mathrm{d}V$

In this section we deduce a formula for the integral of traces of words only in case of $\mathcal{U}(N)^2$ (instead of $\mathcal{U}(N)^M$) for simplicity. A similar formula was derived in [1]. Since the shape of the combinatorial aspect of the formula is important for the computation of the asymptotics, we derive our formula directly from the preceding section.

Let $w_1, w_2, \dots, w_n$ be reduced words in $F_2 = \langle a, b \rangle$, and let $W_1, W_2, \dots, W_n$ be the corresponding words viewed as functions in the variables $(U,V) \in \mathcal{U}(N)^2$ obtained by substituting (U,V) for (a,b). We describe $\mathrm{Tr}(w_1)\cdots\mathrm{Tr}(w_n)$ in terms of a permutation γ and write $\mathrm{Tr}(w_1)\cdots\mathrm{Tr}(w_n) = \mathrm{Tr}_\gamma(w_1\cdots w_n)$, where γ is described as follows.

Let n be the total number of occurrences of the symbol u in $W_1, W_2, \dots, W_n$. Let m be the total number of occurrences of v. For the integral $\iint_{\mathcal{U}(N)} \mathrm{Tr}(W_1)\cdots\mathrm{Tr}(W_p)\,\mathrm{d}U\,\mathrm{d}V$ to be non-zero it is necessary ([8]) that n equals the number of U^* and that m equals the numbers of V^*. We introduce a set of symbols (indexed by the letters u, v, u^*, v^* respectively)

$$X = \{1_u, \dots, n_u, 1_{u^*}, \dots, n_{u^*}, 1_v, \dots, m_v, 1_{v^*}, \dots, n_{v^*}\}.$$

Thus X is a set with $2(n+m)$ elements.

Definition 3.1. Given $w_1, \dots, w_p$ and X as above we define a permutation γ of X by means of the formula

$$\begin{aligned}\mathrm{Tr}_\gamma(w_1\cdots w_p) &= \mathrm{Tr}(w_1)\cdots\mathrm{Tr}(w_p)\\ &= \sum_{\substack{a_{1_u},\dots,a_{n_u}=1\\ a_{1_{u^*}},\dots,a_{n_{u^*}}=1\\ a_{1_v},\dots,a_{m_v}=1\\ a_{1_{v^*}},\dots,a_{m_{v^*}}=1}}^{N} u_{a_{1_u}a_{\gamma(1_u)}}\cdots u_{a_{n_u}a_{\gamma(n_u)}} u^*_{1_{u^*}\gamma(1_{u^*})}\cdots u^*_{n_{u^*}\gamma(n_{u^*})}\\ &\qquad \cdots v_{1_v\gamma(1_v)}\cdots v^*_{m_{v^*}\gamma(m_{v^*})}\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

We denote the term on right hand side by $\Phi_\gamma(U,V)$, where $U, V \in \mathcal{U}(N)$.

Note that since the words are reduced γ has no fixed points.

With these notations we have:

Theorem 3.2. *The integral of* $\mathrm{Tr}(w_1)\cdots\mathrm{Tr}(w_p)$ *over* $U(N)^2$ *is*

$$\iint_{\mathcal{U}(N)^2} \mathrm{Tr}_\gamma(U,V)\,\mathrm{d}U\,\mathrm{d}V = \sum_{\substack{\sigma_u,\theta_u\in S(1_u,\dots,n_u)\\ \sigma_v,\theta_v\in S(1_v,\dots,n_v)}} W^N_{\sigma_u\circ\theta_u^{-1}} W^N\sigma_v\circ\theta_v^{-1} N^{\sharp R(\gamma,\sigma_u,\dots,\theta_v}, $$

where $R(\gamma,\sigma_u,\theta_u,\sigma_v,\theta_v)$ *is the equivalence relation on* X *generated by*

$$\begin{aligned} t_{u^*} &= \gamma(\sigma_u(t_u)), \quad \gamma(t_{u^*}) = \theta_u(t_u), \quad t = 1,2,\dots,n,\\ s_{v^*} &= \gamma(\sigma_v(s_v)), \quad \gamma(s_{v^*}) = \theta_v(t_v), \quad s = 1,2,\dots,m.\end{aligned}$$

Here $\sharp R(\gamma,\sigma_u,\theta_u,\sigma_v,\theta_v)$ *is the number of classes in the equivalence relation.*

Proof. Indeed, we have that

$$\begin{aligned}&\iint_{\mathcal{U}(N)^2} \mathrm{Tr}_\gamma(U,V)\,\mathrm{d}U\,\mathrm{d}V\\ &= \Bigg(\sum_{\substack{a_{1_u},\dots,a_{n_u}=1\\ a_{1_{u^*}},\dots,a_{n_{u^*}}=1}} \int_{\mathcal{U}(N)} u_{a_{1_u}a_{\gamma(1_u)}}\cdots u_{a_{n_u}a_{\gamma(n_u)}} u^*_{a_{1_{u^*}}a_{\gamma(1_{u^*})}}\cdots u^*_{a_{n_{u^*}}a_{\gamma(n_{u^*})}}\,\mathrm{d}U\Bigg)\\ &\quad\cdot\Bigg(\sum_{\substack{a_{1_v},\dots,a_{n_v}=1\\ a_{1_{v^*}},\dots,a_{n_{v^*}}=1}} \int_{\mathcal{U}(N)} v_{a_{1_v}a_{\gamma(1_v)}}\cdots v_{a_{n_v}a_{\gamma(n_v)}} v^*_{a_{1_{v^*}}a_{\gamma(1_{v^*})}}\cdots v^*_{a_{n_{v^*}}a_{\gamma(n_{v^*})}}\,\mathrm{d}V\Bigg)\end{aligned}$$

We now apply the formula from the preceding section, interchange the summation formula with the summation after $\sigma_u, \theta_u, \sigma_v, \theta_v$, where σ_u, θ_u are the permutations that appear in the integrals for the u's and σ_v, θ_v are the permutations that appear in the summations for the θ's. □

Remark 3.3. Since the words are allays reduced, the equivalence relation $R(\gamma, \sigma_u, \theta_u, \sigma_v, \theta_v)$ has no singleton classes and hence $\sharp R(\gamma, \sigma_u, \theta_u, \sigma_v, \theta_v) \leqslant n+m$.

4. THE ASYMPTOTICS FOR $\iint (\mathrm{Tr}(W_1))^{\alpha_1} (\overline{\mathrm{Tr}(W_1)})^{\beta_1} \cdots (\mathrm{Tr}(W_p))^{\alpha_p} (\overline{\mathrm{Tr}(W_P)})^{\beta_p} \, \mathrm{d}U \, \mathrm{d}V$

In this section we show that for all words $w_1, \dots, w_p$ in F_2 by taking $W_1, \dots, W_p$ to be the corresponding functions on $\mathcal{U}(N)^2$ we have that

$$\iint_{\mathcal{U}(N)} (\mathrm{Tr}(W_1))^{\alpha_1} (\overline{\mathrm{Tr}(W_1)})^{\beta_1} \cdots (\mathrm{Tr}(W_p))^{\alpha_p} (\overline{\mathrm{Tr}(W_P)})^{\beta_p} \, \mathrm{d}U \, \mathrm{d}V$$

is $\mathrm{O}\left(\frac{1}{N}\right)$ unless $\alpha_1 = \beta_1, \dots, \alpha_p = \beta_p$ in which case the integral is $\alpha_1! \cdots \alpha_p! \cdot (j(w_1))^{\alpha_1} \cdots (j(w_p))^{\alpha^p}$, where $j_i = j(w_i)$ is the numbers of cyclic rotations of the word w_i which leave w_i it invariant. As in [4] this means that the asymptotic distribution of the variables $\mathrm{Tr}(W_1), \dots, \mathrm{Tr}(W_p)$ as $M \to \infty$ is that of $\sqrt{j_1} Z_1, \dots, \sqrt{j_p} Z_p$, where $Z_1, \dots, Z_p$ are independent gaussian variables.

Theorem 4.1. *Let $w_1, \dots, w_p$ be the words on F_2 and $W_1, \dots, W_p$ be the corresponding functions on $\mathcal{U}(N)^2$. An integral of the form*

$$\iint (\mathrm{Tr}(W_1))^{\alpha_1} (\overline{\mathrm{Tr}(W_1)})^{\beta_1} \cdots (\mathrm{Tr}(W_p))^{\alpha_p} (\overline{\mathrm{Tr}(W_P)})^{\beta_p} \, \mathrm{d}U \, \mathrm{d}V$$

is non-zero (modulo $\mathrm{O}\left(\frac{1}{N}\right)$) if and only if it can be written in the form

$$\iint_{\mathcal{U}(N)^2} |\,\mathrm{Tr}(W_1)|^{2\alpha_1} \cdots |\,\mathrm{Tr}(W_p)|^{2\alpha_p} \, \mathrm{d}U \, \mathrm{d}V$$

in which case it is equal to $\alpha_1! \cdots \alpha_p! j_1^{\alpha_1} \cdots j_p^{\alpha_p}$, with $j_i = j(w_i)$ the number of cyclic rotations of the word w_i, that are leaving w_i invariant.

Consequently, $\mathrm{Tr}(W_1), \dots, \mathrm{Tr}(W_s)$ have the asymptotic moment distribution (as $N \to \infty$) of $\sqrt{j_1} Z_1, \dots, \sqrt{j_s} Z_s$, where $Z_1, \dots, Z_s$ are independent normal gaussian variables.

Proof. We rewrite the formula from the preceding section as follows.

$$\begin{aligned}
&\iint_{\mathcal{U}(N)^2} \mathrm{Tr}(W_1) \cdots \mathrm{Tr}(W_p) \, \mathrm{d}U \, \mathrm{d}V \\
&= \sum_{\beta_1 \cdots \beta_{n_1} \beta_{n_1+1} \cdots \beta_{n_s} = 1} \iint a^{(1)}_{\beta_1 \beta_2} \cdots a^{(n_1)}_{\beta_{n_1} \beta_1} \cdots a^{(n_{p-1}+1)}_{\beta_{n_{p-1}+1} \beta_{n_{p-1}+2}} \cdots \\
&\qquad \cdots a_{\beta_{n_p}} {\beta^{(n_p)}_{n_{p-1}+1}} \, \mathrm{d}U \, \mathrm{d}V
\end{aligned} \tag{5}$$

where the symbols $a^{(1)} \cdots a^{(n_s)}$ belong to the set $\{U, V, U^*, V^*\}$. Here $n_i - n_{i-1}$ is the lenght of the word w_i, $i = 1, 2, \dots, s$.

Denote by $\widetilde{U}$ the set of all symbols $a^{(i)}$ that are equal to the letter u, and similarly for $\widetilde{U^*}, \widetilde{V}, \widetilde{V^*}$. Because of [8], unless $\operatorname{card} \widetilde{U} = \operatorname{card} \widetilde{U}^* = n$, $\operatorname{card} \widetilde{V} = \operatorname{card} \widetilde{V}^* = m$, the integral is $\mathrm{O}(\frac{1}{N})$.

According to the formula in the preceding paragraph the integral will be the summation over all bijection $\sigma_u : \widetilde{U} \to \widetilde{U}^*$, $\theta_u : \widetilde{U}^* \to \widetilde{U}$, $\sigma_v : \widetilde{V} \to \widetilde{V}^*$, $\theta_v : \widetilde{V}^* \to \widetilde{V}$, of

$$\sum_{\sigma_u, \theta_u, \sigma_v, \theta_v} W_{\sigma_u \circ \theta_v^{-1}} \cdot W_{\sigma_v \circ \theta_v^{-1}} N^{\sharp R(\sigma_u, \theta_u, \sigma_v, \theta_v)} \tag{6}$$

where $R(\sigma_u, \theta_u, \sigma_v, \theta_v)$ is the equivalence relation generated by

$$\sigma_u(i) + 1 \sim i, \quad \theta_u(i) \sim i + 1, \quad \sigma_v(i) + 1 \sim i, \quad \theta_v(i) \sim i + 1$$

where by the operation $i + 1$, we mean successively (when $i = n_1, n_2, \ldots, n_s$)

$$n_1 + 1 = 1, \quad n_2 + 1 = n_1 + 1, \quad \ldots, \quad n_s + 1 = n_{s-1} + 1$$

(coresponding to the cycle $(1, \ldots, n_1)(n_1 + 1, \ldots, n_2)(n_{s-1} + 1, \ldots, n_s))$.

The equivalence relation has no singletons and hence $N^{\sharp R(\sigma_u, \theta_u, \sigma_v, \theta_v)}$ is at most N^{n+m}, while the term $W_{\sigma_u \circ \theta_v^{-1}} \cdot W_{\sigma_v \circ \theta_v^{-1}}$ is of the order N^k, where $k \leqslant n + m$, with equality if and only if $\sigma_u = \theta_v^{-1}$, $\sigma_v = \theta_v^{-1}$.

This means that the only non zero terms will come from equivalence relations of the form

$$\sigma_u(i) + 1 \sim i, \quad \sigma_u^{-1}(i) \sim i + 1, \quad \sigma_v(i) + 1 \sim i, \quad \sigma_v^{-1}(i) \sim i + 1.$$

To obtain the number of terms m the sum (6) in this case we need to determine the permutations σ_u, σ_v for which this equivalence relation has all the classes of two elements.

Denote by γ the permutation with the property that $\gamma|\widetilde{U} = \sigma_u$, $\gamma|\widetilde{U}^* = \sigma_u^{-1}$, $\gamma|\widetilde{V} = \sigma_v$, $\gamma|\widetilde{V}^* = \sigma_v^{-1}$. Then γ is an involution and the equivalence relation is described as

$$\gamma(i) + 1 \sim i, \quad \gamma(i) \sim i + 1.$$

But $i \sim \gamma(i) + 1$ and $i = (i-1) + 1 \sim \gamma(i-1)$ and hence $\gamma(i-1) = \gamma(i) + 1$ for all i (since the equivalence relations have only singletons). This means that if i runs over the elements in a word w_1, then $\gamma(i)$ must run in the opposite direction over the elements of the conjugate word w_1^{-1}.

In consequence, the integral in the statement is non zero (modulo $\mathrm{O}(\frac{1}{N})$) only if it is of the form $\iint_{\mathcal{U}(N)} |\operatorname{Tr}(W_1)|^{2\alpha_1} \cdots |\operatorname{Tr}(W_p)|^{2\alpha_p} \, \mathrm{d}u \, \mathrm{d}v$ and in this case the integral is equal to the number of possible pairing between a word and cyclic rotation of its inverse. This completes the proof. □

5. A COMBINATORIAL FORMULA FOR $\iint_{\mathcal{U}(N)^2} |\operatorname{Tr}(W_1)|^2 \cdots |\operatorname{Tr}(W_p)|^2 \, \mathrm{d}U \, \mathrm{d}V$

In this section we establish a formula that is specifically adapted for integrals of products of absolute values of traces. Indeed a positive answer for the Connes's embedding conjecture would require the joint distribution of the variables $|\operatorname{Tr}(W_1)| \cdots |\operatorname{Tr}(W_p)|$ as functions on $\mathcal{U}(N)$.

Thus let $w_1, \ldots, w_p$ be reduced words in F_2, and let X be the total set of symbols of U, U^*, V, V^* in $|\operatorname{Tr}(w_1)|^2 \cdots |\operatorname{Tr}(w_p)|^2$, where each element in X corresponds to a specific occurence of the corresponding sysmbol in $|\operatorname{Tr}(w_1)|^2 \cdots |\operatorname{Tr}(w_p)|^2$.

Assume there are n occurences for the symbol U, m occurences for the symbol V, and hence that the set X has $2(n+m)$ elements. As in the preceding section X is partitioned as $\widetilde{U}\cup\widetilde{U}^*\cup\widetilde{V}\cup\widetilde{V}^*$, where $\widetilde{U},\widetilde{U}^*,\widetilde{V},\widetilde{V}^*$ are the set of symbols of u,u^*,v,v^* respectively in X.

Let Ψ be the map which associates to each symbol a in X, which comes from a word w_1 its corresponding symbol a^* in w_1^{-1} and viceversa for a symbol a in w_1^{-1} it associates the corresponding symbol a^* in w_1. Then Ψ is an involution, Ψ maps $\widetilde{U}$ onto $\widetilde{U}^*$ and $\widetilde{V}$ onto $\widetilde{V}^*$. Let I be the map associating to each symbol a the successor of $\Psi(a)$ in the inverse word.

Let $S_{\widetilde{U}}$ (and respectively $S_{\widetilde{U}^*},S_{\widetilde{V}},S_{\widetilde{V}^*}$) be the set of permutations of the sets $\widetilde{U}$ (respectively $\widetilde{U}^*,\widetilde{V},\widetilde{V}^*$). For each $\sigma_u\in S_{\widetilde{U}}$, $\theta_u\in S_{\widetilde{U}^*}$, $\sigma_v\in S_{\widetilde{V}}$, $\theta_v\in S_{\widetilde{V}^*}$ let $(\sigma_u,\theta_u,\sigma_v,\theta_v)$ be the concatanation of these permutations to a permutation of X.

With the above notations we have:

Proposition 5.1. *Let $w_1,\dots,w_p$ be words in F_2 and let $W_1,\dots,W_p$ be the corresponding words as functions on $\mathcal{U}(N)^2$.*

Let R be the following element in

$$\mathbb{C}(S_{\widetilde{U}})\otimes\mathbb{C}(S_{\widetilde{U}^*})\otimes\mathbb{C}(S_{\widetilde{V}})\otimes\mathbb{C}(S_{\widetilde{V}^*}):$$
$$R=\sum_{\substack{\sigma_u\in S_{\widetilde{U}},\,\theta_u\in S_{\widetilde{U}^*}\\ \sigma_v\in S_{\widetilde{V}},\,\theta_v\in S_{\widetilde{V}^*}}} N^{\sharp(I\circ(\sigma_u,\theta_u,\sigma_v,\theta_v))}\sigma_u\otimes\theta_u\otimes\sigma_v\otimes\theta_v.$$

Note that R depends only of the cardinalities of the sets $I(\widetilde{U})\cap U^*$, $I(\widetilde{U})\cap\widetilde{V}$, $I(\widetilde{U})\cap\widetilde{V}^*,\dots,I(\widetilde{V^*})\cap\widetilde{V}$.

Let Φ be the linear map from $\mathbb{C}(S_{\widetilde{U}})\otimes\mathbb{C}(S_{\widetilde{U}^*})\otimes\mathbb{C}(S_{\widetilde{V}})\otimes\mathbb{C}(S_{\widetilde{V}^*})$ into $\mathbb{C}$ which associates to $\sigma_1\otimes\theta_1\otimes\sigma_2\otimes\theta_2$ the number $W^N_{\sigma_1\Psi^{-1}\theta_1\Psi}W^N_{\sigma_2\Psi^{-1}\theta_2\Psi}$. Then

$$\iint_{\mathcal{U}(N)^2}|\operatorname{Tr}(W_1)|^2\cdots|\operatorname{Tr}(W_p)|^2\,\mathrm{d}U\,\mathrm{d}V=\Phi(R).$$

Proof. Introduce an indexing of the elements in X so that the symbol corresponding to a term $u_{\beta_i\beta_{i+1}}$ in a word w_i to correspond to a $u_{\beta_{\overline{i+1}}\beta^*_{\overline{i}}}$ for a word in w_1^{-1}. Here we use the convention that the elements in w_i have indexing after $\beta_1,\beta_2,\dots$. Then compating the integral

$$\iint_{\mathcal{U}(N)^2}|\operatorname{Tr}(W_1)|^2\cdots|\operatorname{Tr}(W_p)|^2\,\mathrm{d}U\,\mathrm{d}V$$

will amount to compute integrals of the form

$$\int_{\mathcal{U}(N)}u_{\beta_s\beta_{s+1}}\cdots u_{\beta_{\overline{r}}\beta_{\overline{r-1}}}\cdots u^*_{\beta_{\overline{s+1}}\beta_{\overline{s}}}\cdots u^*_{\beta_{r-1}\beta_r}\,\mathrm{d}U.$$

Thus here

$$U=\{\dots,s,\dots,\overline{r}\},\quad U^*=\{\dots,\overline{s+1},\dots,r-1,\dots\}.$$

Then the map Ψ will map $s,\overline{r}$ onto $\overline{s+1}$, $r-1$ respectively and I will map s,r into $\overline{s},\overline{r}$ respectively.

We use the formula from Section 2 and we have the sum over permutations σ_u of the symbols $\{\dots,s,\dots,\overline{r}\}$, and permutations θ_u of the symbols $\{\dots,\overline{s+1},\dots,r-1,\dots\}$.

Hence the corresponding equivalence relation corresponding to these permutations (and the similar permutations for θ will be exactly

$$\overline{s} \sim \sigma_u(s), \quad \overline{r} \sim \sigma_u(r) \quad \text{and} \quad \overline{s+1} \sim \theta_u(s+1), \quad r-1 \sim \theta_u(\overline{r}-1),$$

which is exactly the equivalence relation corresponding to $I \circ (\sigma_u, \theta_u, \sigma_v, \theta_v)$. This completes the proof. □

6. An example for the computation of $\iint_{\mathcal{U}(N)} |\operatorname{Tr}(U^{p_1^1} V^{\varepsilon_1} U^{p_2^1} V^{\varepsilon_2^1} \cdots U^{p_{s_1}^1} V^{\varepsilon_{s_1}^1})|^2 \cdots |\operatorname{Tr}(U^{p_1^t} V^{\varepsilon_1^t} \cdots U^{p_{s_t}^t} V^{\varepsilon_{s_t}^t})|^2 \,\mathrm{d}U \,\mathrm{d}V$

We apply the algorithm in the preceding section for the calculation of a product of words in which between powers of u of degree at least 3 are intercolated powers of v of degree ± 1. We will describe the structure of the element R in such a case since Ψ is easy to be descried in this situation.

Proposition 6.1. *For the integral,* $|p_a^s| \geqslant 3$, $\varepsilon_a^s = \pm 1$,

$$\iint_{\mathcal{U}(N)} |\operatorname{Tr}(U^{p_1^1} V^{\varepsilon_1} U^{p_2^1} V^{\varepsilon_2^1} \cdots U^{p_{s_1}^1} V^{\varepsilon_{s_1}^1})|^2 \cdots |\operatorname{Tr}(U^{p_1^t} V^{\varepsilon_1^t} \cdots U^{p_{s_t}^t} V^{\varepsilon_{s_t}^t})|^2 \,\mathrm{d}U \,\mathrm{d}V$$

the element R *is described as follows:*

Let n *be the total number of* u*'s and* m *the total number of* v*'s.*

The structure of the element R*, viewed as an element of* $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_m) \otimes \mathbb{C}(S_m)$ *is described as follows:*

Since $p_i^a \geqslant 1$ *we have that* $n \geqslant m$*. Then we have a set*

$$X = \{1_x, 2_x, \ldots, (n-m)_x\} \cup \{1_y, 2_y, \ldots, m_y\} = X_0 \cup Y$$

and the first factor S_n *is identified with* $S(x)$ *(permutations of* X*), while the second* S_n *is identified with* $S(\overline{X})$ *where*

$$\overline{X} = \{\overline{1}_x, \overline{2}_x, \ldots, \overline{(n-m)}_x\} \cup \{\overline{1}_y, \overline{2}_y, \ldots, \overline{m}_y\} = \overline{X}_0 \cup \overline{Y}.$$

We consider also two sets $A = \{1_a, \ldots, m_a\}$ *and* $\overline{A} = \{\overline{1}_a, \ldots, \overline{m}_a\}$*. Then the first factor* S_m *is identified with* $S(A)$*, while the second with* $S(\overline{A})$*.*

The map I *acts on* $X_0 \cup \overline{X}_0$ *by mapping* i_x *into* $\overline{i}_x$ *and* $\overline{i}_x$ *into* i_x*, while on the set* $Y \cup \overline{Y}$*,* I *maps* i_y *into* i_a *and* $\overline{i}_y$ *into* $\overline{i}_a$ *(or* i_y *into* $\overline{i}_a$ *and* $\overline{i}_y$ *into* i_a*).* I *is an involution. Then*

$$R = \sum_{\substack{\sigma, \overline{\sigma} \in S(X), S(\overline{X}), \\ \theta, \overline{\theta} \in S(A), S(\overline{A})}} N^{\sharp(I \circ (\sigma, \overline{\sigma}, \theta, \overline{\theta}))} \sigma \otimes \overline{\sigma} \otimes \theta \otimes \overline{\theta}.$$

Proof. This follows by identifying the sets $\widetilde{U}$, $\widetilde{U}^*$ from the preceding proposition with the sets X, $\overline{X}$, while $\widetilde{V}$, $\widetilde{V}^*$ are identified with Y, $\overline{Y}$. □

Remark 6.2. The map Ψ can be explicitely describe as a map from X onto $\overline{X}$, in terms of the alternating signs in $p_1^i, \varepsilon_1^i, \ldots, p_s^i, \varepsilon_s^i$, while on the set Y it simply maps i_a into $\overline{i_a}$.

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We have been informed that R. Speicher, P. Sniady and J. Mingo have obtained independently in a joint paper in preparation the same theorem as our result in Section 4.

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