THE VON NEUMANN ALGEBRA OF THE NON-RESIDUALLY FINITE BAUMSLAG GROUP $\langle a, b \mid ab^3a^{-1} = b^2 \rangle$ EMBEDS INTO R^ω

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Abstract. In this paper we analyze the structure of some sets of non-commutative moments of elements in a finite von Neumann algebra M. If the fundamental group of M is $\mathbb{R}_+ \setminus \{0\}$, then the moment sets are convex, and if M is isomorphic to $M \otimes M$, then the sets are closed under pointwise multiplication. We introduce a class of discrete groups that we call hyperlinear. These are the discrete subgroups (with infinite conjugacy classes) of the unitary group of R^ω. We prove that this class is strictly larger than the class of (i.c.c.) residually finite groups; it contains the Baumslag group $\langle a, b \mid ab^3a^{-1} = b^2 \rangle$. This leads to a previously unknown (non-hyperfinite) type II_1 factor that can be embedded in R^ω. This is positive evidence for Connes's conjecture that any separable type II_1 factor can be embedded into R^ω.

1. Introduction

In this paper we analyze the structure of sets of (non-commutative) moments $\tau(x_1 \cdots x_n)$ of variables $x_1, x_2, \ldots, x_n$ in a type II_1 factor M. We analyze the structure of these sets, for the case of projections and unitaries (see also [Ra] for odd moments of selfadjoint elements). While the understanding of these structures is far from being complete, we prove that any discrete (i.c.c.) group Γ that can be faithfully embedded into the unitary group of R^ω has the property that $\mathcal{L}(\Gamma) \subseteq R^\omega$.

By using techniques pertaining to free probability we prove that the Baumslag group $\Gamma = \langle a, b \mid ab^3a^{-1} = b^2 \rangle$, which is non-residually finite, embeds (faithfully) into $\mathcal{U}(R^\omega)$. Note also that by [CeGr], the algebra $\mathcal{L}(\Gamma)$ does not have property Γ. In particular, it is non-hyperfinite. (We are indebted to P. de la Harpe for bringing this to our attention.)

This is positive evidence towards Connes's conjecture that any separable II_1 factor is embedded into R^ω.

In the second part of the paper we analyze the structure of the sets of first and second order of moments $\tau(e_i)$, $\tau(e_ie_j)$ of finite sets of projections $e_1, \ldots, e_n$ in a type II_1 factor. By work of Kirchberg [Ki], if the closure of the sets of first and second order of moments of unitaries is independent of the type II_1 factor considered, then Connes's conjecture should be true.

It is obvious how to translate this statement in terms of projections. By using methods from [Ra], it follows that the corresponding set of moments, i.e., the set of moments (of order 1 and 2) of projections, is convex and multiplicative for a II_1 factor M such that $M \cong M \otimes M$ and $\mathcal{F}(M) = \mathbb{R}_+ \backslash \{0\}$. We will also analyze the structure of faces of these sets, which gives some additional data on the geometric structure of these sets.

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Definitions and Notations: We recall that for a von Neumann algebra M, the unitary group is denoted by $\mathcal{U}(M)$, while $\mathcal{P}(M)$ stands for the set of selfadjoint projections.

For a type II_1 factor M, the fundamental group $\mathcal{F}(M)$ of M [MvN] [vN] is defined as the multiplicative group of all $t > 0$, such that $M_t \cong M$.

For Γ a countable discrete group, with infinite conjugacy classes (briefly i.c.c.), the algebra $\mathcal{L}(\Gamma)$ is the weak closure of the group algebra $\mathbb{C}(\Gamma)$ embedded (via left regular representation) in $B(\ell^2(\Gamma))$.

If ω is an ultrafilter on $\mathbb{N}$, then following [McD] and [Co], one defines for any II_1 factor M the ultrafilter product M^ω, obtained via G.N.S. construction, by defining the trace of an element (x_n) in the infinite product of copies of M to be $\lim_{n\to\omega} \tau(x_n)$ (in the hypothesis that $\sup \|x_n\| < \infty$). We refer to Connes's [Co] paper on injectivity for full details on this construction.

Finally, we recall a construction from [Po] (see also [Vo]). Consider two von Neumann algebras N_1, N_2 which have a common subalgebra B, containing the unit. Also assume that the algebras N_i have faithful traces whose restriction to B coincides. We assume that we are given conditional expectations E_i from N_i onto B that are trace preserving.

The trace on the reduced amalgamated free product von Neumann algebra $C_1 *_B C_2$ is defined by the requirement that a product $c_{1,1}c_{2,1}c_{1,2}c_{2,2}c_{1,3}c_{2,3}\cdots$, where $c_{1,i}$ belongs to C_1 and $c_{2,i}$ belongs to C_2, has zero trace if $\mathrm{Id} - E_B(c_{ij}) \neq 0$ for all $i = 1, 2$, $j = 1, 2, \ldots$.

2. Moments of unitaries

In this section we define some sets of non-commutative moments of unitaries $\tau(u_1 u_2 \cdots u_p)$ associated with a type II_1 factor M. We will use these sets to check that for any discrete i.c.c. group Γ that can be embedded into the unitary group of R^ω, then also $\mathcal{L}(\Gamma)$ can be embedded into R^ω (as a unital subfactor).

First we consider the set of all possible embeddings (up to order N) of a group-like algebra.

By $\mathcal{V}_{n,p}$ we denote the set of all indices $(i_1, i_2, \ldots, i_k)$ with $1 \leq k \leq p$ and $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, n\}$. By u_I we denote the product $u_{i_1} u_{i_2} \cdots u_{i_k}$ if $I = (i_1, i_2, \ldots, i_k)$.

Definition 2.1. Let M be a separable type II_1 factor and consider the following subset (of $\{0,1\}^{2^{n^p}}$), denoted by $K_M^{n,p}$. We define $K_M^{n,p}$ by requiring that $(\varepsilon_I)_{|I| \leq p}$ belongs to $K_M^{n,p}$ if and only if there exist unitaries $u_1, u_2, \ldots, u_n$ in $\mathcal{U}(M)$ such that $\tau(u_I) = 1$ or 0, and $\varepsilon_I = \tau(u_I)$, for $|I| \leq p$. Here I is an index set $(i_1, i_2, \ldots, i_k)$, with $ij \in \{1, 2, \ldots, n, \}$, $1 \leq k \leq p$ and $|I| = k$.

Remark 2.2. If $M = \mathcal{L}(\Gamma)$ and Γ has non-solvable world problem, then there are 2^{n^p}-uples of 0's and 1's about which it might be undecidable whether they belong to $K_M^{n,p}$.

The above definition might be restrictive for some purposes because it requires that $\tau(u_I)$ is either 0 or 1. In fact $\tau(u_I) = 1$ is equivalent to the fact that $u_I = 1$. (Here by u_I we mean the product $u_{i_1} u_{i_2} \cdots u_{i_k}$ if $I = (i_1, i_2, \ldots, i_k)$.)

Definition 2.3. Let M be a finite von Neumann algebra and let for fixed integers n, p,

$$L_M^{n,p} = \{(\tau(u_I) = \tau(u_{i_1} u_{i_2} \cdots u_k))_{I \in \mathcal{V}_{n,p}} \mid \text{for all} \\ \text{unitaries } u_1, u_2, \ldots, u_n \text{ in } \mathcal{U}(M)\}.$$

The following properties are easy to observe (see also [Ra]).

Proposition 2.4. (a) *Let M_1, M_2 be finite von Neumann algebras. Denote by $\odot$, the pointwise product on $\mathbb{R}^s$, for all s. Then for all positive integers n, p,*

$$L_{M_1}^{n,p} \odot L_{M_2}^{n,p} \subseteq L_{M_1 \otimes M_2}^{n,p}, \qquad K_{M_1}^{n,p} \odot K_{M_2}^{n,p} \subseteq K_{M_1 \otimes M_2}^{n,p}.$$

(b) *In particular if M is such that $M \cong M \otimes M$, then $K_M^{n,p}$ and $L_M^{n,p}$ are closed under pointwise multiplication.*

(c) *If $\lambda \in \mathcal{F}(M)$, then $\lambda L_M^{n,p} + (1-\lambda) L_M^{n,p} \subseteq L_M^{n,p}$ for all integers $n, p \geq 1$. In particular if $\mathcal{F}(M) = \mathbb{R}_+ \backslash \{0\}$, then $L_M^{n,p}$ is convex.*

(d) *$\overline{L_M^{n,p}} \subseteq L_{M^\omega}^{n,p}$ and $L_{M^\omega}^{n,p}$ is closed in the product topology of $\mathbb{R}^{|\mathcal{V}_{n,p}|}$.*

(e) *In particular if $M \cong M \otimes M$ and $(\lambda_I)_{I \in \mathcal{V}_{n,p}}$ is an element in $L_M^{n,p}$ such that either $|\lambda_I| < 1$ or $\lambda_I = 1$, then by replacing the components in $L_M^{n,p}$ which are not 1, by zero, we obtain an element in $K_{M^\omega}^{n,p}$.*

(f) *Let Φ_{u_1} be the operation on $L_M^{n,p}$ which replaces in $(\lambda_I)_{I\in L_M^{n,p}}$ any monomial $\lambda_I = \tau(u_I)$, corresponding to a nonzero total power of u_1, by zero. Assume $\mathcal{F}(M) = \mathbb{R}_+\backslash\{0\}$. Then*

$$\Phi_{u_1}(L_M^{n,p}) \subseteq L_M^{n,p}.$$

(g) *For all separable II_1 factors M we have that $L_M^{n,p} \supseteq L_R^{n,p}$. Moreover if $M \subseteq R^\omega$, then $\overline{L_M^{n,p}} = \overline{L_R^{n,p}}$ (and similarly for K).*

Open Question: Does $K^{n,p}_{\mathcal{L}(\mathrm{SL}_3(\mathbb{Z}))} \subseteq K^{n,p}_{\mathcal{L}(F_\infty)}$ for all n, p?

The proof of properties (a)–(c) is obvious and is basically contained in [Ra]. To check property (d) we have only to verify that $L_{M^\omega}^{n,p}$ is closed. But if $(\lambda_I)_{I\in\mathcal{V}_{n,p}}$ is an accumulation point for $L_M^{n,p}$, then take unitaries $(u_i^s)_{i=1}^n$, for all s, such that $\lim_{s\to\infty}\tau(u_I^s) = \lambda_I$, $I \in \mathcal{V}_{n,p}$. Then $u_i = (u_i^s)_{s\in\mathbb{N}}$ are unitaries in R^ω, whose non-commutative moments give $(\lambda_I)_{I\in\mathcal{V}_{n,p}}$.

Property (e) follows from properties (a) and (c). Property (f) follows by convexity, and integration over θ, where gauging u_1 by $e^{2\pi i\theta}$. Property (g) is obvious.

Proposition 2.5. *Let Γ be a discrete i.c.c. group that embeds faithfully into the unitary group of R^ω. Then $\mathcal{L}(\Gamma) \subseteq R^\omega$.*

Proof. Fix n, p and let $u_1, u_2, \ldots, u_n, \ldots$ be a system of generators of Γ. Let ε_I be the traces of $(u_I)_{I\in\mathcal{V}_{n,p}}$ in the left regular representation of $\mathcal{L}(\Gamma)$. By hypothesis there exist unitaries $v_1, v_2, \ldots, v_n$ in R^ω such that $|\tau(v_I)| < 1$ if $u_I \neq 1$ in Γ and $v_I = 1$ if $u_I = 1$ in Γ. Let $(\alpha_I)^s = \tau(v_I^{\otimes s}) = \tau(v_I)^s$. Then $(\alpha_I)^s_{I\in\mathcal{V}_{n,p}}$ belongs to $\overline{L_R^{n,p}}$ and hence so does the limit

$$\varepsilon_I = \lim_{s\to\infty}(\alpha_I)^s \qquad \text{for } I \in \mathcal{V}_{n,p}.$$

Thus $(\varepsilon_I)_{I\in\mathcal{V}_{n,p}} \in \overline{K_R^{n,p}}$ for all n, p. Hence $\mathcal{L}(\Gamma) \subseteq R^\omega$. □

Definition 2.6. We call an i.c.c. group Γ hyperlinear if Γ embeds faithfully into $\mathcal{U}(R^\omega)$. Clearly any residually finite group is hyperlinear. The class of hyperlinear groups is obviously closed under free products.

Theorem 2.7. *The class of hyperlinear groups is strictly larger than the class of residually finite groups. More precisely, the Baumslag group $\langle a, b \mid ab^3a^{-1} = b^2\rangle$* [Ba] [Ma] *is hyperlinear and non-residually finite. (Note that by* [CeGr], *$\mathcal{L}(\Gamma)$ does not have property Γ.)*

Proof. We divide the proof into several steps. We construct first an approximate embedding of the relation $ab^3a^{-1} = b^2$ into $M_n(\mathbb{C})$. We then take the free amalgamated product (over b^2) by a unitary that commutes with b^2 and perturb a with this unitary. This gives an approximate embedding of Γ into the nonscalar unitaries in some free product algebras. Since these algebras are embeddable into R^ω [Vo] [Wa], the result will follow, by Proposition 2.5.

Step I. Construction of an approximate embedding.

There exist unitaries v_n, b_n (of zero trace) in $M_{6n}(\mathbb{C})$ with the following properties:

Property 1)

$$\left\|v_n b_n^3 v_n - b_n^2\right\|_\infty \leq \frac{K_1}{n}\text{for a universal constant } K_1.$$

Property 2)

Denote by B_0^n the abelian algebra generated by b_n^2 and let $E_{B_0^n}$ be the corresponding conditional expectation. Let $\Phi^n = \mathrm{Id} - E_{B_0^n}$. Then for all $\alpha \in \{\pm 1, \pm 2\}$,

$$\|\Phi(v_n b_n^\alpha v_n^*)\|_2 \geq K_2$$

for a universal constant K_2. Here $\| \ \|_2$ is the normalized Hilbert-Schmidt trace on matrices.

Property 3)

$$E_{B_0^n}(b_n^{\pm 1}) = 0,\ E_{B_0^n}(vb^\alpha) = 0,\ E_{B_0^n}(b^\alpha v) = 0, \qquad \alpha \in \mathbb{N}, \alpha \neq 0.$$

We describe first the construction of the unitaries v_n, b_n.

Let $e_0, e_1, \ldots, e_{6n-1}$ be the diagonal algebra of $M_{6n-1}(\mathbb{C})$, and for convenience we think of e_k as being identified with $\chi_{[\frac{k}{6n}, \frac{k+1}{6n})}$, $k = 0, 1, \ldots, 6n-1$.

Let f_k, for $k = 0, 1, 2, \ldots, n-1$, be the projection $\chi_{[\frac{3k}{6n}, \frac{3k+3}{6n})+\{0,\frac{1}{2}\}}$ and let

$$g_k = \chi_{[\frac{2k}{6n}, \frac{2k+2}{6n})+\left\{0,\frac{1}{3},\frac{2}{3}\right\}}.$$

Let b_n be the unitary defined by

$$b_n = \sum_{k=0}^{6n-1} e^{\frac{2\pi i k}{6n}} e_k.$$

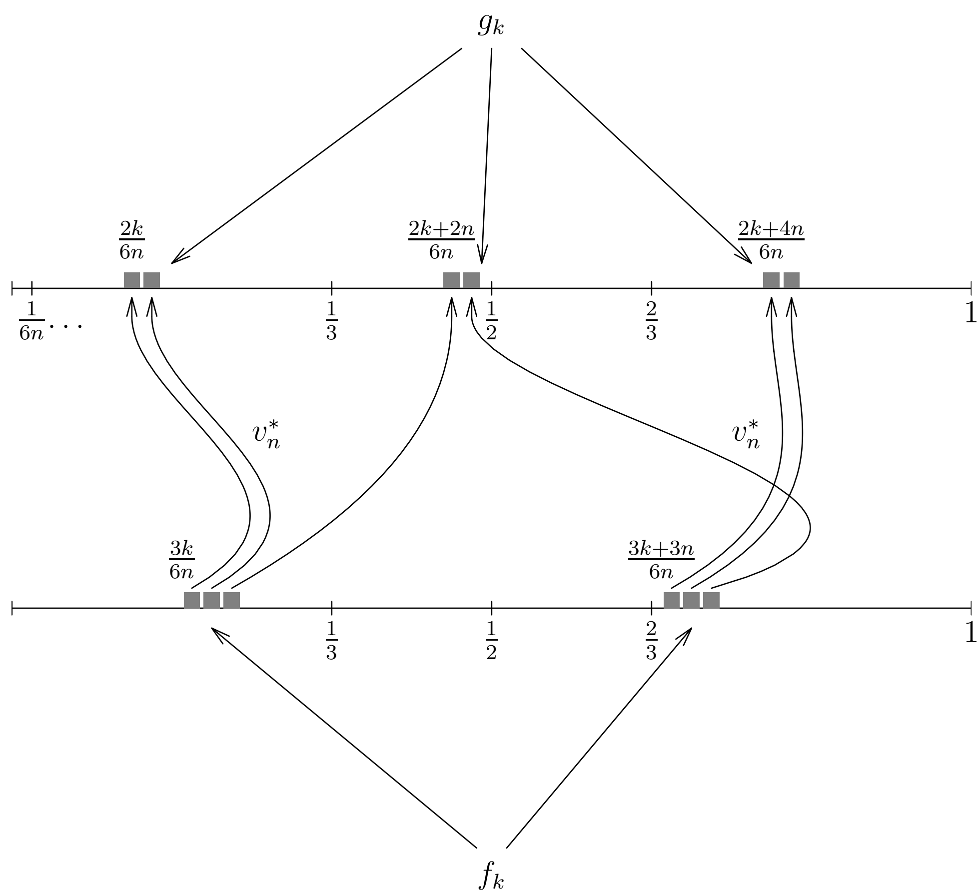

FIGURE 1. Description of v_n^*

Let v_n be a unitary such that $v_n^* f_k = g_k v_n^*$, $k = 0, 1, 2, \ldots, n-1$, and such that

$$(\operatorname{Ad} v_n^*)(e_{3k+\varepsilon}) = e_{2k+\varepsilon},$$
$$(\operatorname{Ad} v_n^*)(e_{3k+3n+\varepsilon}) = (e_{2k+4n+\varepsilon})$$

for all $k = 0, 1, 2, \ldots, n-1$, $\varepsilon = 0, 1$. (See Fig. 1.)

For $\varepsilon = 2$, we define v_n^* by the requirement

$$\operatorname{Ad} v_n^*(e_{3k+2}) = e_{2k+2n},$$
$$\operatorname{Ad} v_n^*(e_{3k+3n+2}) = e_{2k+2n+1}.$$

Observe that the definition of b_n, v_n implies, for a universal constant K_1, that

$$\left\| b_n^2 f_k - e^{\frac{2\pi i k}{n}} f_k \right\|_\infty \le \frac{K_1}{n}, \tag{1}$$

$$\left\| b_n^3 g_k - e^{\frac{2\pi i k}{n}} g_k \right\|_\infty \le \frac{K_1}{n}, \qquad k = 0, 1, \ldots, n-1. \tag{2}$$

Moreover, $v_n^* f_k = g_k v_n^*$, and

(3)

$$(v_n b_n v_n^*) \cdot e_{3k+\varepsilon+\alpha\cdot 3n} = e^{2\pi i(\frac{2k+\varepsilon}{6n} + \alpha\cdot\frac{2}{3})} e_{3k+\varepsilon+\alpha\cdot 3n} \qquad \text{for } \varepsilon = 0, 1,\ \alpha = 0, 1.$$

In the remaining case, we have that

$$(v_n b_n v_n^*) \cdot e_{3k+2+\alpha\cdot 3n} = e^{2\pi i(\frac{2k+\alpha}{6n} + \frac{1}{3})} e_{3k+2+\alpha\cdot 3n} \qquad \text{for } \alpha = 0, 1. \tag{4}$$

We now proceed to the proof of the properties 1), 2), 3).

Since $v_k^* f_k = g_k v_k^*$, $v_k g_k = f_k v_k$, $v_n g_k v_n^* = f_k$, and f_k, g_k commute with b_n, it follows that

$$\begin{aligned} \left\| v_n b_n^3 v_n^* - b_n^2 \right\|_\infty &= \max_{k=0,1\ldots,n-1} \left\| (v_n b_n^3 v_n^* - b_n^2) f_k \right\| \\ &= \max_{k=0,1\ldots,n-1} \left\| v_n (b_n^3 g_k - e^{\frac{2\pi i k}{n}} g_k) v_n^* - (b_n^2 f_k - e^{\frac{2\pi i k}{n}} f_k) \right\|_\infty . \end{aligned}$$

But this quantity is less than $\frac{K_1}{n}$ by (1), (2). This completes the proof of Property 1.

To prove Property 2 we need to describe first $E_{B_0^n}$. But it is obvious that

$$E_{B_0^n}\left(\sum_{k=0}^{6n-1} \lambda_k e_k \right) = \sum_{k=0}^{3n-1} \frac{1}{2} (\lambda_k + \lambda_{k+3n}) (e_k + e_{k+3n}).$$

Consequently

$$\Phi_n \left(\sum_{k=0}^{6n-1} \lambda_k e_k \right) = \sum_{k=0}^{3n-1} \frac{\lambda_k - \lambda_{k+3n}}{2} e_k + \sum_{k=0}^{3n-1} \frac{\lambda_{k+3n} - \lambda_k}{2} e_{k+3n}.$$

We use the above formula for $v_n b_n v_n^*$ and use (3) and (4). We take $P_n = \sum_{k=0}^{n-1} \chi_{[\frac{3k}{6n}, \frac{3k+1}{6n})} = \sum_{k=0}^{n-1} e_{3k}$. Then P_n has trace $\frac{1}{6}$. Since $(v_n b_n v_n^*) e_{3k} = e^{2\pi i \frac{k}{3n}} e_{3k}$ and $(v_n b_n v_n^*) e_{3k+3n} = e^{2\pi i (\frac{k}{3n} + \frac{2}{3})} e_{3k+3n}$ the above formula shows that

$$P_n \Phi_n(vbv^*) = P_n(vbv^* - E_{B_0^n}(vbv^*)) = \sum_{k=0}^{n-1} e^{2\pi i \frac{k}{3n}} \left(1 - e^{2\pi i \frac{2}{3}}\right) e_{3k}.$$

Hence

$$\|P_n\Phi_n(vbv^*)\|_2^2 > \left|1 - e^{2\pi i\frac{2}{3}}\right| \|P_n\|_2^2 = \frac{1}{6}\left|1 - e^{2\pi i\frac{2}{3}}\right|.$$

The computations for $vb_n^{\pm 2}v^*$, $vb_n^{-1}v^*$ are similar, eventually the factor $\frac{2}{3}$ being replaced by $\frac{4}{3}$ or $-\frac{2}{3}$. This completes the proof of Property 2.

It is obvious that $E_{B_0^n}(vb_n^{\alpha})$, $E_{B_0^n}(b_n^{\alpha}v)$, $\alpha \neq 0$, and $E_{B_0^n}(b_n^{\pm 1})$ are vanishing.

Step II. In this step we consider the amalgamated free product of the algebra $\{v_n, b_n\}''$ described above and $\mathcal{L}(\mathbb{Z}) \otimes B_0^n$. The amalgamated free product is considered over B_0^n (the von Neumann algebra generated by b_n^2).

Let $\mathcal{L}(\mathbb{Z})$ have the canonical generator a_1, a Haar unitary. $\mathcal{L}(\mathbb{Z})$ is endowed with the standard trace. Consider the algebra

$$\mathcal{A}_n = (\mathcal{L}(\mathbb{Z}) \otimes B_0^n) *_{B_0^n} \{v_n, b_n\}''$$

with the canonical amalgamated free product trace (see section on definitions, [Po], and [Vo]).

By [Ra] (see also [Dy], [Shly]), we have that $\mathcal{A}_n$ is a free group factor. Using the ultrafilter construction ([Co], [McD]), we construct algebras $\mathcal{A}^{\omega}$, B_0^{ω} and B^{ω} consisting of bounded sequences of elements in the algebras $\mathcal{A}_n$, B_0^n and $\{b_n\}''$ respectively. It is obvious that for $x = (x_n)_n$ in $\mathcal{A}^{\omega}$ we have

$$E_{B_0^{\omega}}((x_n)_n) = \left(E_{B_0^n}(x_n)\right)_n.$$

Let b be the unitary element $(b_n)_n \in B^{\omega}$ and let B_0 be the (abelian) von Neumann algebra generated by b. Let v be the unitary $v = (v_n)_n \in \mathcal{A}^{\omega}$. We identify $\mathcal{L}(\mathbb{Z}) \subseteq \mathcal{A}^{\omega}$ with constant sequences with elements in $\mathcal{L}(\mathbb{Z})$.

Since, by [Wa], [Vo], any type II_1 free group factor embeds into R^{ω}, we obtain that the algebra $\mathcal{A}^{\omega}$ is embedded into R^{ω} and hence

$$\mathcal{A} = \mathcal{L}(\mathbb{Z}) \otimes B_0 *_{B_0} \{v, b\}'' \subseteq R^{\omega}.$$

The trace on $\mathcal{A}$ is the amalgamated free product trace and coincides with the restriction of the trace on R^{ω}. Let Φ be the identity minus the conditional expectation E_{B_0} from $\mathcal{A}$ (or $\mathcal{A}^{\omega}$) onto B_0. The following properties hold true:

1) v, b are Haar unitaries, $vb^3v^* = b^2$.
2) $\|\Phi(vb^{\alpha}v^*)\|_2 \geq \frac{1}{6}$, $\alpha \in \{\pm 1, \pm 2\}$; $E_{B_0}(vb^k) = 0$, $k \in \mathbb{N}$, $E_{B_0}(b^{\pm 1}) = 0$.

To prove Property 2, note that $\left\|E_{B_0^\omega}\left(vb^\alpha v^*\right)\right\|_2 \leq \frac{1}{6}$ because of the corresponding property for $v_n b_n^\alpha v_n^*$. Since $B_0 \subseteq B_0^\omega$ it follows also that $\left\|E_{B_0}\left(vb^\alpha v^*\right)\right\|_2 \leq \frac{1}{6}$.

Let a_1 be the standard generator of $\mathcal{L}(\mathbb{Z})$ and let $A = a_1 v$, $B = b$.

Step III. Let A, B be the unitaries defined in Step II. Clearly $AB^3A^{-1} = B^2$, as a_1 commutes with b^2. Let

$$W = A^{\alpha_1}B^{\beta_1}A^{\alpha_2}\cdots A^{\alpha_n}B^{\beta_{n+1}}$$

be a word in A, B such that $\beta_1 \neq 0, \ldots, \beta_n \neq 0$, $\alpha_2 \neq 0, \ldots, \alpha_n \neq 0$. Consider the following assumption on the sequence of the indices α_i.

Assumption on $W = A^{\alpha_1}B^{\beta_1}A^{\alpha_2}\cdots A^{\alpha_n}B^{\beta_{n+1}}$. One of the following possibilities occurs (about consecutive indices):

A1) Either α_i, α_{i+1} are both positive or negative (except for the case when $\alpha_1 = 0$).
A2) If $\alpha_i < 0$, $\alpha_{i+1} > 0$, then $\beta_i \in \{\pm 1\}$.
A3) If $\alpha_i > 0$, $\alpha_{i+1} < 0$, then $\beta_i \in \{\pm 1, \pm 2\}$.

Claim. If the word W is subject to the conditions A1, A2, A3, then W is not a multiple of a scalar (and hence $|\tau(W)| < 1$).

Proof of the claim in Step III. We use the following property of an amalgamated free product $\mathcal{B} = E *_C F$ where E, F are finite algebras with faithful traces τ_1, τ_2 whose restrictions coincide on the common unital subalgebra C.

Assume $w = e_1 f_1 e_2 f_2 \cdots e_n f_{n+1}$ is a word in $E *_C F$, $e_i \in E$, $f_i \in F$, such that $\mathrm{Id} - E_C(f_1)$, $\mathrm{Id} - E_C(e_2) \neq 0, \ldots, \mathrm{Id} - E_C(e_n) \neq 0$. Then w is not a scalar multiple of the identity. This follows for example from the construction in [Po].

Then for the word $W = A^{\alpha_1}B^{\beta_1}A^{\alpha_2}\cdots A_g^{\alpha_{n+1}}$, we use the fact that $A = a_1 v$, $B = b$. Since

$$E_{B_0}(b^\theta v^*),\ E_{B_0}(vb^\theta)$$

are always zero for all θ, the only instances in the product in W where we could have elements with $\mathrm{Id} - E_{B_0}$ nonzero are in subsequences of the form

$$\cdots avb^{\pm\alpha}v^*a\cdots, \qquad \alpha \in \{1, 2\} \text{ (in } \cdots AB^{\pm\alpha}A^{-1}\cdots),$$

or

$$\cdots v^*a^*b^{\pm 1}av\cdots \qquad (\text{in } \cdots A^{-1}B^{\pm 1}A\cdots).$$

But in these cases $\Phi = \mathrm{Id} - E_{B_0}$ applied to the elements $vb^{\pm 1}v^*$, $vb^{\pm 2}v^*$, and $b^{\pm 1}$ is nonzero. The remaining two cases correspond to subsequences involving $A^n B^\theta A^m$ with $\theta \neq 0$ and n, m both strictly positive or both strictly negative. The case $n, m > 0$ corresponds to a subsequence of the form $\cdots a_1 v b^\alpha a_1 v \cdots$ or $\cdots a_1 b^\alpha v^* a_1 v \cdots$. In either case we use the fact that $E_{B_0}(b^\alpha v) = 0$, $E_{B_0}(v^* b^\alpha) = 0$, for $\alpha \neq 0$. The case $n, m < 0$ is similar.

Hence the property of the amalgamated free product applies, and W is non-scalar.

Step IV. Any word (except the identity) in the Baumslag group $\langle a, b \mid ab^3a^{-1} = b^2 \rangle$, of total degree zero in a, is equal to one of the words

$$a^{\alpha_1} b^{\beta_1} a^{\alpha_2} b^{\beta_2} \cdots b^{\beta_n} a^{\alpha_{n+1}}$$

for which all the Assumptions A1–A3 on consecutive indices, described in Step III, hold. Note that by Proposition 2.4(f) we can reduce the proof of the theorem to words of total degree 0 in a.

To prove the claim of Step IV, the following two lemmas, dealing with easier situations, will be used.

Lemma 2.8. *Let $n \geq 1$ and $k \geq 2$. Then $a^{-n} b^k a^n$ is equal to a product of the form*

$$b^{\theta_0} a^{-i_1} b^{\theta_1} a^{-i_2} \cdots a^{-i_p} b^{\theta_p} a^{i_{p+1}}$$

for some strictly positive numbers $i_1, \ldots, i_{p+1} \in \{1, 2, \ldots, n\}$, $i_{p+1} \leq n$, and θ_0 is nonzero, $|\theta_0| \geq 3$, $\theta_1, \ldots, \theta_p = 1$.

For example $a^{-n} b^2 a^n = b^3 a^{-1} b a^{-1} b \cdots b a^{-1} b a^{n-1}$, where the product involves n occurrences of the letter b (not counting powers).

Proof. We start with $k = 2^l \cdot q$, q odd. Then $a^{-n} b^k a^n = a^{-(n-l)} b^{3^l q} a^{n-l}$. We then split this product as:

$$\left(a^{-(n-l)} b^{3^l q - 1} a^{n-l}\right) \cdot \left(a^{-(n-l)} b a^{(n-l)}\right).$$

We repeat this procedure with

$$3^l q - 1 = 2^{l_1} q_1, \qquad q_1 \text{ odd}$$

and will obtain

$$a^{-n} b^k a^n = a^{-(n-l-l_1)} b^{3^{l_1} q_1} a^{(n-l-l_1)} a^{-(n-l)} b a^{(n-l)}.$$

By repeating this procedure and stopping when running out of powers of a, we get the required result. $\square$

Similarly one proves:

Lemma 2.9. *Let $n \geq 1$, $k \geq 3$. Then $a^n b^k a^{-n}$ is equal to a product of the form*

$$b^{\theta_0} a^{\varepsilon_1} b^{\theta_1} a^{\varepsilon_2} b^{\theta_2} \cdots a^{\varepsilon_s} b^{\theta_s} a^{-\varepsilon_{s+1}},$$

where $0 < \varepsilon_{s+1} \leq n$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s > 0$. Moreover, $\theta_0 \geq 2$ and $\theta_s \in \{1,2\}$ for $s \geq 1$.

Proof of the claim in Step IV. We start with one arbitrary word

$$W = b^{k_0} a^{n_1} b^{k_1} a^{n_2} b^{k_2} \cdots, \qquad n_i \neq 0,\ k_i \neq 0,\ i \geq 1,$$

where no obvious cancellations are possible. By moving from the left to the right we look at the first change in sign in the sequence $n_1, n_2, \ldots$. Say this occurs when $i = i_0$. At that point, if $|k_{i_0}| \geq 2$ when $n_{i_0} < 0$, $n_{i_0+1} > 0$, or if $|k_{i_0}| \geq 3$ when $n_{i_0} > 0$, $n_{i_0+1} < 0$, we apply one of the two preceding lemmas to replace $a^{n_{i_0}} b^{k_{i_0}} a^{n_{i_0+1}}$ by one of the sequences described in the lemmas. More precisely, if, for example, $n_{i_0} < 0$, $n_{i_0+1} > 0$, we apply Lemma 2.8 for

$$x = a^{-\min\left(-n_{i_0}, n_{i_0+1}\right)} b^{k_{i_0}} a^{\min\left(-n_{i_0}, n_{i_0+1}\right)}.$$

By replacing x in the word by the form given in Lemma 2.8, the structure of the word up to the next power of a following $b^{k_{i_0}}$ would fulfill the requirements of the claim. The only case, when in doing this replacement, a change of structure could occur in the structure of the word, before $a^{n_{i_0}}$, is when $n_{i_0-1} > 0$. But in this case $|k_{i_0-1}| \leq 2$ so $b^{k_{i_0-1}}$ won't cancel the b^3 appearing at the beginning of the word from Lemma 2.8. Here we reiterate the procedure. A similar argument works for $n_{i_0} > 0$, $n_{i_0+1} < 0$.

By induction, this completes the proof of Step IV. By Steps III and IV we conclude the proof of our theorem. □

3. Extremal finite von Neumann algebras

In this section we consider the structure of the set of moments of families of projections in a finite von Neumann algebra. Note that by Kirchberg's technique [Ki], for Connes's conjecture to be true, one should prove that the closure of this set is independent of the finite von Neumann algebra for which we consider the set of moments.

Definition 3.1. Let M be a finite separable von Neumann algebra and let τ be a faithful, normalized trace on M. For any integer $n \geq 1$, let K_M^n be the subset of $[0,1]^{\frac{n(n+1)}{2}}$ consisting of the following ordered pairs:

$$K_M^n = \{(\tau(e_i e_j))_{1 \leq i \leq j \leq n} \mid (e_1, e_2, \ldots, e_n) \in (\mathcal{P}(M))^n\}.$$

Proposition 3.2. *Let M be a type II_1 factor with trace τ. Then for all integers $n \geq 1$,*

1. *K_M^n is convex if $\mathcal{F}(M) = \mathbb{R}_+ \setminus \{0\}$,*
2. *K_M^n is closed under pointwise multiplication, if $M \cong M \otimes M$,*
3. *$K_{M^\omega}^n$ is closed in the standard topology of $[0,1]^{\frac{n(n+1)}{2}}$,*
4. *$K_M^n \supseteq K_R^n$, where R is the hyperfinite II_1 factor.*

The proof of this proposition is identical to the proof of the properties for the set of moments associated with unitaries in a II_1 factor. Note that by Kirchberg's results [Ki], $\overline{K_M^n} = \overline{K_R^n}$ for all n, if and only if $M \subseteq R^\omega$.

It is very easy to describe the geometry of a diffuse abelian von Neumann algebra. Indeed,

Proposition 3.3. *Let $Y_n \subseteq [0,1]^{\frac{n(n+1)}{2}}$ consist of all $(\varepsilon_{ij})_{1 \leq i \leq j \leq n}$ such that there are sets $A_1, \ldots, A_n \subseteq X$, X nonvoid, $A_i = \varnothing$ or $A_i = X$ such that $\varepsilon_{ij} = 1$ if $A_i \cap A_j = X$ and $\varepsilon_{ij} = 0$ if $A_i \cap A_j = \varnothing$. Then*

$$K_{L^\infty([0,1])}^n = \operatorname{co} Y_n.$$

In this section we analyze the structure of the closed convex subsets $K_{M^\omega}^n \subseteq [0,1]^{\frac{n(n+1\}}{2}}$. To determine completely this set it would be sufficient to know, for all choices of real numbers $(a_{ij})_{1 \leq i \leq j \leq n}$ of the value of

$$\max_{1 \leq i \leq j \leq n} \left\{ \sum a_{ij} \lambda_{ij} \mid (\lambda_{ij}) \in K_M^n \right\}.$$

This is difficult to handle, but we are able to prove at least one geometrical property related to this maximum value: a type of separation of variables at maximum points in K_M^n.

The following lemma is an easy consequence of the fact that whenever a maximum point is attained at $(e_1^0, \ldots, e_n^0)$, then for any other projection $e_1 \leq e_1^0$ or $e_1 \geq e_1^0$ we get a lower value.

Lemma 3.4. *Fix*

$$(\tau(e_i^0 e_j^0))_{1 \leq i \leq j \leq n}$$

a maximum point for the fixed functional

$$L(\lambda_{ij}) = \sum_{1 \leq i \leq j \leq n} a_{ij} \lambda_{ij} \qquad \text{on } K_M^n.$$

For all $i = 1, 2, \ldots, n$, let

$$\Omega_i = \Omega_i(e_1^0, \ldots, e_0^n) = \sum_{j \neq i} a_{ij} e_j^0 + a_{ii} e_i^0.$$

Then $e_i^0 \Omega_i e_i^0 \geq 0$ and $(1 - e_i^0)\Omega_i(1 - e_i^0) \leq 0$ for all $i = 1, 2, \ldots, n$.

Proof. Fix i in $\{1, 2, \dots, n\}$ and let e_i be any projection less than e_i^0. The fact that

$$\sum_{1 \le i \le j \le n} a_{ij} \tau\left(e_i^0 e_j^0\right)$$

is a maximum value for L on K_M^n, implies that

$$\sum_{j \ne i} a_{ij} \tau\left((e_i^0 - e_i) e_j^0\right) + a_{ii} \tau(e_i^0 - e_i) \ge 0.$$

Thus for any projection e less than e_i^0 we have that

$$\tau\left(e\left(\sum_{j \ne i} a_{ij} e_j + a_{ii} \operatorname{Id}\right)\right) \ge 0.$$

But this gives exactly that

$$e_i^0 \Omega_i e_i^0 \ge 0.$$

Similarly for $1 - e_i^0$. □

Corollary 3.5. *If $\left(\lambda_{ij}^0\right)_{1 \le i \le j \le n}$ in K_M^n is a maximum point for*

$$(\lambda_{ij})_{1 \le i \le j \le n} \in K_M^n \longrightarrow \sum_{1 \le i \le j \le n} a_{ij} \lambda_{ij},$$

then for all $i = 1, 2, \dots, n$ we have that

$$0 \le \sum_{j \ne i} a_{ij} \lambda_{ij}^0 + a_{ii} \lambda_{ii}^0 \le \sum_{j \ne i} a_{ij} \lambda_{jj}^0 + a_{ii}.$$

Proof. This follows by writing down explicitly that

$$\tau(e_i^0 \Omega_i) \ge 0, \qquad \tau\left(\left(1 - e_i^0\right) \Omega_i\right) \le 0.$$

□

We will use a method similar to the method of Lagrange multipliers to determine the finer structure of a set of projections $e_i^0, \dots, e_n^0$ at which a maximum point is attained.

To do this we need to show that the grassmanian manifold associated with a type II_1 factor is large enough.

Lemma 3.6. *Let M be a II_1 factor, e be a non-trivial projection in M and $\mathcal{T}_e$ be the linear space consisting of all Z in M, such that $Z = Z^*$ and $eZe = 0$, $(1-e)Z(1-e)$. Let $\mathring{\mathcal{T}}_e$ be the set of all Z in $\mathcal{T}_e$ such that there exists a one-parameter family $e(t)$ of projections in M, weakly differentiable at 0, such that*

$$e(0) = e, \qquad \dot{e}(0) = Z.$$

Then the space of $\overset{\circ}{\mathcal{T}}_e$ *is weakly dense in* $\mathcal{T}_e$*.*

Proof. Assume first that $\tau(e) = \frac{1}{2}$, and let v be any partial isometry mapping e onto $1-e$. We will show that $Z = v + v^*$ belongs to $\overset{\circ}{\mathcal{T}}_e$.

Indeed $\{e, v\}''$ can be identified with $M_2(\mathbb{C})$ in such a way that

$$v + v^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

But then we take

$$e(\theta) = \begin{pmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}.$$

If the trace of e is different from $\frac{1}{2}$, we may then assume that $\tau(e) < \frac{1}{2}$. By the above argument, any partial isometry v mapping e into a projection under $1-e$, determines an element $v + v^*$ in $\overset{\circ}{\mathcal{T}}_e$.

Thus $\overset{\circ}{\mathcal{T}}_e$ contains $v+v^*$ for any partial isometry v, such that $v^*v = e$, $vv^* \leq 1-e$. Let u be any unitary in eMe and let w be any unitary in $(1-e)\,M\,(1-e)$. The same argument shows that $wvu + u^*v^*w^*$ belongs to $\overset{\circ}{\mathcal{T}}_e$. Since any element in eMe and $(1-e)\,M\,(1-e)$ is a linear combination of unitaries, this shows that $yvx + x^*v^*y^*$ is always in the linear span of $\overset{\circ}{\mathcal{T}}_e$ for all x in eMe and y in $(1-e)\,M\,(1-e)$. This set is obviously weakly dense in $\mathcal{T}_e$. $\square$

Corollary 3.7. *Fix* $n \geq 1$ *and real numbers* $(a_{ij})_{1\leq i\leq j\leq n}$*. Let*

$$(e_1^0, e_2^0, \ldots, e_n^0)$$

be a family of projections in $\mathcal{P}(M)$ *such that the maximum of*

$$L\left((\lambda_{ij})_{1\leq i\leq j\leq n}\right) = \sum_{1\leq i\leq j\leq n} a_{ij}\lambda_{ij}$$

for (λ_{ij}) *in* K_M^n *is attained at* $(\tau(e_i^\circ e_j^\circ))$*. Let*

$$\Omega_i^0 = \Omega_i^0(e_i^0, \ldots, e_n^0, a_{ij}) = a_{ii}\,\mathrm{Id} + \sum_{j\neq i} a_{ij}e_j^0.$$

Then $[e_i^0, \Omega_i^0] = 0$*. By using Lemma* 3.4 *it follows that*

$$e_i^0 \geq \mathrm{supp}(\Omega_i^0)_+,$$
$$1 - e_i^0 \geq \mathrm{supp}\left(\Omega_i^0\right)_-.$$

Proof. Indeed if

$$(\tau(e_i^0 e_j^0))_{1\leq i\leq j\leq n}$$

is such a maximum point for the functional L on K_M^n, then for all Z_i in $\mathring{\mathcal{T}}_{e_i^0}$ we have that

$$\tau(\Omega_i^0 Z_i) = 0.$$

But then this will give that

$$\tau(\Omega_i^0 Z_i) = 0$$

for all $Z_i = (1-e_i^0)Y_1e_i^0 + e_i^0Y_1(1-e_i^0)$, $Y_1 \in M_{sa}$. Thus for all $Y = Y^*$ in M we have

$$\tau\left(\Omega_i^0 e_i^0 Y(1-e_i^0) + \Omega_i^0(1-e_i^0)Ye_i^0\right) = 0$$

and hence

$$\tau\left([(1-e_i^0)\Omega_i^0 e_i^0 + e_i^0\Omega_i^0(1-e_i^0)]Y\right) = 0$$

for all Y selfadjoint in M. Since Ω_i^0 is also selfadjoint, it follows that

$$(1-e_i^0)\Omega_i^0 e_i^0 + (1-e_i^0)\Omega_i^0 e_i^0 = 0$$

and hence that Ω_i^0 commutes with e_i^0. $\square$

Remark 3.8. The above proposition suggests that for Connes's embedding problem, it is sufficient to consider finite von Neumann algebras (which we call *extremal finite von Neumann algebras*) that are generated by families of projections $e_1, e_2, \dots, e_n$ such that there exists a matrix of real numbers $(a_{ij})_{1\leq i\leq j\leq n}$ with the following property.

For each i, let

$$\Omega_i = \sum_{j\neq i} a_{ij}e_j + a_{ii}\,\mathrm{Id}\,.$$

Let s_+^i be the init of the positive part of $(\Omega_i)_+$ and s_-^i be the projection onto the init space of $(\Omega_i)_-$.

Then $1 - s_-^i \geq e_i \geq s_+^i$; in particular, e_i commutes with Ω_i, for all i.

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