MATRIX OPERATORS AND HYPERINVARIANT SUBSPACES

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ABSTRACT

In this paper we study the super-decomposability of some matrix operators as well as some other special properties. These matrix operators are derived from non-analytic functional calculi. As by-products, we obtain statements concerning the existence of (non-trivial) hyperinvariant subspaces.

1. *Introduction*

Let X be a complex Banach space and let $\mathscr{L}(X)$ be the algebra of all bounded linear operators acting on X. For each $S \in \mathscr{L}(X)$ we denote its spectrum by $\sigma(S)$.

For a fixed integer $n \geqslant 1$, let X^n denote the Banach space $X \oplus \ldots \oplus X$ (n copies). Every operator $T \in \mathscr{L}(X^n)$ can be represented as a matrix $(T_{jk})_{j,k=1}^n$, where $T_{jk} \in \mathscr{L}(X)$ for each pair of indices j, k. We shall study in the sequel a class of operators $T \in \mathscr{L}(X^n)$ with the property that the operators T_{jk} from the matrix representation of T mutually commute. To present this class, we need some preliminaries.

Let Ω be a compact topological space, let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω, and let $A \subset C(\Omega)$ be a (not necessarily closed) subalgebra. We recall that A is said to be *normal* if, for every open cover $\{G_1, \ldots, G_m\}$ of Ω, there are positive functions $f_1, \ldots, f_m$ in A such that $\operatorname{supp}(f_p) \subset G_p$ ($p = 1, \ldots, m$) and

$$f_1(\omega) + \ldots + f_m(\omega) = 1$$

for all $\omega \in \Omega$. In particular, $1 \in A$ (the positivity of the functions $f_1, \ldots, f_m$ will play no role in what follows).

1.1 DEFINITION. For an algebra $A \subset C(\Omega)$ we shall consider the following properties:

(i) A is a normal algebra;

(ii) for every pair $f, h \in A$ such that

$$\operatorname{supp}(h) \subset \{\omega \in \Omega : f(\omega) \neq 0\},$$

the function $\omega \to h(\omega)/f(\omega)$, extended with zero outside the set $\operatorname{supp}(h)$, is an element of A;

(iii) A has a Banach algebra structure which makes the inclusion $A \subset C(\Omega)$ continuous.

We shall indicate at the beginning of each section which of these hypotheses on the algebra A are going to be used.

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It is clear that $C(\Omega)$ has the properties (i), (ii) and (iii). If Ω is the closure of a relatively compact open subset of $\mathbb{R}^m$, then the algebra $C^r(\Omega)$ of all functions r-times differentiable in the interior of Ω whose partial derivatives up to order r have continuous extensions to Ω, also has the properties (i), (ii) and (iii). These are, in fact, the most significant examples that we have in mind.

If A is an arbitrary commutative unital algebra, we denote by $M_n(A)$ the algebra of $n\times n$-matrices whose entries are elements of A. The algebra $M_n(A)$ will sometimes be regarded as an A-module. Every unital algebra morphism $\Phi\colon A\to\mathscr{L}(X)$ induces a unital algebra morphism $\Phi_n\colon M_n(A)\to\mathscr{L}(X^n)$, defined by the equality $\Phi_n(\alpha)=(\Phi(\alpha_{jk}))_{j,k=1}^n$, where $\alpha=(\alpha_{jk})_{j,k=1}^n\in M_n(A)$.

1.2 Definition. Let $A\subset C(\Omega)$ be an algebra with the properties (i) and (ii) from Definition 1.1. An operator $T\in\mathscr{L}(X^n)$ will be called *(A,n)-scalar* if there exists a unital algebra morphism $\Phi\colon A\to\mathscr{L}(X)$ and an element $\tau\in M_n(A)$ such that $T=\Phi_n(\tau)$.

This concept extends the concept of n-spectral operator, introduced in [**9**], which in turn extends that of n-normal operator [7]. When A is an admissible algebra of continuous functions, then Definition 1.2 also provides an extension of the concept of A-scalar operator [**5**, **12**] (with respect to this class of admissible algebras).

One of the main purposes of this paper is to prove that every (A,n)-scalar operator is super-decomposable [**8**] (in particular decomposable [**5**, **12**]). Specifically, we shall show that if $T\in\mathscr{L}(X^n)$ is (A,n)-scalar and $\{U_1,U_2\}$ is an open cover of $\sigma(T)$, then there exists an operator $R\in\mathscr{L}(X^n)$ such that $RT=TR$, $\sigma(T|\overline{R(X^n)})\subset U_1$ and $\sigma(T|\overline{(1_n-R)(X^n)})\subset U_2$ (where 1_n is the identity of X^n; we use the same notation for the identity of $M_n(A)$). With the terminology of [**8**], this means precisely that an (A,n)-scalar operator is super-decomposable (see Theorem 3.9).

The decomposability of an (A,n)-scalar operator $T\in\mathscr{L}(X^n)$ can be used to derive the existence of a proper hyperinvariant subspace (that is, invariant under each operator commuting with T), when $\sigma(T)$ contains at least two points. This explains one of the main results of [**9**] (see Corollary 3.7 and Remark 3.8).

By analysing the spectrum of an (A,n)-scalar operator T (Theorem 4.6), we shall obtain the existence of hyperinvariant subspaces of T, even if $\sigma(T)$ contains only one point, provided that T is not a multiple of the identity; thus we obtain a complete extension of [7, Theorem 5.3] (see Corollary 4.8).

In connection with this subject, we also refer to [**2**, **6**, **10**, **11**]. We can apply our methods to a large enough class of matrix operators, including matrices of generalized scalar operators given by a specral distribution [**5**].

2. *A spectral capacity*

Let $A\subset C(\Omega)$ be a normal algebra. We also fix a unital algebra morphism $\Phi\colon A\to\mathscr{L}(X)$ and denote by Φ_n the corresponding morphism of $M_n(A)$ into $\mathscr{L}(X^n)$ induced by Φ.

Since a matrix $\alpha=(\alpha_{jk})_{j,k=1}^n\subset M_n(A)$ can be regarded as a function $\alpha\colon\Omega\to M_n$ (where $M_n=M_n(\mathbb{C})\subset M_n(A)$), the notation $\alpha(\omega)=(\alpha_{jk}(\omega))_{j,k=1}^n$ $(\omega\in\Omega)$ and $\operatorname{supp}(\alpha)$ makes sense. Moreover,

$$\operatorname{supp}(\alpha'\cdot\alpha'')\subset\operatorname{supp}(\alpha')\cap\operatorname{supp}(\alpha'')$$

for each pair $\alpha',\alpha''\in M_n(A)$.

For every $f \in A$ we denote by $\delta(f) \in M_n(A)$ the matrix $\delta(f) = (\delta_{jk} f)_{j,k=1}^n$, where δ_{jk} is the Kronecker symbol. Notice that δ is, in fact, a unital algebra morphism of A into $M_n(A)$ and that $\delta(A)$ is in the centre of $M_n(A)$.

The set $\operatorname{supp}(\Phi)$ (that is the *support* of Φ) is defined as the intersection of all closed sets $F \subset \Omega$ such that $\Phi(f) = 0$ whenever $\operatorname{supp}(f) \subset \Omega \backslash F$ ($f \in A$). The set $\operatorname{supp}(\Phi_n)$ is defined in a similar way. It is easily seen that $\operatorname{supp}(\Phi_n) = \operatorname{supp}(\Phi)$. (Note that $\operatorname{supp}(\alpha) = \bigcup\{\operatorname{supp}(\alpha_{jk}) \colon 1 \leqslant j, k \leqslant n\}$ for each $\alpha = (\alpha_{jk})_{j,k=1}^n \in M_n(A)$.)

2.1 PROPOSITION. *For every closed set $F \subset \Omega$ we define the space*

$$X_\Phi^n(F) = \bigcap\{\ker(\Phi_n(\alpha)) \colon \operatorname{supp}(\alpha) \cap F = \varnothing\}. \tag{2.1}$$

Then the assignment $F \to X_\Phi^n(F)$ is a spectral capacity [3, 12].

Proof. We follow some lines from the proof of [12, Theorem IV.7.3] (see also [1]).

It is clear that $X_\Phi^n(F)$ is a closed linear subspace of X^n. It is easily seen that $X_\Phi^n(\varnothing) = \{0\}$, $X_\Phi^n(\Omega) = X^n$ and $X_\Phi^n(F_1) \subset X_\Phi^n(F_2)$ whenever $F_1 \subset F_2$.

Let $\{G_1, \dots, G_m\}$ be an open cover of Ω. Since A is normal, we can find functions $f_1, \dots, f_m$ in A such that $\operatorname{supp}(f_p) \subset G_p$ ($p = 1, \dots, m$) and $f_1 + \dots + f_m = 1$. Let $\alpha_p = \delta(f_p)$; therefore $\operatorname{supp}(\alpha_p) \subset G_p$ and $\alpha_1 + \dots + \alpha_m = 1_n$. It is then clear that

$$X^n = \Phi_n(\alpha_1) X^n + \dots + \Phi_n(\alpha_m) X^n.$$

We have only to note that

$$\Phi_n(\alpha_p) X^n \subset X_\Phi^n(\operatorname{supp}(\alpha_p)) \subset X_\Phi^n(\bar{G}_p)$$

for every p, and therefore

$$X^n = X_\Phi^n(\bar{G}_1) + \dots + X_\Phi^n(\bar{G}_m). \tag{2.2}$$

Now, let $\{F_\gamma\}_{\gamma \in \Gamma}$ be an arbitrary family of closed subsets of Ω. We shall prove that

$$X_\Phi^n(\bigcup_{\gamma \in \Gamma} F_\gamma) = \bigcup_{\gamma \in \Gamma} X_\Phi^n(F_\gamma). \tag{2.3}$$

Since the mapping $F \to X_\Phi^n(F)$ is increasing, it suffices to prove that the right-hand side of (2.3) is contained in the left-hand side. Let $x \in X_\Phi^n(F_\gamma)$ for all $\gamma \in \Gamma$ and let $F_0 = \bigcap\{F_\gamma \colon \gamma \in \Gamma\}$. Let also $\alpha \in M_n(A)$ be such that $\operatorname{supp}(\alpha) \cap F_0 = \varnothing$. Since $\operatorname{supp}(\alpha)$ is compact, we can choose open sets $H_q = \Omega \backslash F_{\gamma_q}$ ($q = 1, \dots, r$) such that $\operatorname{supp}(\alpha) \subset H_1 \cup \dots \cup H_r$. If $H_0 = \Omega \backslash \operatorname{supp}(\alpha)$, there are functions $h_0, h_1, \dots, h_r$ in A such that $h_0 + h_1 + \dots + h_r = 1$ and $\operatorname{supp}(h_q) \subset H_q$ ($q = 0, 1, \dots, r$). Let $\beta_q = \delta(h_q) \in M_n(A)$. Then

$$\Phi_n(\alpha) x = \Phi_n(\alpha\beta_0) x + \Phi_n(\alpha\beta_1) x + \dots + \Phi_n(\alpha\beta_r) x.$$

Since $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta_0) = \varnothing$ and $\operatorname{supp}(\alpha\beta_q) \cap F_{\gamma_q} = \varnothing$ ($1 \leqslant q \leqslant r$), we have $\Phi_n(\alpha\beta_q) x = 0$ for all $q = 0, 1, \dots, r$. Consequently $\Phi_n(\alpha) x = 0$, so that x is contained in the left-hand side of (2.3). The proof of Proposition 2.1 is complete.

2.2 COROLLARY. *Let $f \in A$ be such that* $\operatorname{supp}(f) \cap \operatorname{supp}(\Phi) = \varnothing$. *Then* $\Phi(f) = 0$.

Proof. Consider first a closed set $F \subset \Omega$ such that if $h \in A$ and $\operatorname{supp}(h) \cap F = \varnothing$, then $\Phi(h) = 0$. In this case we must have $X_\Phi^n(F) = X^n$, by (2.1). Indeed, if $\alpha \in M_n(A)$ and $\operatorname{supp}(\alpha) \cap F = \varnothing$, then $\Phi_n(\alpha) = 0$, that is $\ker(\Phi_n(\alpha)) = X^n$.

Now, let $\{F_\gamma\}_{\gamma\in\Gamma}$ be the family of all closed subsets of Ω sharing the property of F. Then $\bigcap\{F_\gamma : \gamma\in\Gamma\} = \operatorname{supp}(\Phi)$. Since $\operatorname{supp}(\delta(f)) \cap \operatorname{supp}(\Phi) = \emptyset$, it follows that

$$\ker(\Phi_n(\delta(f))) \supset X^n_\Phi(\operatorname{supp}(\Phi)) = \bigcap_{\gamma\in\Gamma} X^n(F_\gamma) = X^n,$$

by (2.3) and the first part of the proof. Consequently $\Phi(f) = 0$.

2.3 COROLLARY. *For every closed $F \subset \Omega$ we have the equality*

$$X^n_\Phi(F) = X^n_\Phi(F \cap \operatorname{supp}(\Phi)).$$

Proof. As we have noted in the proof of Corollary 2.2,

$$X^n_\Phi(\operatorname{supp}(\Phi)) = X^n.$$

Therefore

$$X^n_\Phi(F) = X^n \cap X^n_\Phi(F) = X^n_\Phi(F \cap \operatorname{supp}(\Phi)),$$

by (2.3).

2.4 REMARK. We have not used so far the fact that the functions of A are continuous.

A supplementary condition on the algebra $A \subset C(\Omega)$ makes the mapping $\Phi_n : M_n(A) \to \mathscr{L}(X^n)$ injective on its support.

2.5 LEMMA. *Assume that the algebra A also has the property* (ii) *of Definition* 1.1. *If $\Phi_n(\alpha) = 0$ for some $\alpha \in M_n(A)$, then $\alpha(\omega) = 0$ for every $\omega \in \operatorname{supp}(\Phi)$.*

Proof. Note first that if $\Phi(f) = 0$ for some $f \in A$, then $f(\omega) = 0$ for every $\omega \in \operatorname{supp}(\Phi)$.

Indeed, if $h \in A$ is such that $\operatorname{supp}(h) \subset G = \{\omega\in\Omega : f(\omega) \neq 0\}$, then the extension h_1 of the function $\omega \mapsto h(\omega)/f(\omega)$ belongs to A and we have $\Phi(h) = \Phi(h_1)\Phi(f) = 0$. Therefore $\operatorname{supp}(\Phi) \cap G = \emptyset$.

Now, if $\Phi_n(\alpha) = 0$ and $\alpha = (\alpha_{jk})^n_{j,k-1}$, then $\Phi(\alpha_{jk}) = 0$ for each pair (j, k). By the previous remark, it follows that $\alpha(\omega) = 0$ for all $\omega \in \operatorname{supp}(\Phi)$.

3. *Decomposability*

In this section A will denote a subalgebra of $C(\Omega)$ with the properties (i) and (ii) from Definition 1.1. Let $\Phi: A \to \mathscr{L}(X)$ be a fixed initial algebra morphism. We also fix an element $\tau = (\tau_{jk})^n_{j,k-1} \in M_n(A)$. Let $T = \Phi_n(\tau) \in \mathscr{L}(X^n)$, that is, T is (A, n)-scalar. From the defining relation (2.1), it follows easily that $TX^n_\Phi(F) \subset X^n_\Phi(F)$ for all closed subsets $F \subset \Omega$.

3.1 LEMMA. *For every closed $F \subset \Omega$ we have the inclusion*

$$\sigma(T \mid X^n_\Phi(F)) \subset \bigcup_{\omega\in F} \sigma(\tau(\omega)),$$

and the set on the right-hand side is closed.

Proof. We use the well-known equality

$$\sigma(\tau(\omega)) = \{z \in \mathbb{C}: \det(z1_n - \tau(\omega)) = 0\}, \qquad \omega \in \Omega,$$

where det stands for *determinant*. It is also an elementary fact that there exists a matrix $\tau_*(z) \in M_n(A)$ such that

$$(z1_n - \tau)\tau_*(z) = \tau_*(z)(z1_n - \tau) = \delta(\det(z1_n - \tau)) \tag{3.1}$$

for each $z \in \mathbb{C}$.

Now, let $z \in \mathbb{C}$ be such that $\det(z1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$. We take a function $h \in A$ such that $h = 1$ in a neighbourhood of F and

$$\operatorname{supp}(h) \subset \{\omega \in \Omega: \det(z1_n - \tau(\omega)) \neq 0\}.$$

Since $\det(z1_n - \tau) \in A$ and A has the property (ii) of Definition 1.1, the function $g(\omega) = h(\omega)(\det(z1_n - \tau(\omega)))^{-1}$ (equal to zero outside the set $\operatorname{supp}(h)$) is an element of A. From (3.1) we deduce that

$$(z1_n - T)\Phi_n(g\tau_*(z)) = \Phi_n(g\tau_*(z))(z1_n - T) = \Phi_n(\delta(h)).$$

Since we have $\operatorname{supp}(1-h) \cap F = \varnothing$, it is clear that $\Phi_n(\delta(h)) \mid X^n_\Phi(F)$ is the identity on $X^n_\Phi(F)$. Therefore

$$\Phi_n(g\tau_*(z)) \mid X^n_\Phi(F) = ((z1_n - T) \mid X^n_\Phi(F))^{-1},$$

that is $z \notin \sigma(T \mid X^n_\Phi(F))$.

Finally, if $\det(z1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$, then there exists a neighbourhood V of z such that if $w \in V$, then $\det(w1_n - \tau(\omega)) \neq 0$ for all $\omega \in F$. Consequently, the set $\bigcup\{\sigma(\tau(\omega)): \omega \in F\}$ is closed.

3.2 Remark. The inclusion in the statement of Lemma 3.1 can be written as

$$\sigma(T \mid X^n_\Phi(F)) \subset \bigcup\{\sigma(\tau(\omega)): \omega \in F \cap \operatorname{supp}(\Phi)\},$$

using Corollary 2.3.

3.3 Lemma. *Let $L \subset \mathbb{C}$ be a closed set and let*

$$\theta(L) = \{\omega \in \Omega: \sigma(\tau(\omega)) \cap L \neq \varnothing\}.$$

Then $\theta(L)$ is a compact subset of Ω with the property that $\sigma(T \mid X^n_\Phi(F)) \cap L = \varnothing$ whenever $\theta(L) \cap F = \varnothing$, F closed in Ω.

Proof. If $\omega_0 \notin \theta(L)$, then $\sigma(\tau(\omega_0)) \cap L = \varnothing$. Thus, by the upper semicontinuity of the spectrum, there exists a neighbourhood W_0 of ω_0 such that $\sigma(\tau(\omega)) \cap L = \varnothing$ for each $\omega \in W_0$. Hence $\Omega \backslash \theta(L)$ is open.

Now, let $F = \bar{F} \subset \Omega$ be such that $\theta(L) \cap F = \varnothing$. If z were a point of $\sigma(T \mid X^n_\Phi(F)) \cap L$, then, by virtue of Lemma 3.1, there would exist a point $\omega \in F$ such that $z \in \sigma(\tau(\omega))$. Therefore $\omega \in \theta(L) \cap F$, which contradicts the choice of F.

3.4 Lemma. *The operator T satisfies the condition (β) of Bishop* [**4**].

Proof. We have to show that if $U \subset \mathbb{C}$ is an arbitrary open set and $\{g_p\}_{p=1}^\infty$ is a sequence of X^n-valued functions, analytic in U, such that $(z1_n - T)g_p(z) \to 0$ $(p \to \infty)$ uniformly on the compact subsets of U, then it follows that $g_p(z) \to 0$ $(p \to \infty)$ uniformly on the compact subsets of U.

Let $\{g_p\}_{p=1}^{\infty}$ be a sequence as above and let $\Delta \subset U$ be a fixed closed disc. We show that $g_p(z) \to 0$ $(p \to \infty)$ uniformly on Δ.

We consider the set $\theta(\Delta) \subset \Omega$ (defined in Lemma 3.3) and fix a point $\omega_0 \in \theta(\Delta)$. Let $D_0 \subset \bar{D}_0 \subset U$ be an open disc containing Δ and let $V_0 \subset \mathbb{C}$ be an open set such that $\bar{D}_0 \cap \bar{V}_0 = \varnothing$ and $\sigma(\tau(\omega_0)) \subset D_0 \cup V_0$, which is obviously possible. By the upper semicontinuity of the spectrum, we infer the existence of an open neighbourhood W_0 of ω_0 in Ω such that if $\omega \in W_0$, then $\sigma(\tau(\omega)) \subset D_0 \cup V_0$. This procedure can be applied to any point ω of $\theta(\Delta)$. By the compactness of $\theta(\Delta)$ (Lemma 3.3), we obtain a finite open cover $\{W_1, \ldots, W_m\}$ of $\theta(\Delta)$, open discs $D_1, \ldots, D_m$ whose closures are in U and open sets $V_1, \ldots, V_m$ in $\mathbb{C}$ such that $D_q \supset \Delta$, $\bar{D}_q \cap \bar{V}_q = \varnothing$ and $\sigma(\tau(\omega)) \subset D_q \cup V_q$ for every $\omega \in W_q$ $(q = 1, \ldots, m)$. Let $W_{m+1} = \Omega \backslash \theta(\Delta)$. We take the functions $h_1, \ldots, h_m, h_{m+1}$ from A such that $h_1 + \ldots + h_m + h_{m+1} = 1$ and $\operatorname{supp}(h_q) \subset W_q$ $(q = 1, \ldots, m+1)$. Then consider the matrices $\alpha_q = \delta(h_q)$. Note that

$$\Phi_n(\alpha_q)\, g_p(z) \in X^n_\Phi(\operatorname{supp}(\alpha_q)), \qquad q = 1, \ldots, m, m+1,$$

and that $\sigma(T \mid X^n_\Phi(\operatorname{supp}(\alpha_q)) \subset \bar{D}_q \cup \bar{V}_q$ $(q = 1, \ldots, m)$. Since $\bar{D}_q \cap \bar{V}_q = \varnothing$, we can take another open disc $D'_q \supset \bar{D}_q$ in U such that $D'_q \cap \bar{V}_q = \varnothing$ $(1 \leqslant q \leqslant m)$. Note that the operator

$$(z1_n - T) \mid X^n_\Phi(\operatorname{supp}(\alpha_q))$$

is invertible for $z \in D'_q \backslash \bar{D}_q$ and that $\Phi_n(\alpha_q)$ commutes with T. Therefore, as $p \to \infty$ $\Phi_n(\alpha_q)\, g_p(z) \to 0$ uniformly on the compact subsets of D'_q. By the maximum principle, we deduce that $\Phi_n(\alpha_q)\, g_p(z) \to 0$ $(p \to \infty)$ uniformly on $\bar{D}_q$, in particular on Δ, for every $q = 1, \ldots, m$.

From Lemma 3.3 we obtain that

$$\sigma(T \mid X^n_\Phi(\operatorname{supp}(\alpha_{m+1})) \cap \Delta = \varnothing.$$

Hence $\Phi_n(\alpha_{m+1})\, g_p(z) \to 0$ $(p \to \infty)$ uniformly on Δ, and therefore

$$g_p(z) = \Phi_n((\alpha_1)\, g_p(z) + \ldots + \Phi_n(\alpha_{m+1})\, g_p(z) \to 0 \quad (p \to \infty)$$

uniformly for $z \in \Delta$.

The general assertion now follows by covering an arbitrary compact subset $L \subset U$ with a finite number of closed discs and applying the previous argument to each of these discs.

Since T satisfies the condition (β), it follows that T has the single-valued extension property. In particular, we can discuss the spectral spaces

$$X^n_T(L) = \{x \in X^n : \gamma_T(x) \subset L\}, \tag{3.2}$$

where $L \subset \mathbb{C}$ is an arbitrary closed set and $\gamma_T(x)$ is the local spectrum of T at x (see **[5]** or **[12]** for details). In addition, the space $X^n_T(L)$ is closed (which is an easy consequence of the condition (β)), $X^n_T(L)$ is invariant under every operator that commutes with T (that is $X^n_T(L)$ is hyperinvariant) and $\sigma(T \mid X^n_T(L)) \subset L$ **[5, 12]**.

3.5 Lemma. *The operator T is decomposable.*

Proof. Let $\{U_1, U_2\}$ be an open cover of $\mathbb{C}$, and fix a point $\omega_0 \in \Omega$. Then we can choose two open sets V^1_0 and V^2_0 in $\mathbb{C}$ such that $\sigma(\tau(\omega_0)) \subset V^1_0 \cup V^2_0$, $\bar{V}^q_0 \subset U_q$ $(q = 1, 2)$ and $\bar{V}^1_0 \cap \bar{V}^2_0 = \varnothing$. Let $W_0 \subset \Omega$ be an open set such that $\omega \in \bar{W}_0$ implies that $\sigma(\tau(\omega)) \subset V^1_0 \cup V^2_0$. Since Ω is compact, the previous remark shows that we can find an open cover $\{W_1, \ldots, W_m\}$ of Ω and open sets $\{V_p^q\colon 1 \leqslant p \leqslant m,\ q = 1,2\}$ in $\mathbb{C}$ such that

(a) $\bar{V}_p^q \subset U_q$, $\bar{V}_p^1 \cap \bar{V}_p^2 = \varnothing$;

(b) if $\omega \in \bar{W}_p$ then $\sigma(\tau(\omega)) \subset \bar{V}_p^1 \cup \bar{V}_p^2$

for all $p = 1, \ldots, m$ and $q = 1,2$. From Lemma 3.1 and the property (b) we deduce that

$$\sigma(T \mid X_\Phi^n(\bar{W}_p)) \subset \bigcup_{\omega \in \bar{W}_p} \sigma(\tau(\omega)) \subset \bar{V}_p^1 \cup \bar{V}_p^2.$$

Therefore

$$X_\Phi^n(\bar{W}_p) \subset X_T^n(\bar{V}_p^1 \cup \bar{V}_p^2)$$

by the fact that the space from the right-hand side is spectral maximal [**5**, **12**]. Let us also note the decomposition

$$X_T^n(\bar{V}_p^1 \cup \bar{V}_p^2) = X_T^n(\bar{V}_p^1) + X_T^n(\bar{V}_p^2), \tag{3.3}$$

which follows from (a), the decomposition of the space with respect to separate parts of the spectrum (see, for instance, [**12**, Theorem III.3.11]) and the fact that all involved spaces are spectral maximal.

According to (2.2) and the above considerations, we can write

$$X^n = \sum_{p=1}^{m} X_\Phi^n(\bar{W}_p) = \sum_{p=1}^{m} X_T^n(\bar{V}_p^1) + \sum_{p=1}^{m} X_T^n(\bar{V}_p^2) = X_T^n(\bar{U}_1) + X_T^n(\bar{U}_2),$$

which proves the decomposability of T, by virtue of [**12**, Theorem IV.4.28].

3.6 REMARK. Lemma 3.5 is also stated in [**2**] as Corollary 3.14, in a framework different from ours. It seems to us that the proof of the above-mentioned corollary needs our slightly stronger condition (ii) of Definition 1.1 rather than condition (b) from [**2**, Definition 3.1] (see the construction of the function f [**2**, p. 304]). We are indebted to the referee for drawing our attention to this result from [**2**].

3.7 COROLLARY. *If $\sigma(T)$ contains more than one point, then T has at least one proper hyperinvariant subspace.*

This fact is well known in the theory of decomposable operators and is based on the existence of a compact subset $L \subset \sigma(T)$ such that $X_T^n(L)$ is neither zero nor the whole space. As we have already mentioned, $X_T^n(L)$ is a hyperinvariant subspace of T.

3.8 REMARK. If $A = C(\Omega)$ and Φ is obtained via a spectral measure on Ω, then the operator T is n-spectral [**9**]. If $\sigma(T)$ has more than one point, then T has a proper hyperinvariant subspace, as proved in [**9**]. Consequently, Corollary 3.7 provides an extension of this result.

3.9 THEOREM. *Every (A, n)-scalar operator is super-decomposable.*

Proof. We use the notation and the discussion from the proof of Lemma 3.5.

Let $\{f_1, \ldots, f_m\} \subset A$ be such that $f_1 + \ldots + f_m = 1$ and $\operatorname{supp}(f_p) \subset W_p$ $(p = 1, \ldots, m)$. Let also Q_p^q be the spectral projection of the space $X_T^n(\bar{V}_p^1 \cup \bar{V}_p^2)$ onto $X_T^n(\bar{V}_p^q)$

($q = 1, 2; p = 1, \ldots, m$), which is obtained from the decomposition (3.3) via the analytic functional calculus of the restriction of T to $X^n_T(\bar{V}^1_p \cup \bar{V}^2_p)$ (see [12, Theorem III.3.11]). Since

$$\Phi_n(\delta(f_p))X^n \subset X^n_\Phi(\operatorname{supp}(f_p)) \subset X^n_\Phi(\bar{W}_p) \subset X^n_T(\bar{V}^1_p \cup \bar{V}^2_p),$$

we may define the operators

$$R_q = \sum_{p=1}^{m} Q^q_p \Phi_n(\delta(f_p)) \in \mathscr{L}(X^n), \qquad q = 1, 2.$$

It is easily seen that $R_1 + R_2 = 1_n$. Moreover,

$$\begin{aligned} TR_q &= T\left(\sum_{p=1}^{m} Q^q_p \Phi_n(\delta(f_p))\right) \\ &= \sum_{p=1}^{m} (T \mid X^n_T(\bar{V}^1_p \cup \bar{V}^2_p)) Q^q_p \Phi_n(\delta(f_p)) \\ &= \sum_{p=1}^{m} Q^q_p (T \mid X^n_T(\bar{V}^1_p \cup \bar{V}^2_p)) \Phi_n(\delta(f_p)) \\ &= \sum_{p=1}^{m} Q^q_p T \Phi_n(\delta(f_p)) = R_q T, \end{aligned}$$

since every operator commutes with its analytic functional calculus and $\delta(f_p)$ is in the centre of $M_n(A)$. We also have

$$R_q(X^n) \subset \sum_{p=1}^{m} X^n_T(\bar{V}^q_p) \subset X^n_T(\bar{U}_q),$$

which insures, by virtue of [8, Theorem 1.4], that T is super-decomposable. The proof of the theorem is complete.

The authors thank the referee for suggesting the use of [8, Theorem 1.4] to shorten the original proof of Theorem 3.9.

3.10 Remark. Let $\Omega_\tau = \tau(\Omega)$ and set $A_\tau = \{f \in C(\Omega_\tau) : f \circ \tau \in A\}$, which is a subalgebra of $C(\Omega_\tau)$. Then the map $\Phi_\tau : A_\tau \to \mathscr{L}(X)$ given by $\Phi_\tau(f) = \Phi(f \circ \tau)$ is a unital algebra morphism. Suppose that A_τ has the properties (i) and (ii) from Definition 1.1 (this happens, for instance, when $A = C(\Omega)$). Then the morphism Φ_τ can be used instead of Φ. In this case there is no loss of generality in assuming that Ω is a compact subset of $\mathbb{C}^{n^2}$ and that τ is the matrix of the coordinate functions on $\mathbb{C}^{n^2}$, restricted to Ω.

4. *More about the spectrum*

In this section we assume that $A \subset C(\Omega)$ has the properties (i), (ii) and (iii) of Definition 1.1. As in the previous section, we fix a unital algebra morphism $\Phi : A \to \mathscr{L}(X)$, an element $\tau = (\tau_{jk})^n_{j,k=1} \in M_n(A)$, and consider the (A, n)-scalar operator $T = \Phi_n(\tau) \in \mathscr{L}(X^n)$.

For every closed set $F \subset \Omega$ we define the set

$$S_{\tau, F} = \bigcup_{\omega \in F} \sigma(\tau(\omega)) \subset \mathbb{C}. \tag{4.1}$$

The set $S_{\tau,F}$ is closed (in fact compact), by Lemma 3.1. When $F=\operatorname{supp}(\Phi)$, the set $S_{\tau,F}$ will be denoted simply by S_τ.

4.1 LEMMA. *For every $h\in A$, there is an analytic function $\phi_h\colon \mathbb{C}\backslash S_{\tau,F}\to M_n(A)$ such that $(z1_n-\tau)\phi_h(z)=\delta(h)$ for all $z\notin S_{\tau,F}$, where $F=\operatorname{supp}(h)$.*

Proof. Consider the Banach space $Y=A^n$ and the map $\Psi\colon A\to\mathscr{L}(Y)$ given by

$$\Psi(h)f_1\oplus\ldots\oplus f_n=hf_1\oplus\ldots\oplus hf_n,\qquad h,f_1,\ldots,f_n\in A.$$

Clearly, Ψ is a unital algebra morphism. Let $\Psi_n\colon M_n(A)\to\mathscr{L}(Y^n)$ be the unital algebra morphism induced by Ψ. If we identify Y^n with $M_n(A)$, then, with this identification, $\Psi_n(\alpha)\beta=\alpha\beta$ for all $\alpha,\beta\in M_n(A)$. In particular $\Psi_n(\tau)$ is the multiplication by the matrix τ, which will also be denoted by τ. The operator τ is (A,n)-scalar, and therefore it has the properties described in the previous section.

It is easily seen that $\delta(h)\in Y^n_\Psi(\operatorname{supp}(h))$ (which is defined by (2.1)). According to Lemma 3.1,

$$\sigma(\tau\mid Y^n_\Psi(F))\subset S_{\tau,F},$$

where $F=\operatorname{supp}(h)$. Consequently, we may take

$$\phi_h(z)=((z1_n-\tau)\mid Y^n_\Psi(F))^{-1}\delta(h),\qquad z\notin S_{\tau,F}.$$

4.2 LEMMA. *Assume that there exists a compact subset $L\subset S_\tau\backslash\sigma(T)$ such that $S_\tau\backslash L$ is also compact. Then $L=\varnothing$.*

Proof. Let us assume that $L\neq\varnothing$. Let $V_1\supset L$ and $V_2\supset S_\tau\backslash L$ be open sets such that $\bar V_1\cap\bar V_2=\varnothing$. Then there is an open neighbourhood W of $\operatorname{supp}(\Phi)$ such that $S_{\tau,\bar W}\subset V_1\cup V_2$. We may also assume that $\Gamma=\partial V_1$ is a finite system of Jordan rectifiable curves, positively oriented.

Let $h\in A$ be such that $h=1$ in a neighbourhood of $\operatorname{supp}(\Phi)$ and $\operatorname{supp}(h)\subset W$. Also let ϕ_h be the analytic function given by Lemma 4.1, which is defined outside the set $S_{\tau,\bar W}$. Then we may consider the element

$$e=\frac{1}{2\pi i}\int_\Gamma\phi_h(z)\,dz\in M_n(A).$$

Set $F_1=\{\omega\in\Omega\colon h(\omega)=1\}$. Since $\delta(h)(\omega)=1_n$ for $\omega\in F_1$, $\phi_h(z)(\omega)=(z1_n-\tau(\omega))^{-1}$. It follows from our assumption on the algebra A (Definition 1.1(iii)) that the point evaluations are continuous. Hence

$$e(\omega)=\frac{1}{2\pi i}\int_\Gamma(z1_n-\tau(\omega))^{-1}\,dz,\qquad \omega\in F_1,$$

which shows that $e(\omega)^2=e(\omega)$ (in fact $e(\omega)$ is a spectral projection of $\tau(\omega)$). Since F_1 is a neighbourhood of $\operatorname{supp}(\Phi)$, it follows that $\Phi_n(e)$ is an idempotent. In addition, $\Phi_n(e)$ commutes with T because of the equality $\tau(\omega)e(\omega)=e(\omega)\tau(\omega)$ $(\omega\in F_1)$.

Consider now the integral

$$e_w=\frac{1}{2\pi i}\int_\Gamma(w-z)^{-1}\phi_h(z)\,dz,\qquad w\notin\bar V_1.$$

It is clear that

$$(w1_n - \tau(\omega))\, e_w(\omega) = e_w(\omega)\,(w1_n - \tau(\omega)) = e(\omega) \tag{4.2}$$

for all $\omega \in F_1$ and $w \notin \bar{V}_1$.

Since $\Phi_n(e)$ is idempotent, it follows that $Z = \Phi_n(e)(X^n)$ is a closed subspace of X^n, invariant under T and also under $\Phi_n(e_w)$. Moreover, from (4.2) we deduce that

$$((w1_n - T)\,|\,Z)(\Phi_n(e_w)\,|\,Z) = (\Phi_n(e_w)\,|\,Z)((w1_n - T)\,|\,Z) = 1_Z,$$

where 1_Z is the identity of Z. This shows that $\sigma(T|Z) \subset \bar{V}_1$. On the other hand, $\sigma(T) \subset V_2$, by Remark 3.2 and the property of L. Therefore $\sigma(T) \cap \sigma(T|Z) = \varnothing$, which is not possible unless $Z = \{0\}$. This shows that $\Phi_n(e) = 0$, so that $e(\omega) = 0$ for each $\omega \in \operatorname{supp}(\Phi)$, by virtue of Lemma 2.5, which contradicts our assumption. Indeed, if $z_0 \in L$, then there exists $\omega_0 \in \operatorname{supp}(\Phi)$ such that $z_0 \in \sigma(\tau(\omega_0))$. Then V_1 contains at least one point from the spectrum of the matrix $\tau(\omega_0)$, whence $e(\omega_0) \neq 0$. Consequently we must have $L = \varnothing$.

4.3 LEMMA. *Let $F \subset \Omega$ be closed and let*

$$X_\Phi(F) = \bigcap\{\ker(\Phi(f)) : \operatorname{supp}(f) \cap F = \varnothing\}.$$

Then the space $X_\Phi(F)^n$ is invariant under T and the restriction $T|X_\Phi(F)^n$ is (A, n)-scalar.

Proof. It is easily seen that $X_\Phi(F)^n = X_\Phi(F) \oplus \ldots \oplus X_\Phi(F)$ (n copies) is invariant under T. Since $X_\Phi(F)$ is invariant under $\Phi(f)$ for every $f \in A$, we may define the map

$$A \ni f \longrightarrow \Phi_F(f) = \Phi(f)\,|\,X_\Phi(F) \in \mathscr{L}(X_\Phi(F)), \tag{4.3}$$

which is a unital algebra morphism. If $\Phi_{F,n}$ is the unital algebra morphism from $M_n(A)$ into $\mathscr{L}(X_\Phi(F)^n)$ induced by Φ_F, then $T|X_\Phi(F)^n = \Phi_{F,n}(\tau)$, which is precisely our assertion.

4.4 REMARK. With the notation of Lemma 4.3, we have the inclusion $\sigma(T|X_\Phi(F)^n) \subset \sigma(T)$. Indeed, if $z \notin \sigma(T)$, then the space $X_\Phi(F)^n$ is invariant under $(z1_n - T)^{-1}$, since

$$\Phi_n(\delta(f))(z1_n - T)^{-1}x = (z1_n - T)^{-1}\Phi_n(\delta(f))x = 0$$

for every $f \in A$ with $\operatorname{supp}(f) \cap F = \varnothing$, and each $x \in X_\Phi(F)^n$.

4.5 LEMMA. *The morphism Φ_F from* (4.3) *has the following property:*

$$\operatorname{int}(F) \cap \operatorname{supp}(\Phi) \subset \operatorname{supp}(\Phi_F) \subset F \cap \operatorname{supp}(\Phi)$$

for each closed F.

Proof. Let X_F be the space $X_\Phi(F)$, defined in Lemma 4.3. Let also $f \in A$ be such that $\operatorname{supp}(f) \cap F \cap \operatorname{supp}(\Phi) = \varnothing$. By using the normality of the algebra A, we can write $f = f_1 + f_2$, where $f_1, f_2 \in A$, $\operatorname{supp}(f_1) \cap F = \varnothing$ and $\operatorname{supp}(f_2) \cap \operatorname{supp}(\Phi) = \varnothing$. Then

$$\Phi_F(f) = \Phi(f_1)\,|X_F + \Phi(f_2)|\,X_F = 0,$$

which shows that $\operatorname{supp}(\Phi_F) \subset F \cap \operatorname{supp}(\Phi)$.

Conversely, let $\omega_0 \in \operatorname{int}(F) \cap \operatorname{supp}(\Phi)$, let W_0 be an open neighbourhood of ω_0 such that $\bar{W}_0 \subset \operatorname{int}(F)$, let $W_1 = \operatorname{int}(F)$ and let $W_2 \subset \Omega$ be open such that $\bar{W}_2 \cap \bar{W}_0 = \varnothing$ and $W_1 \cup W_2 = \Omega$. Then, by Proposition 2.1 (with $n = 1$),

$$X = X_{\bar{W}_1} + X_{\bar{W}_2} = X_F + X_{\bar{W}_2}.$$

If $f \in A$ and $\operatorname{supp}(f) \subset W_0$, then $\Phi(f) \mid X_{\bar{W}_2} = 0$. Since $\omega_0 \in \operatorname{supp}(\Phi)$, this shows that $\omega_0 \in \operatorname{supp}(\Phi_F)$.

4.6 THEOREM. *Let $T \in \mathscr{L}(X^n)$ be an (A, n)-scalar operator such that $T = \Phi_n(\tau)$. Then we have the equality*

$$\sigma(T) = \bigcup\{\sigma(\tau(\omega)) : \omega \in \operatorname{supp}(\Phi)\}.$$

Proof. The inclusion $\sigma(T) \subset S_\tau$ has already been noted (see Remark 3.2).

Conversely, assume that there exists a point $z_0 \in S_\tau \backslash \sigma(T)$. Let $\omega_0 \in \Omega$ be such that $z_0 \in \sigma(\tau(\omega))$. Let V_1, V_2 be open sets in $\mathbb{C}$ such that $V_1 \ni z_0$, $V_2 \supset \sigma(T)$, $\bar{V}_1 \cap \bar{V}_2 = \varnothing$ and $\sigma(\tau(\omega_0)) \subset V_1 \cup V_2$. Then there exists an open set $W_0 \ni \omega_0$ in Ω such that $\sigma(\tau(\omega)) \subset V_1 \cup V_2$ for every $\omega \in F = \bar{W}_0$. According to Remark 4.4, we have the inclusion $\sigma(T_F) \subset \sigma(T) \subset V_2$, where $T_F = T \mid X_\Phi(F)^n$. On the other hand,

$$\bigcup\{\sigma(\tau(\omega)) : \omega \in \operatorname{supp}(\Phi_F)\} \subset S_{\tau,\Phi} \subset V_1 \cup V_2,$$

by virtue of Lemma 4.5. From the same lemma it also follows that $\omega_0 \in \operatorname{supp}(\Phi_F)$. This shows that the set

$$L = \bigcup\{\sigma(\tau(\omega)) : \omega \in \operatorname{supp}(\Phi_F) \cap \bar{V}_1\}$$

is non-empty, which contradicts Lemma 4.2, applied to T_F. Therefore $S_\tau \backslash \sigma(T) = \varnothing$.

4.7 DEFINITION. The map $\Phi_n : M_n(A) \to \mathscr{L}(X^n)$ is said to be of *finite algebraic order* if there exists an integer $m \geqslant 1$ such that from the fact that $\alpha(\omega) = 0$ for all $\omega \in \operatorname{supp}(\Phi_n)$ and a certain $\alpha \in M_n(A)$, it follows that $\Phi_n(\alpha^m) = 0$.

If $A = C^r(\Omega)$ and $\Phi_n : M_n(A) \to \mathscr{L}(X^n)$ is continuous, then for every $\beta \in M_n(A)$ which is null on $\operatorname{supp}(\Phi_n)$ together with its partial derivatives up to order r, we have $\Phi_n(\beta) = 0$. This fact is well known for scalar distributions and can be extended to vector distributions as well; an outline of the proof can be found in [**12**, Lemma IV.8.8]. This shows, in particular, that Φ_n is of finite algebraic order $\leqslant r+1$.

We can now complete Corollary 3.7 with the following statement.

4.8 COROLLARY. *If $\sigma(T) = \{z_0\}$ and the morphism $\Phi_n : M_n(A) \to \mathscr{L}(X^n)$ is of finite algebraic order, then $z_0 1_n - T$ is nilpotent.*

In particular, if T is not a multiple of the identity, then T has a proper hyperinvariant subspace.

Proof. It follows from Theorem 4.6 that $\sigma(\tau(\omega)) = \{z_0\}$ for every $\omega \in \operatorname{supp}(\Phi)$. In other words, the matrix $z_0 1_n - \tau(\omega)$ is nilpotent for each $\omega \in \operatorname{supp}(\Phi)$, that is $(z_0 1_n - \tau(\omega))^n = 0$ ($\omega \in \operatorname{supp}(\Phi)$).

Since the map Φ_n is of finite algebraic order, $\Phi_n((z_0 1_n - \tau)^{mn}) = 0$ for some integer $m \geqslant 1$, that is $z_0 1_n - T$ is nilpotent. If T is not a multiple of the identity, then $\ker(z_0 1_n - T)$ is a proper hyperinvariant subspace of T.

Note added in proof. E. Albrecht (in a private communication) has shown that [**2**, Definition 3.1, conditions (a) and (b)] imply Definition 1.1(i) and (ii) above. The converse is also true (see Remark 3.6).

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