Proposed problem

Dear Oscar, I would like to submit the following problem

Let α and β be real numbers such that for any positive integer k

$$\alpha \neq -2k, \quad \alpha \neq (2k-1), \quad \beta \neq 2k+2, \quad \beta \neq -2k-1.$$

Define $\quad a_n \doteq \prod_{k=1}^{n} \frac{\alpha + (-1)^k k}{\beta + (-1)^{k+1} + (-1)^{k+1}(k+1)}$ *and*

$$T(\alpha,\beta) \doteq \lim_{N\to\infty} \sum_{n=1}^{N} a_n, \qquad U(\alpha,\beta) \doteq \lim_{N\to\infty} \sum_{n=1}^{N} n a_n, \qquad S(\alpha,\beta) \doteq \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^n a_n$$

a) *Prove that if $-1 < \alpha + \beta < 1$, the limits exist.*
b) *Find the values of (α, β) such that $(\beta+\alpha)S(\alpha,\beta) - 3T(\alpha,\beta) - 2U(\alpha,\beta) > 0$*

Proof **a)**. Define $c_n = (-1)^n a_n$. We start with the limit defining $S(\alpha,\beta)$ and prove that asymptotically c_n has definite sign.

$$\frac{c_n}{c_{n-1}} = -\frac{\alpha + (-1)^n n}{\beta + (-1)^{n+1} + (-1)^{n+1}(n+1)}$$

For n even the ratio is $\frac{c_n}{c_{n-1}} = -\frac{\alpha + n}{\beta - 1 - (n+1)}$ and asymptotically we have

$$\frac{c_n}{c_{n-1}} = 1 + \frac{\beta+\alpha-2}{n} + O(n^{-2}), \qquad \frac{p_n}{p_{n-1}} = 1 - \frac{1}{n} - \frac{2}{n\ln n} + O(n^{-2}).$$

If $\beta + \alpha - 2 < -1$ that is $\beta + \alpha < 1$, it follows $c_n/c_{n-1} < p_n/p_{n-1}$.

For n odd the ratio is $\frac{c_n}{c_{n-1}} = -\frac{\alpha - n}{\beta + 1 + (n+1)}$ and asymptotically we have

$$\frac{c_n}{c_{n-1}} = 1 + \frac{-\beta-\alpha-2}{n} + O(n^{-2}),$$

If $-\beta - \alpha - 2 < -1$ that is $\beta + \alpha > -1$, it follows $c_n/c_{n-1} < p_n/p_{n-1}$.

Moreover since $+1$ is the leading term of the asymptotic expansions of the ratios c_n/c_{n-1}, we can affirm that c_n has definitively constant sign. Combining the two cases, we obtain the convergence of the limit $S(\alpha,\beta)$ provided that $-1 < \beta + \alpha < 1$.

Moreover we have

$$\frac{q_n}{q_{n-1}} = 1 - \frac{1}{n} - \frac{1}{n\ln n} + O(1/n^2)$$

Thus if $\beta+\alpha>1$ we would have for n even $c_n/c_{n-1}\geq q_n/q_{n-1}$ yielding divergence since we know that $\sum q_{2n}$ diverges. If $\beta+\alpha\leq -1$ the same occurs for n odd letting us to conclude that the series defining $S(\alpha,\beta)$ converges if and only if $-1<\beta+\alpha<1$.

• The convergence of the limit $T(\alpha,\beta)$ for $|\alpha+\beta|<1$ follows immediately since it is an alternating series ($a_n=(-1)^n c_n$) converging absolutely as showed although we have not found its character for $|\alpha+\beta|\geq 1$

• As for the convergence of the limit $U(\alpha,\beta)$, it defines also a Leibnitz–series. Since $b_n=(-1)^n n c_n$, it suffices to show that $|b_n|$ goes to zero monotonically. To this end we employ the theorem whose proof we omit since it is well known, just apply Cauchy–property of any convergent sequence and use monotonicity

Theorem *If* $\sum x_n$ *is a convergent series of positive, monotone terms, then* $nx_n\to 0$

Thus we have $nc_n\to 0$.

The monotonicity of nc_n amounts to show

$$\frac{c_n}{c_{n-1}}\frac{n}{n-1}\leq 1.$$

For n even we have

$$\frac{c_n}{c_{n-1}}\frac{n}{n-1}=\left(1+\frac{1}{n}\right)\left(1+\frac{\alpha+\beta-2}{n}+O(1/n^2)\right)=$$
$$=1+\frac{\beta+\alpha-1}{n}+O(n^{-2})\leq 1\qquad \text{for}\quad \beta+\alpha<1$$

For n odd we have

$$\frac{c_n}{c_{n-1}}\frac{n}{n-1}=\left(1+\frac{1}{n}\right)\left(1+\frac{-\alpha-\beta-2}{n}+O(1/n^2)\right)=$$
$$=1+\frac{-\beta-\alpha-1}{n}+O(n^{-2})\leq 1\qquad \text{for}\quad \beta+\alpha>-1$$

To prove **b)** we show that

$$(\beta+\alpha)S(\alpha,\beta)-3T(\alpha,\beta)-2U(\alpha,\beta)+\alpha c_1+b_1+2a_1=0,\qquad |\beta+\alpha|<1$$

Since $S(\alpha,\beta)$ is finite for any $|\alpha+\beta|<1$, and $\alpha c_1+b_1+a_1=\alpha-1$, **b)** follows.

To this end let's rewrite the ratio c_n/c_{n-1} as

$$c_n\beta+\alpha c_{n-1}+c_n(-1)^{n+1}+c_n(-1)^{n+1}(n+1)+(-1)^n n c_{n-1}=0$$

or

$$c_n(\beta+\alpha)+\alpha(c_{n-1}-c_n)+c_n(-1)^{n+1}+c_n(-1)^{n+1}(n+1)+(-1)^n n c_{n-1}=0$$

By summing between $n=2$ and $n=N$ and telescoping we get

$$(\beta+\alpha)\sum_{n=2}^{N} c_n + \alpha c_1 - \alpha c_N - \sum_{n=2}^{N} a_n - \sum_{n=2}^{N}(a_n+b_n) - \sum_{n=2}^{N}(a_{n-1}+b_{n-1})$$

The limit $N\to\infty$ yields

$$(\beta+\alpha)S(\alpha,\beta) - 3T(\alpha,\beta) - 2U(\alpha,\beta) + \alpha c_1 + 2a_1 + b_1 = 0$$

or

$$(\beta+\alpha)S(\alpha,\beta) - 3T(\alpha,\beta) - 2U(\alpha,\beta) = \frac{(\alpha-1)(\alpha-3)}{\beta+3}$$

The answer to **b**) is thus $\alpha<1$ or $\alpha>3$ and $\beta>-3$ and clearly $|\alpha+\beta|<1$. The second possible condition $1<\alpha<3$ and $\beta<-3$ is actually forbidden by $|\alpha+\beta|<1$.

References

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[3] Prob. num. 11473, Amer.Math.Monthly, Vol.116, num.10, 2009

[4] Prob. num. 3583, Crux Mathematicorum with Math. Mayhem, Vol.36–7, Nov.2010

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Best regards
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