Proposed problem

Dear Editors in Chief,

I would like to propose the attached problem to "Mathproblems" which was inspired by num. 11639 of Americam Mathematical Monthly, Aplil–2012 (Omran Kouba). That problem asked to evaluate $\int_0^{\pi/2} (\ln(\cos x))^2\, dx$

Evaluate $\displaystyle\int_0^{\pi/2} 4(\cos x)^2 (\ln(\cos x))^2\, dx$

Answer: $-\pi \ln 2 + \pi \ln^2 2 - \dfrac{\pi}{2} + \dfrac{\pi^3}{12}$

Proof Setting $x = \arctan t$ we get

$$\int_0^{+\infty} \frac{\left(\ln(1+t^2)\right)^2}{(1+t^2)^2} dt$$

We write

$$\begin{aligned}\frac{d}{da}\int_0^{+\infty} \frac{\left(\ln(1+at^2)\right)^2}{(1+t^2)^2} dt =& \int_0^{+\infty} \frac{2\ln(1+at^2)t^2}{(1+t^2)^2(1+at^2)} dt \\ =& 2\sqrt{a}\int_0^{+\infty} \frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2} dt = \sqrt{a}\int_{-\infty}^{+\infty} \frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2} dt\end{aligned}$$

Standard theorem on the exchange between integral and derivatives allow us to differentiate under the integral.

By introducing the complex function

$$\frac{z^2 \mathrm{Ln}(z^2+1)}{(1+z^2)(z^2+a)^2} = \frac{z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2} + \frac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2}$$

we write the two integrals

$$\sqrt{a}\int_{-\infty}^{+\infty} \frac{z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2} + \sqrt{a}\int_{-\infty}^{+\infty} \frac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2}$$

where $\mathrm{Ln}(z) = \ln(|z|) + i\vartheta$.

Let's consider the complex function $f_1(z) = \dfrac{z^2 \mathrm{Ln}(z+i)}{(1+z^2)(z^2+a)^2}$ and cut the complex plane along the set $Rez = 0$, $Imz = [-i, -\infty)$. We perform the counterclockwise integral over the two curves

$$\begin{aligned}\gamma_1(t) &= \{z \in \mathbf{C} : z = t,\ -r \le t \le r\}, \\ \gamma_2(t) &= \{z \in \mathbf{C} : z = re^{it},\ 0 \le t \le \pi\},\end{aligned}$$

$$\int_{\gamma_1 \cup \gamma_2} f_1(z)dz = 2\pi i(Resf_1(i) + Resf_1(i\sqrt{a}))$$

Moreover by defining $f_2(z) = \dfrac{z^2 \mathrm{Ln}(z-i)}{(1+z^2)(z^2+a)^2}$ and cutting the complex plane along the set $Rez = 0$, $Imz = [i, +\infty)$, we perform the clockwise integral over the two curves

$$\begin{aligned}\gamma_1(t) &= \{z \in \mathbf{C} : z = t,\ -r \le t \le r\}, \\ \gamma_2(t) &= \{z \in \mathbf{C} : z = re^{-it},\ -2\pi \le t \le -\pi\},\end{aligned}$$

$$\int_{\gamma_1\cup\gamma_2} f_2(z)dz = -2\pi i(Resf_2(-i) + Resf_2(-i\sqrt{a}))$$

Now we compute the various residues.

$$2\pi i\left(Resf_1(i) + Resf_2(-i)\right) = \frac{-2\pi i\text{Ln}(2i)}{2i(a-1)^2} - \frac{2\pi i\text{Ln}(-2i)}{2i(a-1)^2} = \frac{-\pi 2\ln 2}{(a-1)^2}$$

$$Resf_1(i\sqrt{a}) = \lim_{z\to i\sqrt{a}} \frac{d}{dz}\frac{z^2\text{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^2} =$$
$$= \lim_{z\to i\sqrt{a}}\Big(\frac{z^2}{(z+i)(1+z^2)(z+i\sqrt{a})^2} + \frac{2z\text{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^2} +$$
$$- \frac{2z^3\text{Ln}(z+i)}{(1+z^2)^2(z+i\sqrt{a})^2} - \frac{2z^2\text{Ln}(z+i)}{(1+z^2)(z+i\sqrt{a})^3}\Big) \doteq a_3 + a_4 + a_5 + a_6$$

$$Resf_2(-i\sqrt{a}) = \lim_{z\to -i\sqrt{a}} \frac{d}{dz}\frac{z^2\text{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^2} =$$
$$= \lim_{z\to -i\sqrt{a}}\Big(\frac{z^2}{(z-i)(1+z^2)(z-i\sqrt{a})^2} + \frac{2z\text{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^2} +$$
$$- \frac{2z^3\text{Ln}(z-i)}{(1+z^2)^2(z-i\sqrt{a})^2} - \frac{2z^2\text{Ln}(z-i)}{(1+z^2)(z-i\sqrt{a})^3}\Big) \doteq a_7 + a_8 + a_9 + a_{10}$$

$$a_3 - a_7 = \frac{1}{2i(1-a)((1+\sqrt{a})}$$

$$a_4 - a_8 = \frac{-i\ln(1+\sqrt{a})}{\sqrt{a}(1-a)}$$

$$a_5 - a_9 = \frac{-i\sqrt{a}\ln(1+\sqrt{a})}{(1-a)^2}$$

$$a_6 - a_{10} = \frac{i\ln(1+\sqrt{a})}{\sqrt{a}(1-a)^2}$$

and then

$$\sqrt{a}\int_{-\infty}^{+\infty}\frac{z^2\text{Ln}(z+i)}{(1+z^2)(z^2+a)^2} + \sqrt{a}\int_{-\infty}^{+\infty}\frac{z^2\text{Ln}(z-i)}{(1+z^2)(z^2+a)^2} =$$
$$\frac{-2\pi\ln 2\sqrt{a}}{(a-1)^2} + \frac{\pi\sqrt{a}}{(1-a)(1+\sqrt{a})} + \frac{\pi\ln(1+\sqrt{a})}{(1-a)} + \frac{2\pi a\ln(1+\sqrt{a})}{(1-a)^2}$$

Now we must integrate respect to the variable a (we assume $0 < a < 1$ but $a > 1$ would yield of course the same result) and we get

$$
\begin{aligned}
&\sqrt{a}\int_{-\infty}^{+\infty}\frac{\ln(1+t^2)t^2}{(1+t^2)(a+t^2)^2}dt = \\
&-2\pi\ln 2\left(\frac{1}{2(1-\sqrt{a})}-\frac{1}{2(1+\sqrt{a})}+\frac{1}{2}\ln\frac{1-\sqrt{a}}{1+\sqrt{a}}\right)+ \\
&\pi\left(-\frac{1}{1+\sqrt{a}}-\frac{3}{2}\ln(1+\sqrt{a})-\frac{1}{2}\ln(1-\sqrt{a})\right)+ \\
&\pi\left(-\frac{1}{2}\ln^2(1+\sqrt{a})-\ln 2\cdot\ln(1-\sqrt{a})+\mathrm{Li}_2\left(\frac{1-\sqrt{a}}{2}\right)\right)+ \\
&+2\pi\left(\frac{1}{4}\ln(1-\sqrt{a})+\frac{1}{4}\frac{(1+\sqrt{a})\ln(1+\sqrt{a})}{1-\sqrt{a}}+\ln 2\cdot\ln(1-\sqrt{a})-\mathrm{Li}_2\left(\frac{1-\sqrt{a}}{2}\right)+\right. \\
&\left.+\frac{1}{2}\frac{\ln(1+\sqrt{a})}{1+\sqrt{a}}+\frac{1}{2}\frac{1}{1+\sqrt{a}}+\frac{1}{2}\ln^2(1+\sqrt{a})\right)+C
\end{aligned}
$$

where as usual $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^2} = -\int_0^z \ln(1-w)\frac{dw}{w}$, $w \in \mathbf{C}\backslash\{w \in \mathbf{C}\colon Imw = 0, Rew \in [1,+\infty)\}$. The integrations are standard so we explain only the third integral

$$
\begin{aligned}
&\int\frac{\ln(1+\sqrt{a})}{a-1}da \underbrace{=}_{a=y^2} \int\left(\frac{\ln(1+y)}{1+y}dy+\int\frac{\ln(1+y)}{y-1}\right)dy = \\
&\frac{1}{2}\ln^2(1+y)+\int\left(\frac{\ln 2}{y-1}-\frac{\ln(1-\frac{1-y}{2})}{1-y}\right)dy = \qquad (y=\sqrt{a}) \\
&\frac{1}{2}\ln^2(1+\sqrt{a})+\ln 2\cdot\ln(1-\sqrt{a})-\mathrm{Li}_2(\frac{1-\sqrt{a}}{2})
\end{aligned}
$$

For $a = 0$ we have

$$0 = -\pi\mathrm{Li}_2(\frac{1}{2})+C = 0 \implies C = \frac{\pi^3}{12}-\frac{\pi}{2}\ln^2$$

Now we select the terms diverging for $a \to 1^-$ and obtain

$$
\begin{aligned}
&-2\pi\ln 2\left(\frac{1}{2}\ln(1-\sqrt{a})+\frac{1}{2(1-\sqrt{a})}\right)-\frac{\pi}{2}\ln(1-\sqrt{a})-\pi\ln 2\cdot\ln(1-\sqrt{a})+ \\
&+2\pi\left(\frac{1}{4}\ln(1-\sqrt{a})+\frac{1}{4}\frac{(1+\sqrt{a})\ln(1+\sqrt{a})}{1-\sqrt{a}}+\ln 2\cdot\ln(1-\sqrt{a})\right) \to \frac{-\pi}{2}\ln 2-\frac{\pi}{2}
\end{aligned}
$$

The terms not diverging for $a \to 1^-$ contribute

$$-\frac{\pi}{2}\ln 2+\frac{3}{2}\ln^2 2$$

and finally we get

$$-\frac{\pi}{2}\ln 2+\frac{3}{2}\ln^2 2-\frac{\pi}{2}\ln 2-\frac{\pi}{2}+\frac{\pi^3}{12}-\frac{\pi}{2}\ln^2 2 = -\pi\ln 2+\pi\ln^2 2-\frac{\pi}{2}+\frac{\pi^3}{12}$$